Assessing financial model risk

and an application to electricity prices

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Outline

- ”Model risk in risk management and the Basel multiplier
- ”Absolute” measure of model risk
- ”Relative” measure of model risk
- ”Local” measure of model risk
- Application to electricity prices

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- **Model risk in risk management and the Basel multiplier**
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Model risk

- Model risk is a recurrent theme in Economics and Finance.
- It broadly refers to the (bad) impact the choice of a **wrong model** can have.
- It is difficult to define or measure. Also, it is questionable what the correct model is (and whether it exists)
- It certainly affects areas such as **portfolio choice**, **pricing**, **hedging** and measurement of **risk**
- A distinction should be made between model (or **misspecification**) risk and **estimation** risk.
- In general, two broad approaches have been pursued: **model averaging** (Bayesian or not) and **worst-case** approach.

Model risk

- Model risk has a strong impact when assessing the risk of a portfolio.
- Several **model assumptions** affect the final VaR (or ES) figure
  - Volatilities
  - Other marginal distributions
  - Correlations
  - Other joint distributions (copulae)
  - Pricing models for derivatives and choice of the relevant factors
- **Worst-case risk measures** under different sets of models (incomplete information) have been intensively studied, also in connection with robust portfolio optimization. See Ghaoui, Oks and Oustry (2003) among many others.
- Kerkhof, Melenberg and Schuhmacher (2010) introduce a measure of model risk which is based on the worst-case risk figure. It is different from what we propose here.

A motivation: the Basel multiplier

- Within the Basel framework, financial institutions are allowed to use internal models to assess the capital requirement due to market risk.

- The term that measures risk in *usual* conditions is given by:

$$CC = \max\left\{ VaR^{(0)}, \frac{\lambda}{60}\sum_{i=1}^{60} VaR^{(-i)} \right\},$$

  where
  - $VaR^{(0)}$ is the portfolio's **Value-at-Risk** (order 1% and 10-day horizon) computed/estimated today
  - $VaR^{(-i)}$ is the VaR we obtained i days ago
  - λ is the *multiplier*

- Remind:

$$\begin{aligned} VaR_\alpha(X) &= -F_X^{-1}(\alpha) \qquad \text{if } F_X \text{ is invertible} \\ &= -\inf\{x \,:\, F_X(x) \geqslant \alpha\} \qquad \text{more in general} \end{aligned}$$

A motivation: the Basel multiplier

- Remind
$$CC = \max\{VaR_0, \lambda \overline{VaR}\}$$
($\overline{VaR}$ average of the last 60 VaR's)
- The **multiplier** (λ) is assigned to each institution by the regulator
- It depends on back-testing performances of the system (poor performance yields higher λ) and it is revised on a periodical basis
- It is in the interval $[3, 4]$
- As $\lambda \geqslant 3$, it is apparent that in normal conditions the second term is the leading one in the maximum giving the capital charge CC

A motivation: the Basel multiplier

- Stahl (1997) offered a simple theoretical justification for λ to be in the range $[3, 4]$
- Let X be the portfolio Profits-and-Losses (r.v.) due to market risk.
- As the time-horizon is short, we can assume $\mathbb{E}[X] = 0$. From **Chebishev** inequality:

$$P(X \leqslant -q) \leqslant P(|X| \geqslant q) \leqslant \frac{\sigma^2}{q^2}, \quad q > 0.$$

- It immediately follows

$$VaR_\alpha(X) \leqslant \frac{\sigma}{\sqrt{\alpha}}$$

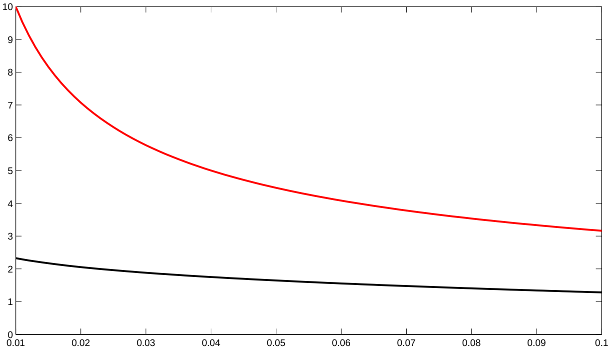
- The r.h.s. provides an **upper bound** for the VaR of a r.v. having mean 0 and variance σ^2

A motivation: the Basel multiplier

- This bound can be compared with the VaR we obtain under the **normal** hypothesis ($\alpha < 0.5$)

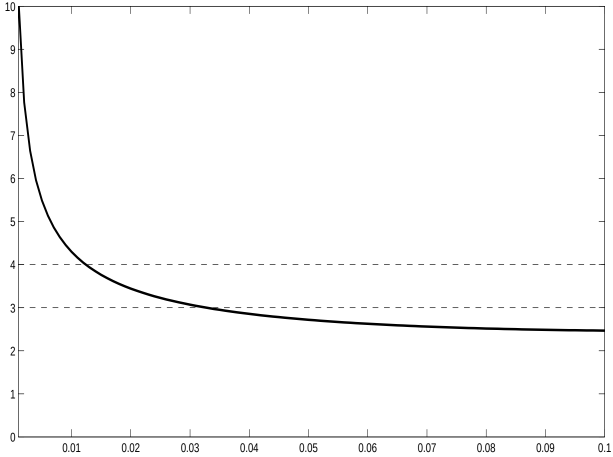
$$VaR_\alpha(X) = \sigma|z_\alpha| \qquad (z_\alpha = \Phi^{-1}(\alpha))$$

- Here are the two VaRs (normal: black, upper bound: red, $\sigma = 1$)



A motivation: the Basel multiplier

- Here is the ratio (upper bound/normal)



A Chebishev bound for the Expected Shortfall

- The Chebishev inequality can be used to obtain an upper bound for the **Expected Shortfall**

$$ES_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_u(X)\,du$$

under $\mathbb{E}[X] = 0$ and $\sigma(X) = \sigma$.

- Integrating we have

$$ES_\alpha(X) \leqslant \frac{1}{\alpha}\int_0^\alpha \frac{\sigma}{\sqrt{u}}\,du = \frac{2\sigma}{\sqrt{\alpha}}$$

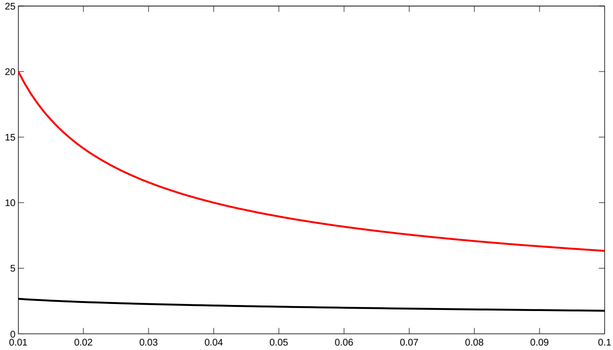
A Chebishev bound for the Expected Shortfall

- Under the **normal** hypothesis for X we have

$$ES_\alpha(X) = \frac{\sigma\varphi(z_\alpha)}{\alpha}$$

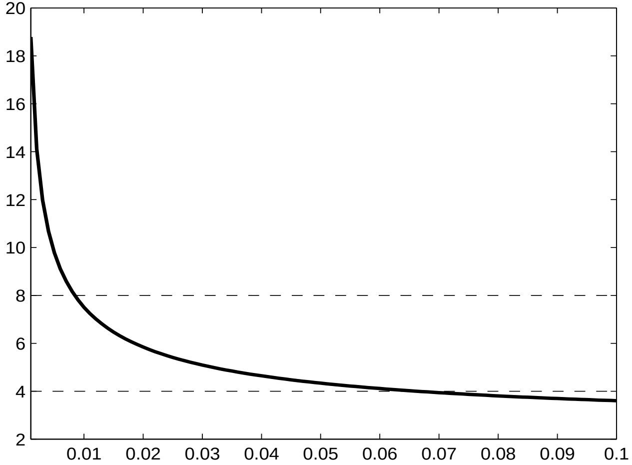
where φ is the density of a standard normal.

- Here are the two ESs (normal: black, upper bound: red, $\sigma = 1$)



A Chebishev bound for the Expected Shortfall

- Here is the ratio (upper bound/normal)



Cantelli upper bounds

- Even though Chebishev inequalities are sharp, the upper bounds on VaR and ES are not.
- A single-tail **sharp** inequality is the **Cantelli**'s one:

$$P(X \leqslant -q) \leqslant \frac{\sigma^2}{\sigma^2 + q^2} \qquad (q < 0)$$

- It follows a **sharp bound** for VaR

$$VaR_\alpha(X) \leqslant \sigma \sqrt{\frac{1-\alpha}{\alpha}}$$

and a slightly improved (but still not sharp) bound for ES

$$ES_\alpha(X) \leqslant \sigma \left(\sqrt{\frac{1-\alpha}{\alpha}} + \frac{1}{\alpha} \arctan \sqrt{\frac{1-\alpha}{\alpha}} \right)$$

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- Model risk in risk management and the Basel multiplier
- **”Absolute” measure of model risk**
- ”Relative” measure of model risk
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An absolute measure of model risk: definition

- We want to generalize the notion of multiplier as the **ratio** between the worst-case risk and the risk computed under a reference model.
- So, we need to consider:
  - a risk measure
  - a **reference** model
  - a set of **alternative** models

An absolute measure of model risk: definition

- A risk measure ρ is given, defined on some set of random variables.
- We assume ρ is
  - **law-invariant**, i.e. $\rho(X) = \rho(Y)$ whenever $X \sim Y$
  - **positive homogeneous**: $\rho(aX) = a\rho(X)$ for any $a \geqslant 0$
  - **translation invariant**: $\rho(X + b) = \rho(X) - b$ for any $b \in \mathbb{R}$
- Both VaR and ES satisfy these properties (but also spectral and other r.m.)

An absolute measure of model risk: definition

- Let X_0 be a reference r.v. Assume $\rho(X_0) > 0$.
- By law-invariance, what really matters is the **distribution** of X_0.
- Let $\mathcal{L}$ be a set of alternative r.v.'s, with $X_0 \in \mathcal{L}$.
- Define

$$\underline{\rho}(\mathcal{L}) = \inf_{X \in \mathcal{L}} \rho(X), \qquad \overline{\rho}(\mathcal{L}) = \sup_{X \in \mathcal{L}} \rho(X)$$

and assume they are finite.

An absolute measure of model risk: definition

- The **absolute measure of model risk** is defined as

$$AM = AM(\rho, X_0, \mathcal{L}) = \frac{\overline{\rho}(\mathcal{L})}{\rho(X_0)} - 1.$$

- $AM \geqslant 0$ with equality if and only if X_0 has already a worst-case distribution, i.e. $\rho(X_0) = \overline{\rho}(\mathcal{L})$

- We see that $AM + 1$ can be interpreted as a *generalized* multiplier

- The larger is $\mathcal{L}$, the higher is AM (hence *absolute* measure)

An absolute measure of model risk: properties

- (**scale invariance**) For any $a > 0$ it holds

$$AM(aX_0, a\mathcal{L}) = AM(X_0, \mathcal{L})$$

where $a\mathcal{L} = \{aX \,:\, X \in \mathcal{L}\}$.

- (**translation**) For $b \in \mathbb{R}$ it holds

$$AM(X_0 + b, \mathcal{L} + b) \begin{cases} > AM(X_0, \mathcal{L}), & \text{for } b > 0 \\ < AM(X_0, \mathcal{L}), & \text{for } b < 0 \end{cases}$$

where $\mathcal{L} + b = \{X + b \,:\, X \in \mathcal{L}\}$.

An absolute measure of model risk: an example

- For X having mean μ and variance σ^2, consider the set of alternative models

$$\mathcal{L}_{\mu,\sigma} = \{X \,:\, \mathbb{E}[X] = \mu,\ \sigma(X) = \sigma\}$$

- Set, as before, $\mu = 0$.
- By scale invariance, w.l.o.g. we concentrate on the case $\sigma = 1$.
- If $X_0 \in \mathcal{L}_{0,1}$, we have

$$AM = \frac{\overline{\rho}(\mathcal{L}_{0,1})}{\rho(X_0)} - 1$$

An absolute measure of model risk: an example

- We already know (sharp Cantelli ineq.) that

$$\sup_{X \in \mathcal{L}_{0,1}} VaR_\alpha(X) = \sqrt{\frac{1-\alpha}{\alpha}}$$

- Bertsimas et al (2004), using convex programming techniques, proved that

$$\sup_{X \in \mathcal{L}_{0,1}} ES_\alpha(X) = \sqrt{\frac{1-\alpha}{\alpha}}$$

  (a **much lower bound** than that derived using Cantelli)

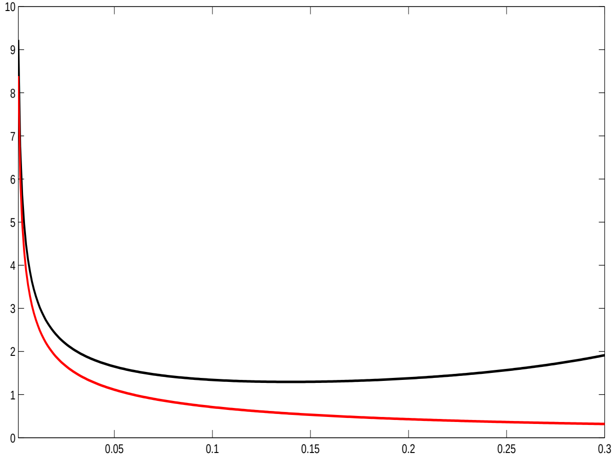
- Therefore

$$AM = \frac{1}{\rho(X_0)} \sqrt{\frac{1-\alpha}{\alpha}} - 1$$

  for $\rho = VaR_\alpha$ or ES_α

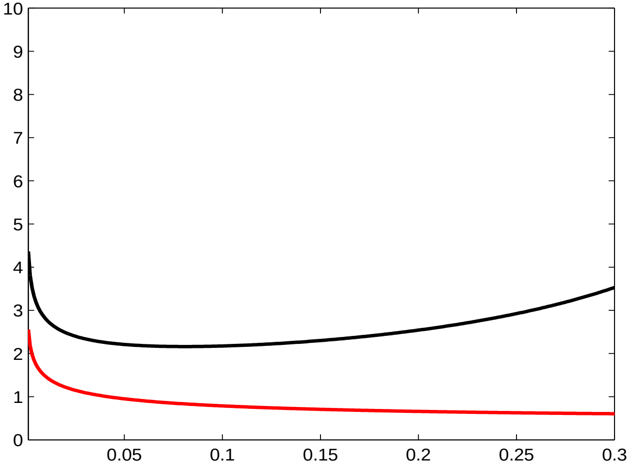
An absolute measure of model risk: an example

- X_0 standard **normal**. Black: VaR, red: ES.



An absolute measure of model risk: an example

- X_0 **student-t** with $\nu = 3$ degrees of freedom. Black: VaR, red: ES.



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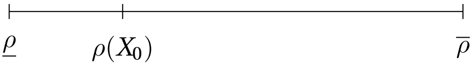
A relative measure of model risk: definition

- The **relative measure of model risk** is defined as

$$RM = RM(\rho, X_0, \mathcal{L}) = \frac{\overline{\rho}(\mathcal{L}) - \rho(X_0)}{\overline{\rho}(\mathcal{L}) - \underline{\rho}(\mathcal{L})}.$$

- For instance

$$RM = 0.75$$



- $0 \leqslant RM \leqslant 1$ and $RM = 0$ or 1 precisely when $\rho(X_0) = \overline{\rho}(\mathcal{L})$ (**no model risk**) or $\rho(X_0) = \underline{\rho}(\mathcal{L})$ (**full model risk**)
- RM need not be increasing in $\mathcal{L}$, thus providing a *relative* assessment of model risk

A relative measure of model risk: properties

- (**scale and translation invariance**) For any $a > 0$ and $b \in \mathbb{R}$ it holds
$$RM(aX_0 + b, a\mathcal{L} + b) = RM(X_0, \mathcal{L}).$$

- As $\mathcal{L}_{\mu,\sigma} = \sigma\mathcal{L}_{0,1} + \mu$ it follows
$$RM(X_0, \mathcal{L}_{\mu,\sigma}) = RM(\widetilde{X}_0, \mathcal{L}_{0,1}),$$
where
$$\widetilde{X}_0 = \frac{X - \mu}{\sigma}$$

- A more general result holds for $\mathcal{L} \subset \mathcal{L}_{\mu,\sigma}$, with $\widetilde{\mathcal{L}} = \{\widetilde{X} : X \in \mathcal{L}\}$ replacing $\mathcal{L}_{0,1}$
- Therefore, w.l.o.g. we can concentrate on **standard** r.v. (i.e. in $\mathcal{L}_{0,1}$)

A relative measure of model risk: an example

- We already know that

$$\sup_{X \in \mathcal{L}_{0,1}} VaR_\alpha(X) = \sup_{X \in \mathcal{L}_{0,1}} ES_\alpha(X) = \sqrt{\frac{1-\alpha}{\alpha}}$$

- Using bi-atomic distributions it is quite easy to see that

$$\inf_{X \in \mathcal{L}_{0,1}} VaR_\alpha(X) = -\sqrt{\frac{\alpha}{1-\alpha}} \qquad (\textit{negative}!)$$

- Using tri-atomic distributions we can also prove that

$$\inf_{X \in \mathcal{L}_{0,1}} ES_\alpha(X) = 0$$

(see also Bertsimas et al, 2004)

A relative measure of model risk: an example

- Let X_0 standard.
- For VaR we immediately find

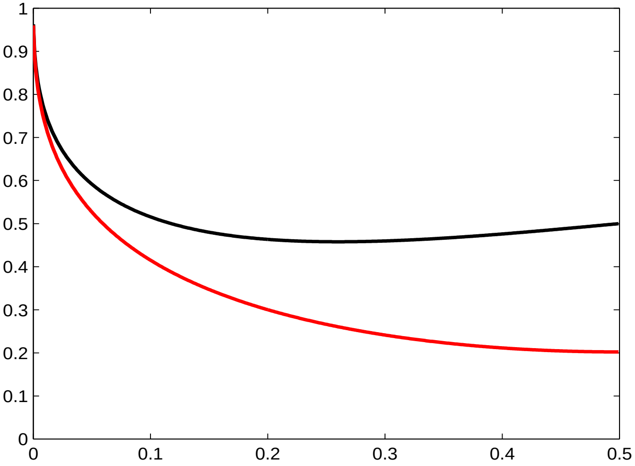
$$RM = 1 - \alpha - \sqrt{\alpha(1-\alpha)}VaR_{\alpha}(X_0)$$

- In a similar way, for ES

$$RM = 1 - \sqrt{\frac{\alpha}{1-\alpha}}ES_{\alpha}(X_0)$$

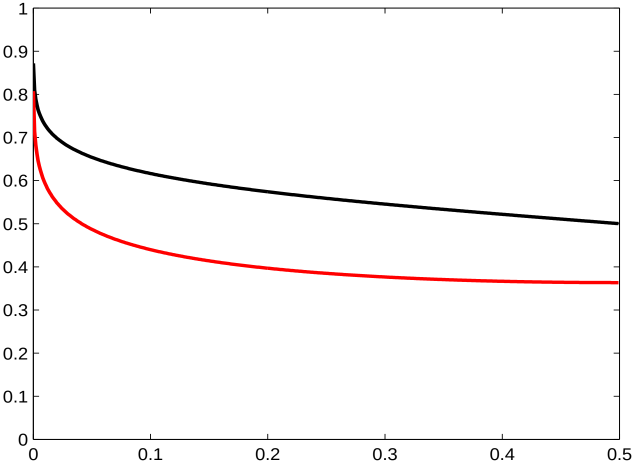
A relative measure of model risk: an example

- X_0 standard normal. Black: VaR, red: ES.



A relative measure of model risk: an example

- X_0 student-t with $\nu = 3$ degrees of freedom. Black: VaR, red: ES.



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A local measure of model risk: definition

- Finally, we want to assess model risk **locally** around X_0
- Let $(\mathcal{L}_\varepsilon)_{\varepsilon>0}$ a decreasing (as ε decreases) family of alternative distributions sets, meaning that
  - $\mathcal{L}_\varepsilon$ is a set of r.v. for any $\varepsilon > 0$
  - if $\varepsilon < \varepsilon'$, then $\mathcal{L}_\varepsilon \subset \mathcal{L}_{\varepsilon'}$
  - $\cap_\varepsilon \mathcal{L}_\varepsilon = \{X_0\}$
- Examples (assume $X_0 \in \mathcal{L}_{0,1}$)
  - If d is some **distance** between distributions (Levy, Kolmogorov, Kullback-Leibler divergence, etc) consider

    $$\mathcal{L}_\varepsilon = \{X : d(X, X_0) \leqslant \varepsilon\}$$

    In particular, Kullback-Leibler (or relative entropy) is considered by Alexander and Sarabia (2012) and by Glasserman and Xu (2013)
  - As before, with all X in $\mathcal{L}_{0,1}$
  - if F_0 is the distribution of X_0,

    $$\mathcal{L}_\varepsilon = \{X : F_X = (1-\theta)F_0 + \theta G,\ G \in \mathcal{L}_{0,1},\ \theta \in (0, \varepsilon)\}$$

A local measure of model risk: definition

- The **local measure of model risk** is

$$LM = \lim_{\varepsilon \to 0} RM(\mathcal{L}_\varepsilon) = \lim_{\varepsilon \to 0} \frac{\overline{\rho}(\mathcal{L}_\varepsilon) - \rho(X_0)}{\overline{\rho}(\mathcal{L}_\varepsilon) - \underline{\rho}(\mathcal{L}_\varepsilon)}$$

- The limit is in the form $0/0$

- If it exists, then it is in $[0, 1]$

- It describes the **relative position** of $\rho(X_0)$ w.r.t. the worst and best case for infinitesimal perturbations.

A local measure of model risk: an example

- Consider again the set of ε**-mixtures**

$$\mathcal{L}_\varepsilon = \{X \,:\, F_X = (1-\theta)F_0 + \theta G,\ G \in \mathcal{L}_{0,1},\ \theta \in (0,\varepsilon)\}$$

- It is immediate to see that $\mathcal{L}_\varepsilon \subset \mathcal{L}_{0,1}$
- Using results from the theory of Markov-Chebishev extremal distributions we can prove that for $\rho = VaR_\alpha$ we have

$$\underline{\rho}(\mathcal{L}_\varepsilon) = \inf_{X \in \mathcal{L}_\varepsilon} VaR_\alpha(X) = VaR_{\frac{\alpha}{1-\varepsilon}}(X_0)$$

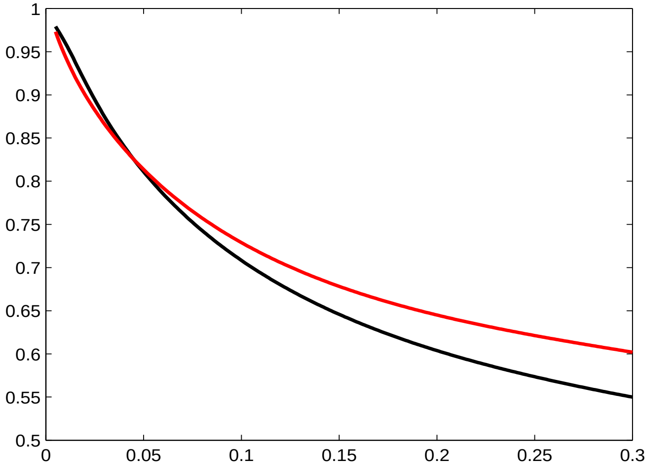
  provided α is small enough ($\alpha < (1-\varepsilon)F_0(0)$)
- Also, $r = \overline{\rho}(\mathcal{L}_\varepsilon)$ is solution of the following equation

$$(1-\varepsilon)F_0(-r) + \frac{\varepsilon}{1+r^2} = \alpha$$

  which can easily be treated numerically.

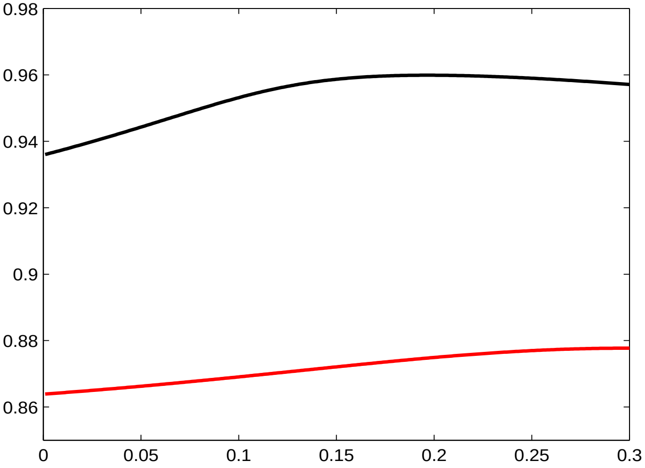
A local measure of model risk: an example

- X_0 standard normal, $\rho = VaR_\alpha$. Black: $\varepsilon = 0.2$, red: $\varepsilon = 0.05$. α on the x-axis



A local measure of model risk: an example

- X_0 standard normal, $\rho = VaR_\alpha$. Black: $\alpha = 1\%$, red: $\alpha = 3\%$. ε on the x-axis



A local measure of model risk: an example

- Using de l'Hôpital and the particular form of the extremal distribution we can also explicitly compute (remind: $\rho = VaR_\alpha$)

$$LM = \lim_{\varepsilon \to 0} RM(\mathcal{L}_\varepsilon) = 1 - \alpha(1 + q_\alpha(X_0)^2)$$

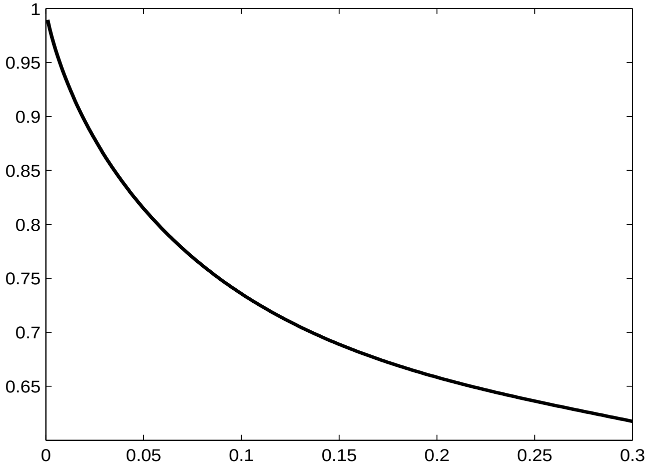
provided X_0 is a.c.

- If X_0 is standard normal

$$LM = 1 - \alpha(1 + z_\alpha^2)$$

A local measure of model risk: an example

- X_0 standard normal, $\rho = VaR_\alpha$. LM as a function of α on the x-axis



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The portfolio

- As a numerical example, we apply the relative measure of model risk to daily Value-at-Risk (1% and 5%) estimation for a portfolio investing in the (German) electricity market
- In particular, let P_t be the price at day t for 1 MWh in the day-ahead market (this is considered as a spot market). Notice that P_t may become negative in Germany!
- At every day, the portfolio invests in 1 unit (i.e. 1 MWh), so that its daily Profit and Loss is

$$PL_{t+1} = \Delta_{t+1} P = P_{t+1} - P_t.$$

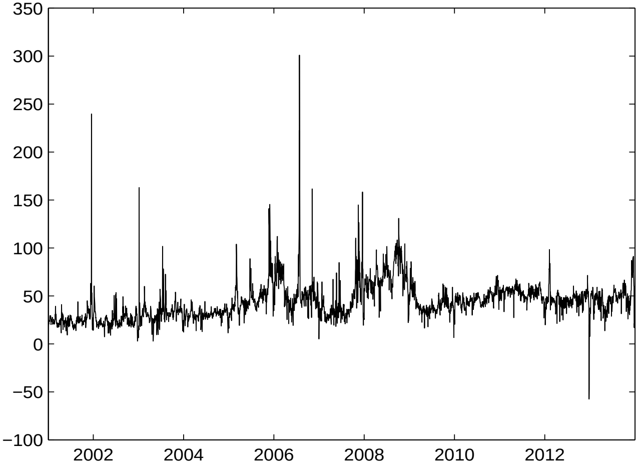
- If $\mathcal{F}_t$ is the information up to time t, we want to estimate

$$VaR_{\alpha,t+1} = VaR_{\alpha}(PL_{t+1} \,|\, \mathcal{F}_t),$$

where $\alpha = 1\%$ or 5%.

Electricity prices

- Day-ahead price for 1 MWh (in Euro) in the German market



GARCH modeling

- Price differences $X_t = \Delta_t P$ are usually assumed to be nearly stationary. However, contrarily to equity prices, the mean component is not negligible.
- We estimate (daily) an AR(5)-GARCH(1,1) model for X_t, meaning that

$$X_t = \mu_t + \varepsilon_t = \mu_t + \sigma_t Z_t,$$

  where
  - $\mu_t = \mathbb{E}\left[X_t \mid \mathcal{F}_{t-1}\right]$ is defined according to the AR(5)

$$\mu_t = c + \phi_1 X_{t-1} + \ldots + \phi_5 X_{t-5}$$

  - $\sigma_t = \sigma(X_t \mid \mathcal{F}_{t-1})$ is defined according to the GARCH(1,1) model

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

  - the innovations (Z_t) are IID $\sim D(0,1)$ (i.e. standard)
- According to this specification

$$VaR_\alpha(PL_{t+1} \mid \mathcal{F}_t) = -\mu_{t+1} + \sigma_{t+1} VaR_\alpha(Z)$$

Reference and alternative distributions

- In the classical "normal GARCH", we assume $Z \sim N(0,1)$. This is our *reference distribution.*
- Then we consider as alternative distributions:
  - Skew Normal
  - t-Student and Skew t-Student
  - Generalized Error Distributions (GED) and Skew GED
  - Johnson's SU;
  - Normal Inverse Gaussian (NIG);
  - Generalized Hyperbolic (a superclass including some of the previous classes).
- These are distributions of common use in risk management. Note that some allow for asymmetry, some for heavy tails and for both.
- The parameters of alternative distributions are fitted with ML using observed innovations (i.e. the series $\widehat{Z}_t = (X_t - \mu_t)/\sigma_t$)

Relative measure of model risk

- Day by day we compute

$$\overline{VaR}_{\alpha,t} = \sup_{Z \in \text{Models}} VaR_\alpha(PL_t \,|\, \mathcal{F}_{t-1}) = -\mu_t + \sigma_t \sup_{Z \in \text{Models}} VaR_\alpha(Z),$$

  where "Models" is a suitable class of models for Z.

- We define $\underline{VaR}_{\alpha,t}$ similarly.

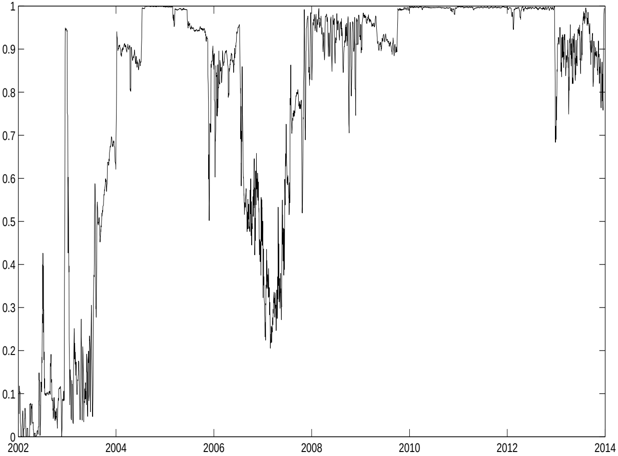
- Therefore, the empirical measure of relative model risk at date t is

$$RM_t = \frac{\overline{VaR}_{\alpha,t} - VaR^{Normal}_{\alpha,t}}{\overline{VaR}_{\alpha,t} - \underline{VaR}_{\alpha,t}} = \frac{\sup_Z VaR_\alpha(Z) - VaR_\alpha(N(0,1))}{\sup_Z VaR_\alpha(Z) - \inf_Z VaR_\alpha(Z)}$$

- Notice that the rhs depends on day t as the parameters for the distributions of innovations are fitted to past observations.

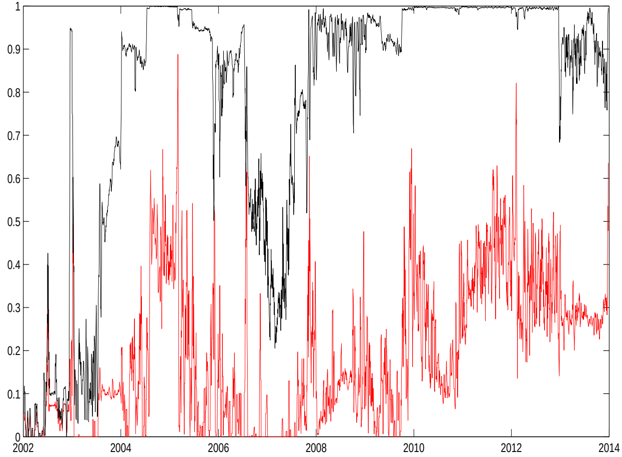
Results

- Relative measure of model risk for $VaR_{1\%}$



Results

- Rel. measure of model risk for $VaR_{5\%}$ (red) and $VaR_{1\%}$ (black)



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