Mathematical Aspects of Local vs. Global Risk Assessment

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Risk =

"known unknowns"

$\sim P$

(Knightian)
Uncertainty =

"unknown unknowns"

$\sim$ model ambiguity

$$\underbrace{\mathcal{P}}_{\text{a whole class of ``plausible'' probabilistic models}}$$

A case study:

the quantification of financial risk

— Beyond "Value at Risk"

$$\underset{\substack{\uparrow \\ \text{financial position} \\ X:\,\Omega \to \mathbb{R}}}{VaR_\alpha(X)} := \inf\{m \,/\, \underbrace{P[X+m<0]\le\alpha}_{\text{criterion of acceptability}}\}$$

Axiomatic approach

(what do we really want?)

$\Downarrow$

theory of monetary

(convex / coherent)

risk measures

- initiated by
Artzner, Delbaen, Eber, Heath (1999)
- extended by
F. Schied
Frittelli, Rosazza-Gianin (2002)

Monetary Risk as a Capital Requirement:

(regulatory perspective, cf. Basel II, III, ...)

$$\rho(X) = \inf\{m \,/\, X + m \in \mathcal{A}\}$$

"acceptable" positions

$\mathcal{A}$ saturated from above

convex
(diversification is not penalized)

$\Downarrow$

a convex risk measure

$$\rho : \mathcal{X} \to (-\infty, \infty]$$

a linear space of measurable functions X on $(\Omega, \mathcal{F})$
"financial positions"

- monotone:
$$X \geq Y \Rightarrow \rho(X) \leq \rho(Y)$$
- cash-invariant:
$$\rho(X+m) = \rho(X) - m$$
- convex:
$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y)$$

("coherent": also pos. homogeneous
i.e. $\mathcal{A}$ is convex cone 2)

$\Downarrow$ Fenchel - Moreau
(+ some continuity)

robust / dual representation

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \Big(\underbrace{E_Q[-X]}_{\text{expected loss under } Q} - \alpha(Q) \Big)$$

where

$\mathcal{Q}$ = a class of probability measures on $(\Omega, \mathcal{F})$

$$\alpha(Q) = \sup_{X \in \mathcal{A}} E_Q[-X] \quad \text{"penalty"}$$

In general (beyond "law-invariance"): a case study in "Knightian uncertainty" "law-invariance":

$\rho(X)$ depends on

$\mu_X :=$ law of X under given probability measure P on $(\Omega, \mathcal{F})$

$\sim$ a functional R on "lotteries"

+ "elicitability" ($\Rightarrow$ convex level sets)

$\Longrightarrow$ "shortfall risk measures" ("expectiles")

S. Weber 2006
(Gneiting, Ziegel)
F. Delbaen

utility-based shortfall risk (UBSR)

X acceptable $:\Longleftrightarrow$

$$E_P[u(X)] \geq u_0$$

or

$$E_P[\ell(-X)] \leq \ell_0$$

$\ell(x) = -u(-x)$
loss function

penalty function:

$$\alpha(Q) = \inf_{\lambda > 0} \frac{1}{\lambda}\left(\ell_0 + E_P\left[\ell^*\left(\lambda \frac{dQ}{dP}\right)\right]\right)$$

Example: "entropic" risk measure

$u(x) = 1 - e^{-\beta x} \Rightarrow \rho(X) = \frac{1}{\beta} \log E_P[e^{-\beta X}]$

$$\alpha(Q) = \frac{1}{\beta} H(Q|P)$$

from a mathematical point of view:

rich "prehistory" / many interfaces

- Choquet integrals (50's)
- Actuarial premium principles
  Deprez, Gerber (1985)
  $$\pi(X) = \rho(-X)$$
  ρ = convex risk measure
- Robust Statistics
  P. Huber (1981)
  ~ coherent case

- Preferences in the face of Risk and Uncertainty

$$X \succsim Y$$

$\Longleftrightarrow$ mild conditions (Debreu, Gerstenhaber-Milnor, ...)

$$U(X) \geq U(Y)$$

i.e., numerical representation by some "utility functional"

von Neumann - Morgenstern, Savage, ... :
classical "axioms of rationality"

$$\Downarrow$$

$$U(X) = E_P[u(X)]$$

for some
- utility function u
- probability measure P

$\underbrace{\qquad\qquad\qquad\qquad}$

paradigm of "expected utility"

more flexible "axioms of rationality":

- Gilboa - Schmeidler (1989)
- Maccheroni, Marinacci, Rustichini (2006)

$$\Downarrow$$

$$U(X) = - \underset{\uparrow}{\rho}(u(X))$$

convex* risk measure

(Gilboa-Schmeidler: coherent)

$$= \inf_{Q \in \mathcal{Q}} \left(E_Q[u(X)] + \alpha(Q) \right)$$

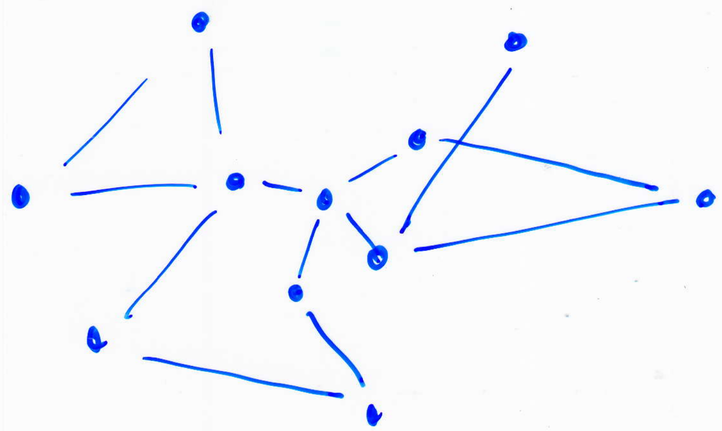
* reflects model uncertainty aversion while concavity of u reflects classical risk aversion

local vs. global

risk assessment

in terms of convex risk measures

— in a large financial network



I = set of "sites" / nodes

S = set of possible states for each site (e.g. $\mathbb{R}^d$)

space of configurations:

$$\Omega := S^I = \text{all } \omega : I \to S$$

$$\bar{\mathcal{X}} := \text{all } \bar{X} = (X_i)_{i \in I}$$

e.g.: P&L at i

"systemic" risk measure

$\bar{\rho}$ = a convex risk measure on $\bar{\mathcal{X}}$ (Banach space e.g. $(L^p)^I$)

typically:

"structural decomposition"

$$\boxed{\bar{\varrho} = \varrho \circ \Lambda}$$

where

i) $\Lambda : \bar{\mathcal{X}} \to \mathbb{R}^1$

is a real-valued <u>concave</u> <u>"aggregation function"</u>

ii) ϱ is a <u>"global"</u> <u>convex risk measure</u> on $\mathcal{X} \supseteq \Lambda(\bar{\mathcal{X}})$

cf. Chen, Moallemi (2013)

Kromer, Overbeck, Zilch (2014)

---

a general structure theorem!

dual representation:

$$\bar{\rho}(\bar{X}) = \rho(+1(\bar{X}))$$

$$= \sup_{Q} \left(E_Q[-\underbrace{1(\bar{X})}_{\text{convex}}] - \alpha(Q) \right)$$

$$= \sup_{\ell} \left((\ell, \bar{X}) - (-1)^*(\ell) \right)$$

$$= \sup_{Q,\ell} \left(E_Q[(\ell, \bar{X})] - \bar{\alpha}(Q, \ell) \right)$$

cf. Overbeck et al (2014)

more detailed

financial picture ?

S. Weber, Z. Feinstein, B. Rudloff

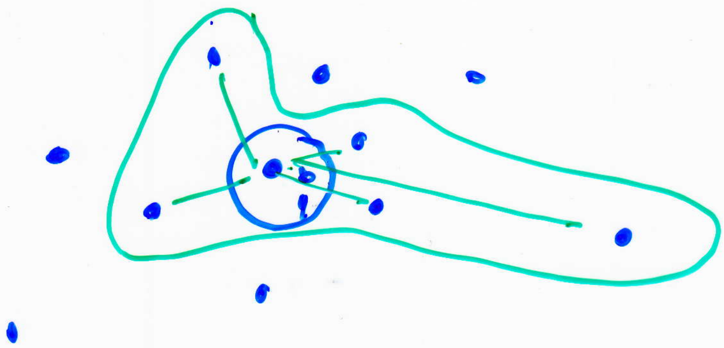
(2015 / Lectures Nov. 2014
Isaac Newton Institute)

- specification / set-valued extension
- contains as special case

M. Brunnenmeier, P. Cheridito
(2014)
"Measuring and allocating financial risk":

global ρ = UBSR

Local risk assessment:



N(i) = "neighbors" of i

$$\rho_{\{i\}}(\omega, X_i)$$

= regular convex risk kernel from $(\Omega, \mathcal{F}_{I-\{i\}})$ to $(\Omega, \mathcal{F})$

– assesses the risk at i, conditional on the "environment" of i

= situation on $I-\{i\}$ (or $N(i)$)

– conditional

(not marginal)

approach

cf. "CoVaR"
(Adrian, Brunnermeier)

more generally:

for any $V \in \mathcal{V} :=$ all finite $V \subseteq I$

$\rho_V(\omega, \cdot) =$ regular convex risk kernel from $(\Omega, \mathcal{F}_{V^c})$ to $(\Omega, \mathcal{F})$

Definition: The family $(\rho_V)_{V \in \mathcal{V}}$ is called a local specification (of a convex risk measure) if it is spatially consistent, i.e.,

$$\rho_W = \rho_W(-\rho_V) \quad \forall V \subseteq W$$

or, equivalently:

$$\rho_V(\cdot, X) \geq \rho_V(\cdot, Y) \Longrightarrow \rho_W(\cdot, X) \geq \rho_W(\cdot, Y)$$

special "linear" case:

Gibbs measures

(Dobrushin, Lanford, Ruelle)

$$\varrho_V(\omega, X) = \int -X(y)\, \pi_V(\omega, dy)$$

where

$(\pi_V)_{V \in \mathcal{O}}$ = local specification of a random field

i.e.,

- $\pi_V(\omega, dy)$ is a stochastic kernel from $(\Omega, \mathcal{F}_{V^c})$ to $(\Omega, \mathcal{F})$
- $\pi_W \pi_V = \pi_W \quad \forall\, W \supseteq V$

(spatial consistency)

a non-linear example:

$$\rho_V(\omega, X) = \frac{1}{\beta} \log \int e^{-\beta X(\eta)} \pi_V(\omega, d\eta)$$

$$= \sup_{Q(\omega,\cdot) \approx \pi_V(\omega,\cdot)} \left(E_{Q(\omega,\cdot)}[-X] - \frac{1}{\beta} H\left(Q(\omega,\cdot) \middle| \pi_V(\omega,\cdot)\right)\right)$$

— "entropic" local specification with parameter $\beta \in [0, \infty)$

$\supset$ the linear case

($\beta = 0$: limiting case)

$$\rho_V(\omega, X) = \int (-X) \, d\pi_V(\omega, \cdot)$$

Problem I :

Construction of a local specification $\{\rho_V \mid V \in \mathcal{O}\}$ from the one-site risk measures $\{\rho_{\{i\}} \mid i \in I\}$

How arbitrarily can we prescribe the conditional risk measures at the single sites ?

"local consistency" ?

solution in the "linear" case

(Dobrushin, Spitzer):

$(\pi_{\{i\}})_{i \in I}$ admits a consistent extension to a local specification $(\pi_V)_{V \in \mathcal{O}}$ $\Longleftrightarrow$

S discrete

"Markov" w.r.t. graph structure

$\exists$ "interaction potential"

$(U_C)_{C \in \mathcal{C}}$:= the cliques of the graph

$\mathcal{F}_C$ - measurable

$$\pi_{\{i\}}(\omega, s) \sim \exp\Big(\sum_{\substack{C \in \mathcal{C} \\ i \in C}} U_C(\omega_{\{i\}^c}, s)\Big)$$

In the non-linear case :

- wide open in general

- solved in the "locally law-invariant" case (cf. below)

Problem II:

Given a local specification

$$(\rho_V)_{V \in \mathcal{D}},$$

what is the structure of

$\mathcal{R} :=$ all global convex risk measures ρ such that

$$\rho(-\rho_V) = \rho \quad \forall V \in \mathcal{D}$$

?

- uniqueness vs. "phase transition"
- $\sim$ integral representation (cf. Gibbs measures)

solution in the "linear" case of

<u>Gibbs measures:</u>

$\mathcal{P} :=$ all P on $(\Omega, \mathcal{F})$
consistent with $(\pi_V)_{V \in \mathcal{O}}$,
i.e., $P\pi_V = P \quad \forall V \in \mathcal{O}$

<u>"phase transition"</u> $:\Leftrightarrow |\mathcal{P}| > 1$

non-uniqueness

<u>structure of $\mathcal{P}$</u>:

$$\mathcal{P}_e := \{ P \in \mathcal{P} \mid P = 0\text{-}1 \text{ on } \mathcal{F}_\infty \text{ "ergodic"} \}$$

where

$$\mathcal{F}_\infty := \bigcap_{V \in \mathcal{O}} \mathcal{F}_{V^c}$$

"tail field"

integral representation coupled to the tail field

– adapting Dynkin's construction of entrance boundaries for Markov processes:

$\exists$ stochastic kernel π_∞ from $(\Omega, \mathcal{F}_\infty)$ to $(\Omega, \mathcal{F})$:

i) $\pi_\infty(\omega, \cdot) \in \mathcal{P}_e \qquad \forall \omega \in \Omega$

ii) $P = P\pi_\infty \qquad \forall P \in \mathcal{P}$

i.e.

$$P = \int \pi_\infty(\omega, \cdot)\, P\big|_{\mathcal{F}_\infty}(d\omega)$$

$$= \int_{\mathcal{P}_e} Q\, \mu_P(dQ)$$

cf. F., "Phase transition and Martin boundary" (1975)

essential part
(in analogy to "Martin boundary"):

$$(\Omega, \hat{\mathcal{F}}) = \text{"Dynkin boundary"}$$

where

$$\hat{\mathcal{F}} := \sigma \begin{pmatrix} \omega \to \pi_\infty(\omega, \cdot) \\ \Omega \to \mathcal{P}_e \end{pmatrix}$$

Thus:

Gibbs measures $P \in \mathcal{P}$

$\overset{1-1}{\longleftrightarrow}$ probability measures on Dynkin boundary analogue in the "non-linear" case of

risk measures

$\rho \in \mathcal{R}$?

A) solved in the case of local law-invariance:

$\rho_V(\omega, \cdot)$ law-invariant w.r.t. $\pi_V(\omega, \cdot)$

A.F., Statistics and Risk Modeling (2014)

B) general case

$\rho_V(\omega, \cdot) \ll \pi_V(\omega, \cdot)$

A.F., C. Klüppelberg

Excursion:

a general result on

consistency & law-invariance

cf. Kupper, Schachermayer (2009)
Cerreia-Voglio, Maccheroni, Marinacci, Montrucchio (2011)
– with a new twist

ρ = "nice" law-invariant risk measure on $L^\infty(\Omega, \mathcal{F}, P)$
atomless

$\downarrow$

partial order on "lotteries", i.e., on $\mathcal{M}_1(\mathbb{R}^1)$:

$$\mu \succeq \nu \;:\Longleftrightarrow\; \rho(X) \leq \rho(Y)$$

where $\operatorname{law}(X) = \mu$, $\operatorname{law}(Y) = \nu$

i.e.

$$\mu \succsim \nu \iff C(\mu) \geq C(\nu)$$

$$C(\mu) := -\rho(X)$$

We know:

If $\succsim$ satisfies the von Neumann-Morgenstern axioms then

$\exists\ u \nearrow$ continuous:

$$\mu \succsim \nu \iff \int u\,d\mu \geq \int u\,d\nu$$

"expected utility"

$$\Rightarrow \quad C(\mu) = u^{-1}\left(\int u\,d\mu\right)$$

$=$ "certainty equivalent"

But:

$C(\cdot)$ = certainty equivalent w.r.t. u

+ cash-invariant, i.e.,

$C(\mu_{\text{shifted by } m}) = C(\mu) + m$

$\Downarrow$ de Finetti: Sul concetto di media (1931)

(cf. FS, 3^d ed., Prop. 2.46)

u exponential ($\supseteq$ linear)

i.e.

ρ = entropic risk measure

crucial condition:

"independence axiom"

$\mu \succsim \nu$, $\lambda \in (0,1)$, $\forall$ lottery γ:

$$\lambda\mu + (1-\lambda)\gamma \succsim \lambda\nu + (1-\lambda)\gamma$$

— follows from consistency!

More precisely:

for $\mathcal{F}_0 \subseteq \mathcal{F}$, ρ is called $\mathcal{F}_0$-consistent if

$$\rho = \rho(-\rho_0)$$

for some conditional risk measure ρ_0 w.r.t. $\mathcal{F}_0$

In particular:

$$\rho_0(X I_{A_0} + Y I_{A_0^c}) = \rho_0(X) I_{A_0} + \rho_0(Y) I_{A_0^c}$$

("locality")

<u>Theorem:</u> Suppose that the law-invariant risk measure ρ is $\mathcal{F}_0$-consistent for some $\mathcal{F}_0$ such that

i) $(\Omega, \mathcal{F}_0, P)$ is atomless

ii) $(\Omega, \mathcal{F}, P)$ is conditionally atomless given $\mathcal{F}_0$

($\exists\, U_1, U_2, \dots$ i.i.d. uniform, $\perp \mathcal{F}_0$)

Then

ρ is <u>entropic</u>

(hence "fully consistent", i.e., $\mathcal{F}_1$-consistent $\forall\, \mathcal{F}_1 \subseteq \mathcal{F}$)

Proof: Check the independence axiom:

(C-V, M, M, M + conditional twist)

$\mu, \nu, \gamma, \quad \lambda \in (0,1), \quad \mu \preceq \nu$

to show:

$$\lambda\mu + (1-\lambda)\gamma \preceq \lambda\nu + (1-\lambda)\gamma$$

$\exists\ X, Y, Z \sim \mu, \nu, \gamma, \quad \perp \mathcal{F}_0$

$A_0 \in \mathcal{F}_0 : \quad P[A_0] = \lambda$

$$\Rightarrow \quad \lambda\mu + (1-\lambda)\gamma \sim I_{A_0} X + I_{A_0^c} Z$$
$$\lambda\nu + (1-\lambda)\gamma \sim I_{A_0} Y + I_{A_0^c} Z$$

Thus:

$$C(\lambda\mu + (1-\lambda)\gamma) = -\rho(I_{A_0} X + I_{A_0^c} Z)$$

to show: $\overset{!}{\geq} C(\lambda\nu + (1-\lambda)\gamma)$

$$= -\rho(I_{A_0} Y + I_{A_0^c} Z)$$

Indeed:

$$\rho(X I_{A_0} + Z I_{A_0^c})$$

$$\underset{\text{consistent}}{=} \rho(-\underbrace{\rho_0(X I_{A_0} + Z I_{A_0^c})})$$

$$\underset{\text{locality}}{=} I_{A_0} \underbrace{\rho_0(X)}_{\substack{= \rho(X) \\ X \perp \mathcal{F}_0}} + I_{A_0^c} \underbrace{\rho_0(Y)}_{\substack{= \rho(Y) \\ Y \perp \mathcal{F}_0}}$$

$$\underset{\mu \succsim \nu}{\leq} \rho(Y)$$

hence

$$\rho(X I_{A_0} + Z I_{A_0^c}) \underset{\text{monotone}}{\leq} \rho(Y I_{A_0} + Z I_{A_0^c})$$

i.e.

$$\lambda\mu + (1-\lambda)\gamma \succsim \lambda\nu + (1-\lambda)\gamma$$

q.e.d.

Corollary (Kupper-Schachermayer):

Law-invariant and
dynamically consistent
convex risk measures are
entropic

here:

alternative "static" proof
- without auxiliary Markov chain/
Skorohod's embedding principle
(does not work in spatial setting)

Back to our

spatial setting:

Given:

i) $(\pi_V)_{V \in \mathcal{V}}$

= local specification of a random field (à la Dobrushin)

ii) $(\rho_V)_{V \in \mathcal{V}}$

= local specification of a convex risk measure

— tied to $(\pi_V)_{V \in \mathcal{V}}$:

A) <u>local law-invariance</u>:

$\rho_V(\omega,\cdot)$ is law-invariant w.r.t. $\pi_V(\omega,\cdot)$

+ consistency $\rho_W(-\rho_V)=\rho_W$

$\Longrightarrow$ <u>locally entropic</u>:

Theorem

$$\rho_V(\omega,X)=\frac{1}{\beta_V(\omega)}\log\int e^{-\beta_V(\omega)X(\eta)}\,\pi_V(\omega,d\eta)$$

Due to consistency:

$$\beta_V(\omega)=\beta_W(\omega)\quad P\text{-a.s. } \forall P\in\mathcal{P}$$
$$=:\beta_\infty(\omega)$$

tail-measurable

In this case:

straightforward consistent extension of (ρ_V) to the tail field:

$$\rho_\infty(\omega, X) = \frac{1}{\beta_\infty(\omega)} \log \int e^{-\beta_\infty(\omega) X(y)} \pi_\infty(\omega, dy)$$

Dynkin's kernel

= key to structure problem (II)

(cf. general case below)

B) Beyond (local) law-invariance:

Assume only:

$$\rho_V(\omega,\cdot) \ll \pi_V(\omega,\cdot)$$

i.e.

$$\rho_V(\omega,X) = \rho_V(\omega,Y) \quad \text{if} \quad X = Y \quad \pi_V(\omega,\cdot)\text{-a.s.}$$

$\Longrightarrow$ Detlefsen-Scandolo

$\forall P \in \mathcal{P}$:

$$\rho_V(\cdot,X) = \operatorname*{ess.sup}_{\substack{Q \ll P \\ Q \approx P \text{ on } \mathcal{F}_{V^c}}} \left(E_Q[-X \mid \mathcal{F}_{V^c}] - \alpha_V(Q) \right)$$

extension to

tail field $\mathcal{F}_\infty = \bigcap_n \mathcal{F}_{V_n^c}$?

trivial extension:

Fix $V_n \uparrow I$, $\mathcal{F}_{V_n^c} \downarrow \mathcal{F}_\infty$.

Define

$$\rho_\infty(\omega, X) := \overline{\lim_n} \, \rho_{V_n}(\omega, X)$$

- a regular convex risk kernel from $(\Omega, \mathcal{F}_\infty)$ to $(\Omega, \mathcal{F})$
- consistent with $(\rho_V)_{V \in \mathcal{O}}$

i.e.

$$\rho_\infty(-\rho_V) = \rho_\infty$$

properties under $P \in \mathcal{P}$?

– via non-linear extension of backwards martingale convergence:

Fix $V_n \in \mathcal{V}$, $V_n \uparrow I$, $\mathcal{F}_{V_n^c} \downarrow \mathcal{F}_\infty$

Consistency along $(\mathcal{F}_{V_n^c})$

$\Longrightarrow$

Delbaen, Kupper, Cheridito
Bion-Nadal
F., Penner

$\forall P \in \mathcal{P}$:

$$V_n(P,X) := \rho_{V_n}(X) + \alpha_{V_n}(P)$$

$(n = 1, 2, \ldots)$

backwards supermartingale w.r.t. P along $(\mathcal{F}_{V_n^c})$, hence

convergent $\mathcal{P}$-a.s.

i.e. P-a.s. $\forall P \in \mathcal{P}$

Proposition (F., Irina Penner):

$\forall P \in \mathcal{P}$

- $\rho_\infty(\cdot, X) = \lim_n \rho_{V_n}(\cdot, X)$ P-a.s. for any $X \in \mathcal{M}$
- ρ_∞ has Fatou property w.r.t. P
- dual representation with penalty function $\alpha_\infty(Q) = \lim_n \alpha_{V_n}(Q)$ P-a.s.

---

key to structure problem (II)

regularization:

$$\hat{\rho}_\infty(\omega, X) := \int \rho_\infty(\eta, X)\, \pi_\infty(\omega, d\eta)$$

- a regular convex risk kernel from Dynkin boundary $(\Omega, \hat{\mathcal{F}})$ to $(\Omega, \mathcal{F})$
- consistent with $(\rho_V)_{V \in \mathcal{V}}$
- $\hat{\rho}_\infty(\omega, \cdot)$ has Fatou property w.r.t. $\pi_\infty(\omega, \cdot)$
- $\hat{\rho}_\infty(\cdot, X) = \rho_\infty(\cdot, X)$ $\pi_\infty(\omega, \cdot)$-a.s. $\forall \omega \in \Omega$ (by ergodicity), hence P-a.s.

$\mathcal{R}_L$ = all global risk measures $\rho \in \mathcal{R}$ which have the <u>Lebesgue property w.r.t. P</u>, i.e.,

$$\rho(X_n) \to \rho(X)$$

for any uniformly bounded sequence $(X_n) \subseteq \mathcal{X}$ such that

$$\lim_n X_n = X \quad P\text{-a.s.}$$

$\hat{\mathcal{R}}_L$ = all $\rho \in \hat{\mathcal{R}}$ (convex risk measures on the Dynkin boundary, i.e. on $\hat{M} := M_b(\Omega, \mathcal{F})$) with Lebesgue property w.r.t. P (i.e., w.r.t. pointwise convergence)

Theorem. Any $\rho \in \mathcal{R}_L$ has the form

$$\rho = \hat{\rho}(-\hat{\rho}_\infty)$$

with some $\hat{\rho} \in \hat{\mathcal{R}}_L$, namely

$\hat{\rho} := \rho|_{\hat{M}}$ = restriction of ρ to the Dynkin boundary

Proof. $\rho \in \mathcal{R} \Rightarrow$

$$\rho(-\underbrace{\rho_{V_n}(X)}) = \rho(X)$$

$$\xrightarrow[P\text{-a.s.}]{} \rho_\infty(X) \underset{P\text{-a.s.}}{=} \hat{\rho}_\infty(X)$$

$$\xrightarrow[\rho \in \mathcal{R}_L]{} \rho(-\underbrace{\hat{\rho}_\infty(X)}_{\in \hat{M}}) = \hat{\rho}(-\hat{\rho}_\infty(X))$$

Corollary. If each $\hat{\rho}_\infty(\omega, \cdot)$ has Lebesgue property w.r.t. $\pi_\infty(\omega, \cdot)$ (as in entropic case!) then

$$\mathcal{R}_L = \{ \hat{\rho}(-\hat{\rho}_\infty) \mid \underbrace{\hat{\rho} \in \hat{\mathcal{R}}_L}_{\text{on the Dynkin boundary}} \}$$

Proof. "$\subseteq$": ✓

"$\supseteq$": $\hat{\rho} \in \hat{\mathcal{R}}_L \Rightarrow \rho := \hat{\rho}(-\hat{\rho}_\infty) \in \mathcal{R}$

Lebesgue property:

$X_n \to X$ P-a.s. $\Rightarrow$ $\pi_\infty(\omega, \cdot)$-a.s. $\forall \omega \in \Omega$

$\underset{\text{Assumption}}{\Longrightarrow}$ $\hat{\rho}_\infty(\omega, X_n) \to \hat{\rho}_\infty(\omega, X)$

$$\Rightarrow \rho(X_n) = \hat{\rho}(-\hat{\rho}_\infty(\cdot, X_n)) \xrightarrow[\hat{\rho} \in \hat{\mathcal{R}}_L]{} \hat{\rho}(-\hat{\rho}_\infty(\cdot, X)) = \rho(X)$$

In this case:

Corollary.

$$|\mathcal{R}_L| = 1 \iff |\mathcal{O}| = 1$$

i.e., a global risk measure $\rho \in \mathcal{R}_L$ is uniquely determined by the local specification if and only if there is no probabilistic phase transition

Proof. 1) $|\mathcal{O}| = 1 \Rightarrow \hat{f}$ trivial

$\hat{\mathcal{M}}$ = the constants

$\hat{\rho}(m) = -m$

unique

2) $|\mathcal{O}| > 1 \Rightarrow \pi_\infty(\omega_1, \cdot) \not\perp \pi_\infty(\omega_2, \cdot)$ (ergodicity)

$\Rightarrow \hat{\rho}_\infty(\omega_i, \cdot) \ll \pi_\infty(\omega_i, \cdot)$

are different $\Rightarrow |\mathcal{R}_L| > 1$

However:

$$\underbrace{|\mathcal{P}| = 1}_{\text{no probabilistic phase transition}} \quad \not\Rightarrow \quad |\mathcal{R}| = 1$$

i.e.

multiplicity of global risk measures in $\mathcal{R}$ (without Lebesgue property)

– illustrated in entropic case:

entropic case:

$$\hat{\rho}_\infty(\omega, X) = \frac{1}{\hat{\beta}(\omega)} \log \int e^{-\hat{\beta}(\omega) X(\eta)} \pi_\infty(\omega, d\eta)$$

$$\hat{\beta}(\omega) := \int \beta_\infty(\eta) \pi_\infty(\omega, d\eta)$$

$\hat{\mathcal{F}}$-measurable

$$|\Theta| = 1 \implies \begin{array}{l} \pi_\infty(\omega, \cdot) \equiv P \\ \hat{\beta}(\omega) \equiv \beta \end{array}$$

i.e., there is only one $\rho \in \mathcal{R}_L$, namely the entropic risk measure w.r.t. P and β

But:

$$\rho_\infty(\omega, X) := \overline{\lim_n}\, \rho_{V_n}(\omega, X) \in \mathcal{R}$$

$$= -X(\omega) \quad \forall \text{ tail-measurable } X$$

i.e. multiplicity in $\mathcal{R}$!

we have solved

Problem (II)

= a purely mathematical structure problem somehow motivated by the much broader issue of risk analysis ("systemic risk") in large financial networks

– an example for

Math $\longleftarrow$ Finance

$\longrightarrow$

?

possible message:

local

(even conditional, vs. just marginal)

risk analysis

may not suffice to capture all sources of the aggregate / "systemic" risk

– in non-linear analogy to the probabilistic analysis of phase transitions

Second message:

The consistency requirement is too strong
(even though it seems plausible at first sight)

open problem:

What happens in the spatial setting, if we use related notions of consistency ?
(S. Weber 2006, S. Tutsch 2007)

Thanks

for listening!