Stochastic analysis for Markov processes

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1. Markov processes: trivia.
2. Stochastic analysis for additive functionals.
3. Applications to geometry.

Markov processes

X locally compact separable metric space.
A stochastic process $Y = (Y_t)_{t\geq 0}$ is *Markov process* with state space X if (very loosely speaking !)
there is a family $(\mathbb{P}^x)_{x\in X}$ of p.m.'s on $(\Omega, \mathcal{F})$ such that

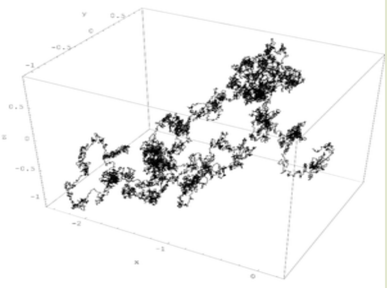
- $x \mapsto \mathbb{P}^x(Y_t \in A)$ is a Borel function for all Borel sets $A \subset X$ and all $t \geq 0$,
- with $\mathcal{F}_t := \sigma(Y_s : s \leq t)$ we have

$$\mathbb{P}^x\left[Y_{t+s} \in A | \mathcal{F}_t\right] = \mathbb{P}^{Y_t}[Y_s \in A]$$

for all $s, t \geq 0$ and $A \subset X$ Borel
('process forgets past, given present')

Example

d-dim. Brownian motion $(B_t)_{t\geq 0}$ (with varying starting points) is a Markov process with state space $\mathbb{R}^d$.



Consider suitable volume measure m on X ('speed measure').

Y is *m-symmetric* if

$$\mathbb{E}^m[f(Y_t)g(Y_0)] = \mathbb{E}^m[f(Y_0)g(Y_t)]$$

for all $t > 0$ and bounded Borel f, g.

Here $\mathbb{P}^m = \int_X \mathbb{P}^x \, m(dx)$ and $\mathbb{E}^m$ expectation w.r.t. $\mathbb{P}^m$.

There is a probability kernel $P_t(x, dy)$ such that

$$\mathbb{P}^x(Y_t \in A) = \int_A P_t(x, dy).$$

By *m*-symmetry

$$P_t f(x) := \mathbb{E}^x[f(Y_t)]$$

defines a strongly continuous *Markovian semigroup* $(P_t)_{t\geq 0}$ of symmetric operators on $L_2(X, m)$ with *generator*

$$Lf := \lim_{t\to 0} \frac{1}{t}(P_t f - f), \quad f \in dom\, L.$$

L non positive definite self-adjoint on $L_2(X, m)$.

Example

For d-dim. Brownian motion $(B_t)_{t\geq 0}$ have

$$P_t f(x) = \int_{\mathbb{R}^d} p(t, x-y) f(y) dy$$

with

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x|^2}{2t}\right),$$

symmetric on $L_2(\mathbb{R}^d)$. Generator is

$$\frac{1}{2}\Delta = \frac{1}{2}\sum_i \frac{\partial^2 f}{\partial x_i^2}$$

(Friedrichs extension $(\frac{1}{2}\Delta, H^2(\mathbb{R}^d))$.

Connect with martingale theory:

Theorem

(Doob, Kakutani, Dynkin)
If $f \in$ dom L (and nice) then for q.e. $x \in X$

$$f(Y_t) - f(Y_0) - \int_0^t (Lf)(Y_s)ds$$

is a $\mathbb{P}^x$-martingale (w.r.t. 'natural filtration').

Example

If $(B_t)_{t\geq 0}$ Brownian motion on $\mathbb{R}^d$ and f is C^2 then Itô formula holds,

$$f(B_t) - f(B_0) - \frac{1}{2}\int_0^t (\Delta f)(B_s)ds = \sum_i \int_0^t \frac{\partial f}{\partial x_i}(B_s)dB_s^i.$$

If h harmonic then $h(B_t)$ forms martingale for any $\mathbb{P}^x$.

Energy and additive functionals

Relax hypotheses by using *energy forms*. Consider the unique symmetric positive definite bilinear form $(Q, dom\, Q)$ on $L_2(X,m)$ such that

$$Q(f,g) := -(Lf,g)_{L_2(X,m)},\quad f \in dom\, L,\ g \in dom\, Q.$$

(Dirichlet form).

Examples

$(B_t)_{t\geq 0}$ d-dim Brownian motion, then

$$Q(f,g) = \frac{1}{2}\int_{\mathbb{R}^d} \nabla f \nabla g\, dx,$$

$f,g \in H^1(\mathbb{R}^d) \supsetneqq dom\,\Delta = H^2(\mathbb{R}^d)$.

Theorem

(Fukushima)
If $f \in$ dom Q and nice, then

$$M_t^{[f]} = f(Y_t) - f(Y_0) - N_t^{[f]} \quad (\textit{uniquely})$$

where $(M_t^{[f]})_{t\geq 0}$ a continuous 'martingale additive functional' of Y of finite energy, and $(N_t^{[f]})_{t\geq 0}$ an continuous 'additive functional' of Y of zero energy.

This is sth. like a semimartingale decomposition.
Problem: family $(\mathbb{P}^x)_{x\in X}$ of p.m.'s.

Additive functionals:

Examples

If B Brownian motion on $\mathbb{R}^d$ then

$$A_t = \int_0^t g(B_s)ds$$

is a continuous additive functional of B, additivity property is

$$\int_0^{t+s} g(B_r)dr = \int_0^s g(B_r)dr + \int_0^t g(B_{r+s})dr \quad a.s.$$

Space of continuous AF's of zero energy ('analytically nice'):

$$\mathcal{N}_c := \{N : N \text{ finite continuous AF of } Y \text{ with } \mathbf{e}(N) = 0 \text{ and such that } \mathbb{E}_x(|N_t|) < +\infty \text{ q.e. for each } t > 0\},$$

where

$$\mathbf{e}(M) = \lim_{t\to 0} \frac{1}{2t}\mathbb{E}^m(M_t^2).$$

('finite quadratic variation part')

Space of martingale additive functionals of finite energy ('probabilistically nice'):

$$\mathring{\mathcal{M}} = \{M : M \text{ AF of } Y \text{ with } \mathbf{e}(M) < \infty \text{ such that} \\ \text{for q.e. } x \in X,\ \mathbb{E}^x(M_t^2) < \infty \text{ and } \mathbb{E}^x(M_t) = 0,\ t > 0\Big\},$$

The space $(\mathring{\mathcal{M}}, \mathbf{e})$ is Hilbert.

To each $M \in \mathring{\mathcal{M}}$ assign *energy measure* $\mu_{\langle M \rangle}$ ... *Revuz measure* of its sharp bracket $\langle M \rangle$:

For q.e. $x \in X$, $M^2 - \langle M \rangle$ is a $\mathbb{P}^x$-martingale (*Doob-Meyer version*).

For $h \geq 0$ Borel and $f \in dom\, Q$ (nice) have

$$\mathbb{E}_{hm}\left(\int_0^t f(Y_s) d\left\langle M \right\rangle_s\right) = \int_0^t \int_X \mathbb{E}_x h(Y_s) f(x)\, \mu_{\langle M \rangle}(dx) ds, \quad t > 0.$$

('Fubini with trading strange scaling (time change) between time and space')

Examples

If B is BM on $\mathbb{R}^d$ and $\mu(dx) = g(x)dx$ then μ is Revuz measure of

$$A_t = \int_0^t g(B_s)ds.$$

Examples

If B is BM on $\mathbb{R}$ and δ_y Dirac at y, then up to a constant, δ_y is the Revuz measure of Brownian local time $L(t,y)$,

$$\int_0^t \mathbf{1}_E(B_s)ds = 2\int_E L(t,y)dy, \quad E \subset \mathbb{R} \text{ Borel}.$$

Stochastic integrals

For $f \in L_2(X, \mu_{\langle M \rangle})$ can define the *stochastic integral* $f \bullet M \in \mathring{\mathcal{M}}$ of f with respect to $M \in \mathring{\mathcal{M}}$ by

$$\mathbf{e}(f \bullet M, N) = \frac{1}{2} \int_X f d\mu_{\langle M,N \rangle}, \quad N \in \mathring{\mathcal{M}}.$$

The integral $f \bullet M$ is an L_2-limit of sums

$$\sum_i f(Y_{t_i})(M_{t_{i+1}} - M_{t_i})$$

(Itô type). Not known how to use 'general integrands'.

Example

If $B = (B^1, \ldots, B^d)$ is the d-dim. Brownian motion, seen as Markov process, then

$$\mathring{\mathcal{M}} = \left\{ \sum_{i=1}^{d} f_i \bullet B^i : f_i \in L_2(\mathbb{R}^d), i = 1, \ldots, d \right\}$$

and

$$\mathbf{e}\left(\sum_{i=1}^{d} f_i \bullet B^i \right) = \frac{1}{2} \sum_{i=1}^{d} \|f_i\|^2_{L_2(\mathbb{R}^d)} .$$

Definition

(Motoo/Watanabe, Hino)

The *martingale dimension* of $(Y_t)_{t\geq 0}$ is the smallest natural number p such that there exist $M^{(1)}, ..., M^{(p)} \in \mathring{\mathcal{M}}$ allowing the representation

$$M_t = \sum_{i=1}^{p} (h_i \bullet M^{(i)})_t, \quad t > 0, \ \mathbb{P}^x\text{-a.e. for q.e. } x \in X,$$

with suitable $h_i \in L_2(X, \mu_{\langle M^{(i)} \rangle})$ for every $M \in \mathring{\mathcal{M}}$. If no such p exists, we define the martingale dimension to be infinity.

Examples

Martingale dimension of d-dim. Brownian motion is d.

'Additive functional version of martingale representation'. Exact relation between the formulations is not yet understood.

$(B_t)_{t\geq 0}$ one dim. Brownian motion on a p. space $(\Omega, \mathcal{F}, \mathbb{P})$,
$\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$, $\mathcal{F}_\infty := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$.

Lemma

For all random variables $F \in L_2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ there exists a unique predictable process H which is in L_2 and satisfies

$$F = \mathbb{E}F + \int_0^\infty H_s dB_s \;\; \mathbb{P} - a.s.$$

('space of stochastic integrals is large').

Now based on d-dim. Brownian motion $(B^1, \ldots, B^d)$:

Theorem

Let $(M_t)_{t\geq 0}$ be an d-dim. L_2-integrable $(\mathcal{F}_t)_{t\geq 0}$-martingale. Then there are a constant C and predictable processes H^i, $i = 1, \ldots, d$ in L_2 such that

$$M_t = C + \sum_{i=1}^{d} \int_0^t H_s^i dB_s^i \ a.s.$$

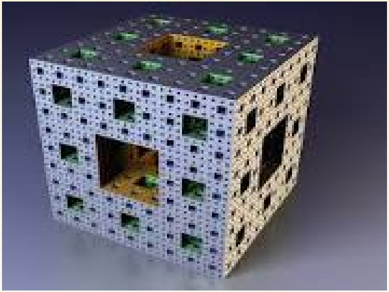
Think of d as 'degree of freedom' for 'heat particle'.

Geometry of rough spaces

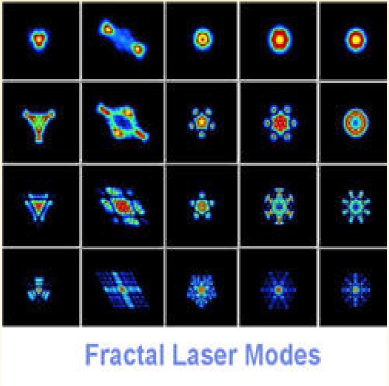
Lungs.

Artificial fern.

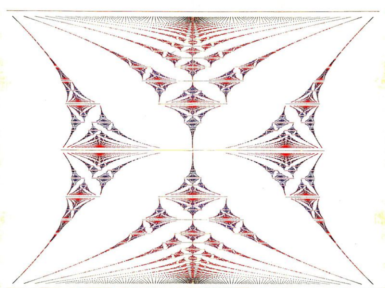
Sponge.



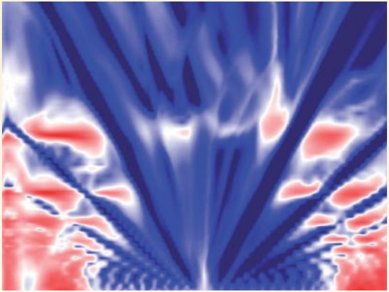
Menger sponge.



Refraction patterns in Laser optics.



Hofstadter Butterfly (energy spectra, magnetic field on square lattice).



Hofstadter Butterfly observed on Graphene structure.

Interest:

Geometry, analysis, stochastic processes, math. physics

on rough spaces

(no rectifiability or curvature dimension bounds, 'fractals')

- Study microstructure ... complement homogenization.

Problem:

- Classical differentiation unavailable.
- Diffusion processes exist and can be used.
- Dimension issues (topological, Hausdorff, martingale, ...)

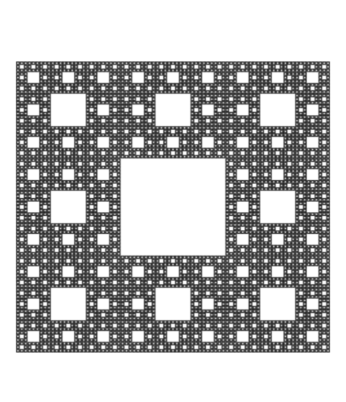
Credo:

- 'Diffusion does not need smoothness.'

Some applications / motivations:

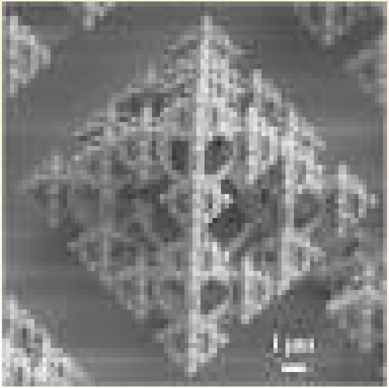
- Waveguides for optical high frequency signals.
- Fractal antennas
- 'Fractal structuring': Separating layers between polymer films.
- Ultra light weight materials.
- Networks at different scales.
- 'Fractal microcavities'.
- Nanotubes.
- Geometric learning and pattern recognition.
- Space-time scaling in models for quantum gravity.

Sierpinski carpet



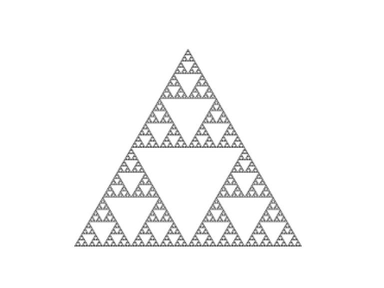
Barlow/Bass ’89 (existence of Brownian motion),
Barlow/Bass/Kumagai/Teplyaev ’10 (uniqueness).

Honeycomb structure (stable ultra light weight material, US patent).



Pyramid structure with huge surface.

Sierpinski gasket *SG*



Barlow/Perkins ’88, Kigami ’89 (ex. and uniqueness of Brownian motion).

- $d_H = \frac{\log 3}{\log 2}$ *Hausdorff dimension* of *SG*
- $d_w = \frac{\log 5}{\log 2} > 2$ *walk index*,

$$c_1 t^{2/d_w} \leq \mathbb{E}^x |Y_t - Y_0|^2 \leq c_2 t^{2/d_w}$$

('particle moves slower than normal')

- $d_S = 2d_H/d_w < 2$ *spectral dimension*, short time exponent
- diffusion is sub-Gaussian, i.e.

$$p(t,x,y) \sim ct^{-d_s/2} \exp\left(-c \left(\frac{d_R(x,y)^{d_w}}{t}\right)^{1/(1-d_w)}\right).$$

- log-scale fluctuations in on-diagonal behaviour $t^{d_s/2} p(t,x,x)$ (*Kajino*)

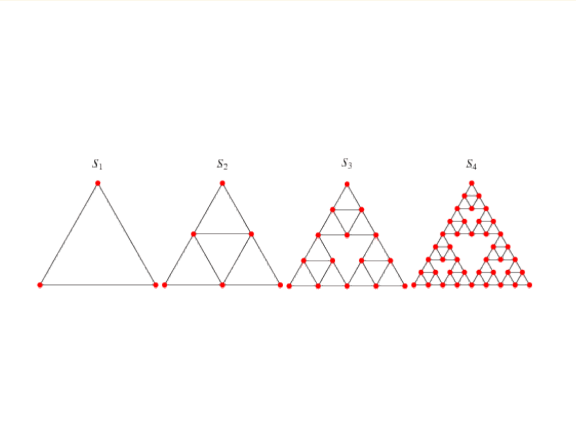
Construct energy functional

$$\mathcal{E}(f) = \text{''} \int |f'(x)|^2 dx \text{''}$$

as the (rescaled) limit

$$\mathcal{E}(f) = \lim_n \left(\frac{5}{3}\right)^n \sum_{p,q \in V_n,\ q \sim p} (f(p) - f(q))^2$$

of discrete energy forms on approximating graphs (*Kigami '89, '93, Kusuoka '93*)


S_1
S_2
S_3
S_4

Get a space $\mathcal{F}$ of functions on SG with finite energy, i.e.

$$\mathcal{E} : \mathcal{F} \to [0, +\infty).$$

Simultaneously get a (resistance) metric d_R on SG so that

$$\mathcal{F} \subset C(SG)$$

(Sobolev embedding theorem).

Construction is purely combinatorial.

With 'any reasonable' finite Borel measure μ on *SG* the pair $(\mathcal{E},\mathcal{F})$ becomes a *Dirichlet form* on $L_2(SG,\mu)$.

Integration by parts also yields Laplacian (generator) Δ_μ for (speed) measure μ,

$$\mathcal{E}(f,g)=-\int_{SG} f\Delta_\mu g\,d\mu.$$

(*'Second derivative on fractals'*)

Fukushima's theory yields associated diffusion (*'Brownian motion on SG'*)

Analytic counterpart

Recall $P_t f(x) = \mathbb{E}^x[f(Y_t)]$, where $(Y_t)_{t\geq 0}$ diffusion on X. Then

$$Q(f,g) := \lim_{t\to 0} \frac{1}{2t}(f - P_t f, g)_{L_2(X,m)}.$$

$(Q, dom\ Q)$ strongly local regular symmetric Dirichlet form on $L_2(X,m)$.

The core $\mathcal{C} := C_c(X) \cap dom\ Q$ is an algebra.

On $\mathcal{C} \otimes \mathcal{C}$ consider the nonnegative def. symmetric bilinear form

$$\langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} := Q(bda, c) + Q(a, bdc) - Q(ac, bd).$$

Factoring out zero seminorm elements yields *Hilbert space* $\mathcal{H}$ *of differential* 1*-forms / vector fields*.

(Mokobodzki, LeJan, Nakao, Lyons/Zhang, Eberle, Cipriani/Sauvageot, etc.)

Close to algebra and NCG.

- $\mathcal{H}$ can be given module structure
- the operator $\partial : \mathcal{C} \to \mathcal{H}$ with

$$\partial f := f \otimes \mathbf{1}$$

is a bounded derivation
(∂f is $\mathcal{H}$-class universal derivation / Kähler differential of f).

Examples

M compact Riemannian manifold, $(Y_t)_{t\geq 0}$ Brownian motion on M,

$$Q(f,g) = \int_M \langle df, dg\rangle_{T^*M}\, dvol, \quad f,g \in H^1(M),$$

$dvol$ Riemannian volume, d exterior derivative. Then
$\mathcal{H} = L_2(M, dvol, T^*M)$ and ∂ coincides with d.

Theorem

There are a suitable measure ν and suitable Hilbert spaces $\mathcal{H}_x$ such that $\mathcal{H}$ may be written as direct integral,

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x \nu(dx).$$

The fibers $\mathcal{H}_x$ may be regarded as (co)tangent spaces at x to X.

Examples

Manifold case: $\mathcal{H}_x \cong T_x M$ for *dvol*-a.e. x.

Theorem

The spaces $\mathcal{H}$ and $\mathring{\mathcal{M}}$ are isometrically isomorphic under $g\partial f \mapsto g \bullet M^{[f]}$.

(Nakao: manifolds, H./Teplyaev/Röckner: fractals)

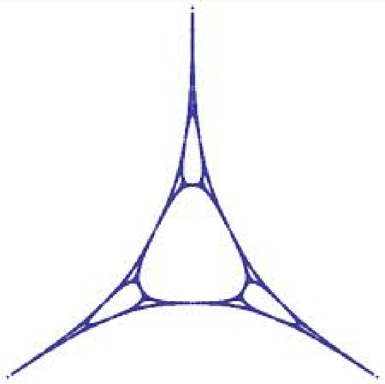
Theorem

(Hino)
The martingale dimension of $(Y_t)_{t\geq 0}$ equals $\operatorname{ess\,sup}_{x\in X} \dim \mathcal{H}_x$.

('maximal degree of freedom for diffusing particle is essentially given by maximal tangent space dimension')

Examples

The (harmonic) Sierpinski gasket has tangent spaces of dimension one a.e.



Play with this correspondence:

- gradient ∂f ... martingale AF $M^{[f]}$
- divergence $\partial^* v$ ... Revuz measure (density) of Nakao functional
- vector field $g\partial f$ ... stochastic integral $g \bullet M^{[f]}$

etc.

Some results

Theorem

$$P_t^{a,v}f(x) := \mathbb{E}_x[e^{i\int_{Y([0,t])} a - \int_0^t v(Y_s)ds} f(Y_t)]$$

with Stratonovich integral

$$\int_{Y([0,t])} a := \Theta(a) + \int_0^t (\partial^* a)(Y_s)ds$$

is semigroup for magnetic Hamiltonian

$$H^{a,v} = -(\partial + ia)^*(\partial + ia) + v.$$

$\Theta : \mathcal{H} \to \mathring{\mathcal{M}}$ *Nakao isomorphism.*

('Feynman-Kac-Itô', H.'14)

Theorem

Hodge theorem in topo dim one:

$$'\mathcal{H} = \mathit{Im}\,\partial \oplus \text{(locally) harmonic forms}'.$$

(Ionescu/Rogers/Teplyaev '11, H./Teplyaev '12)

Theorem

'Harmonic forms give Čech cohomology'

(Ionescu/Rogers/Teplyaev '11, H./Teplyaev '12)

Theorem

'If topo dimension is one (but Hausdorff dim 10 000), Navier-Stokes system reduces to Euler equation'.

(H./Teplyaev '12)

Theorem

If topo dimension is one, then either martingale dimension is one or exterior derivation is not closable.

(H./Teplyaev '15) (unprecedented in diff. geo)

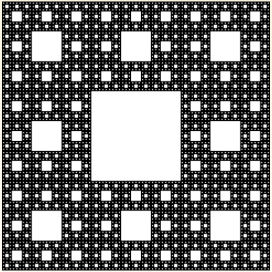


FIGURE 1. $S_{1/3}$

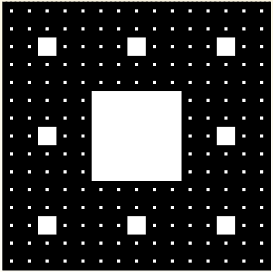


FIGURE 2. $S_{(1/3,1/5,1/7,\ldots)}$

THANK YOU.