Pricing Interest Rate Derivatives
under
Volatility Uncertainty

Julian Hölzermann

Center for Mathematical Economics, Bielefeld University
Collaborative Research Center 1283

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Outline

Volatility in Mathematical Finance

- Traditional models (Black and Scholes, 1973; Merton, 1973):

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

- Market data: The volatility is neither constant nor deterministic.
- Stochastic volatility models (Heston, 1993):

$$dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dB_t,$$
$$d\nu_t = \theta(\tilde{\mu} - \nu_t)dt + \sigma\sqrt{\nu_t} d\tilde{B}_t.$$

- Epstein and Ji (2013): This approach leads to model uncertainty.
- Alternative: We consider a family of probability measures.
- Goal: Models are robust with respect to the volatility.

Mathematical Framework

- Let $(\Omega, \mathcal{F}, P_0)$ be the Wiener space.
- The canonical process $(B_t)_t$ is a Brownian motion under P_0.
- For each $[\underline{\sigma}, \overline{\sigma}]$-valued, $(\mathcal{F}_t)_t$-adapted process $\sigma = (\sigma_t)_t$,

$$P^\sigma := P_0 \circ \left(\int_0^\cdot \sigma_t dB_t \right)^{-1}.$$

- The collection of all such measures is denoted by $\mathcal{P}$.
- We define the sublinear expectation

$$\hat{\mathbb{E}}[\cdot] := \sup_{P \in \mathcal{P}} \mathbb{E}_P[\cdot].$$

- $\hat{\mathbb{E}}$ corresponds to the G-expectation on $L_G^1(\Omega)$, i.e., $(B_t)_t$ is a G-Brownian motion under $\hat{\mathbb{E}}$ (Denis, Hu, and Peng, 2011).

G-Brownian Motion

- G-expectation: $u(t,x) := \hat{\mathbb{E}}[\varphi(x+B_t)]$ is the viscosity solution to

$$\partial_t u + G(\partial^2_{xx} u) = 0, \quad u(0,x) = \varphi(x),$$

where $G(a) := \frac{1}{2}\sup_{\sigma \in [\underline{\sigma},\overline{\sigma}]}(\sigma^2 a)$.

- Then $(\langle B \rangle_t)_t$ is an uncertain process, satisfying

$$\overline{\sigma}^2 t \geq \langle B \rangle_t \geq \underline{\sigma}^2 t \quad \textit{quasi-surely}.$$

- The space of admissible random variables is given by

$$L^p_G(\Omega) = \Big\{ \xi \in L^p(\Omega) \,\Big|\, \xi \text{ has a } \textit{quasi-continuous} \text{ version}, \\ \lim_{n\to\infty} \hat{\mathbb{E}}[|\xi|^p 1_{\{|\xi|>n\}}] = 0 \Big\}.$$

- The "same" holds for the space of admissible processes, $M^p_G(0,T)$.

Related Literature

Mathematical approaches to volatility uncertainty:

- Denis and Martini (2006), Peng (2007, 2008, 2019),...

Volatility uncertainty in asset markets:

- Avellaneda, Levy, and Parás (1995), Lyons (1995),...

Volatility uncertainty in interest rate models:

- Fadina, Neufeld, and Schmidt (2019), Hölzermann (2021, 2022)

Outline

The Heath-Jarrow-Morton (HJM) Methodology

- For $t \leq T \leq \bar{T}$, the forward rate satisfies (quasi-surely)

$$f_t(T) = f_0(T) + \int_0^t \alpha_u(T)du + \int_0^t \beta_u(T)dB_u + \int_0^t \gamma_u(T)d\langle B\rangle_u.$$

- Zero-coupon bonds and the money-market account are defined by

$$P_t(T) := \exp\Big(-\int_t^T f_t(s)ds\Big),$$
$$M_t := \exp\Big(\int_0^t r_s ds\Big),$$

  respectively, where the short rate is given by $r_t := f_t(t)$.
- We restrict to the discounted bonds, $\tilde{P}_t(T) := M_t^{-1} P_t(T)$.

Underlying Assumptions

Assumption (No-Arbitrage).

We assume that

$$\alpha(T) = 0, \quad \gamma(T) = \beta(T)b(T),$$

where $b(T)$ is defined by $b_t(T) := \int_t^T \beta_t(s)ds$.

Assumption (Regularity).

We assume that $\beta : [0, \bar{T}] \times [0, \bar{T}] \to \mathbb{R}$ is continuous.

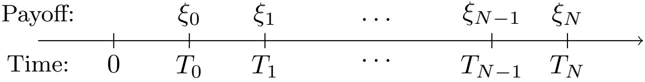
Assumption (Hedging).

We assume that $b(T)$ is strictly positive (invertible).

Outline

Additional Contract

- We consider an additional contract with the following payoff.



- Then the discounted payoff is given by

$$\tilde{X} := \sum_{i=0}^{N} M_{T_i}^{-1} \xi_i.$$

No-Arbitrage Pricing

- In the classical case, no-arbitrage prices are given by

$$M_t \hat{\mathbb{E}}_t[\tilde{X}] = M_t \mathbb{E}_{P_0}[\tilde{X} | \mathcal{F}_t].$$

- In the presence of volatility uncertainty, $\hat{\mathbb{E}}$ is sublinear, implying

$$\hat{\mathbb{E}}[\tilde{X}] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[\tilde{X}] \geq \inf_{P \in \mathcal{P}} \mathbb{E}_P[\tilde{X}] = -\hat{\mathbb{E}}[-\tilde{X}].$$

- No-arbitrage prices are determined by $M_t \hat{\mathbb{E}}_t[\tilde{X}]$ and $-M_t \hat{\mathbb{E}}_t[-\tilde{X}]$, i.e., there is (possibly) a range of no-arbitrage prices. Details

Outline

Pricing Single Cashflows

- In order to price a (single) cashflow ξ, we need to compute

$$M_t \hat{\mathbb{E}}_t[M_T^{-1}\xi], \quad -M_t \hat{\mathbb{E}}_t[-M_T^{-1}\xi].$$

- We define the T-forward sublinear expectation, $\hat{\mathbb{E}}^T$, such that

$$M_t \hat{\mathbb{E}}_t[M_T^{-1}\xi] = P_t(T)\hat{\mathbb{E}}_t^T[\xi].$$

- $\hat{\mathbb{E}}^T$ corresponds to the expectation under the forward measure.
▸ Details
- We obtain further results for pricing typical cashflows.

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Pricing Bond Options

- Most cashflows correspond to bond options, that is,

$$\xi := \varphi\Big(\big(P_{T_0}(T_i)\big)_{i=1}^{N}\Big)$$

for a function $\varphi : \mathbb{R}^N \to \mathbb{R}$.

- The pricing reduces to solving a nonlinear PDE if φ satisfies

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \tag{1}$$

for a positive integer m and a constant $C > 0$. Details

- If φ is also convex, then we have

$$\hat{\mathbb{E}}_t^{T_0}[\xi] = \mathbb{E}_{P^{\overline{\sigma}}}^{T_0}[\xi|\mathcal{F}_t], \quad -\hat{\mathbb{E}}_t^{T_0}[-\xi] = \mathbb{E}_{P^{\underline{\sigma}}}^{T_0}[\xi|\mathcal{F}_t].$$

Details

Outline

Pricing a Stream of Cashflows

- We cannot price the cashflows separately, since

$$\hat{\mathbb{E}}[\tilde{X}] \leq \sum_{i=0}^{N} \hat{\mathbb{E}}[M_{T_i}^{-1}\xi_i], \quad -\hat{\mathbb{E}}[-\tilde{X}] \geq \sum_{i=0}^{N} -\hat{\mathbb{E}}[-M_{T_i}^{-1}\xi_i].$$

- If ξ_i, for all i, satisfies $\hat{\mathbb{E}}_t^{T_i}[\xi_i] = -\hat{\mathbb{E}}_t^{T_i}[-\xi_i]$, then

$$M_t\hat{\mathbb{E}}_t[\tilde{X}] = \sum_{i=0}^{N} P_t(T_i)\hat{\mathbb{E}}_t^{T_i}[\xi_i] = -M_t\hat{\mathbb{E}}_t[-\tilde{X}].$$

- In general, $\hat{\mathbb{E}}[\tilde{X}] = \tilde{Y}_0^+$ and $-\hat{\mathbb{E}}[-\tilde{X}] = -\tilde{Y}_0^-$, where

$$\tilde{Y}_i^{\pm} := P_{T_{i-1}}(T_i)\hat{\mathbb{E}}_{T_{i-1}}^{T_i}[\pm\xi_i + \tilde{Y}_{i+1}^{\pm}]$$

for $i = 0, 1, ..., N$ and $\tilde{Y}_{N+1}^{\pm} := 0$.

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A Stream of Bond Options

- Most interest rate derivatives consist of bond options, i.e.,

$$\xi_i := \varphi_i\big(P_{T_i}(T_{i+1})\big)$$

  for a function $\varphi_i : \mathbb{R} \to \mathbb{R}$ for $i = 0, 1, ..., N-1$ and $\xi_N := 0$.
- The pricing reduces to solving nonlinear PDEs if for all i,

$$|\varphi_i(x) - \varphi_i(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \tag{2}$$

  for a positive integer m and a constant $C > 0$. Details
- If φ_i is also convex for all i, then we have

$$\tilde{Y}_0^+ = \sum_{i=0}^{N-1} P_0(T_i)\mathbb{E}_{P^{\overline{\sigma}}}^{T_i}[\xi_i], \quad -\tilde{Y}_0^- = \sum_{i=0}^{N-1} P_0(T_i)\mathbb{E}_{P^{\underline{\sigma}}}^{T_i}[\xi_i]$$

Details

Outline

Common Interest Rate Derivatives

- Fixed-coupon bonds, floating rate notes, and interest rate swaps have simple (linear) payoffs. ▸ Details
- Swaptions correspond to convex bond options. ▸ Details
- Caps and floors and typical in-arrears contracts can be written as a stream of convex bond options. ▸ Details
- Other contracts can be priced by solving nonlinear PDEs.

Practical Relevance

- The theoretical results offer an explanation for empirical findings (unspanned stochastic volatility).
- The robust pricing procedure allows for stress testing by considering different levels of uncertainty.
- One can infer the level of uncertainty from observable spreads to price other instruments.
- Alternatively, one can infer the level of uncertainty from the historical volatility as confidence intervals.

Thank you for your attention.

Supplemental Material

Supplemental Material

Symmetric and Asymmetric Contracts

- We consider two contracts with discounted payoffs $\tilde{X}^S$ and $\tilde{X}^A$.
- $\tilde{X}^S$ has a symmetric payoff and $\tilde{X}^A$ has an asymmetric payoff:

$$\hat{\mathbb{E}}[\tilde{X}^S] = -\hat{\mathbb{E}}[-\tilde{X}^S], \quad \hat{\mathbb{E}}[\tilde{X}^A] > -\hat{\mathbb{E}}[-\tilde{X}^A].$$

- We assume that the discounted prices, $(\tilde{X}_t^S)_t$ and $\tilde{X}_0^A$, satisfy

$$\tilde{X}_t^S = \hat{\mathbb{E}}_t[\tilde{X}^S], \quad \hat{\mathbb{E}}[\tilde{X}^A] > \tilde{X}_0^A > -\hat{\mathbb{E}}[-\tilde{X}^A].$$

Extended Bond Market

Definition (Market Strategy).

A market strategy $(\pi, \pi^S, \pi^A, \tau)$ consists of processes $\pi = (\pi_t^1, ..., \pi_t^n)_t$ and $\pi^S = (\pi_t^S)_t$, a constant $\pi^A \in \mathbb{R}$, and a vector $\tau \in [0, \bar{T}]^n$ for some $n \in \mathbb{N}$. The portfolio value at terminal time is given by

$$\tilde{v}(\pi, \pi^S, \pi^A, \tau) := \sum_{i=1}^{n} \int_0^{\tau_i} \pi_t^i d\tilde{P}_t(\tau_i) + \int_0^{\bar{T}} \pi_t^S d\tilde{X}_t^S + \pi^A(\tilde{X}^A - \tilde{X}_0^A).$$

Definition (Arbitrage).

A market strategy $(\pi, \pi^S, \pi^A, \tau)$ is called arbitrage strategy if

$$\tilde{v}(\pi, \pi^S, \pi^A, \tau) \geq 0 \quad \text{quasi-surely,}$$
$$P\big(\tilde{v}(\pi, \pi^S, \pi^A, \tau) > 0\big) > 0 \quad \text{for at least one } P \in \mathcal{P}.$$

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Absence of Arbitrage

Proposition (No-Arbitrage).

The extended bond market is arbitrage-free.

Remark (Hedging).

The pricing-hedging duality under volatility uncertainty (Vorbrink, 2014) shows that other pricing procedures lead to arbitrage.

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Supplemental Material

Risk-Neutral Valuation

Pricing Single Cashflows

Pricing a Stream of Cashflows

Common Interest Rate Derivatives

Forward Sublinear Expectation

Definition (Forward Sublinear Expectation)

For $\xi \in L_G^p(\Omega_T)$ with $p > 1$ and $T \leq \bar{T}$, we define the T-forward sublinear expectation $\hat{\mathbb{E}}^T$ by $\hat{\mathbb{E}}_t^T[\xi] := Y_t^{T,\xi}$, where $Y^{T,\xi}$ solves

$$Y_t^{T,\xi} = \xi - \int_t^T b_u(T) Z_u d\langle B \rangle_u - \int_t^T Z_u dB_u - (K_T - K_t).$$

- Then $\hat{\mathbb{E}}^T$ is a time consistent sublinear expectation and

$$Y_t^{T,\xi} = (X_t^T)^{-1} \hat{\mathbb{E}}_t[X_T^T \xi],$$

where $X_t^T := \frac{\tilde{P}_t(T)}{P_0(T)}$ (Hu, Ji, Peng, and Song, 2014).

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Pricing Bond Options

Proposition (Bond Options).
If φ satisfies (1), then for $t \le T_0$,

$$\hat{\mathbb{E}}_t^{T_0}[\pm\xi] = u^{\pm}\Big(t, \big(\tfrac{P_t(T_i)}{P_t(T_0)}\big)_{i=1}^{N}\Big),$$

where $u^{\pm} : [0, T_0] \times \mathbb{R}^N \to \mathbb{R}$ is the unique viscosity solution to

$$\partial_t u + G\big(\sigma(t,x) D_{xx}^2 u \, \sigma(t,x)'\big) = 0, \quad u(T_0, x) = \pm\varphi(x)$$

where $x = (x_i)_{i=1}^N$ and $\sigma(t,x) := ((b_t(T_i) - b_t(T_0))x_i)_{i=1}^N$.

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Pricing Convex Bond Options

Proposition (Convex Bond Options).

If φ is convex and satisfies (1), then for $t \leq T_0$,

$$\hat{\mathbb{E}}_t^{T_0}[\xi] = u^{\overline{\sigma}}\Big(t, \big(\tfrac{P_t(T_i)}{P_t(T_0)}\big)_{i=1}^N\Big), \quad -\hat{\mathbb{E}}_t^{T_0}[-\xi] = u^{\underline{\sigma}}\Big(t, \big(\tfrac{P_t(T_i)}{P_t(T_0)}\big)_{i=1}^N\Big),$$

where $u^\sigma : [0, T_0] \times \mathbb{R}^N \to \mathbb{R}$, for $\sigma > 0$, is defined by

$$u^\sigma(t, x) := \mathbb{E}_{P_0}\big[\varphi\big((X_{T_0}^i)_{i=1}^N\big)\big]$$

and the process $X^i = (X_s^i)_{t \leq s \leq T_0}$, for $i = 1, ..., N$, is given by

$$X_s^i = x_i - \int_t^s \big(b_u(T_i) - b_u(T_0)\big) X_u^i \sigma dB_u.$$

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Pricing a Stream of Bond Options

Proposition (Stream of Bond Options).

If φ_i satisfies (2) for all $i = 0, 1, ..., N-1$, then

$$\tilde{Y}_0^{\pm} = P_0(T_0) u_1^{\pm}\Big(0, \tfrac{P_0(T_1)}{P_0(T_0)}, \big(\tfrac{P_0(T_k)}{P_0(T_{k-1})}, \tfrac{P_0(T_{k+1})}{P_0(T_k)}\big)_{k=1}^{N-1}\Big),$$

where $u_i^{\pm} : [0, T_{i-1}] \times \mathbb{R}^{2(N-i)+1} \to \mathbb{R}$, for $i = 1, ..., N$, is the unique viscosity solution to the nonlinear PDE

$$\partial_t u + G\big(H_i(t, x_i, D_{x_i} u, D^2_{x_i x_i} u)\big) = 0, \quad u(T_{i-1}, x_i) = f_i^{\pm}(x_i),$$

where $x_i := (\hat{x}_i, (\tilde{x}_k, \hat{x}_k)_{k=i+1}^N)$ for $i = 1, ..., N-1$ and $x_N := \hat{x}_N$ and

$$f_i^{\pm}(x_i) := \pm\varphi_{i-1}(\hat{x}_i) + \tilde{x}_{i+1} u_{i+1}(T_{i-1}, x_{i+1})$$

for $i = 1, ..., N-1$ and $f_N(x_N) := \pm\varphi_{N-1}(\hat{x}_N)$.

Pricing a Stream of Bond Options

$$H_i(t, x_i, D_{x_i} u, D^2_{x_i x_i} u) := \sigma_i(t, x_i)' D^2_{x_i x_i} u \, \sigma_i(t, x_i) + 2 D_{x_i} u \, \mu_i(t, x_i),$$

$$\sigma_i(t, x_i) := \operatorname{diag}(x_i) \Big(b_t(T_i) - b_t(T_{i-1}),$$
$$\big(b_t(T_k) - b_t(T_{k-1}), b_t(T_{k+1}) - b_t(T_k) \big)_{k=i}^{N-1} \Big)',$$

$$\mu_i(t, x_i) := \operatorname{diag}\big(\sigma_i(t, x_i)\big) \Big(0, \big(b_t(T_{i-1}) - b_t(T_{k-1}),$$
$$b_t(T_{i-1}) - b_t(T_k) \big)_{k=i}^{N-1} \Big)'$$

for $i = 1, ..., N-1$ and

$$H_N(t, x_N, D_{x_N} u, D^2_{x_N x_N} u) := \big(b_t(T_N) - b_t(T_{N-1}) \big)^2 x_N^2 \partial^2_{x_N x_N} u.$$

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Pricing a Stream of Convex Bond Options

Proposition (Stream of Convex Bond Options).

If φ_i is convex and satisfies (2) for all $i = 1, ..., N$, then

$$\bar{Y}_0^+ = \sum_{i=1}^{N} P_0(T_{i-1}) u_i^{\overline{\sigma}}\big(0, \tfrac{P_0(T_i)}{P_0(T_{i-1})}\big),$$

$$-\bar{Y}_0^- = \sum_{i=1}^{N} P_0(T_{i-1}) u_i^{\underline{\sigma}}\big(0, \tfrac{P_0(T_i)}{P_0(T_{i-1})}\big),$$

where $u_i^\sigma : [0, T_{i-1}] \times \mathbb{R} \to \mathbb{R}$, for $i = 1, ..., N$ and $\sigma > 0$, is defined by

$$u_i^\sigma(t, \hat{x}_i) := \mathbb{E}_{P_0}[\varphi_i(X^i_{T_{i-1}})]$$

and the process $X^i = (X^i_s)_{t \le s \le T_{i-1}}$ is given by

$$X^i_s = \hat{x}_i - \int_t^s \big(b_u(T_i) - b_u(T_{i-1})\big) X^i_u \sigma dB_u.$$

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Supplemental Material

Pricing Fixed Coupon Bonds

- The cashflows of a fixed coupon bond are given by

$$\xi_i = 1_{\{N\}}(i) + 1_{\{1,\dots,N\}}(i)(T_i - T_{i-1})K \tag{3}$$

for $i = 0, 1, ..., N$.

Proposition (Fixed Coupon Bonds).

Let ξ_i be given by (3) for $i = 0, 1, ..., N$. Then for $t \leq T_0$,

$$M_t \hat{\mathbb{E}}_t[\tilde{X}] = P_t(T_N) + \sum_{i=1}^{N} P_t(T_i)(T_i - T_{i-1})K = -M_t \hat{\mathbb{E}}_t[-\tilde{X}].$$

Pricing Floating Rate Notes

- The cashflows of a floating rate note are given by

$$\xi_i = 1_{\{N\}}(i) + 1_{\{1,...,N\}}(i)(T_i - T_{i-1})L_{T_{i-1}}(T_i) \tag{4}$$

for $i = 0, 1, ..., N$, where

$$L_{T_{i-1}}(T_i) := \tfrac{1}{T_i - T_{i-1}}(\tfrac{1}{P_t(T_i)} - 1).$$

Proposition (Floating Rate Notes).

Let ξ_i be given by (4) for $i = 0, 1, ..., N$. Then for $t \leq T_0$,

$$M_t\hat{\mathbb{E}}_t[\tilde{X}] = P_t(T_0) = -M_t\hat{\mathbb{E}}_t[-\tilde{X}].$$

Pricing Interest Rate Swaps

- The cashflows of an interest rate swap are given by

$$\xi_i = 1_{\{1,...,N\}}(i)(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big) \tag{5}$$

  for $i = 0, 1, ..., N$.

Proposition (Interest Rate Swaps).
Let ξ_i be given by (5) for $i = 0, 1, ..., N$. Then for $t \leq T_0$,

$$M_t \hat{\mathbb{E}}_t[\tilde{X}] = P_t(T_0) - P_t(T_N) - \sum_{i=1}^{N} P_t(T_i)(T_i - T_{i-1})K = -M_t \hat{\mathbb{E}}_t[-\tilde{X}].$$

Swaptions

- The payoff is determined by Proposition (Interest Rate Swaps).
- The cashflows of a swaption are given by

$$\xi_i = 1_{\{0\}}(i)\left(1 - P_{T_0}(T_n) - \sum_{j=1}^{N} P_{T_0}(T_j)(T_j - T_{j-1})K\right)^+ \tag{6}$$

for $i = 0, 1, ..., N$.

Pricing Swaptions

Theorem (Swaptions).

Let ξ_i be given by (6) for $i = 0, 1, ..., N$. Then it holds

$$\hat{\mathbb{E}}[\tilde{X}] = P_0(T_0)u^{\overline{\sigma}}\Big(0, \big(\tfrac{P_0(T_i)}{P_0(T_0)}\big)_{i=1}^{N}\Big),$$
$$-\hat{\mathbb{E}}[-\tilde{X}] = P_0(T_0)u^{\underline{\sigma}}\Big(0, \big(\tfrac{P_0(T_i)}{P_0(T_0)}\big)_{i=1}^{N}\Big),$$

where the function $u^\sigma : [0, T_0] \times \mathbb{R}^N \to \mathbb{R}$, for $\sigma > 0$, is defined by

$$u^\sigma(t, x) := \mathbb{E}_{P_0}\left[\left(1 - X_{T_0}^N - \sum_{i=1}^{N} X_{T_0}^i (T_i - T_{i-1})K\right)^+\right]$$

and the process $X^i = (X_s^i)_{t \leq s \leq T_0}$, for all $i = 1, ..., N$, is given by

$$X_s^i = x_i - \int_t^s \sigma_u(T_0, T_i)X_u^i \sigma dB_u.$$

Caps and Floors

- The cashflows of a Cap are called caplets and are given by

$$\xi_i = 1_{\{1,...,N\}}(i)(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+ \tag{7}$$

  for $i = 0, 1, ..., N$.
- The cashflows of a Floor are called floorlets and are given by

$$\xi_i = 1_{\{1,...,N\}}(i)(T_i - T_{i-1})\big(K - L_{T_{i-1}}(T_i)\big)^+ \tag{8}$$

  for $i = 0, 1, ..., N$.

Pricing Caps

Theorem (Caps).

Let ξ_i be given by (7) for $i = 0, 1, ..., N$. Then it holds

$$\hat{\mathbb{E}}[\tilde{X}] = \sum_{i=1}^{N} P_0(T_{i-1}) u_i^{\overline{\sigma}}\big(0, \tfrac{P_0(T_i)}{P_0(T_{i-1})}\big),$$

$$-\hat{\mathbb{E}}[-\tilde{X}] = \sum_{i=1}^{N} P_0(T_{i-1}) u_i^{\underline{\sigma}}\big(0, \tfrac{P_0(T_i)}{P_0(T_{i-1})}\big),$$

where $u_i^{\sigma} : [0, T_{i-1}] \times \mathbb{R} \to \mathbb{R}$, for $i = 1, ..., N$ and $\sigma > 0$, is defined by

$$u_i^{\sigma}(t, x_i) := \tfrac{1}{K_i} \mathbb{E}_{P_0}[(K_i - X^i_{T_{i-1}})^+]$$

for $K_i := \frac{1}{1+(T_i - T_{i-1})K}$ and the process $X^i = (X^i_s)_{t \leq s \leq T_{i-1}}$ is given by

$$X^i_s = x_i - \int_t^s \sigma_u(T_{i-1}, T_i) X^i_u \sigma dB_u.$$

Pricing Floors

Theorem (Floors).

Let ξ_i be given by (8) for $i = 0, 1, ..., N$. Then it holds

$$\hat{\mathbb{E}}[\tilde{X}] = \sum_{i=1}^{N} P_0(T_{i-1}) u_i^{\overline{\sigma}}\big(0, \tfrac{P_0(T_i)}{P_0(T_{i-1})}\big),$$

$$-\hat{\mathbb{E}}[-\tilde{X}] = \sum_{i=1}^{N} P_0(T_{i-1}) u_i^{\underline{\sigma}}\big(0, \tfrac{P_0(T_i)}{P_0(T_{i-1})}\big),$$

where $u_i^\sigma : [0, T_{i-1}] \times \mathbb{R} \to \mathbb{R}$, for $i = 1, ..., N$ and $\sigma > 0$, is defined by

$$u_i^\sigma(t, x_i) := \tfrac{1}{K_i} \mathbb{E}_P[(X^i_{T_{i-1}} - K_i)^+]$$

and K_i and the process X^i are given as in Theorem (Caps).

In-Arrears Contracts

- Now the floating rate is reset each time the contract pays off.
- As a representative contract, we consider in-arrears swaps.
- Other in-arrears contracts can be priced in a similar way.
- The cashflows of an in-arrears swap are given by

$$\xi_i = 1_{\{0,1,\dots,N-1\}}(i)(T_{i+1} - T_i)\big(L_{T_i}(T_{i+1}) - K\big) \tag{9}$$

for $i = 0, 1, ..., N$.

Pricing In-Arrears Swaps

Theorem (In-Arrears Swaps).

Let ξ_i be given by (9) for $i = 0, 1, ..., N$. Then it holds

$$\hat{\mathbb{E}}[\tilde{X}] = \sum_{i=1}^{N} P_0(T_i) u_i^{\overline{\sigma}}\big(0, \tfrac{P_0(T_{i-1})}{P_0(T_i)}\big),$$

$$-\hat{\mathbb{E}}[-\tilde{X}] = \sum_{i=1}^{N} P_0(T_i) u_i^{\underline{\sigma}}\big(0, \tfrac{P_0(T_{i-1})}{P_0(T_i)}\big),$$

where $u_i^\sigma : [0, T_{i-1}] \times \mathbb{R} \to \mathbb{R}$, for $i = 1, ..., N$ and $\sigma > 0$, is defined by

$$u_i^\sigma(t, x_i) := \mathbb{E}_{P_0}[X^i_{T_{i-1}}(X^i_{T_{i-1}} - \tfrac{1}{K_i})]$$

for K_i as in Theorem (Caps) and $X^i = (X^i_s)_{t \leq s \leq T_{i-1}}$ is given by

$$X^i_s = x_i - \int_t^s \sigma_u(T_i, T_{i-1}) X^i_u \sigma dB_u.$$

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