A Fourier approach to the computation of risk measures and risk contributions

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Outline

Motivation

The financial crisis has higlighted the importance of reliable risk assessment

Relevance: Economically sound (Diversification, Aggregation, Basel)

Performance: Numerically efficient (real time, huge number of assets)

Motivation

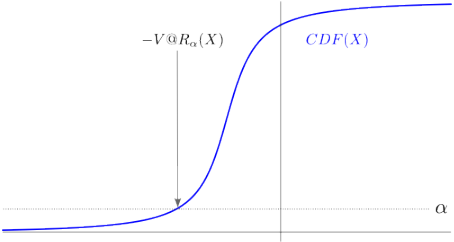
Value at Risk (V@R)

V@R

$$V@R_\alpha(X) := -q_X(\alpha)$$

- Widely used (Basel II)
- Straightforward interpretation
- Easy / efficient implementation
  $\rightsquigarrow$ A single root finding

Illustration



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Drawback:

Non-consistent aggregation of risks

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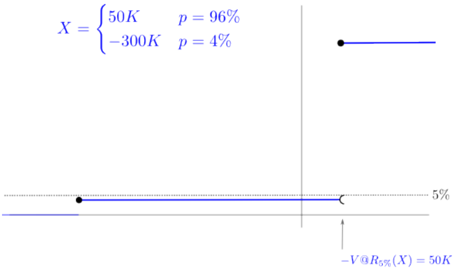

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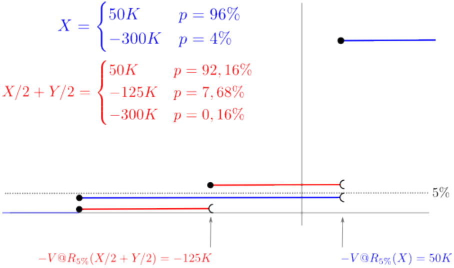

$X \sim Y$ and X independent of Y

$$-100K = V@R_\alpha(X) + V@R_\alpha(Y) < V@R(X+Y) = 250K$$

Motivation

Monetary Risk Measures: Artzner, Delbaen, Eber, Heath (1999); Föllmer, Schied (2002)

"Diversification should not increase risk"

Definition (Monetary Risk Measure)

A monetary risk measure ρ is

- **Diversifying:** for any two assets X and Y the diversified asset profile $\lambda X + (1-\lambda)Y$ is less risky than the worse outcome

$$\rho(\lambda X + (1-\lambda)Y) \leq \sup\{\rho(X), \rho(Y)\}$$

- **Monotone:** $\rho(X) \geq \rho(Y)$ if loss $-X$ is greater than loss $-Y$;
- **Monetary:** $\rho(X+m) = \rho(X) - m$ for every amount of cash m.

Monetary and diversification implies that ρ is **convex**

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$$

Examples: CV@R, Entropic Risk Measure, Shortfall Risk Measure, etc.

Motivation

Monetary Risk Measures: Conditional Value at Risk (CV@R)

CV@R (Artzner et al.)

$$CV@R_\alpha(X) := -\frac{1}{\alpha}\int_0^\alpha q_X(s)ds$$

- Consistent aggregation
- Basel III / Swiss Solvency Test
- Good interpretation

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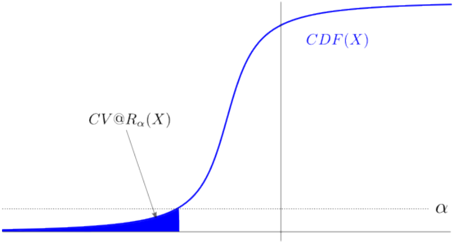

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Drawbacks:

- Increased numerical complexity
- Backtesting – Elicitability
- Statistical robustness

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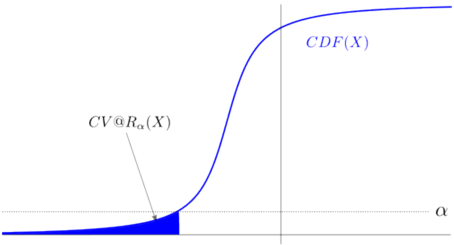


Optimized Certainty Equivalents

Definition

OCE (Ben-Tal, Teboulle [86,06])

- Today's expected loss of X

$$E\left[l(-X)\right]$$

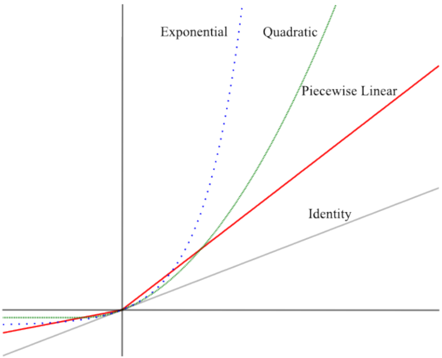
- Allocating cash η

$$E\left[l\left(\eta - X\right)\right] - \eta$$

- Optimal allocation $\rightsquigarrow$ OCE

$$\rho(X) := \inf_{\eta} \left\{E\left[l\left(\eta - X\right)\right] - \eta\right\}$$

Typical Loss Functions l



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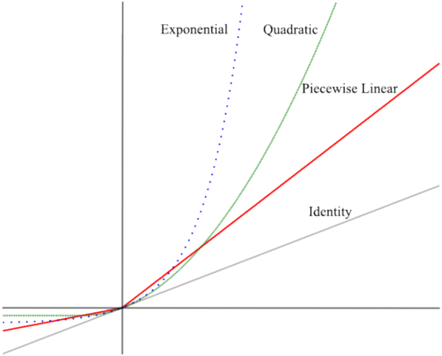

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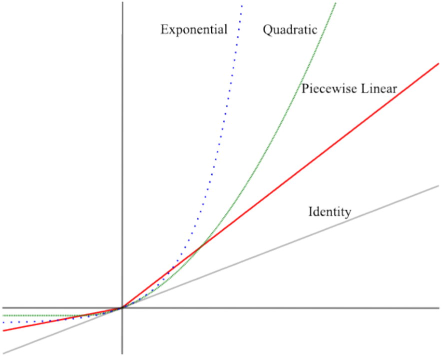

Optimized Certainty Equivalents

OCE as a Risk Measure

Theorem

- The optimized certainty equivalent ρ is a monetary risk measure
- Optimal allocation

$$\rho(X) = \inf_{\eta} \{E[l(\eta - X)] - \eta\} = E[l(\eta^* - X)] - \eta^*$$

where η^* fulfills

$$E\left[l'(\eta^* - X)\right] = 1$$

- Robust Representation

$$\rho(X) = \sup_{Q} \left\{ E_Q[-X] - E\left[l^*\left(\frac{dQ}{dP}\right)\right]\right\} = E_{Q^*}[-X] - E\left[l^*\left(\frac{dQ^*}{dP}\right)\right]$$

where l^* is the convex conjugate of l and $dQ^*/dP = l'(\eta^* - X)$

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Optimized Certainty Equivalents

OCE as a Risk Measure

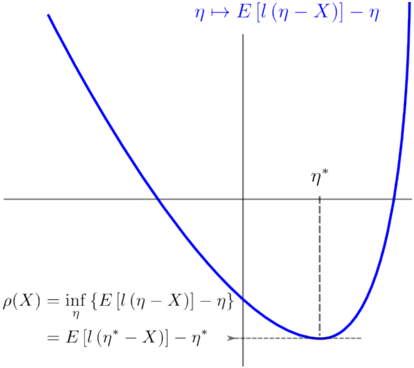
Sketch of Proof

- The function $\eta \mapsto E\left[l\left(\eta - X\right)\right] - \eta$ is real-valued, convex and coercive.
- Optimal allocation: first order conditions imply that η^* fulfills

$$E\left[l'\left(\eta^* - X\right)\right] = 1$$

and then

$$\rho(X) := \inf_{\eta} \left\{E\left[l\left(\eta - X\right)\right] - \eta\right\} = E\left[l\left(\eta^* - X\right)\right] - \eta^*$$



Optimized Certainty Equivalents

In a Nutshell

- Easy interpretation: optimal allocation of losses
- Adequate for financial optimization problems (Cherny, Kupper)
- Wide class of monetary risk measures by specifying the loss function l

l	RM	η^*	$\rho(X)$
$e^x - 1$	Entropic	$-\ln E[e^{-X}]$	$\ln E[e^{-X}]$
x^+/α	$CV@R_\alpha$	$q_X(\alpha)$	$-\frac{1}{\alpha}\int_0^\alpha q_X(s)ds$
$x + x^2/2$	Quadratic RM	$\rightsquigarrow$ monotone mean variance	
$\frac{([x+1]^+)^n - 1}{n}$	Polynomial RM		

Optimized Certainty Equivalents

In a Nutshell

An easy two step computation

1. Find the root η^* of

$$\eta \mapsto E\left[l'\left(\eta - X\right)\right] - 1 \tag{1}$$

2. Compute an integral

$$\rho(X) = E\left[l\left(\eta^* - X\right)\right] - \eta^* \tag{2}$$

Numerical Computation: Transform Methods

General Result

Ingredients to compute the risk of X

- Moment generating function: $M_X(u) = E[e^{uX}]$
- Fourier transform $\hat{f}(u) = \int e^{iux} f(x)dx$

Theorem

The optimal allocation η^* is the unique root of

$$\eta \longmapsto \frac{1}{2\pi} \int_{\mathbb{R}} e^{(a-iu)\eta} M_X(iu-a)\hat{l'}(u+ia)du - 1,$$

and the Optimized Certainty Equivalent is given by

$$\rho(X) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{(b-iu)\eta^*} M_X(iu-b)\hat{l}(u+ib)du - \eta^*$$

Here a, b are adequate constants.

Numerical Computation: Transform Methods

Conditional Value at Risk

Proposition

Given $\eta^* = q_X(\alpha)$ it holds

$$CV@R_\alpha(X) = -\frac{1}{2\pi\alpha}\int_{\mathbb{R}} \frac{e^{(b-iu)\eta^*}}{(u+ib)^2} M_X(iu-b)du - \eta^*,$$

Alternative representations

- Quantile Integration

$$CV@R_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha q_X(s)ds$$

  Increased complexity.

- Rockafellar and Uryasev

$$CV@R_\alpha(X) = \frac{1}{\alpha}E\left[[q_X(\alpha) - X]^+\right] - q_X(\alpha)$$

  Either Monte Carlo or direct integration if density $P_X(dx) = f_X(x)dx$ is known.

Numerical Computation: Transform Methods

Polynomial loss functions

Proposition

The optimal allocation is the unique root of

$$\eta \mapsto \frac{(\gamma-1)!}{2\pi}\int_{\mathbb{R}} M_X(iu-R)\frac{e^{(R-iu)(1+\eta)}}{(R-iu)^{\gamma}}du - 1. \tag{3}$$

Once η^* is determined, then

$$\rho(X) = \frac{(\gamma-1)!}{2\pi}\int_{\mathbb{R}} M_X(iu-R)\frac{e^{(R-iu)(1+\eta^*)}}{(R-iu)^{\gamma+1}}du - \frac{1}{\gamma} - \eta^*. \tag{4}$$

Remark

The same method applies to other risk measures, e.g. expectiles or shortfall risk measures

$$\rho_{SR} \mapsto E[l(-\rho_{SR} - X)] - \lambda$$

Numerical Computation: Transform Methods

Scenarios

This approach is particularly flexible for the following reasons:

- We only need the moment generating function of X
- A whole class of risk instruments parametrized by $\hat{l}$;
- We can immediately aggregate portfolios if the factors are independent, e.g. $X = \sum_{i=1}^{N} X_i$ where N is a random variable and X_i independent, then

$$M_X(v) = \sum_{k=1}^{\infty} P[N = k] \prod_{i=1}^{k} M_{X_i}(v)$$

- Weighted portfolios and loss models à la Dembo et al.
- Linear mixture models for dependence: $Y_1, \ldots, Y_m$ independent random variables, then the dependent factors $U = (U_1, \ldots, U_n)$ are defined via $U = AY$ for $A \in \mathbb{R}^{n \times m}$. The moment generating function of the risk factor $X = \sum_{i=1}^{n} U_i$ is provided by

$$M_X(u) = \prod_{l=1}^{m} M_{Y_l}(u\alpha_l), \qquad \alpha_l := \sum_{i=1}^{n} A_{il}$$

- Elliptical distributions, ...

Risk Contribution

OCE and Risk Contributions

OCE $\rightsquigarrow$ straightforward expression for risk contributions

Theorem

The contribution of a factor Y to the risk of a financial position X is given by

$$RC\,(X;Y) := \lim_{\varepsilon \downarrow 0} \frac{\rho(X+\varepsilon Y) - \rho(X)}{\varepsilon} = E\left[l'\left(\eta^* - X\right) Y\right]$$

where η^* fulfills

$$E\left[l'\left(\eta^* - X\right)\right] = 1$$

- Once again a two step computation with a root finding and an expectation.
- The optimal allocation η^* is already computed!

Risk Contribution

Risk Contribution with CV@R

- $X = \sum_{i=1}^{N} X_i$ aggregation for N lines of independent risks X_i
- $\eta^* = q_X(\alpha)$
- Goal: Contribution of the line X_i to the aggregated risk X.

Example (CV@R Risk Contribution)

$$RC(X; X_i) = \frac{1}{4\pi^2\alpha} \int_{\mathbb{R}^2} M(R + iu) \frac{e^{-(R_1+iu_1)\eta^*}}{(R_1 + iu_1)(u_1 - u_2 - iR_1 - iR_2)^2} du$$

where

$$M(u_1, u_2) = M_{X_i}(u_2) \prod_{j \neq i}^{n} M_{X_j}(u_1),$$

for adequate constants $R = (R_1, R_2)$.

2 dimensional integration! Monte Carlo needs to simulate N random variables.

Remark

In the linear mixture model: replace the moment generating function.

Numerical Results

NIG

The Moment generating function of NIG is given by

$$M_X(u) = \exp\left(u\mu + \delta\left[\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2}\right]\right),$$

where α = shape, β = skewness and δ = scaling parameters (zero mean)

	Parameters		
	α	β	δ
NIG_1	106.00	-26.00	0.0110
NIG_2	26.00	-10.60	0.0070
NIG_3	6.20	-3.90	0.0011
NIG_4	1.00	0.00	1.0000

Table : Parameter sets for NIG distributions.

Numerical Results

CV@R

Programmed in Matlab, R and Num/Sci-Py (same results)

	V@R		CV@R		
	Value	CT	Value	CT (F)	CT (S)
NIG_1	0.0210	92 ms	0.0298	99 ms	212 ms
NIG_2	0.0311	87 ms	0.0585	94 ms	359 ms
NIG_3	0.0073	88 ms	0.0352	97 ms	636 ms
NIG_4	1.5914	89 ms	2.2872	97 ms	197 ms

Table : Numerical results for V@R and CV@R at the 5% level.

	V@R		CV@R		
	Value	CT	Value	CT (F)	CT (S)
NIG_1	0.0350	95 ms	0.0444	104 ms	211 ms
NIG_2	0.0737	92 ms	0.1108	99 ms	360 ms
NIG_3	0.0369	88 ms	0.1162	100 ms	507 ms
NIG_4	2.7019	94 ms	3.4503	99 ms	194 ms

Table : Numerical results for V@R and CV@R at the 1% level.

Numerical Results

Polynomial loss function

Polynomial loss function, Fourier vs Stochastic Root Finding (Weber et al.)

	Fourier			SRF
	η^*	$\rho(X)$	CT(F)	CT
NIG_1	0.0028	-0.0027	62 ms	455 ms
NIG_2	0.0031	-0.0029	71 ms	449 ms
NIG_3	0.0008	-0.0007	129 ms	443 ms
NIG_4	-0.0957	0.4380	39 ms	448 ms

Table : Polynomial risk measure with $\gamma = 2$.

	$\gamma = 4$			$\gamma = 5$		
	η^*	$\rho(X)$	CT(F)	η^*	$\rho(X)$	CT(F)
NIG_1	0.0027	-0.0026	70 ms	0.0026	-0.0026	30 ms
NIG_2	0.0028	-0.0026	40 ms	0.0026	-0.0025	32 ms
NIG_3	0.0006	-0.0005	39 ms	0.0005	-0.0004	27 ms
NIG_4	-1.0283	1.4994	124 ms	-1.8095	2.3915	103 ms

Table : Polynomial risk measures with $\gamma = 4$ and $\gamma = 5$.

Conclusions

Optimized certainty equivalents and Fourier methods offer

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- very competitive computational times
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