Law Invariance vs. Heterogeneity in Risk Sharing

Risk Measures and Uncertainty in Insurance

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Overview

1. The risk sharing problem
2. Uniqueness of reference measures
3. Optimal risk sharing under heterogeneous beliefs

The risk sharing problem

Optimal risk sharing

- Consider classical risk sharing problem:

$$\sum_{i=1}^{n} \rho_i(X_i) \to \min \quad \text{subject to } \mathbf{X} = (X_1, \dots, X_n) \in \mathbb{A}_X$$

- Agents $i \in [n] := \{1, ..., n\}$
- $X, X_1, \dots, X_n \in L^\infty$ over (atomless) probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- Set of allocations of $X \in L^\infty$: $\mathbb{A}_X := \{\mathbf{X} = (X_1, \dots, X_n) \mid \sum_{i=1}^n X_i = X\}$
- **Important:** Do not impose feasibility constraints!
- $\rho_i \colon L^\infty \to \mathbb{R}$ monetary risk measures
  - **Normalisation:** $\rho_i(0) = 0$
  - **Monotonicity:** $X \leq Y \implies \rho_i(X) \leq \rho_i(Y)$
  - **Cash-additivity:** $X \in L^\infty,\ m \in \mathbb{R} \implies \rho_i(X + m) = \rho_i(X) + m$
- Want to find optimal allocation(s) $\mathbf{X}^{opt} \in \mathbb{A}_X$ solving risk sharing problem

Optimal risk sharing

- Agents entertain individual beliefs: $\forall\, i \in [n]\, \exists$ probability measure $\mathbb{Q}_i$ on $(\Omega, \mathcal{F})$ s.t.

$$\mathbb{Q}_i \circ X^{-1} = \mathbb{Q}_i \circ Y^{-1} \quad \Longrightarrow \quad \rho_i(X) = \rho_i(Y) \qquad (\star)$$

- **Well known:** $\exists$ comonotone optimal allocations (under mild additional conditions on ρ_i's) in homogeneous situation: Can choose the $\mathbb{Q}_i$'s with property $(\star)$ such that

$$\forall\, i, j \in [n]: \quad \mathbb{Q}_i = \mathbb{Q}_j$$

- **Problem:** $\mathbb{Q}_i$'s can be heterogeneous, i.e., for some $i, j \in [n]$, $\mathbb{Q}_i \neq \mathbb{Q}_j$!
- **Typically:** $\mathbb{Q}_i \not\approx \mathbb{Q}_j$ for some $i, j \implies \nexists$ optimal allocations
- Natural question:

When can we find $\mathbb{Q}^\star$ having property $(\star)$ for all $i \in [n]$?

- ... i.e., when does heterogeneity resolve?

Uniqueness of reference measures

Reference measures have to be equivalent

- $(\Omega, \mathcal{F})$ mb. space
- $\mathcal{X}$ bounded measurable random variables $f\colon \Omega \to \mathbb{R}$
- $\mathcal{X}_0 \subset \mathcal{X}$ simple random variables
- Given functional $\varphi\colon \mathcal{X}_{(0)} \to \mathbb{R}$, $\mathfrak{Ref}(\varphi)$ is set of all reference probability measures $\mathbb{P}$, i.e.,

$$\mathbb{P} \circ f^{-1} = \mathbb{P} \circ g^{-1} \quad \implies \quad \varphi(f) = \varphi(g)$$

Proposition (L., '22)

Suppose $\mathbb{Q} \not\approx \mathbb{P}$ and $\mathbb{P}$ atomless. For any functional $\varphi\colon \mathcal{X} \to \mathbb{R}$,

$$\mathbb{P}, \mathbb{Q} \in \mathfrak{Ref}(\varphi) \quad \iff \quad \varphi \text{ is constant}$$

Reference measures have to be equivalent

- For $f \in \mathcal{X}$,

$$M(f) := \inf\{x \in \mathbb{R} \mid \mathbb{P}(f \leq x) = 1\}$$
$$m(f) := \sup\{x \in \mathbb{R} \mid \mathbb{P}(f \leq x) = 0\}$$

- Clear: $\mathfrak{Ref}(M) = \mathfrak{Ref}(m) = \{\mathbb{Q} \approx \mathbb{P}\}$

Theorem (L., '22)

- $\mathbb{P} \approx \mathbb{Q}$ *nonatomic probability measures on* $\mathcal{F}$
- $\varphi \colon \mathcal{X} \to \mathbb{R}$ *is l.s.c., monotone, satisfies* $\mathbb{P}, \mathbb{Q} \in \mathfrak{Ref}(\varphi)$

Then one of the following alternatives holds:

1. $\mathbb{P} = \mathbb{Q}$.
2. $\varphi = G \circ (m, M)$ *for a unique function* $G\colon \mathbb{R}^2 \to \mathbb{R}$.

Consequences for risk measures

Corollary

For a monetary risk measure $\rho\colon \mathcal{X} \to \mathbb{R}$ *with Fatou property, one of the following alternatives holds:*

1. $\rho = M$
2. $|\mathfrak{Ref}(\rho)| \leq 1$

Consequence:

Heterogeneity in risk sharing above typically does not resolve!

$\implies$ Need to impose conditions on $\mathbb{Q}_i$'s, ρ_i's, and their interplay!

Key lemma

Proofs heavily rely on Lyapunov's Convexity Theorem and:

Lemma (L., '22)

- $\varphi\colon \mathcal{X}_0 \to \mathbb{R}$
- $\mathbb{P} \approx \mathbb{Q}$ *atomless*, $\mathbb{P} \neq \mathbb{Q}$, *and* $\mathbb{P}, \mathbb{Q} \in \mathfrak{Ref}(\varphi)$

For all $f, g \in \mathcal{X}_0$*:*

$$\{x \in \mathbb{R} \mid \mathbb{P}(f = x) > 0\} = \{x \in \mathbb{R} \mid \mathbb{P}(g = x) > 0\} \quad \Longrightarrow \quad \varphi(f) = \varphi(g)$$

Illustration for Bernoulli-distributed f, g ...

Optimal risk sharing under heterogeneous beliefs

Consistent risk measures

- $\mathbb{Q} \ll \mathbb{P}$
- For $X, Y \in L^\infty$:

$$X \preceq_{ssd}^{\mathbb{Q}} Y \quad \iff \quad \forall\, v\colon \mathbb{R} \to \mathbb{R} \text{ convex \& nondecreasing:} \quad \mathbb{E}[v(X)] \le \mathbb{E}[v(Y)]$$

- ρ $\mathbb{Q}$-consistent (cf. Mao & Wang, '20) if

$$X \preceq_{ssd}^{\mathbb{Q}} Y \quad \Longrightarrow \quad \rho(X) \le \rho(Y)$$

- Mao & Wang, '20: ρ consistent $\iff$ ρ dilatation monotone:

$$\text{For all sub-}\sigma\text{-algebras } \mathcal{G} \subset \mathcal{F},\ \rho\big(\mathbb{E}_{\mathbb{Q}}[X|\mathcal{G}]\big) \le \rho(X)$$

- **Example:** ρ convex and $\mathbb{Q}$-law invariant risk measure $\Longrightarrow$ ρ $\mathbb{Q}$-consistent
- **Important:** ρ $\mathbb{Q}$-consistent $\not\Longrightarrow$ ρ convex!

Compatible dual elements

- Acceptance set $\mathcal{A}_\rho := \{X \in L^\infty \mid \rho(X) \leq 0\}$
- Asymptotic cone of $\mathcal{A}_\rho$:

$$\mathcal{A}_\rho^\infty = \{\lim_n t_n Y_n \mid (t_n)_{n\in\mathbb{N}} \subset (0,\infty) \text{ null sequence}, \ (Y_n)_{n\in\mathbb{N}} \subset \mathcal{A}_\rho\}$$

Definition

1. Probability density $Z^* \in L^1_+$ compatible if
   a. $\rho^*(Z^*) = \sup_{Y\in\mathcal{A}_\rho} \mathbb{E}_\mathbb{P}[Z^*Y] < \infty$
   b. $\forall\, U \in \mathcal{A}_\rho^\infty$:
   $$\mathbb{E}_\mathbb{P}[Z^*U] = 0 \quad \Longrightarrow \quad U = 0.$$

   $\mathbf{C}(\rho)$ set of all compatible elements.
2. ρ admissible if $\mathbf{C}(\rho) \neq \emptyset$.

Existence of compatible dual elements/admissibility

Remark

$\mathbb{Q}$-consistent risk measure ρ admissible $\implies \mathbb{Q} \approx \mathbb{P}$

Proposition (L., '22)

$\mathbb{Q} \approx \mathbb{P}$, $\rho\colon L^\infty \to \mathbb{R}$ $\mathbb{Q}$-consistent risk measure. The following are equivalent:

1. *ρ is admissible*
2. $\frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathbf{C}(\rho)$
3. $|\mathrm{dom}(\rho^*)| \geq 2$
4. $|\mathrm{dom}(\rho^*) \cap L^1| \geq 2$

They all imply:

$$\nexists \beta > 0 : \quad \rho \leq \mathbb{E}_{\mathbb{Q}}[\cdot] + \beta$$

Admissibility: The star-shaped case

Admissibility even milder for star-shaped risk measures:

Proposition (L. & Munari, '22)

Suppose a $\mathbb{Q}$-consistent risk measure is star shaped (cf. Castagnoli et al., '21):

$$\forall s \in [0,1] \, \forall Y \in \mathcal{A}_\rho : \quad sY \in \mathcal{A}_\rho.$$

Then

$$\rho \text{ admissible} \iff \rho \neq \mathbb{E}_\mathbb{Q}[\cdot]$$

Existence theorem

$(\mathbb{Q}_1, \dots, \mathbb{Q}_n)$ vector of reference measures equivalent to $\mathbb{P}$

Assumption (SIM)

$\forall\, i \in [n]: \quad \frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}}$ is a simple function.

$\rho_i : L^\infty \to \mathbb{R}$ $\mathbb{Q}_i$-consistent risk measures, $i \in [n]$

Assumption (COMP)

$\forall\, 1 \le j \le n-1\ \exists\, Z_j \in \mathbf{C}(\rho_j)$ s.t. $Z_j \in \bigcap_{i\in[n]} \mathrm{dom}(\rho_i^*)$

Theorem (L., '22)

- $(\mathbb{Q}_1, \dots, \mathbb{Q}_n)$ *checks assumption* (SIM)
- $(\rho_i)_{i\in[n]}$ *check assumption* (COMP)

Then, for all $X \in L^\infty$ *there is* $\mathbf{X}^{opt} \in \mathbb{A}_X$ *solving the risk sharing problem.*

More on assumption (SIM)

Theorem (L., '22)

$\rho_i : L^\infty \to \mathbb{R}$ $\mathbb{Q}_i$-consistent risk measures. Then the following are equivalent:

1. $(\mathbb{Q}_1, \dots, \mathbb{Q}_n)$ *can be chosen to satisfy* (SIM).
2. *There is a common finite σ-algebra $\mathcal{H} \subset \mathcal{F}$ s.t. for all $i \in [n]$, sub-σ-algebras $\mathcal{G} \subset \mathcal{F}$, and for all $X \in L^\infty$:*

$$\mathcal{H} \subset \mathcal{G} \quad \Longrightarrow \quad \rho_i\big(\mathbb{E}_{\mathbb{P}}[X|\mathcal{G}]\big) \le \rho_i(X).$$

Interpretation: If agents have enough information to decide if certain shocks occur or not, they can agree on $\mathbb{P}$. Else, they retract to $\mathbb{Q}_i$.

More on assumption (SIM)

- Assumption (SIM) not exotic:
  - Present in Cambou & Filipović ’17
  - Abstraction of the setting of Marshall ’92 (first investigation of risk sharing under belief heterogeneity)
  - $\implies$ ρ_i’s are special case of scenario-based risk measures characterised in Wang & Ziegel ’21
- Interpretable as “random variable translation” of Anscombe-Aumann framework

More on assumption (COMP)

- All ρ_i's admissible and $\rho_i^*\left(\frac{d\mathbb{Q}_j}{d\mathbb{P}}\right) < \infty$, $i, j \in [n] \implies$ (COMP)
- ρ_n does not have to be admissible, e.g., $\rho_n = \mathbb{E}_{\mathbb{Q}_n}[\cdot]$ possible
- In that case: $\text{dom}(\rho_n^*) = \{\frac{d\mathbb{Q}_n}{d\mathbb{P}}\}$, hence:

$$(\text{COMP}) \iff \frac{d\mathbb{Q}_n}{d\mathbb{P}} \in \bigcap_{i=1}^{n-1} \mathbf{C}(\rho_i)$$

- **Open question:** How is (COMP) related to condition $\bigcap_{i=1}^{n} \text{dom}(\rho_i^*) \neq \emptyset$? (Farkas' Lemma)

Proposition (L., unpublished)

- *All ρ_i's admissible*
- *Marshall's setting: $\exists$ event $A \in \mathcal{F}$ such that $\frac{d\mathbb{Q}_i}{d\mathbb{P}} = q_i \mathbf{1}_A + r_i \mathbf{1}_{A^c}$*

Then:

$$\bigcap_{i=1}^{n} \text{dom}(\rho_i^*) \neq \emptyset \iff \bigcap_{i=1}^{n} \mathbf{C}(\rho_i) \neq \emptyset \iff (\text{COMP})$$

Bibliography

Thank you for your attention!

Cambou, M., D. Filipović (2017), Model uncertainty and scenario aggregation. *Mathematical Finance* 27(2):534–567.

Castagnoli, E., G. Cattelan, F. Maccheroni, C. Tebaldi, and R. Wang (2021), Star-shaped risk measures. Preprint.

Liebrich, F.-B. (2022), Are reference measures of law-invariant functionals unique?. Preprint.

Liebrich, F.-B. (2022), Risk sharing under heterogeneous beliefs without convexity. Preprint.

Liebrich, F.-B., and C. Munari (2022), Law-invariant functionals that collapse to the mean: Beyond convexity. Forthcoming in *Mathematics & Financial Economics*.

Mao, T., and R. Wang (2020), Risk aversion in regulatory capital principles. *SIAM Journal on Financial Mathematics* 11(1):169–200.

Marshall, J. M. (1992), Optimum insurance with deviant beliefs. In: *Contributions to Insurance Economics*, pp. 255–274. Kluwer.

Wang, R., J. F. Ziegel (2021), Scenario-based risk evaluation. *Finance & Stochastics* 25:725–756.