

Jointly Exchangeable Collective Risk Models: Interaction, Structure, and Limit Theorems

Daniel Gaigall^{1,2}, Stefan Weber²

¹*FH Aachen - University of Applied Sciences*

²*House of Insurance, Leibniz University Hannover*

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Abstract

We introduce a framework for systemic risk modeling in insurance portfolios using jointly exchangeable arrays, extending classical collective risk models to account for interactions. Joint exchangeability is a more general probabilistic symmetric than de Finetti's exchangeability, characterized by the Aldous-Hoover-Kallenberg representation. We establish central limit theorems that asymptotically capture total portfolio losses, providing a theoretical foundation for approximations in large portfolios and over long time horizons. These approximations are validated through simulation-based numerical experiments. Additionally, we analyze the impact of dependence on portfolio loss distributions, with a particular focus on tail behavior.

Keywords: Central limit theorem; Collective risk model; Joint Exchangeability; Systemic risk; Contagion; Dependence; Insurance mathematics; Loss distributions

1 Introduction

Systemic risk is a major concern in finance and insurance. Idiosyncratic and systematic risks account for only part of the probabilistic variations in outcomes, while systemic interactions play a crucial role in financial markets, cyber networks, power grids, and other infrastructures. This paper focuses on modeling insurance portfolio losses within a generalized collective model, or frequency-severity model. Classical risk theory does not inherently capture systemic effects, highlighting the well-recognized need for more sophisticated models.

We introduce a probabilistic model for total losses, such as those of an insurance company, using the theory of jointly exchangeable arrays, a weaker notion than classical exchangeability. This approach extends beyond idiosyncratic and systematic risk by incorporating a specific systemic component: network interaction. Specifically, our framework generalizes de Finetti's exchangeability by preserving network interactions while allowing for relabeling, unlike conventional models that assume homogeneity after adjusting for covariates. Example specifications include random graphs and graphon models – such as Erdős-Rényi random networks – to describe transmissions and infections within an insurance portfolio. Additionally, conditional loss expenditures are modeled as dependent through jointly exchangeable arrays. The classical collective risk model emerges as a special case. Within this framework, we establish several limit theorems that asymptotically characterize the distribution of total portfolio losses.

E-Mail: gaigall@fh-aachen.de (Daniel Gaigall); stefan.weber@insurance.uni-hannover.de (Stefan Weber)

In insurance, systemic risk naturally arises in networked systems. Relevant examples include power grids and digital infrastructures, operating both at local and global scales, as well as contagion-type dynamics generated by cyber threats. Market sentiment can likewise be modeled within this framework. Recent surveys addressing these phenomena include [Awiszus et al. \(2023\)](#) and [ENISA \(2024\)](#). Our approach is most closely related to the systemic perspective of [Zeller & Scherer \(2022\)](#), while placing particular emphasis on underlying probabilistic symmetries and on the derivation of corresponding limit theorems.

For a systematic presentation of probabilistic symmetries, including joint exchangeability, we refer to [Aldous et al. \(1985\)](#) and [Kallenberg \(2005\)](#). Under the additional assumption that jointly exchangeable arrays are dissociated, limit results in distribution for related statistics are derived in [Silverman \(1976\)](#), with further results in [Eagleson & Weber \(1978\)](#). Jointly exchangeable distributions admit an ergodic decomposition that extends de Finetti's theorem, see [de Finetti \(1937\)](#). These structural results are developed in [Hoover \(1979\)](#), [Aldous \(1981\)](#), [Kallenberg \(1988\)](#), [Kallenberg \(1989\)](#), and [Kallenberg \(2005\)](#). The Choquet representation of the joint distribution as a mixture of dissociated arrays provides a natural reduction to this case.

Limit theorems for exchangeable arrays have been studied from various perspectives. [Barbour & Eagleson \(1985\)](#) examines dissociated arrays and generalizations of central limit theorems to U -statistics. [Davezies, D'Haultfœuille & Guyonvarch \(2021\)](#) extends these results to empirical processes. [Austern & Orbanz \(2022\)](#) develops a general theory of limit theorems within the framework of distributional symmetries defined by invariance under group transformations. Although this broader theory applies in our setting, we use the classical results of [Silverman \(1976\)](#) and [Eagleson & Weber \(1978\)](#), which are sufficient for our analysis.

Building on these theoretical foundations, we now turn to the implications for risk management and loss distribution modeling. Of particular interest are the properties of the underlying loss distribution. In our generalized collective model with interacting losses, the expectation and variance of the total loss of the insurance company can be explicitly computed. We establish asymptotic characterizations of the total loss distribution as the number of contracts or the time horizon grows. The central limit theorems we establish provide a useful tool for risk management and include Anscombe-type results; see [Anscombe \(1952\)](#). Simulations demonstrate that these asymptotic approximations remain accurate even for moderate portfolio sizes and relatively short time horizons.

The use of probabilistic interacting particle models to capture systemic risk in economic applications dates back to [Föllmer \(1974\)](#). In credit risk, [Giesecke & Weber \(2004, 2006\)](#) asymptotically characterize total portfolio losses under local interaction, specifically employing a voter model. In insurance, the collective model with systemic effects is particularly relevant for cyber risk and associated losses. For surveys on cyber insurance, see [Awiszus et al. \(2023\)](#), [Dacorogna & Kratz \(2023\)](#), [Eling \(2020\)](#), [ENISA \(2024\)](#), and [Knispel et al. \(2024\)](#). Closely related to our approach, [Zeller & Scherer \(2022\)](#) study a cyber collective model incorporating covariates but without network interactions. Other forms of local and global interactions in actuarial cyber risk models are studied in [Fahrenwaldt, Weber & Weske \(2018\)](#), [Hillairet & Lopez \(2021\)](#), [Hillairet et al. \(2022\)](#), [Bessy-Roland, Boumezoued & Hillairet \(2021\)](#), [Hillairet, Réveillac & Rosenbaum \(2023\)](#), and [Awiszus et al. \(2024\)](#), though without addressing limit theorems under joint exchangeability.

The main contributions of this paper are

- (i) a novel framework for systemic risk modeling in insurance portfolios based on jointly exchangeable arrays, extending classical collective risk models to incorporate interactions,
- (ii) asymptotic characterizations of total losses via central limit theorems, providing theoret-

ical justification for approximations in large portfolios and over long time horizons,

- (iii) simulation-based validation of the model and its approximations, demonstrating the practical relevance of the approach for risk management applications.

The structure of the paper is as follows. Section 2 introduces the extended collective model, discusses joint exchangeability, and provides illustrative examples. Section 3 presents limit theorems for total losses and characterizes their relation to classical collective models. All proofs are deferred to Appendix A. In Section 4, we conduct case studies, validate the accuracy of approximations, and analyze the impact of dependence. Finally, Section 5 concludes with a discussion of open research questions.

2 A collective loss model with interaction

In this paper, we are interested in a frequency-severity model of insurance losses, also called collective model, which includes a special form of contagious interactions. We will first explain the type of interaction on which we focus before, we describe the full dynamic model. The underlying probability space will be denoted by $(\Omega, \mathfrak{A}, P)$ and assumed to be atomless.

2.1 Contagious losses

We consider a collection of agents or, in a collective model, of loss occurrences, enumerated by the natural numbers \mathbb{N} . In the latter case, unlike some common conventions, losses of zero are also admissible. Throughout the paper, the terms agent and entity will be used interchangeably. We assume that agents can trigger losses of others, a phenomenon that can be captured by a graph as illustrated in Figure 1. Vertices represent entities, while edges correspond to idiosyncratic infections and transmission pathways. Infections and transmissions occur according to a random mechanism.

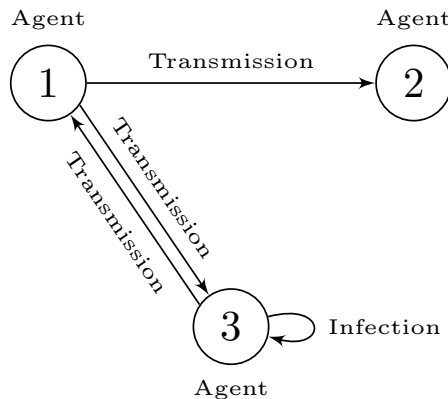


Figure 1: Agents, transmissions and infections as a directed graph.

The random graph can be encoded by an infinite matrix (also called array) $I = (I_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ with entries that are $\{0, 1\}$ -valued random variables. For $i \neq j$, entity j causes a loss at entity i , if $I_{i,j} = 1$; otherwise, $I_{i,j} = 0$. This definition does not require entity j itself to be infected. An infection of agent i is represented by $I_{i,i} = 1$; otherwise, $I_{i,i} = 0$. Loss expenditures resulting from infections and transmission are specified by an array $Z = (Z_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ whose entries are $[0, \infty)$ -valued random variables. For $i, j \in \mathbb{N}$, the random variable $Z_{i,j}$ captures the loss expenditure of policy holder i caused either by a transmission from entity j to agent i (if $i \neq j$) or by an infection of the entity itself (if $i = j$).

The resulting losses are given by the array $G = (G_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$, where

$$G_{i,j} = I_{i,j} \cdot Z_{i,j}, \quad i, j \in \mathbb{N}. \quad (2.1)$$

In other words, $G_{i,j}$ represents the actual loss incurred by entity i due to a transmission from agent j to i (if $i \neq j$) or due to an infection of the entity itself (if $i = j$).

2.2 Jointly exchangeable arrays

In this paper, we impose structure on the random mechanisms that generate the losses. Instead of relying on the classical notion of independence, we consider weaker concepts that allow for dependence. A fundamental, classical example is exchangeability, as introduced and characterized by Bruno de Finetti. While independence corresponds to the joint distribution being a product measure, exchangeability requires the joint distribution to be invariant under all finite permutations. De Finetti's theorem shows that exchangeability is equivalent to conditional independence, or, equivalently, to the joint distribution being a mixture of product measures. This framework aligns precisely with the setting of classical Bayesian statistics.

We introduce an even weaker probabilistic symmetry, tailored to our context. By using the collection of edges as an index set, we consider joint exchangeability, which requires invariance of the joint distribution only under a subgroup of finite permutations. This relaxation accommodates more general dependence structures than exchangeability while preserving a tractable framework for analysis. For a comprehensive exposition on this and other distributional symmetries, see [Kallenberg \(2005\)](#).

Although we focus on arrays of real-valued random variables, many definitions and results can often be stated in a more general form for random elements in measurable spaces. To ensure the existence of regular conditional distributions and simplify the application of standard results without additional verification, we assume that these spaces are Polish and maintain this assumption throughout.

Definition 1. Consider random elements $X_{i,j}$, $i, j \in \mathbb{N}$ in the Borel space (R, \mathfrak{G}) , defined on $(\Omega, \mathfrak{A}, P)$. The array $X = (X_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ is called jointly exchangeable if it has the same distribution as $(X_{\pi(i), \pi(j)})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ for all bijective $\pi : \mathbb{N} \rightarrow \mathbb{N}$.

The definition can equivalently be expressed in terms of finite permutations. As in the exchangeable case, joint exchangeability also allows for a characterization of the structure of joint distributions in terms of mixtures of ergodic measures. An expository discussion can be found in [Orbanz & Roy \(2015\)](#), while a detailed analysis is provided in [Kallenberg \(2005\)](#). Unlike product measures, the ergodic measures can be characterized through sampling schemes. This characterization is most naturally expressed through functional representations.

To clarify the distinction between exchangeability and joint exchangeability, we begin by considering an exchangeable array $X = (X_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$, meaning that its joint distribution remains invariant under permutations of the edges $(i, j) \in \mathbb{N} \times \mathbb{N}$. A functional representation of De Finetti's theorem states that there exist i.i.d. random variables $\xi, \xi_{i,j}$, $i, j \in \mathbb{N}$, uniformly distributed in $[0, 1]$, along with a measurable function $f : [0, 1]^2 \rightarrow R$, such that almost surely

$$X_{i,j} = f(\xi, \xi_{i,j}), \quad (i, j) \in \mathbb{N} \times \mathbb{N}.$$

Conditionally on ξ , the random elements $X_{i,j}$, $i, j \in \mathbb{N}$ are independent, with conditional distribution given by the probability kernel $\nu(\xi, B) = \int_{[0,1]} I(f(\xi, z) \in B) dz$, $B \in \mathfrak{G}$. In terms of this kernel and the distribution of ξ , the joint distribution can be expressed as a mixture of product measures, specifically, $\int_{[0,1]} \nu(z, \cdot)^{\otimes \infty} dz$.

The jointly exchangeable case (along with related symmetries such as contractibility or separate exchangeability) imposes a weaker invariance condition on the joint distribution, leading to a more general functional representation. As in the exchangeable case, we introduce i.i.d. random variables $\xi_{i,j}$, $i, j \in \mathbb{N}$, though their role will now differ slightly. Consider a jointly exchangeable array $X = (X_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$, meaning that its joint distribution remains invariant under permutations of the vertices $i \in \mathbb{N}$. The array X is not required to be symmetric. According to the Aldous-Hoover-Kallenberg theorem (see Hoover (1979), Aldous (1981), Kallenberg (1988), Kallenberg (1989)) such an array admits the following functional representation:

There exist i.i.d. random variables $\xi, \xi_i, \xi_{i,j}$, $i \leq j$, all uniformly distributed in $[0, 1]$, along with a measurable function $h : [0, 1]^4 \rightarrow R$, such that almost surely

$$X_{i,j} = h(\xi, \xi_i, \xi_j, \xi_{i,j}), \quad (i, j) \in \mathbb{N} \times \mathbb{N},$$

where we set $\xi_{j,i} := \xi_{i,j}$ for $i \leq j$. The latter symmetry assumption is an essential part of the functional representation, which characterizes the structure of the corresponding joint distribution through a sampling scheme. Conversely, any array that admits an Aldous-Hoover-Kallenberg representation is jointly exchangeable.

The distribution of the array can be decomposed as a mixture. To this end, we consider the conditional distribution of X given ξ , represented by a kernel $\bar{\nu}$ from $[0, 1]$ to $R^{\mathbb{N} \times \mathbb{N}}$. The original joint distribution is then characterized by the Choquet representation $\int_{[0,1]} \bar{\nu}(z) dz$. For each $z \in [0, 1]$, $\bar{\nu}(z)$ is the distribution of the array $(h(z, \xi_i, \xi_j, \xi_{i,j}))_{(i,j) \in \mathbb{N} \times \mathbb{N}}$, corresponding to the ergodic extremal points. These arrays are precisely those that are dissociated, a concept that we recall in the next definition.

Definition 2. Consider an array of random elements $X = (X_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ taking values in (R, \mathfrak{S}) . We say that X is dissociated if the subarrays $(X_{i,j})_{(i,j) \in A \times A}$ and $(X_{i,j})_{(i,j) \in B \times B}$ are independent whenever $A, B \subseteq \mathbb{N}$ and $A \cap B = \emptyset$.

Remark 3. A special case of jointly exchangeable arrays is given by infinite graphs (see Orbanz & Roy (2015)). In this case, we set $R = \{0, 1\}$. Focusing only on the case of undirected graphs, we obtain a functional representation by

$$X_{i,j} = I(\xi_{i,j} \leq W(\xi, \xi_i, \xi_j)), \quad i, j \in \mathbb{N},$$

where $W : [0, 1]^3 \rightarrow [0, 1]$ is a measurable function that is symmetric in its last two arguments. In the dissociated case, there is no dependence on ξ , and the corresponding function is of the form $W : [0, 1]^2 \rightarrow [0, 1]$. This function is often referred to as a graphon.

2.3 Mixtures and limit theorems

For sums of jointly exchangeable arrays,

$$U(n) = \sum_{1 \leq i < j \leq n} X_{i,j},$$

where $X = (X_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ takes values in the real numbers and $n \in \mathbb{N}$, limit results as $n \rightarrow \infty$ in distribution are available in Silverman (1976) and Eagleson & Weber (1978). Limit theorems for the jointly exchangeable case follow directly from those for the dissociated case. Consider the functional representation $X_{i,j} = h(\xi, \xi_i, \xi_j, \xi_{i,j})$, and define $\tilde{X}_{i,j}(z) = h(z, \xi_i, \xi_j, \xi_{i,j})$, for $i, j \in \mathbb{N}$, with joint distribution $\tilde{\nu}(z)$. Then, setting $\tilde{U}(n, z) = \sum_{1 \leq i < j \leq n} \tilde{X}_{i,j}(z)$, we obtain

$$P(U(n) \leq x) = \int_{[0,1]} P(\tilde{U}(n, z) \leq x) dz.$$

This shows that distributional properties of sums related to jointly exchangeable arrays can be derived from those of jointly exchangeable and dissociated arrays via integration or mixture distributions. The same applies to limit results as $n \rightarrow \infty$ in distribution, provided the dominated convergence theorem is used as an additional tool.

To simplify the presentation of our results, we introduce two assumptions that will be used repeatedly.

Assumption 1. *The array of random variables G is jointly exchangeable and satisfies $E(G_{i,j}^2) < \infty$ for all $i, j \in \mathbb{N}$.*

Assumption 2. *The array of random variables G is dissociated.*

2.4 Examples

Due to the Aldous-Hoover-Kallenberg representation, constructing examples is straightforward. According to equation (2.1), the array G is determined by infections and transmissions I and by conditional expenditures Z . If I and Z are independent of each other and satisfy either joint exchangeability, dissociatedness, or both, then G inherits the respective property. We provide specifications for I in Example 4 and for Z in Example 5; these arrays are jointly separable and dissociated. The functional representation naturally extends to the non-dissociated case by introducing a common factor variable. As discussed, the general joint distribution can be decomposed into ergodic components – i.e., dissociated distributions – via a Choquet representation.

Example 4. *a) Standard case: Infections (in the absence of transmissions) are*

$$I = \begin{pmatrix} J_1 & 0 & 0 & \dots \\ 0 & J_2 & 0 & \dots \\ 0 & 0 & J_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where J_i , $i \in \mathbb{N}$, are independent and identically distributed $\{0, 1\}$ -valued random variables with $P(J_i = 1) = p_J$ for some $p_J \in [0, 1]$. This example includes only infections but no transmissions. It corresponds to a classical insurance loss model. An illustration is provided in Figure 2a.

b) Erdős-Rényi model: Transmissions (in the absence of infections) are modeled by

$$I = \begin{pmatrix} 0 & K_1 & K_2 & \dots \\ K_1 & 0 & K_3 & \dots \\ K_2 & K_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Here, the $\{0, 1\}$ -valued random variables K_i , $i \in \mathbb{N}$, are independent and identically distributed with $P(K_i = 1) = p_K$ for some $p_K \in [0, 1]$. This example includes only symmetric transmissions, corresponding to an undirected random graph: the Erdős-Rényi model. An illustration is provided in Figure 2b.

c) Countermonotonic Erdős-Rényi model: This directed random graph is constructed from the undirected Erdős-Rényi model by tossing independent fair coins to assign a direction to each edge. Transmissions are given by

$$I = \begin{pmatrix} 0 & J_1 K_1 & J_2 K_2 & \dots \\ (1 - J_1) K_1 & 0 & J_3 K_3 & \dots \\ (1 - J_2) K_2 & (1 - J_3) K_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Here, $J_i, K_i, i \in \mathbb{N}$, are independent $\{0, 1\}$ -valued random variables. As before, $K_i, i \in \mathbb{N}$, are identically distributed with $P(K_i = 1) = p_K$ for some $p_K \in [0, 1]$. The “fair coins” $J_i, i \in \mathbb{N}$, are identically distributed with $P(J_i = 1) = 1/2$. An illustration is presented in Figure 2c.

d) Erdős-Rényi model with infections: This case combines the standard case of infections with undirected Erdős-Rényi transmissions, resulting in the array

$$I = \begin{pmatrix} J_1 & K_1 & K_2 & \dots \\ K_1 & J_2 & K_3 & \dots \\ K_2 & K_3 & J_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The independent J_i and $K_i, i \in \mathbb{N}$, are $\{0, 1\}$ -valued with $P(J_i = 1) = p_J$ for some $p_J \in [0, 1]$ and $P(K_i = 1) = p_K$ for some $p_K \in [0, 1]$. An illustration is provided in Figure 2d.

e) Contagion model: In this case, entities can be infected idiosyncratically. Additionally, transmissions from one entity to another can independently occur, if the former is infected. This is captured by the array

$$I = \begin{pmatrix} J_1 & J_2 K_{1,2} & J_3 K_{1,3} & \dots \\ J_1 K_{2,1} & J_2 & J_3 K_{2,3} & \dots \\ J_1 K_{3,1} & J_2 K_{3,2} & J_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The independent J_i and $K_{i,j}, i, j \in \mathbb{N}, i \neq j$, are $\{0, 1\}$ -valued with $P(J_i = 1) = p_J$ for some $p_J \in [0, 1]$ and $P(K_{i,j} = 1) = p_K$ for some $p_K \in [0, 1]$. An illustration is provided in Figure 2e.

Next, we consider specifications of conditional loss expenditures, focusing again on jointly exchangeable and dissociated examples.

Example 5. a) Independent conditional losses: If the loss expenditures $Z_{i,j}, i, j \in \mathbb{N}$, are independent and identically distributed, the model closely resembles the classical collective risk model. However, systemic risk typically requires incorporating dependencies, generalizing independence. This can be achieved, for example, by imposing only joint exchangeability, based on the Aldous-Hoover-Kallenberg representation.

b) Comonotonic conditional loss transmission: The random variable $Z_{i,j}$ represents the conditional loss expenditure from entity j to i . A specific type of contagion can be modeled by assuming that these losses depend only on their origin, i.e., for each i , the losses $Z_{i,j}$ are comonotonic. This means that there exists positive random variables $\tilde{Z}_j, j \in \mathbb{N}$, assumed to be i.i.d., and an increasing, measurable function $h : [0, \infty) \rightarrow [0, \infty)$ such that $Z_{i,j} = h(\tilde{Z}_j)$. In this simplest case $h(x) = x$ for all x , so that the conditional loss expenditures satisfy $Z_{i,j} = \tilde{Z}_j$ for all $i, j \in \mathbb{N}$.

c) Positively dependent conditional loss expenditure transmissions: More generally, we consider a measurable function $h : [0, \infty) \times [0, 1] \rightarrow [0, \infty)$ that is increasing in the first argument. Conditional losses are given by

$$Z_{i,j} = h(\tilde{Z}_j, \varepsilon_{i,j}), \quad i, j \in \mathbb{N},$$

where the i.i.d. and positive random variables $\tilde{Z}_j, j \in \mathbb{N}$, model the contagious component of the loss expenditures, while the independent shocks $\varepsilon_{i,j}, i, j \in \mathbb{N}$, uniformly distributed in $[0, 1]$, capture the idiosyncratic properties of loss expenditure transmissions. Of course, in specific models, one could alternatively choose different functions h and distributions for $\varepsilon_{i,j}$, where appropriate.

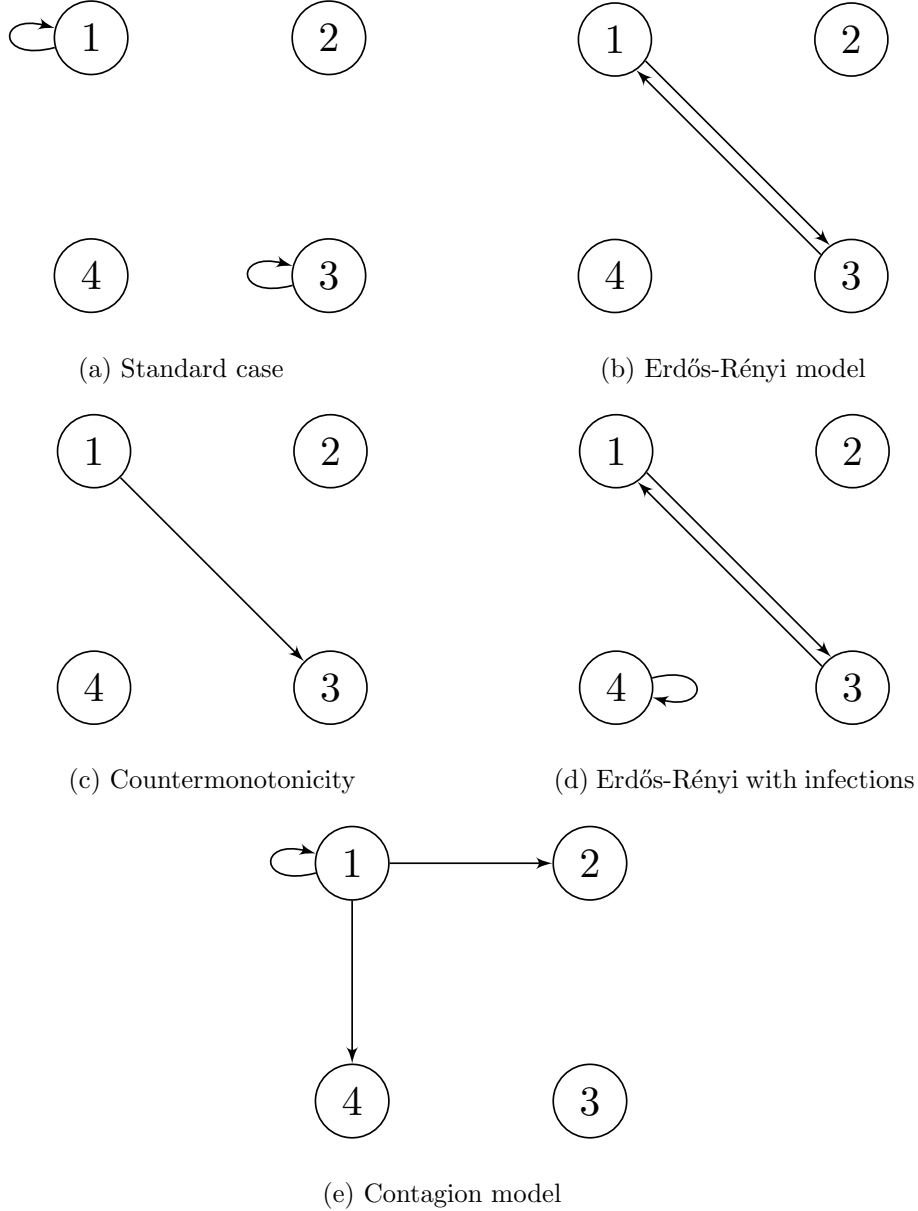


Figure 2: Transmissions and infections in the model specifications in Example 4.

2.5 The number of loss events

To model the number of loss events in a dynamic insurance portfolio, we introduce a counting process $N = (N(t))_{t \in [0, \infty)}$, i.e, a stochastic process indexed by $[0, \infty)$ with values in \mathbb{N}_0 , satisfying $N(0) = 0$ and having non-decreasing sample paths. For a given time $t \in [0, \infty)$, $N(t)$ represents the number of loss events up to time t . For technical reasons, we state the following moment assumption.

Assumption 3. *The counting process N satisfies $E(N(t)^2) < \infty$ for all $t \in [0, \infty)$.*

Various concrete specifications of N are possible.

Example 6. *a) The simplest case is a constant number of loss events, i.e.,*

$$N(t) \equiv m \text{ for some } m \in \mathbb{N}_0.$$

b) A common example is a homogeneous Poisson process with intensity $\lambda \in (0, \infty)$ that

satisfies $N(t) \sim \text{Poi}(\lambda t)$, $t \in [0, \infty)$, i.e., the number of loss events at time t follows a Poisson distribution with parameter λt . In this case, we have $\frac{N(t)}{t} \xrightarrow{\text{a.s.}} \lambda$, and, in particular, $N(t) \xrightarrow{\text{a.s.}} \infty$ as $t \rightarrow \infty$.

c) More generally, we may consider Cox processes. To construct these, we let T be a random time change, i.e., a stochastic process on $[0, \infty)$ with strictly increasing sample paths satisfying $\lim_{t \rightarrow \infty} T(t) = \infty$. Let \tilde{N} be an independent homogeneous Poisson process with rate $\lambda \in (0, \infty)$. The Cox process is then given by $N(t) = \tilde{N}(T(t))$, $t \in [0, \infty)$. The special case of an inhomogeneous Poisson process corresponds to the situation where T is deterministic and absolutely continuous with respect to Lebesgue measure.

2.6 The loss of the insurance company

To specify our full model, we introduce dynamic losses, extending equation (2.1).

Assumption 4. The arrays (G, I, Z) satisfy condition (2.1). For $k \in \mathbb{N}$, the tuples (G_k, I_k, Z_k) , $k \in \mathbb{N}$, are independent copies of (G, I, Z) . Moreover, the counting process N introduced in Section 2.5, the array (G, I, Z) , and the sequence of array tuples $(G_k, I_k, Z_k)_{k \in \mathbb{N}}$ are jointly independent.

To fix the notation, we adopt the convention that the loss event counter k is the last subscript, i.e., $G_{i,j,k} = I_{i,j,k} \cdot Z_{i,j,k}$, $i, j \in \mathbb{N}$.

We denote the loss incurred by entity i up to time t , caused either by transmissions from entity j (if $i \neq j$) or by infections (if $i = j$), by

$$X_{i,j}(t) = \sum_{k=1}^{N(t)} G_{i,j,k}.$$

Considering an insurance portfolio of entities indexed by $\{1, 2, \dots, n\}$, where $n \in \mathbb{N}$, the total loss experienced by entity $i = 1, \dots, n$ up to time t is

$$Y_i(n, t) = \sum_{j=1}^n X_{i,j}(t).$$

Finally, the total loss of the insurance company up to time t is given by the sum of the individual losses in the insurance portfolio:

$$S(n, t) = \sum_{i=1}^n Y_i(n, t).$$

Alternatively, this can be expressed as

$$S(n, t) = \sum_{k=1}^{N(t)} L_k(n),$$

where $L_k(n) = \sum_{i,j=1}^n G_{i,j,k}$ denotes the total loss incurred during loss event $k \in \mathbb{N}$.

3 Limit theorems

In this section, we derive several limit theorems for large portfolios, long-term horizons, and their combination. To be more specific, Section 3.1 provides preliminaries: the computations of the expectation and variance of the insurance company's loss. Section 3.2 establishes central limit

theorems for the total loss $S(n, t)$ as the number of insurance contracts $n \rightarrow \infty$ increases, while keeping the time point $t \in [0, \infty)$ fixed. Section 3.3 extends these results to large-portfolio limits over long time horizons. Section 3.4 presents central limit theorems for the total loss $S(n, t)$ when considering fixed portfolio sizes over extended time horizons. Finally, in Section 3.5, we discuss the relation of our model to the classical collective risk model.

3.1 Expectation and variance

Applying Wald's equations, we derive formulas for the expectation and variance of the insurance company's total loss. These results will be used in the characterization of asymptotic distributions in the limit theorems that we prove next.

Theorem 7. *Suppose that Assumptions 1–4 hold.*

a) *The expected total loss satisfies*

$$\mu_S(n, t) = \mathbb{E}(S(n, t)) = \mu_N(t)\mu_L(n),$$

where $\mu_N(t) = E(N(t))$, $\mu_L(n) = E(L_1(n))$ and

$$\mathbb{E}(L_1(n)) = n\mathbb{E}(G_{1,1}) + n(n-1)\mathbb{E}(G_{1,2}).$$

b) *The variance of the total loss is given by*

$$\sigma_S^2(n, t) = \text{Var}(S(n, t)) = \sigma_N^2(t)\mu_L(n)^2 + \mu_N(t)\sigma_L^2(n),$$

where $\sigma_N^2(t) = \text{Var}(N(t))$ and $\sigma_L^2(n) = \text{Var}(L_1(n))$ with

$$\begin{aligned} \text{Var}(L_1(n)) &= n\text{Var}(G_{1,1}) \\ &+ n(n-1) \\ &\times (2\text{Cov}(G_{1,1}, G_{1,2}) + 2\text{Cov}(G_{1,1}, G_{2,1}) + \text{Var}(G_{1,2}) + \text{Cov}(G_{1,2}, G_{2,1})) \\ &+ n(n-1)(n-2) \\ &\times (\text{Cov}(G_{1,2}, G_{1,3}) + \text{Cov}(G_{1,2}, G_{3,1}) + \text{Cov}(G_{2,1}, G_{1,3}) + \text{Cov}(G_{2,1}, G_{3,1})). \end{aligned}$$

3.2 Central limit theorems as the number of insurance contracts increases

Fixing the time horizon t , we first establish a central limit theorem for the large portfolio limit. The total loss of the insurance company up to time t can be expressed as $S(n, t) = \sum_{i,j=1}^n X_{i,j}(t)$ where the array of real-valued random variables $X(t) = (X_{i,j}(t))_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ satisfies $X_{i,j}(t) = \sum_{k=1}^{N(t)} G_{i,j,k}$ for $i, j \in \mathbb{N}$. Introducing the notation $X'_{i,j}(m) = \sum_{k=1}^m G_{i,j,k}$, $i, j \in \mathbb{N}$, we can equivalently write $S'(n, N(t)) = S(n, t)$ and $X'_{i,j}(N(t)) = X_{i,j}(t)$ in what follows.

For illustration, suppose that $N(t) \equiv m$ for some $m \in \mathbb{N}_0$. In this case, Assumptions 1–4 imply that the array of real-valued random variables $X'_{i,j}(m)$, $i, j \in \mathbb{N}$, is jointly exchangeable and dissociated. Consequently, statistical results for $S'(n, m)$ are available. We use this observation and apply the limit results in Silverman (1976) to $S'(n, m)$ by conditioning on the event $\{N(t) = m\}$ for $m \in \mathbb{N}_0$, leading to the following statement.

Theorem 8. *Suppose that Assumptions 1–4 hold and that $\sigma_L^2(n) > 0$ for n sufficiently large.*

a) *For any $m \in \mathbb{N}_0$, we have*

$$\frac{S'(n, m) - m\mu_L(n)}{\sigma_L(n)} \xrightarrow{d} \eta(m) \text{ as } n \rightarrow \infty,$$

where the real-valued random variable $\eta(m)$ has the distribution function

$$P(\eta(m) \leq x) = \Phi_{0,m}(x)$$

for $x \in \mathbb{R}$, with Φ_{μ,σ^2} denoting the distribution function of $N(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma^2 \in (0, \infty)$, and with $\mu_L(n)$ and $\sigma_L(n)$ as defined in Theorem 7.

b) The total portfolio loss $S(n, t) = S'(n, N(t))$ converges in distribution as $n \rightarrow \infty$:

$$\frac{S'(n, N(t)) - N(t)\mu_L(n)}{\sigma_L(n)} \xrightarrow{d} \eta \text{ as } n \rightarrow \infty,$$

where the real-valued random variable η has the distribution function

$$P(\eta \leq x) = \sum_{m=0}^{\infty} P(N(t) = m)\Phi_{0,m}(x)$$

for $x \in \mathbb{R}$.

Remark 9. The previous result can be directly applied in practical settings to approximate the distribution of the total loss $S(n, t)$ by a mixture of normal distributions for sufficiently large n . Suppose that the hypotheses of Theorem 8 are satisfied. Then, for $x \in \mathbb{R}$ and sufficiently large n , the distribution function of the total loss is approximately given by

$$P(S'(n, N(t)) \leq x) \approx \sum_{m=0}^{\infty} P(N(t) = m)\Phi_{m\mu_L(n), m\sigma_L^2(n)}(x).$$

3.3 Central limit theorems as the number of insurance contracts and the point in time increase

A modified version of Theorem 8 also holds in the case where time increases simultaneously with the number of contracts. Dividing both sides of Theorem 8 a) by \sqrt{m} , we obtain convergence to a standard normal distribution as $n \rightarrow \infty$ for all m . This suggests that an extension to the joint limit $m, n \rightarrow \infty$ might seem trivial. However, while the limit theorem for each fixed m ensures convergence as $n \rightarrow \infty$, the simultaneous limit introduces additional challenges. In particular, uniformity in m is not automatic, and without appropriate control, the dependence of n on m could influence the limiting behavior. Nevertheless, in our setting, we show that these issues do not arise, thereby establishing the validity of the joint limit.

To this end, we recall the expression for the total loss of the insurance company, $S(n, t) = \sum_{k=1}^{N(t)} L_k(n)$, where our assumptions ensure that the random variables $L_k(n) = \sum_{i,j=1}^n G_{i,j,k}$, $k \in \mathbb{N}$, are independent and identically distributed. We will verify Lindeberg's condition and subsequently apply the central limit theorem of Lindeberg-Feller.

Theorem 10. Suppose that Assumptions 1–4 hold and that $\sigma_L^2(n) > 0$ for sufficiently large n .

a) Then convergence in distribution jointly in $n, m \rightarrow \infty$ holds:

$$\frac{S'(n, m) - m\mu_L(n)}{\sqrt{m}\sigma_L(n)} \xrightarrow{d} N(0, 1) \text{ as } n, m \rightarrow \infty,$$

where $\mu_L(n)$ and $\sigma_L^2(n)$ are defined in Theorem 7.

b) If, in addition, $N(t) \xrightarrow{a.s.} \infty$ as $t \rightarrow \infty$, then for $S(n, t) = S'(n, N(t))$, we have convergence in distribution:

$$\frac{S'(n, N(t)) - N(t)\mu_L(n)}{\sqrt{N(t)}\sigma_L(n)} \xrightarrow{d} N(0, 1) \text{ as } n, t \rightarrow \infty.$$

Clearly, Theorem 10 can be applied to approximate $S(n, t) = S'(n, N(t))$ for sufficiently large n and t , similarly to Remark 9.

3.4 Central limit theorems as the point in time increases

We now consider central limit theorems for the total loss of the insurance company as the time horizon $t \rightarrow \infty$ increases, while the number of insurance contracts $n \in \mathbb{N}$ remains fixed. As before, these results can be used for loss approximations, as explained in Remark 9.

Theorem 11. *Suppose that Assumptions 1–4 hold.*

a) *If $\sigma_L^2(n) > 0$, then we obtain convergence in distribution*

$$\frac{S'(n, m) - m\mu_L(n)}{\sqrt{m}\sigma_L(n)} \xrightarrow{d} N(0, 1) \text{ as } m \rightarrow \infty,$$

where $\mu_L(n)$ and $\sigma_L^2(n)$ are defined in Theorem 7.

b) *If $\sigma_L^2(n) > 0$ and in addition $N(t) \xrightarrow{a.s.} \infty$ as $t \rightarrow \infty$, then for $S(n, t) = S'(n, N(t))$, we have convergence in distribution:*

$$\frac{S'(n, N(t)) - N(t)\mu_L(n)}{\sqrt{N(t)}\sigma_L(n)} \xrightarrow{d} N(0, 1) \text{ as } t \rightarrow \infty.$$

Of course, Theorem 11 can be used to approximate the loss distribution of $S(n, t) = S'(n, N(t))$ for fixed n and sufficiently large t , analogous to Remark 9.

We now study special cases related to Poisson counting processes. First, we consider a homogeneous Poisson process, exploiting the fact that Poisson processes are Lévy processes, meaning that the increments $N(\lfloor t \rfloor) - N(\lfloor t \rfloor - 1), \dots, N(1) - N(0)$ are independent and identically distributed. This property enables a particularly convenient approximation of the insurance company's loss distribution in applications; see Remark 13. Second, we extend the result to Cox processes, which are defined in terms of increasing random time changes T ; see Example 2.4.

Theorem 12. *Suppose that Assumptions 1–4 hold.*

a) *If $\sigma_S^2(n, t) > 0$ for sufficiently large t and the counting process $N = (N(t))_{t \in [0, \infty)}$ is a homogeneous Poisson process with intensity $\lambda \in (0, \infty)$, then for $S(n, t) = S'(n, N(t))$, we have convergence in distribution:*

$$\frac{S'(n, N(t)) - \lambda t \mu_L(n)}{\sqrt{\lambda t (\mu_L(n)^2 + \sigma_L^2(n))}} \xrightarrow{d} N(0, 1) \text{ as } t \rightarrow \infty,$$

where $\mu_L(n)$ and $\sigma_L^2(n)$ are defined in Theorem 7.

b) *Suppose that $\sigma_S^2(n, t) > 0$ for sufficiently large t and that the jump process $N = (N(t))_{t \in [0, \infty)}$ is of the form $N(t) = \tilde{N}(T(t))$, $t \in [0, \infty)$, where $\tilde{N} = (\tilde{N}(t))_{t \in [0, \infty)}$ is a homogeneous Poisson process with intensity $\lambda \in (0, \infty)$, and $T = (T(t))_{t \in [0, \infty)}$ is a stochastic process with values in $[0, \infty)$ satisfying $\lim_{t \rightarrow \infty} T(t) = \infty$ almost surely. Assume further that the random elements \tilde{N} , T , and G_k , $k \in \mathbb{N}$, are independent. Then, for $S(n, t) = S'(n, \tilde{N}(T(t)))$, we have the convergence in distribution:*

$$\frac{S'(n, \tilde{N}(T(t))) - \lambda T(t) \mu_L(n)}{\sqrt{\lambda T(t) (\mu_L(n)^2 + \sigma_L^2(n))}} \xrightarrow{d} N(0, 1) \text{ as } t \rightarrow \infty.$$

Remark 13. *The previous result can be applied in practice to approximate the distribution of the insurance company's loss $S(n, t)$ for sufficiently large t . Suppose that the hypotheses of*

Theorem 12 hold and consider the setting of part a). Letting Φ_{μ, σ^2} be defined as in Theorem 8 and letting $x \in \mathbb{R}$, we obtain for sufficiently large t :

$$P(S'(n, N(t)) \leq x) \approx \Phi_{\lambda t \mu_L(n), \lambda t (\mu_L(n)^2 + \sigma_L^2(n))}(x).$$

Suppose that the hypotheses of Theorem 12 hold and consider the setting of part b). Denoting by $\mu_{T(t)}$ the distribution of $T(t)$ and letting $x \in \mathbb{R}$, we obtain for sufficiently large t :

$$P(S'(n, \tilde{N}(T(t))) \leq x) \approx \int \Phi_{\lambda s \mu_L(n), \lambda s (\mu_L(n)^2 + \sigma_L^2(n))}(x) d\mu_{T(t)}(s).$$

3.5 Relation to the collective risk model

Before discussing numerical case studies, another important aspect deserves emphasis—namely, that under certain conditions, the proposed model can be reformulated as a classical frequency-severity model, also known as the collective model. These conditions require that the arrays I and Z are independent and that the conditional loss expenditures $Z_{i,j}$, $i, j \in \mathbb{N}$, are independent, as discussed in Example 5 a). To align with the notation of a classical collective model, the frequency component must be adjusted, while the severity component follows directly from the conditional loss expenditures. In this section, we briefly discuss this transformation. Our notation follows Assumption 4.

Suppose that Assumptions 1–4 hold and that the conditional loss expenditures $Z_{i,j}$, $i, j \in \mathbb{N}$, are independent and identically distributed. Let \tilde{Z}_i , $i \in \mathbb{N}$, denote another independent sequence of i.i.d. random variables such that $\tilde{Z}_1 \stackrel{d}{=} Z_{1,1}$. The total number of claims up to time t is given by

$$M(n, t) = \sum_{k=1}^{N(t)} \sum_{i=1}^n \sum_{j=1}^n I_{i,j,k}.$$

Defining $M(n) = (M(n, t))_{t \in [0, \infty)}$, the random elements $M(n)$ and \tilde{Z}_i , $i \in \mathbb{N}$, are jointly independent.

Theorem 14. *Under the assumptions outlined above, we obtain equality in distribution with a collective model:*

$$(S(n, t), M(n, t)) \stackrel{d}{=} \left(\sum_{i=1}^{M(n, t)} \tilde{Z}_i, M(n, t) \right).$$

Remark 15. *In the absence of transmissions and under independent infections, as described in Example 4 a), the results of Theorem 14 simplify further. In this case, the total number of claims is given by $M(n, t) = \sum_{k=1}^{N(t)} \sum_{i=1}^n I_{i,i,k}$. We highlight two special cases: a) If $N(t) \equiv 1$, then the claim count follows a binomial distribution: $M(n, t) \sim \text{Bin}(n, p_J)$. b) If N is a homogeneous Poisson process with intensity $\lambda \in (0, \infty)$, and if $n = 1$ and $p_J = 1$, then the claim count follows a Poisson distribution: $M(n, t) \sim \text{Poi}(\lambda t)$.*

4 Approximations and interaction effects – case studies

In this section, we present numerical experiments implemented in the statistical software R. Section 4.1 outlines the general setup of the case studies and provides moment computations required for the approximations derived in Section 3. In Section 4.2, we assess the quality of selected distributional approximations established in Section 3. Finally, Section 4.3 examines in greater detail the impact of dependencies in loss occurrences and conditional loss exposures on the distribution of total losses, with a particular focus on their tails.

4.1 Implementation of the simulation studies

In the numerical case studies, we will always assume that Assumptions 1–4 hold. Infections and transmissions I will be modeled using the Erdős-Rényi model with infections (Example 4 d)) and the contagion model (Example 4 e)). We focus on positively dependent conditional loss expenditures Z (Example 5 c)). To simplify notation, we do not assume that the independent shocks are uniformly distributed on $[0, 1]$, but instead belong to a parametric family of distributions. Specifically, we set

$$Z_{i,j} = \tilde{Z}_j + \varepsilon_{i,j}, \quad i, j \in \mathbb{N}, \quad (4.1)$$

where the contagious components \tilde{Z}_j , $j \in \mathbb{N}$, are independent, $[0, \infty)$ -valued random variables, each following a gamma distribution $G(\nu, \kappa)$ with a common shape parameter $\nu \in (0, \infty)$ and scale parameter $\kappa \in (0, \infty)$. The gamma distribution forms a rich family that includes the exponential, chi-squared, and Erlang distributions. The shocks $\varepsilon_{i,j}$, $i, j \in \mathbb{N}$, are independent random variables following a half-normal distribution $HN(\sigma^2)$ with parameter $\sigma^2 \in (0, \infty)$. Further replications of all random variables are generated according to Assumption 4, with the loss event counter k added as the last subscript. The counting process N is taken to be a Poisson process with intensity $\lambda \in (0, \infty)$. Using Theorem 7, we explicitly compute the relevant moments for the case studies:

(i) **Erdős-Rényi model with infections** (Example 4 d))

$$\begin{aligned} \mu_L(n) &= np_J \left(\nu\kappa + \sqrt{\frac{2\sigma^2}{\pi}} \right) + n(n-1)p_K \left(\nu\kappa + \sqrt{\frac{2\sigma^2}{\pi}} \right) \\ \sigma_L^2(n) &= n \left(p_J \left(\nu\kappa^2 + \nu^2\kappa^2 + 2\nu\kappa\sqrt{\frac{2\sigma^2}{\pi}} + \sigma^2 \right) - p_J^2 \left(\nu\kappa + \sqrt{\frac{2\sigma^2}{\pi}} \right)^2 \right) \\ &\quad + n(n-1) \left(p_K \left(\nu\kappa^2 + \nu^2\kappa^2 + 2\nu\kappa\sqrt{\frac{2\sigma^2}{\pi}} + \sigma^2 \right) - p_K^2 \left(\nu\kappa + \sqrt{\frac{2\sigma^2}{\pi}} \right)^2 \right) \\ &\quad + 2p_K p_K \nu\kappa^2 + (p_K - p_K^2) \left(\nu\kappa + \sqrt{\frac{2\sigma^2}{\pi}} \right)^2 \\ &\quad + n(n-1)(n-2)p_K^2 \nu\kappa^2 \\ \mu_S(n, t) &= \lambda t \mu_L(n) \\ \sigma_S^2(n, t) &= \lambda t \mu_L^2(n) + \lambda t \sigma_L^2(n) \end{aligned}$$

(ii) **Contagion model** (Example 4 e))

$$\begin{aligned}
\mu_L(n) &= np_J \left(\nu\kappa + \sqrt{\frac{2\sigma^2}{\pi}} \right) + n(n-1)p_J p_K \left(\nu\kappa + \sqrt{\frac{2\sigma^2}{\pi}} \right) \\
\sigma_L^2(n) &= n \left(p_J \left(\nu\kappa^2 + \nu^2\kappa^2 + 2\nu\kappa\sqrt{\frac{2\sigma^2}{\pi}} + \sigma^2 \right) - p_J^2 \left(\nu\kappa + \sqrt{\frac{2\sigma^2}{\pi}} \right)^2 \right) \\
&\quad + n(n-1) \left(p_J p_K \left(\nu\kappa^2 + \nu^2\kappa^2 + 2\nu\kappa\sqrt{\frac{2\sigma^2}{\pi}} + \sigma^2 \right) - p_J^2 p_K^2 \left(\nu\kappa + \sqrt{\frac{2\sigma^2}{\pi}} \right)^2 \right) \\
&\quad + 2p_J p_K \left(\nu\kappa^2 + \nu^2\kappa^2 + 2\nu\kappa\sqrt{\frac{2\sigma^2}{\pi}} + \frac{2\sigma^2}{\pi} \right) - 2p_J^2 p_K \left(\nu\kappa + \sqrt{\frac{2\sigma^2}{\pi}} \right)^2 \\
&\quad + n(n-1)(n-2) \left(p_J p_K^2 \left(\nu\kappa^2 + \nu^2\kappa^2 + 2\nu\kappa\sqrt{\frac{2\sigma^2}{\pi}} + \frac{2\sigma^2}{\pi} \right) - p_J^2 p_K^2 \left(\nu\kappa + \sqrt{\frac{2\sigma^2}{\pi}} \right)^2 \right) \\
\mu_S(n, t) &= \lambda t \mu_L(n) \\
\sigma_S^2(n, t) &= \lambda t \mu_L^2(n) + \lambda t \sigma_L^2(n)
\end{aligned}$$

4.2 Accuracy of the approximations

We investigate the quality of selected distributional approximations suggested by our findings in Section 3 through simulation. We consider the model specifications described in Section 4.1. In our simulations, we fix the model parameters as follows: $p_J = 0.25$, $p_K = 0.35$, $\nu = 0.75$, $\kappa = 1.75$, $\sigma^2 = 1.5$, and $\lambda = 1$.

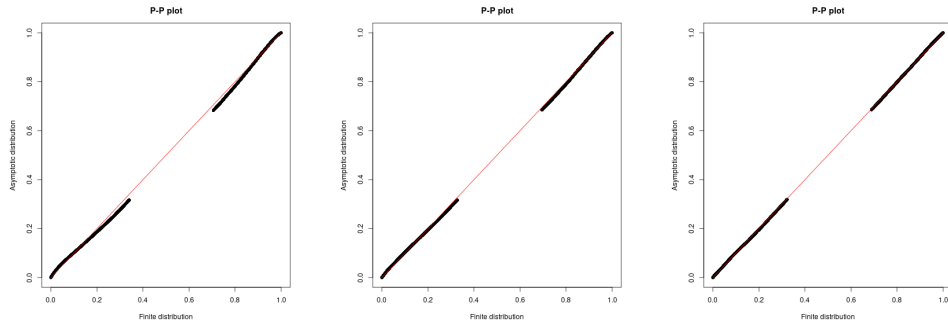
Theorem 8 addresses the case where the number of insurance contracts increases. Here, we fix the time point at $t = 1$ and vary the number of insurance contracts as $n = 15, 50, 200$. Theorem 10 establishes asymptotics when both the number of insurance contracts and the time horizon increase jointly. We examine the approximations for the pairs $(n, t) = (15, 15), (50, 25), (200, 80)$. Finally, we assess the quality of the approximation in Theorem 11, where the number of contracts remains fixed while the time horizon increases. We set the number of insurance contracts to $n = 10$ and vary the time points as $t = 15, 25, 80$. To obtain the distributions under consideration, we perform Monte Carlo simulations with 100,000 replications for each scenario.

Figure 3 presents P-P and Q-Q plots for the distributions in Theorem 8 b), where the Monte Carlo estimate of the finite-sample distribution is plotted on the abscissa against the asymptotic distribution on the ordinate. Overall, the accuracy of the finite-sample approximation improves as the number of insurance contracts n increases, ultimately reaching a high level of precision, confirming the practical applicability of Theorem 8. More specifically, for $n = 15$, both the P-P and Q-Q plots indicate that the approximation is not yet highly accurate. A likely contributor to deviations in the upper tail is the heavier tail of the gamma distribution used for modeling loss expenditures in the finite distribution compared to the normal distributions in the asymptotic case. For $n = 50$, the P-P plots already suggest a good approximation in the central part of the distribution, though small deviations remain in the tails. For $n = 200$, the Q-Q plots indicate a highly precise approximation across the entire distribution. These findings hold for both the Erdős-Rényi model with infections and the contagion model. The empty regions in the P-P plots result from the strictly positive probability mass at zero in the underlying distributions.

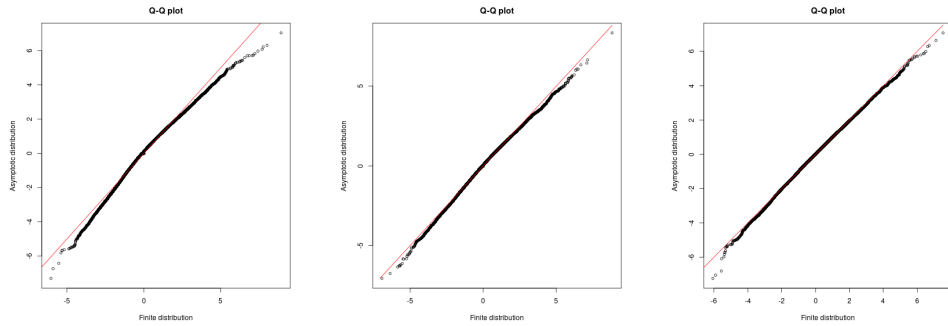
Figure 4 also presents P-P and Q-Q plots for approximations based on Theorem 10 b), where the finite-sample distribution on the abscissa is plotted against the asymptotic distribution on the ordinate. The accuracy of the finite-sample approximation improves as both the number of insurance contracts n and the time horizon t increase, ultimately reaching a high level of precision, confirming the practical applicability of Theorem 10. More specifically, for

$(n, t) = (15, 15)$, the P-P and Q-Q plots indicate small deviations in the approximation. We observe discrepancies in the upper tail of the distribution, likely due to the heavier tail of the gamma distribution compared to the asymptotic normal distribution. For $(n, t) = (50, 25)$, the approximation improves significantly, though minor deviations remain in the tails. Finally, for $(n, t) = (200, 80)$, remaining deviations essentially vanish. These results hold for both the Erdős-Rényi model with infections and the contagion model. Overall, the approximation appears to perform better than in the setting of Theorem 8 for a given portfolio size n . This improvement is due to the fact that in Theorem 10, both n and t tend to infinity.

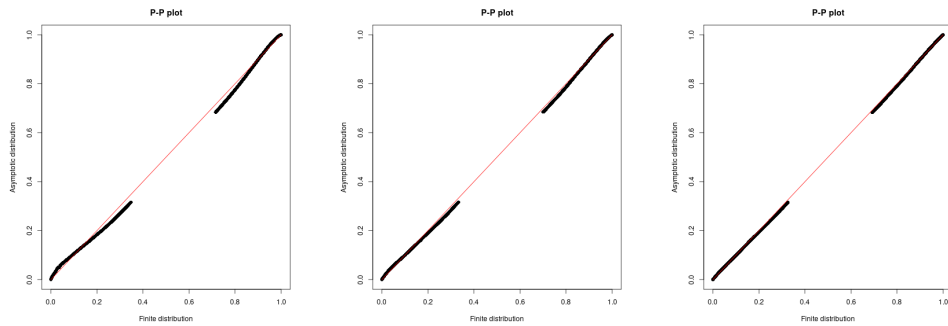
Figure 5 presents P-P and Q-Q plots for the distributions in Theorem 12 a), where the empirical distribution on the abscissa is plotted against the asymptotic distribution on the ordinate. The accuracy of the approximation improves as the time horizon t increases. A closer inspection of the P-P and Q-Q plots for $t = 15$ reveals that the approximation is not yet highly accurate. As before, deviations in the upper tail are likely due to the difference in tail heaviness between the gamma distribution used for modeling loss expenditures in the finite distribution and the asymptotic normal distribution. For $t = 25$, the P-P plots already indicate a good approximation in the central part of the distribution, while the Q-Q plots still show some deviations in the tails. By $t = 80$, these tail deviations are further reduced in the Q-Q plots. In general, the approximation in Theorem 10 appears to perform better compared to Theorem 12 for the same values of t . The likely reason is that in Theorem 10, both n and t tend to infinity, whereas in Theorem 12, n is fixed and only t tends to infinity.



Erdős-Rényi with infections, $n = 15$. Erdős-Rényi with infections, $n = 50$. Erdős-Rényi with infections, $n = 200$.



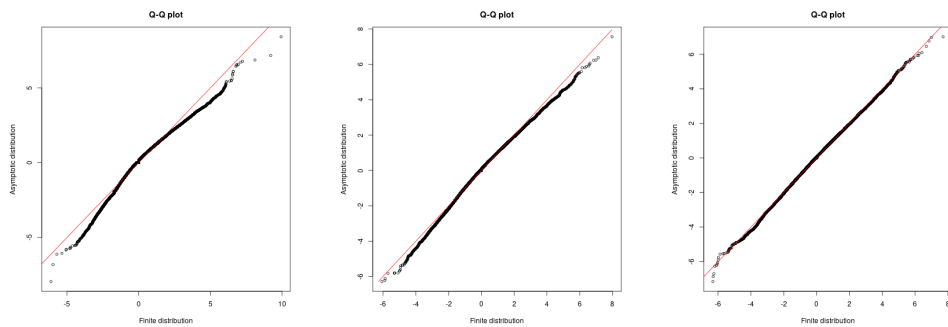
Erdős-Rényi with infections, $n = 15$. Erdős-Rényi with infections, $n = 50$. Erdős-Rényi with infections, $n = 200$.



Contagion case, $n = 15$.

Contagion case, $n = 50$.

Contagion case, $n = 200$.



Contagion case, $n = 15$.

Contagion case, $n = 50$.

Contagion case, $n = 200$.

Figure 3: Plots related to Theorem 8.

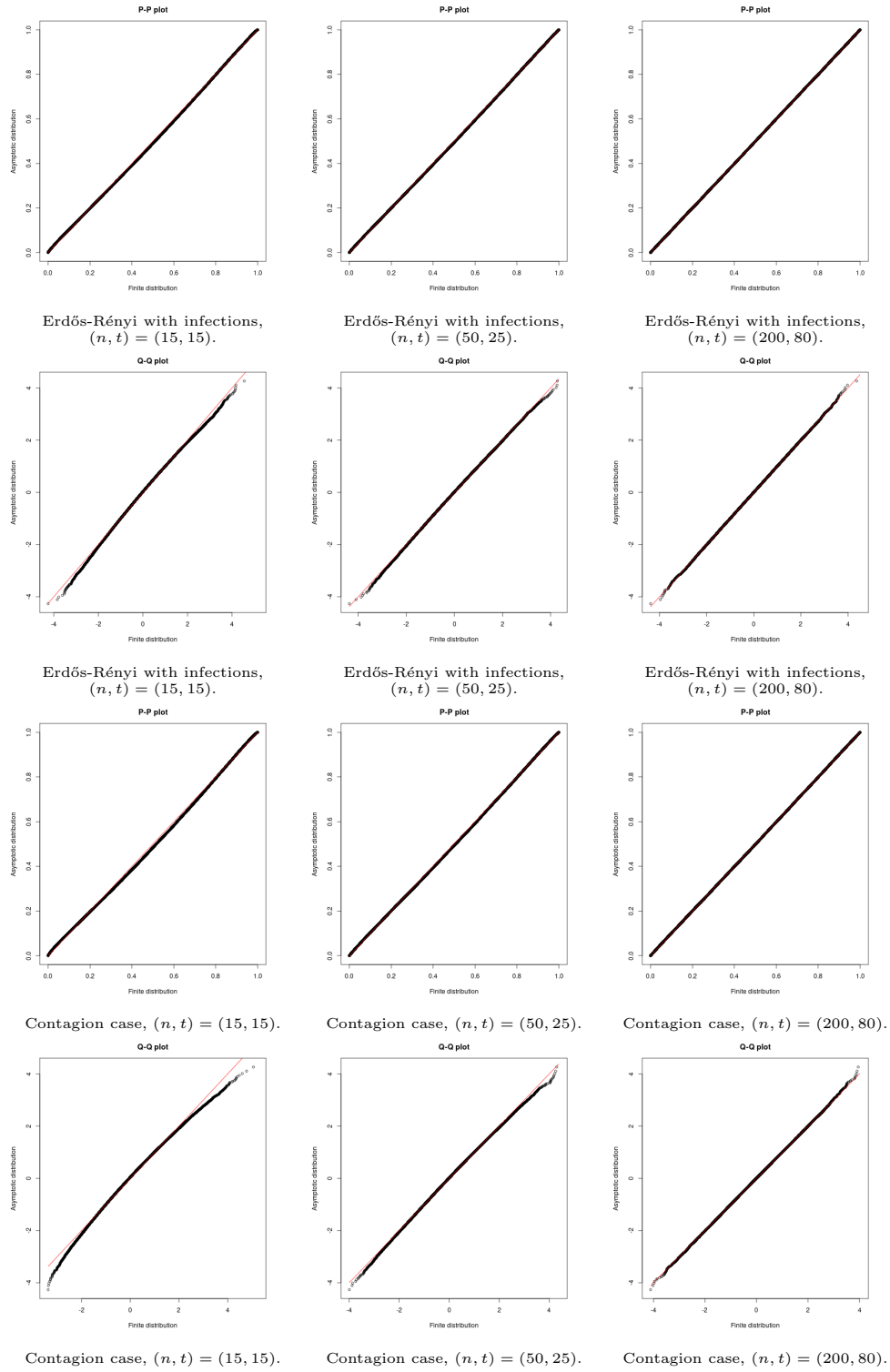


Figure 4: Plots related to Theorem 10.

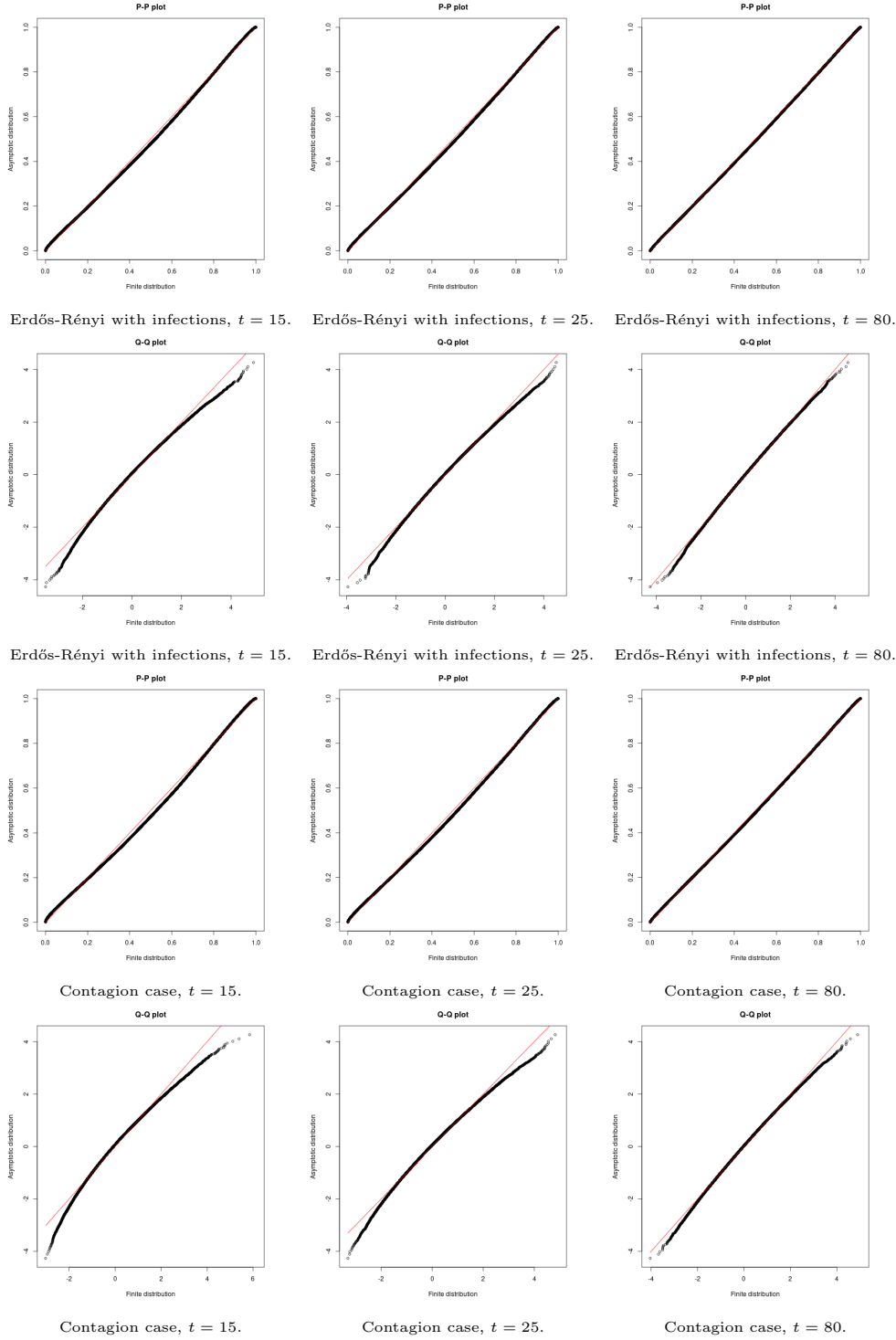


Figure 5: Plots related to Theorem 12.

4.3 The impact of interaction

We numerically compare the interaction-augmented model specified in Section 4.1 with the classical collective risk model. To facilitate a direct comparison with a binomial claim count distribution, as described in Remark 15, we consider a single loss event by conditioning on $N(t) \equiv 1$.

Using Monte Carlo simulations and asymptotic approximations from Section 3, we compare

the model in Section 4.1 to two alternative settings. The setup in Section 4.1 is referred to as the *dependent* case, where infections and transmissions I follow either the Erdős-Rényi model with infections (Example 4 d)) or the contagion model (Example 4 e)). Positively dependent conditional loss expenditures Z are structured according to equation (4.1). Numerical results are presented in Table 1 and Table 2. In both cases, we compare the model from Section 4.1 to a corresponding *independent* case and a *standard* case, where the standard case remains identical across both tables—apart from minor differences due to Monte Carlo sampling.

The *independent* case uses the same model for infections and transmissions I as the dependent case. Independence pertains to the conditional loss expenditures Z , which follow a modified version of equation (4.1). Specifically, we set

$$Z_{i,j} = \tilde{Z}_{i,j} + \varepsilon_{i,j}$$

with independent $\tilde{Z}_{i,j} \stackrel{d}{=} \tilde{Z}_j$, $i, j \in \mathbb{N}$. Thus, the contagious components \tilde{Z}_j , $j \in \mathbb{N}$, are replaced by idiosyncratic loss expenditures with the same distribution. This can be interpreted as a collective model according to Theorem 14, where the claims count process is governed by the infections and transmissions I . The *standard* case further modifies the matrix I in line with Example 5 a), removing all transmissions and assuming independent infections. The corresponding claims count process follows a binomial distribution, as described in Remark 15.

The numerical results of the comparison, rounded to one decimal place, are presented in Table 1 for the Erdős-Rényi model with infections and in Table 2 for the contagion model. For the comparison, we focus on the upper quantiles of the distribution of the total loss of the insurance company, $S'(n, 1)$, which are particularly relevant in risk management. Specifically, we report the quantiles $q_{0.5}$, $q_{0.75}$, $q_{0.95}$, and $q_{0.995}$, corresponding to probability levels 0.5, 0.75, 0.95, and 0.995, respectively. The latter level is of particular importance in the context of the Solvency II directive. Additionally, we provide the associated quantile differences $q_{0.75} - q_{0.5}$, $q_{0.95} - q_{0.5}$, and $q_{0.995} - q_{0.5}$ to capture the dispersion in the tail of the distribution. Results are displayed for portfolio sizes $n = 15, 50, 200$, considering the dependent case, the independent case, and the standard case as defined above. To approximate the quantiles, we employ two methods: first, Monte Carlo simulations based on 100,000 replications in each case; second, asymptotic approximations derived from Theorem 8 a), as discussed in Remark 9.

Comparing the Erdős-Rényi model with infections in Table 1 to the contagion model in Table 2, we observe that, in both the dependent and independent cases, the quantiles are larger in the Erdős-Rényi model. This is primarily due to the fact that contagion in the contagion model occurs only conditionally on an infection, whereas in the Erdős-Rényi model, transmissions directly trigger loss occurrences.

We now focus on observations within each of the two tables, which exhibit qualitatively similar patterns. In both the dependent and independent cases, the medians (quantiles at level 0.5) are of comparable magnitude. However, the difference between the quantiles in the dependent case and their counterparts in the independent case increases as the quantile level rises. Consequently, the quantile distances also increase with higher quantile levels. This behavior is expected, as the dependent case models positive correlations between the losses incurred by different policyholders, leading to greater risk exposure (i.e., higher upper quantiles relative to the median). In contrast, the independent case assumes no dependencies and zero correlations between conditional losses, resulting in lower risk levels.

The portfolio size naturally influences the number of infections and transmissions differently: infections increase linearly in n , while transmissions grow quadratically in n in the Erdős-Rényi model. In the contagion model, these effects are intertwined, adding further complexity. Additionally, diversification effects are present, influencing loss dynamics as portfolio size increases. These combined factors contribute to the overall increase in total losses for larger portfolios.

All findings hold consistently for both the Erdős-Rényi model (Table 1) and the contagion model (Table 2). In the standard case, where neither contagious losses nor transmissions occur, quantiles are significantly smaller.

n	Case	Method	$q_{0.5}$	$q_{0.75}$	$q_{0.95}$	$q_{0.995}$	$q_{0.75} - q_{0.5}$	$q_{0.95} - q_{0.5}$	$q_{0.995} - q_{0.5}$
15	Dependent	Monte-Carlo	189.8	219.0	267.8	323.7	29.2	78.0	133.9
		Asymptotic	193.8	221.8	261.9	300.4	27.9	68.1	106.6
	Independent	Monte-Carlo	192.9	213.0	243.4	274.2	20.1	50.5	81.3
		Asymptotic	193.8	213.6	242.1	269.4	19.8	48.2	75.5
	Standard	Monte-Carlo	8.8	12.7	19.4	26.9	3.9	10.6	18.2
		Asymptotic	9.4	13.1	18.3	23.4	3.7	8.9	14.0
50	Dependent	Monte-Carlo	2172.1	2317.1	2544.8	2785.7	144.9	372.7	613.6
		Asymptotic	2183.1	2324.4	2527.6	2722.6	141.3	344.5	539.5
	Independent	Monte-Carlo	2182.9	2249.7	2347.5	2440.0	66.8	164.7	257.2
		Asymptotic	2183.1	2249.8	2345.8	2438.0	66.7	162.7	254.8
	Standard	Monte-Carlo	30.7	37.7	48.8	60.4	7.0	18.1	29.7
		Asymptotic	31.4	38.0	47.6	56.9	6.68	16.3	25.5
200	Dependent	Monte-Carlo	35042.9	36098.0	37701.4	39299.6	1055.1	2658.5	4256.7
		Asymptotic	35080.4	36123.4	37623.9	39063.4	1043.0	2543.5	3983.0
	Independent	Monte-Carlo	35080.4	35348.4	35734.5	36112.9	268.0	654.2	1032.6
		Asymptotic	35080.4	35348.3	35733.8	36103.6	267.9	653.4	1023.2
	Standard	Monte-Carlo	124.8	138.4	159.1	180.2	13.6	34.3	55.4
		Asymptotic	125.4	138.8	158.0	176.5	13.5	32.6	51.0

Table 1: Quantiles and related distances for the Erdős-Rényi model with infections.

n	Case	Method	$q_{0.5}$	$q_{0.75}$	$q_{0.95}$	$q_{0.995}$	$q_{0.75} - q_{0.5}$	$q_{0.95} - q_{0.5}$	$q_{0.995} - q_{0.5}$
15	Dependent	Monte-Carlo	51.1	74.2	114.8	162.5	23.1	63.8	111.5
		Asymptotic	55.5	77.2	108.5	138.5	21.7	53.0	82.9
	Independent	Monte-Carlo	53.4	73.2	104.4	136.3	19.9	51.1	83.0
		Asymptotic	55.5	78.8	112.2	144.3	23.3	56.7	88.8
	Standard	Monte-Carlo	8.7	12.6	19.3	27.0	3.9	10.6	18.2
		Asymptotic	9.4	13.1	18.3	23.4	3.7	8.9	14.0
50	Dependent	Monte-Carlo	556.6	679.3	874.8	1090.9	122.7	318.2	534.3
		Asymptotic	569.3	686.7	855.6	1017.6	117.4	286.3	448.3
	Independent	Monte-Carlo	564.9	665.0	816.1	974.1	100.1	251.1	409.1
		Asymptotic	569.3	678.1	834.7	984.8	108.8	265.4	415.5
	Standard	Monte-Carlo	30.7	37.7	48.8	60.2	7.0	18.1	29.5
		Asymptotic	31.4	38.0	47.6	56.9	6.8	16.3	25.5
200	Dependent	Monte-Carlo	8813.2	9729.3	11113.5	12531.3	916.2	2300.4	3718.1
		Asymptotic	8864.2	9762.6	11055.1	12295.2	898.4	2190.9	3431.0
	Independent	Monte-Carlo	8844.1	9589.9	10687.7	11772.8	745.7	1843.6	2928.6
		Asymptotic	8864.2	9628.8	10728.8	11784.1	764.6	1864.6	2919.9
	Standard	Monte-Carlo	124.7	138.3	159.0	180.1	13.5	34.3	55.3
		Asymptotic	125.4	138.8	158.0	176.5	13.4	32.6	51.0

Table 2: Quantiles and related distances for the contagion model.

5 Conclusion and outlook

In an insurance context, we extended collective models to incorporate interactions in loss occurrences and expenditures. The key feature of our approach is the joint exchangeability of arrays (which is weaker than de Finetti's exchangeability), which allows us to move beyond the assumption of independence. Within this framework, we established limit theorems as the number of insurance contracts, the time horizon, or both tend to infinity. Furthermore, we rigorously examined the relationship to the classical collective model. To support our findings, we implemented the model in the statistical software R and conducted simulation studies to

assess the accuracy of the approximations and evaluate the impact of network interactions on total loss distributions.

From a more practical perspective, building on [Zeller & Scherer \(2022\)](#), the model could be extended to incorporate additional covariates, thereby enhancing its applicability. Such extensions would enable its use in insurance pricing and risk management, particularly in the context of cyber and power networks. On the theoretical side, further research directions include investigating rates of convergence (Berry-Esseen type results) and assessing the applicability of finite sample corrections to improve approximation accuracy. Additionally, statistical methodologies such as parameter estimation and goodness-of-fit assessment (i.e., model calibration and validation) based on loss data should be explored both theoretically and in practical applications.

Beyond these extensions, several fundamental challenges remain. A key question is the development of interaction mechanisms beyond joint exchangeability. Another direction is incorporating explicit time dependence into the interaction structure, allowing for evolving networks where connections form and dissolve dynamically, leading to more realistic contagion models. Furthermore, extending the framework from univariate to multivariate loss processes would enable the joint modeling of different risk types, such as operational, cyber, and financial losses, along with their interdependencies. Finally, systemic risk modeling remains a major challenge, as losses may propagate not only within a single insurance portfolio but also across multiple insurers and financial institutions, requiring systemic contagion models that account for inter-firm dependencies, capital constraints, and regulatory spillover effects.

A Proofs

Proof of Theorem 7. By Assumptions 1–4, the random variables $L_1(n), L_2(n), \dots$ are independent and identically distributed and independent of the counting process $N(t)$. Applying Wald's equation for expectation, we obtain

$$E(S(n, t)) = E(N(t))E(L_1(n)).$$

Using the definition of $L_1(n)$ and linearity of expectation, it follows that

$$E(L_1(n)) = \sum_{i,j=1}^n E(G_{i,j}) = nE(G_{1,1}) + n(n-1)E(G_{1,2}),$$

which establishes the formula for the expectation.

For the variance, we apply Wald's equation for variance:

$$\text{Var}(S(n, t)) = \text{Var}(N(t))E(L_1(n))^2 + E(N(t))\text{Var}(L_1(n)).$$

To compute $\text{Var}(L_1(n))$, we expand

$$\begin{aligned} \text{Var}(L_1(n)) &= \sum_{i,j,i',j'=1}^n \text{Cov}(G_{i,j,1}, G_{i',j',1}) \\ &= n\text{Var}(G_{1,1}) \\ &\quad + n(n-1)(2\text{Cov}(G_{1,1}, G_{1,2}) + 2\text{Cov}(G_{1,1}, G_{2,1}) \\ &\quad \quad + \text{Var}(G_{1,2}) + \text{Cov}(G_{1,2}, G_{2,1})) \\ &\quad + n(n-1)(n-2)(\text{Cov}(G_{1,2}, G_{1,3}) + \text{Cov}(G_{1,2}, G_{3,1}) \\ &\quad \quad + \text{Cov}(G_{2,1}, G_{1,3}) + \text{Cov}(G_{2,1}, G_{3,1})). \end{aligned}$$

□

Proof of Theorem 8. a) For arbitrary $m \in \mathbb{N}_0$ and $1 \leq i, j \leq n$, $i \neq j$, define

$$\begin{aligned}\tilde{X}_{i,j}(m) &= X'_{i,j}(m) + X'_{j,i}(m) - 2m\mathbb{E}(G_{1,2}), \\ \tilde{X}_{i,i}(m) &= X'_{i,i}(m) - m\mathbb{E}(G_{1,1}).\end{aligned}$$

By Assumptions 1–4, the array $\tilde{X}(m) = (\tilde{X}_{i,j}(m))_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ is centered, symmetric, jointly exchangeable, and dissociated. Moreover, we can decompose

$$S'(n, m) - m\mu_L(n) = U(n, m) + R(n, m),$$

where

$$\begin{aligned}U(n, m) &= \sum_{1 \leq i < j \leq n} \tilde{X}_{i,j}(m), \\ R(n, m) &= \sum_{i=1}^n \tilde{X}_{i,i}(m).\end{aligned}$$

By Theorem A in Silverman (1976), the statistic $U(n, m)$ satisfies

$$\frac{U(n, m)}{n^{3/2}} \xrightarrow{d} N(0, m\tau^2) \text{ as } n \rightarrow \infty,$$

where

$$\tau^2 = \text{Cov}(G_{1,2}, G_{1,3}) + \text{Cov}(G_{1,2}, G_{3,1}) + \text{Cov}(G_{2,1}, G_{1,3}) + \text{Cov}(G_{2,1}, G_{3,1}).$$

Furthermore, we have

$$\frac{n^{3/2}}{\sigma_L(n)} \longrightarrow \frac{1}{\tau} \text{ as } n \rightarrow \infty,$$

By the strong law of large numbers,

$$\frac{R(n, m)}{n} \xrightarrow{a.s.} 0 \text{ as } n \rightarrow \infty.$$

Finally, from

$$\begin{aligned}\frac{S'(n, m) - m\mu_L(n)}{\sigma_L(n)} &= \frac{n^{3/2}}{\sigma_L(n)} \frac{S'(n, m) - m\mu_L(n)}{n^{3/2}} \\ &= \frac{n^{3/2}}{\sigma_L(n)} \frac{U(n, m)}{n^{3/2}} + \frac{n^{3/2}}{\sigma_L(n)} \frac{R(n, m)}{n}\end{aligned}$$

Slutsky's theorem implies that for all $m \in \mathbb{N}_0$,

$$\frac{S'(n, m) - m\mu_L(n)}{\sigma_L(n)} \xrightarrow{d} N(0, m) \text{ as } n \rightarrow \infty.$$

b) For any fixed $x \in \mathbb{R}$, we define, for $m \in \mathbb{N}_0$,

$$h_n(m) = P\left(\frac{S'(n, N(t)) - N(t)\mu_L(n)}{\sigma_L(n)} \leq x \mid N(t) = m\right), \quad h(m) = \Phi_{0,m}(x).$$

Part a) combined with the Portmanteau theorem yields $h_n(m) \rightarrow h(m)$ as $n \rightarrow \infty$. Observing

$$\begin{aligned} P\left(\frac{S'(n, N(t)) - N(t)\mu_L(n)}{\sigma_L(n)} \leq x\right) &= E\left(P\left(\frac{S'(n, N(t)) - N(t)\mu_L(n)}{\sigma_L(n)} \leq x \middle| N(t)\right)\right) \\ &= E\left(h_n(N(t))\right), \end{aligned}$$

it follows from the dominated convergence theorem, since $h_n \geq 0$ and h_n is bounded by 1, that

$$E\left(h_n(N(t))\right) \rightarrow E(h(N(t))) = \sum_{m=0}^{\infty} P(N(t) = m)\Phi_{0,m}(x) \text{ as } n \rightarrow \infty.$$

□

Proof of Theorem 10. a) We show the statement for an arbitrary sequence $m(n) \in (0, \infty)$, $n \in \mathbb{N}$, with $\lim_{n \rightarrow \infty} m(n) = \infty$. It is

$$\begin{aligned} \frac{S'(n, m(n)) - m(n)\mu_L(n)}{\sqrt{m(n)}\sigma_L(n)} &= \frac{\sum_{k=1}^{m(n)} L_k(n) - m(n)\mu_L(n)}{\sqrt{m(n)}\sigma_L(n)} \\ &= \frac{1}{\sqrt{m(n)}} \sum_{k=1}^{m(n)} \frac{L_k(n) - \mu_L(n)}{\sigma_L(n)} \\ &= \frac{S(n)}{s(n)}, \end{aligned}$$

where

$$S(n) = \sum_{k=1}^{m(n)} H_k(n), \quad s^2(n) = \text{Var}(S(n)) = m(n)$$

with the triangular array of standardized (centered and normalized) independent and identically distributed random variables

$$H_k(n) = \frac{L_k(n) - \mu_L(n)}{\sigma_L(n)}, \quad k = 1, \dots, m(n).$$

We will apply the central limit theorem of Lindeberg-Feller. For this purpose, we verify Lindeberg's condition. We have for all $u \in (0, \infty)$

$$\frac{1}{s(n)^2} \sum_{k=1}^{m(n)} E(H_k(n)^2 I(|H_k(n)| > s(n)u)) = E(H_1(n)^2 I(|H_1(n)| > \sqrt{m(n)}u)).$$

Applying again Theorem A in [Silverman \(1976\)](#), we obtain analogously to the proof of Theorem 8

$$H_1(n) \xrightarrow{d} \eta \text{ as } n \rightarrow \infty.$$

where $\eta \sim N(0, 1)$. By the Skorokhod representation theorem, we may replace $(H_1(n))_n$ and η by new random variables (with notation unchanged) on a common probability space such that $H_1(n) \xrightarrow{a.s.} \eta$ as $n \rightarrow \infty$. From

$$H_1(n)^2 I(|H_1(n)| > \sqrt{m(n)}u) \xrightarrow{a.s.} 0 \text{ as } n \rightarrow \infty$$

and $H_1(n)^2 I(|H_1(n)| > \sqrt{m(n)u}) \leq H_1(n)^2$ as well as $E(H_1(n)^2) = 1$ it follows from the dominated convergence theorem that

$$E(H_1(n)^2 I(|H_1(n)| > \sqrt{m(n)u})) \longrightarrow 0 \text{ as } n \rightarrow \infty,$$

i.e., Lindeberg's condition. The central limit theorem of Lindeberg-Feller implies

$$\frac{S(n)}{s(n)} \xrightarrow{d} N(0, 1) \text{ as } n \rightarrow \infty.$$

b) It is sufficient to show the statement for an arbitrary sequence $t_n \in (0, \infty)$, $n \in \mathbb{N}$, with $\lim_{n \rightarrow \infty} t_n = \infty$. Due to Assumption 4 we may assume without loss of generality that the random variables $L_1(n), L_2(n), \dots$ on the one hand and the process $N = (N(t))_{t \in [0, \infty)}$ on the other hand are defined on different probability spaces $(\Omega_1, \mathfrak{A}_1, P_1)$ and $(\Omega_2, \mathfrak{A}_2, P_2)$, where $(\Omega, \mathfrak{A}, P) = (\Omega_1 \times \Omega_2, \mathfrak{A}_1 \otimes \mathfrak{A}_2, P_1 \otimes P_2)$ is endowed with the product measure. We fix an arbitrary $\omega \in \{\lim_{t \rightarrow \infty} N(t) = \infty\} \in \mathfrak{A}_2$ and set $m(n) = N(t_n)(\omega)$; $\lim_{n \rightarrow \infty} m(n) = \infty$ is satisfied. In addition, we fix arbitrary $x \in \mathbb{R}$. Setting

$$h_n(\omega) = P_1 \left(\frac{S'(n, N(t_n)(\omega)) - N(t_n)(\omega)\mu_L(n)}{\sqrt{N(t_n)(\omega)}\sigma_L(n)} \leq x \right) = P_1 \left(\frac{S'(n, m(n)) - m(n)\mu_L(n)}{\sqrt{m(n)}\sigma_L(n)} \leq x \right),$$

and $h(\omega) = \Phi_{0,1}(x)$ constant in ω , it follows from part a) that $h_n(\omega) \rightarrow h(\omega)$ as $n \rightarrow \infty$. Because $h_n \geq 0$ and h_n is bounded by 1, it follows from the dominated convergence theorem that

$$\begin{aligned} & P \left(\frac{S'(n, N(t_n)) - N(t_n)\mu_L(n)}{\sqrt{N(t_n)}\sigma_L(n)} \leq x \right) \\ &= \int h_n(\omega) dP_2(\omega) \\ &= \int h_n(\omega) I \left(\omega \in \left\{ \lim_{t \rightarrow \infty} N(t) = \infty \right\} \right) dP_2(\omega) \\ &\longrightarrow \int h(\omega) I \left(\omega \in \left\{ \lim_{t \rightarrow \infty} N(t) = \infty \right\} \right) dP_2(\omega) \\ &= \Phi_{0,1}(x) \text{ as } n \rightarrow \infty, \end{aligned}$$

observing that our additional assumption implies $P_2(\{\lim_{t \rightarrow \infty} N(t) = \infty\}) = 1$. This completes the proof of part b). \square

Proof of Theorem 11. a) From the expression

$$S'(n, m) = \sum_{k=1}^{N(t)} L_k(n),$$

where $L_1(n), L_2(n), \dots$ are independent and identically distributed real-valued random variables with expectation $\mu_L(n)$ and variance $\sigma_L^2(n)$, we have

$$\frac{S'(n, m) - m\mu_L(n)}{\sqrt{m}\sigma_L(n)} = \frac{\sum_{k=1}^{N(t)} L_k(n) - m\mu_L(n)}{\sqrt{m}\sigma_L(n)},$$

so the statement in a) follows immediately by the application of the central limit theorem of Lindeberg-Levy.

b) This follows immediately from part a) and with analogous arguments as in the proof of Theorem 10 b). \square

Proof of Theorem 12. a) We recall the expression for the loss of the insurance company:

$$S(n, t) = \sum_{k=1}^{N(t)} L_k(n), \quad L_k(n) = \sum_{i,j=1}^n G_{i,j,k}.$$

The expectation and variance are

$$\begin{aligned} \mu_S(n, t) &= \mathbb{E}(S(n, t)) = \mu_N(t)\mu_L(n) = \lambda t\mu_L(n) \\ \sigma_S^2(n, t) &= \text{Var}(S(n, t)) = \sigma_N^2(t)\mu_L(n)^2 + \mu_N(t)\sigma_L^2(n) = \lambda t(\mu_L(n)^2 + \sigma_L^2(n)). \end{aligned}$$

We can write $S(n, t) - \mu_S(n, t) = \tilde{S}(n, t) + R(n, t)$ with the telescopic sum $\tilde{S}(n, t) = \sum_{k=1}^{\lfloor t \rfloor} H_k(n)$ of the increments

$$H_k(n) = \sum_{k'=1}^{N(k)} L_{k'}(n) - \sum_{k'=1}^{N(k-1)} L_{k'}(n) - \lambda\mu_L(n), \quad k = 1, \dots, \lfloor t \rfloor,$$

and with the remainder term

$$R(n, t) = \sum_{k=N(\lfloor t \rfloor)+1}^{N(t)} L_k(n) - \lambda(t - \lfloor t \rfloor)\mu_L(n).$$

Due to Assumptions 1–4, and in particular the fact that the counting process $N = (N(t))_{t \in [0, \infty)}$ is a homogeneous Poisson process with intensity $\lambda \in (0, \infty)$, we observe that the increments $H_1(n), \dots, H_{\lfloor t \rfloor}(n)$ form a sequence of independent and identically distributed centered random variables. Setting $\sigma_H^2(n) = \text{Var}(H_1(n))$, it follows from the central limit theorem of Lindeberg-Levy that

$$\frac{\tilde{S}(n, t)}{\sqrt{\lfloor t \rfloor}\sigma_H(n)} \xrightarrow{d} N(0, 1) \text{ as } t \rightarrow \infty.$$

Moreover, it is

$$\sigma_{\tilde{S}}^2(n, t) = \text{Var}(\tilde{S}(n, t)) = \lfloor t \rfloor \text{Var}(H_1(n)) = \lfloor t \rfloor \sigma_H^2(n).$$

Due to Assumptions 1–4, and in particular the fact that the counting process $N = (N(t))_{t \in [0, \infty)}$ is a homogeneous Poisson process with intensity $\lambda \in (0, \infty)$, the random variables $\tilde{S}(n, t)$ and $R(n, t)$ are independent. For that reason, we have

$$\sigma_S^2(n, t) = \text{Var}(S(n, t) - \mu_S(n, t)) = \sigma_{\tilde{S}}^2(n, t) + \text{Var}(R(n, t)),$$

and, due to the fact that the distribution of $R(n, t)$ is periodic in t with period 1, it follows that

$$\sigma_S^2(n, t) \sim \sigma_{\tilde{S}}^2(n, t) \text{ as } t \rightarrow \infty.$$

This implies

$$\lfloor t \rfloor \sigma_H^2(n) \sim \sigma_S^2(n, t) \text{ as } t \rightarrow \infty.$$

Furthermore, again due to the fact that the distribution of $R(n, t)$ is periodic in t with period 1, we have $\frac{R(n, t)}{\sqrt{\lfloor t \rfloor}} \xrightarrow{P} 0$ as $t \rightarrow \infty$. In summary, Slutsky's theorem implies

$$\frac{S(n, t) - \mu_S(n, t)}{\sigma_S(n, t)} = \frac{\sqrt{\lfloor t \rfloor}\sigma_H(n)}{\sigma_S(n, t)} \left(\frac{\tilde{S}(n, t)}{\sqrt{\lfloor t \rfloor}\sigma_H(n)} + \frac{1}{\sigma_H(n)} \frac{R(n, t)}{\sqrt{\lfloor t \rfloor}} \right) \xrightarrow{d} N(0, 1) \text{ as } t \rightarrow \infty.$$

This completes the proof of a).

b) The proof is similar to the proof of Theorem 10 b). Due to our assumptions, we can assume without loss of generality that the random variables \tilde{N} , G_k , $k \in \mathbb{N}$, on the one hand and the process T on the other hand are defined on different probability spaces $(\Omega_1, \mathfrak{A}_1, P_1)$ and $(\Omega_2, \mathfrak{A}_2, P_2)$, where $(\Omega, \mathfrak{A}, P) = (\Omega_1 \times \Omega_2, \mathfrak{A}_1 \otimes \mathfrak{A}_2, P_1 \otimes P_2)$ is endowed with the product measure. We fix an arbitrary $\omega \in \{\lim_{s \rightarrow \infty} T(s) = \infty\} \in \mathfrak{A}_2$ and set $t'(t) = T(t)(\omega)$ for $t \in [0, \infty)$; $\lim_{t \rightarrow \infty} t'(t) = \infty$ is satisfied. In addition, we fix arbitrary $x \in \mathbb{R}$. Setting

$$h_t(\omega) = P_1 \left(\frac{S'(n, \tilde{N}(T(t)(\omega))) - \lambda T(t) \mu_L(n)}{\sqrt{\lambda T(t) (\mu_L(n))^2 + \sigma_L^2(n)}} \leq x \right)$$

it is

$$h_t(\omega) = P_1 \left(\frac{S'(n, \tilde{N}(t'(t))) - \lambda t'(t) \mu_L(n)}{\sqrt{\lambda t'(t) (\mu_L(n))^2 + \sigma_L^2(n)}} \leq x \right).$$

Furthermore, $S'(n, \tilde{N}(t'(t)))$ is the total loss at time $t'(t)$ in the situation in part a) with expectation $\mathbb{E}(S'(n, \tilde{N}(t'(t)))) = \lambda t'(t) \mu_L(n)$ and variance $\text{Var}(S'(n, \tilde{N}(t'(t)))) = \lambda t'(t) (\mu_L(n))^2 + \sigma_L^2(n)$. Setting $h(\omega) = \Phi_{0,1}(x)$ (which is constant in ω), part a) implies that $h_t(\omega) \rightarrow h(\omega)$ as $t \rightarrow \infty$. Because $h_t \geq 0$ and h_t is bounded by 1, it follows from the dominated convergence theorem that

$$\begin{aligned} & P \left(\frac{S(n, t) - \lambda T(t) \mu_L(n)}{\sqrt{\lambda T(t) (\mu_L(n))^2 + \sigma_L^2(n)}} \leq x \right) \\ &= \int h_t(\omega) dP_2(\omega) \\ &= \int h_t(\omega) I \left(\omega \in \left\{ \lim_{s \rightarrow \infty} T(s) = \infty \right\} \right) dP_2(\omega) \\ &\rightarrow \int h(\omega) I \left(\omega \in \left\{ \lim_{s \rightarrow \infty} T(s) = \infty \right\} \right) dP_2(\omega) \\ &= \Phi_{0,1}(x) \text{ as } t \rightarrow \infty, \end{aligned}$$

observing that $P_2(\{\lim_{s \rightarrow \infty} T(s) = \infty\}) = 1$ is satisfied. This completes the proof of part b). \square

Proof of Theorem 14. For $t \in [0, \infty)$, we consider the Laplace transform of $(S(n, t), M(n, t))$, i.e., $\psi_{(S(n, t), M(n, t))}(u, v) = \mathbb{E}(e^{-uS(n, t) - vM(n, t)})$ for $(u, v) \in [0, \infty) \times [0, \infty)$, which characterizes the distribution uniquely. With $\eta \in \mathbb{N}_0$, $\iota = (\iota_{i,j,k})_{(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}} \in \{0, 1\}^{\mathbb{N} \times \mathbb{N} \times \mathbb{N}}$, and setting $m = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \iota_{i,j,k}$, we have for $t \in [0, \infty)$ and $(u, v) \in [0, \infty) \times [0, \infty)$:

$$\mathbb{E}(e^{-uS(n, t) - vM(n, t)} | N(t) = \eta, I = \iota) = \psi_Z(u)^m e^{-vm},$$

thus

$$\psi_{(S(n, t), M(n, t))}(u, v) = \mathbb{E}(\mathbb{E}(e^{-uS(n, t) - vM(n, t)} | N(t), I)) = \mathbb{E}(\psi_Z(u)^{M(n, t)} e^{-vM(n, t)})$$

on the one hand; on the other hand, we obtain for $m \in \mathbb{N}_0$:

$$\mathbb{E}(e^{-u \sum_{i=1}^{M(n, t)} Z_i - vM(n, t)} | M(n, t) = m) = \psi_Z(u)^m e^{-vm},$$

thus

$$\psi_{\left(\sum_{i=1}^{M(n, t)} Z_i, M(n, t)\right)}(u, v) = \mathbb{E}(\mathbb{E}(e^{-u \sum_{i=1}^{M(n, t)} Z_i - vM(n, t)} | M(n, t))) = \mathbb{E}(\psi_Z(u)^{M(n, t)} e^{-vM(n, t)}).$$

This proves the assertion. \square

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