A decomposition of general premium principles into risk and deviation

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Motivation

Starting point:

- The axiomatization of premium principles is one of the most classic topics in the field of actuarial mathematics.
- So far, premium principles are usually written as functionals of the probability distribution of losses.
- Implicitly, thus, a probabilistic model is assumed.
- Model uncertainty or Knightian uncertainty is by now widely recognized and crucial for insurance mathematics (Solvency II).

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Aims:

- Axiomatization of premium principles in an ex-ante “probability-free” setting.
- Consistency with existing axiomatizations.

Insurance premia

- The basic ad-hoc approach to premia: expected loss (fair premium) plus some safety loading.
  Simple example: variance principle

$$H(X) = \mathbb{E}_{\mathbb{P}}(X) + \frac{\theta}{2}\mathrm{var}_{\mathbb{P}}(X).$$

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- Aim: Provide an axiomatization of premium principles under Knightian uncertainty and without monotonicity, cf. Filipović & Kupper (2007).
- Main result: Premium principle = risk measure + deviation measure

Setup

Throughout, we consider

- a measurable space $(\Omega, \mathcal{F})$,
- the space $\mathrm{B}_b = \mathrm{B}_b(\Omega, \mathcal{F})$ of all bounded measurable functions $\Omega \to \mathbb{R}$,
- a set $C \subset \mathrm{B}_b$ of insurance claims with $0 \in C$ and $X + m \in C$ for all $X \in C$ and $m \in \mathbb{R}$.

A map $R\colon \mathrm{B}_b \to \mathbb{R}$ is called a *risk measure*, cf. Föllmer and Schied (2004),

- $R(0) = 0$ and $R(X + m) = R(X) + m$ for all $X \in \mathrm{B}_b$ and $m \in \mathbb{R}$,
- $R(X) \leq R(Y)$ for all $X, Y \in \mathrm{B}_b$ with $X \leq Y$.

A *deviation measure*, cf. Rockafellar-Uryasev (2013), is a map $D\colon C \to \mathbb{R}$ with

- $D(X + m) = D(X)$ for all $X \in C$ and $m \in \mathbb{R}$,
- $D(0) = 0$ and $D(X) \geq 0$ for all $X \in C$.

Premium principles, risk and deviation measures

Definition

We say that a map $H\colon C \to \mathbb{R}$ is a *premium principle* if

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Some comments:

- The definition of a premium principle implies that $H(m) = m$, for all $m \in \mathbb{R}$, leading to the common assumption of *no unjustified risk loading*.
- Typically, insurance principles are only defined for nonnegative claims $X \geq 0$ (resembling losses).
- The condition $H(X) \geq 0$, for all $X \in C$ with $X \geq 0$, is a minimal requirement for a sensible notion of a premium principle.

Decomposition theorem

Theorem

A map $H\colon C \to \mathbb{R}$ *is a premium principle if and only if*

$$H(X) = R(X) + D(X) \quad \text{for all } X \in C,$$

where $R\colon \mathrm{B}_b \to \mathbb{R}$ *is a risk measure and* $D\colon C \to \mathbb{R}$ *is a deviation measure.*

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Remarks:

- The monetary risk measure generalizes the expected loss.
- The deviation measure generalizes variance or other measures of fluctuation.
- The decomposition needs not be unique.

The "maximal" decomposition

Theorem

Let $H\colon C \to \mathbb{R}$ *be a premium principle. Define*

$$R_{\mathrm{Max}}(X) := \inf\left\{H(X_0)\,\middle|\,X_0 \in C,\ X_0 \geq X\right\}, \quad \text{for } X \in \mathrm{B}_b,$$
$$D_{\mathrm{Min}}(X) := H(X) - R_{\mathrm{Max}}(X), \quad \text{for } X \in C.$$

Then, $R_{\mathrm{Max}}\colon \mathrm{B}_b \to \mathbb{R}$ *is a risk measure,* $D_{\mathrm{Min}}\colon C \to \mathbb{R}$ *is a deviation measure, and*

$$H(X) = R_{\mathrm{Max}}(X) + D_{\mathrm{Min}}(X) \quad \text{for all } X \in C.$$

For every other decomposition of the form $H = R + D$ *with a risk measure* R *and a deviation measure* D*, we have* $R \leq R_{\mathrm{Max}}$ *and* $D \geq D_{\mathrm{Min}}$*.*

Some intuition

- Notice that R_{Max} can be seen as sort of a superhedging functional in the sense that

$$R_{\mathrm{Max}}(X) = \inf\left\{m \in \mathbb{R} \,\middle|\, \exists X_0 \in C_0,\ m + X_0 \geq X\right\},$$

where $C_0 := \{X_0 \in C \mid H(X_0) = 0\}$ is the set of all claims with zero premium.

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- As R_{Max} is defined analogously to a superhedging functional, it is monotone and cash invariant, thus a risk measure
- Maximality of R_{Max}: If $H(X_0) = R(X_0) + D(X_0) \geq R(X_0)$ for all $X_0 \in C$, then for all $X \in \mathrm{B}_b$ and $X_0 \in C$ with $X_0 \geq X$, we have

$$R(X) \leq R(X_0) \leq H(X_0).$$

  Take the infimum over $X_0 \in C$ with $X_0 \geq X$ to obtain $R(X) \leq R_{\mathrm{Max}}(X)$.

Variance principle

Consider the variance principle

$$H(X) = \mathbb{E}_{\mathbb{P}}(X) + \frac{\theta}{2}\mathrm{var}_{\mathbb{P}}(X), \quad \text{for } X \in \mathrm{B}_b,$$

with a constant $\theta \geq 0$.

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- Here, $R(X) = \mathbb{E}_{\mathbb{P}}(X)$, and $D(X) = \frac{\theta}{2}\mathrm{var}_{\mathbb{P}}(X)$ is *one* possible decomposition of H into risk and deviation.

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- Here, $R(X) = \mathbb{E}_\mathbb{P}(X)$, and $D(X) = \frac{\theta}{2}\mathrm{var}_\mathbb{P}(X)$ is *one* possible decomposition of H into risk and deviation.
- However, for $\theta > 0$, this is not the "maximal" decomposition. For $\theta > 0$, the maximal risk measure R_{Max} is given by

  $$R_{\mathrm{Max}}(X) = \max_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_\mathbb{Q}(X) - \frac{1}{2\theta} G(\mathbb{Q}|\mathbb{P}),$$

  where $\mathcal{P}$ consists of all probability measures $\mathbb{Q}$, which are absolutely continuous w.r.t. $\mathbb{P}$ and satisfy

  $$G(\mathbb{Q}|\mathbb{P}) := \mathrm{var}_\mathbb{P}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) < \infty.$$

  The map G is the *Gini concentration index*, see Maccheroni et al. (2006,2009).

Mean absolute deviation principle

Consider the premium principle

$$H(X) = \mathbb{E}_{\mathbb{P}}(X) + \theta \mathbb{E}_{\mathbb{P}}\big(|X - \mathbb{E}_{\mathbb{P}}(X)|\big), \quad \text{for } X \in \mathrm{B}_b,$$

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with a constant $\theta \geq 0$.

- Here, $R(X) = \mathbb{E}_{\mathbb{P}}(X)$, and $D(X) = \theta \mathbb{E}_{\mathbb{P}}\big(|X - \mathbb{E}_{\mathbb{P}}(X)|\big)$ is a decomposition of H into risk and deviation.
- For $0 \leq \theta \leq \frac{1}{2}$, this is a monotone premium principle. Hence, $R_{\mathrm{Max}}(X) = H(X)$ and $D_{\mathrm{Min}}(X) = 0$ for all $X \in \mathrm{B}_b$.
- More generally, for $p \in [1, \infty)$, one can consider the class of so-called L^p-deviation principles, cf. Filipović & Kupper (2007), given by

  $$H(X) = \mathbb{E}_{\mathbb{P}}(X) + \theta \|X - \mathbb{E}_{\mathbb{P}}(X)\|_p, \quad \text{for } X \in \mathrm{B}_b.$$

  The special case $p = 2$ leads to the standard deviation principle.

Ambiguity indices

Consider a nonempty set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$, and fix a baseline model $\mathbb{P} \in \mathcal{P}$, interpreted as the most plausible model.

Castagnoli et al. (2002) introduce the ambiguity index

$$\mathrm{Amb}_{\mathcal{P}}(X) := \frac{1}{2} \sup_{\mathbb{Q},\mathbb{Q}' \in \mathcal{P}} \Big(\mathbb{E}_{\mathbb{Q}}(X) - \mathbb{E}_{\mathbb{Q}'}(X) \Big), \quad \text{for } X \in \mathrm{B}_b.$$

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Then, $\mathrm{Amb}_{\mathcal{P}}$ as an uncertainty surcharge together with the variance as a risk surcharge leads to the premium principle

$$H(X) = \mathbb{E}_{\mathbb{P}}(X) + \frac{\theta}{2} \mathrm{var}_{\mathbb{P}}(X) + \gamma \mathrm{Amb}_{\mathcal{P}}(X), \quad \text{for } X \in \mathrm{B}_b, \tag{1}$$

with $\gamma, \theta \geq 0$.

Absolute deviation principle

Consider the *absolute deviation principle*, cf. Rolski et al. (1999),

$$H(X) = \mathbb{E}_{\mathbb{P}}(X) + \gamma \mathbb{E}_{\mathbb{P}}\left(\left|X - \mathbb{P}_X^{-1}\left(\tfrac{1}{2}\right)\right|\right), \quad \text{for } X \in \mathrm{B}_b,$$

as a modification of the standard deviation principle. Again, $R(X) = \mathbb{E}_{\mathbb{P}}(X)$ and $D(X) = \gamma \mathbb{E}_{\mathbb{P}}\big(\big|X - \mathbb{P}_X^{-1}(1/2)\big|\big)$ is *one* possible decomposition into risk and deviation.

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$$D(X) = \frac{\gamma}{2} \sup_{\mathbb{Q}, \mathbb{Q}' \in \mathcal{P}} \Big(\mathbb{E}_{\mathbb{Q}}(X) - \mathbb{E}_{\mathbb{Q}'}(X)\Big) = \gamma \mathrm{Amb}_{\mathcal{P}}(X)$$

is (up to a constant) an ambiguity index, where $\mathcal{P}$ consists of all probability measures $\mathbb{Q} \ll \mathbb{P}$ whose density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is $\mathbb{P}$-a.s. bounded by 2.

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$$R_{\mathrm{Max}}(X) = \mathrm{AV@R}_\mathbb{P}^{\frac{1}{1+\gamma}}(X), \quad \text{for } X \in \mathrm{B}_b,$$

where $\mathrm{AV@R}_\mathbb{P}^{\frac{1}{1+\gamma}}$ is the *average value at risk* or *expected shortfall* at level $\frac{1}{1+\gamma}$.

Duality for convex premium principles

We denote the set of all finitely additive probability measures on $(\Omega, \mathcal{F})$ by ba_+^1.

Theorem

Assume that C is a linear space and that H is convex. Then, R_{Max} is convex and

$$R_{\mathrm{Max}}(X) = \max_{\mathbb{P} \in \mathrm{ba}_+^1} \mathbb{E}_{\mathbb{P}}(X) - H^*(\mathbb{P}).$$

Here, H^* is the convex dual of H, given by

$$H^*(\mathbb{P}) = \sup_{X \in C} \mathbb{E}_{\mathbb{P}}(X) - H(X) \in [0, \infty], \quad \textit{for } \mathbb{P} \in \mathrm{ba}_+^1.$$

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Remarks:

- $H^*(\mathbb{P})$ represents the confidence that the insurer puts in a particular model $\mathbb{P} \in \mathrm{ba}^1_+$.
- If H is continuous from above and $\mathbb{P} \in \mathrm{ba}^1_+$, the convex dual $H^*(\mathbb{P})$ is finite if and only if $\mathbb{P}$ is countably additive.

Sublinear premium principles

Assume that C is a linear space and that H is convex. We call

$$\mathcal{P} := \left\{\mathbb{P} \in \mathrm{ba}^1_+ \,\middle|\, H^*(\mathbb{P}) < \infty\right\}$$

the set of all plausible models.

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Theorem

Assume that C is a linear space and that H is a sublinear. Then, $\mathbb{P} \in \mathrm{ba}^1_+$ is a plausible model if and only if H incorporates a safety loading for $\mathbb{P}$, i.e.,

$$H(X) \geq \mathbb{E}_{\mathbb{P}}(X) \quad \text{for all } X \in C.$$

In particular, R_{Max} is a coherent risk measure, cf. Artzner et al. (1999), and

$$R_{\mathrm{Max}}(X) = \max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(X) \quad \text{for all } X \in \mathrm{B}_b.$$

Conclusion

- We provide a “model-free” approach to insurance premia.
- Insurance premia are the sum of a risk measure and a deviation measure.
- Existing models can be subsumed under our framework.
- Duality theory for convex premia.
- Consistency with asset pricing and securization.
- Law invariance carries over from the premium principle H to the maximal risk measure R_{Max}. Under mild conditions, H carries a loading.
- Extension of the model to unbounded claims.

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Thank you very much for your attention!