Pricing and exercising American options in a market-consistent way

Pablo Koch-Medina

Department of Finance and Swiss Finance Institute
University of Zurich, Switzerland
pablo.koch@df.uzh.ch

Cosimo Munari

Department of Economics
University of Verona, Italy
cosimo.munari@univr.it

Gregor Svindland

Institute of Actuarial and Financial Mathematics and House of Insurance
Leibniz University Hannover, Germany
gregor.svindland@insurance.uni-hannover.de

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Abstract

We develop a new approach to pricing European and American payoffs in an arbitrage-free market. Our analysis is grounded on market-consistent prices for off-market transactions, i.e., prices that ensure neither seller nor buyer can achieve better outcomes through direct market trading, rather than arbitrage-free prices, i.e., prices that preclude arbitrage opportunities when the payoff is added to the market as a new security. This approach addresses the theoretically inconsistent assumption that introducing a new security leaves original security prices unchanged — an inconsistency already noted by Kreps in [13]. Our approach avoids complex martingale theory while addressing: what prices rational agents should consider, which exercise strategies are optimal for American option holders, and their equivalent martingale measure representations. Our approach yields a more economically coherent and mathematically streamlined treatment, particularly valuable for understanding pricing and optimal exercise in incomplete markets.

Keywords: American options, arbitrage pricing, market-consistent pricing, exercise strategies

1 Introduction

European and American contracts represent two basic types of financial instruments. European contracts deliver either a single payoff at maturity or a payoff stream with predetermined payments at multiple intermediate dates prior to maturity. By contrast, American contracts, henceforth "American options", are characterized by a stream of potential payoffs with the following special feature: At any date during the lifetime of the contract, the holder can either exercise the option and receive the payment specified for that date, thereby forfeiting all posterior payoffs, or decline the current payment to maintain the right to exercise later. The ability to strategically time the exercise decision introduces analytical complexities for American options that significantly exceed those encountered when studying their European counterparts.

American options have attracted considerable attention in the literature starting with [18] and [14] as well as [2] and [9]; see [4] for an extensive bibliography up to 2005 focusing on continuous-time models. In the discrete-time case, which is the one we are concerned with, the most comprehensive treatment is found in [6, Chapters 6 and 7]. Even a cursory examination of those chapters reveals that the topic is quite technical and makes heavy use of martingale theory in the form of a combination of martingale representation theorems, uniform Doob decompositions, robust Snell envelopes, and stability under pasting.

In this paper we propose a new approach to price and determine the optimal exercise of American options that emphasizes the financial interpretation and deliberately avoids the technical machinery of martingale theory, in particular, eliminating the need for robust Snell envelopes. In fact, our approach provides a novel perspective also for European contracts. This results in a substantially simplified framework compared to the standard methodology. As we explain further down, the new approach provides a better economic foundation for Arbitrage Pricing Theory in general. For American options, this new approach yields two significant benefits: first, it naturally leads to new concepts and results, particularly regarding optimal exercise strategies, that enhance our understanding of American options, particularly in incomplete markets; second, it provides a more accessible pedagogical framework for teaching American options beyond the usual complete market setting.

Our analysis centers on the following three fundamental questions:

(1) *Market-consistent (buyer and seller) prices*: What price ranges should rational buyers and sellers consider when transacting American options?

(2) *Market-consistent exercise strategies*: Which exercise strategies should rational option holders consider adopting?

(3) *Representation of market-consistent exercise strategies*: How can market-consistent prices and exercise strategies be represented in terms of equivalent martingale measures?

To the best of our knowledge, the second question has not been systematically studied outside a complete market setting — a setting which, in discrete time, presupposes a finite state space. Our paper addresses this gap by providing a comprehensive analysis of exercise strategies. However, by grounding our analysis in the concept of market-consistent prices rather than the usual notion of arbitrage-free prices, our approach offers a fresh perspective also for pricing related questions — not only for American options but for European contracts as well. Importantly, our analysis makes a clear distinction between financial concepts and their dual representations which are meant to provide useful mathematical ways to represent these concepts but should ideally not be a surrogate for a sound “primal” formulation. Indeed, despite their mathematical equivalence, arbitrage-free and market-consistent prices reflect fundamentally different economic intuitions. In the next section of this introduction, we elaborate on this critical distinction.

Market-consistent prices

Arbitrage Pricing Theory is essentially about the pricing of financial contracts — European or American — in the presence of a liquid market for basic securities in which no arbitrage opportunities exist. Here, arbitrage opportunities are self-financing dynamic portfolios of basic securities that have a nonzero positive terminal payoff but can be implemented without requiring an initial investment. The absence of arbitrage opportunities is typically justified based on an equilibrium argument: arbitrage opportunities would attract unlimited demand, causing market forces to rapidly eliminate them.

When pricing a payoff that is not traded in the market, the typical narrative is that its price should be *arbitrage-free*, meaning that incorporating a new security with this payoff into the original market at this specific price should not create arbitrage opportunities in the thus extended (hypothetical) market. This reasoning, however, lacks economic coherence. As Kreps observed in [13], the introduction of a new security “creates new economic opportunities for agents and may therefore change the prices of bundles [securities]” in the original market. This reveals a fundamental inconsistency in the standard narrative: it relies on equilibrium considerations to justify arbitrage-free markets, yet disregards these same considerations when defining its core theoretical concept—arbitrage-free prices.[1]

An economically compelling alternative view is that arbitrage theory is concerned with prices at which a payoff could be transacted *outside of the market* between two rational parties with full access to the market. In this case, imposing a “no arbitrage” requirement would be conceptually misplaced: the rationale for the absence of arbitrage was that market forces would eliminate it, but in an isolated transaction between two parties no such mechanism exists. There is, however, a more intuitive alternative requirement, which we postulate as the Principle of Market Consistency:

> *When a financial contract is transacted off-market, only prices at which neither the buyer nor the seller can do better by trading directly in the market should be considered.*

We call prices obtained by applying this principle *market-consistent prices*. In the one-period case, it is easy to see that market-consistent prices and arbitrage-free prices are mathematically equivalent. In the multi-period case this is less obvious and requires proof. We believe that the Principle of Market Consistency provides a better economic foundation for the theory than the requirement that a hypothetically extended market is arbitrage free. At the same time this principle provides strong guidance in formulating the questions that the theory should answer and in identifying the right concepts to study.

In Section 3 we describe the theory of market-consistent prices for European payoffs, which prescribe a single payment at a fixed maturity, and for European payoff streams, which prescribe payments not only at maturity but also at intermediate times. The latter arise naturally when

[1] In [6, Page 23] the following justification for this narrative is offered: “In the previous definition [of an arbitrage-free price], we made the implicit assumption that the introduction of a contingent claim C as a new asset does not affect the prices of primary assets. This assumption is reasonable as long as the traded volume of C is small compared to that of the primary assets”. However, this explanation remains inadequate because the formal model requires a liquid frictionless market where both primary assets and the contingent claim can be freely traded in arbitrary quantities.

studying market-consistent prices of American options in Section 4 because choosing an exercise strategy for an American option results in a payoff stream. In fact, a critical reframing driving our analysis is that an American option can be conceptualized as a basket of European payoff streams, with each exercise strategy generating a distinct payoff stream. This perspective allows us to build upon the results established for European contracts to provide a full characterization of the set of market-consistent prices for American options. We show that this set is either a singleton or an order interval. Notably, this interval always excludes its essential supremum (the *superreplication price*) and sometimes includes its essential infimum (the *subreplication price*), which aligns with the main results in [1] about arbitrage-fee prices. We emphasise, however, that the theory of market-consistent prices, including this last characterization, is developed entirely without reference to dual representations and martingale theory and relies solely on "primal" concepts with a clear financial interpretation.

Market-consistent exercise strategies

The bulk of the literature on optimal exercise strategies for American options does not focus on strategies a rational option holder would actually implement. Instead, it emphasizes exercise strategies yielding subreplication and superreplication prices, which could be more appropriately termed *pricing bounds strategies*. Moreover, with [6] being the most notable exception, they only deal with complete markets. In the literature, pricing bounds strategies are characterized using Snell envelopes. While this ensures an elegant treatment within a complete market setting, its extension to incomplete markets requires introducing more sophisticated mathematical machinery such as upper and lower Snell envelopes. Our approach to exercise strategies is different and recognizes the multiple distinct roles that exercise strategies may play. We differentiate between strategies according to which it is reasonable for the holder to exercise (*market-consistent* exercise strategies), strategies that deliver subreplication or superreplication prices for the American option (*lower pricing-bound*, respectively *upper pricing-bound*, exercise strategies), and strategies that deliver the sets of market-consistent seller or buyer prices (*seller*, respectively *buyer*, exercise strategies). Concrete examples and results demonstrate that these different notions are fundamentally distinct.

Focusing on market-consistent strategies, the fundamental observation underpinning our analysis is that exercising an American option is economically equivalent to selling the residual option (the right to exercise in the future) for a price equal to the current exercise value. Conversely, not exercising is equivalent to purchasing the residual option for a price equal to the current exercise value. From the Principle of Market Consistency we can therefore derive the following general rule:

> *The holder of an American option should* never *exercise when the exercise value is not a market-consistent seller price for the residual option and should* always *exercise when the exercise value is not a market-consistent buyer price for the residual option.*

Section 5 provides a comprehensive characterization of market-consistent exercise strategies and identifies the precise earliest and latest times at which exercising the option is market consistent. Interestingly enough, the key stopping time used in [6, Theorem 6.47] in the context of lower Snell envelopes — the first time when the exercise value coincides with the option's subreplication price — fails to be market consistent. This exemplifies the focus on pricing rather than optimal rational exercise in the literature.

Representation theorems

The core financial concepts of our theory are market-consistent prices and market-consistent exercise strategies. These "primal" concepts can be expressed in equivalent mathematical terms using "dual representations" involving, e.g., equivalent martingale measures or stochastic discount factors. Each of these representations provides access to powerful mathematical machinery that casts new light on the underlying financial concepts, with the preferred choice depending on the specific context. In this paper, we focus exclusively on equivalent martingale measures, which we examine in depth in Section 6.

In the literature, dual representations are often used to introduce core financial concepts, thereby blurring the line between financial notions and their mathematical representations. In contrast, our approach explicitly and intentionally separates these two distinct perspectives. We first develop the theory to its fullest extent using only financially meaningful core concepts before turning to the subject of dual representations. By maintaining this clear distinction, we provide a solid economic foundation for what is, at its heart, a financially motivated theory.

Two final complementary observations

Along the way to developing the theory of market-consistent prices, we make two observations that to the best of our knowledge have not been explicitly highlighted before and which flow very naturally from our approach, even though they could have been made also in the context of arbitrage-free prices. These are:

(a) *The market places no value on the optionality of an American option.* This resembles a situation where an option to select an item from a basket of goods is being transacted: a buyer would only pay a price if at least one item in the basket merits paying that much, while a seller would only accept a price if every item in the basket is worth at least that amount. In both cases and perhaps surprisingly, the flexibility to choose when to exercise — the defining characteristic of American options — adds no additional value beyond what is already captured by the market-consistent prices of the single items in the basket. It is important to note, however, that while the market as a whole places no additional value on this optionality, individual agents with specific preferences or constraints may still value the flexibility to exercise at any time. Nonetheless, they do not need to pay for it. Despite the absence of a premium for optionality, there remain compelling reasons why the purchase of American options may be suboptimal as explained in our next observation.

(b) *Replicable American options do not enrich the investment opportunity set and should never be purchased.* This is because replicability yields a unique market-consistent price and, as will be shown, for this price an investor can always construct a superreplication strategy, i.e., a self-financing portfolio of traded assets whose liquidation value dominates the option's exercise value at every possible exercise date. Such a portfolio provides strictly greater flexibility than the option itself, as it offers the same or better payoff regardless of when the holder chooses to exercise, showing that if timing flexibility is the desired feature, a better alternative to the replicable American option exists at no additional cost. The fact that replicable American options are not worth purchasing suggests reconsidering the theoretical emphasis placed on American options in complete market settings, where all American options are replicable.

Plan of the paper

The paper is organized as follows. In Section 2 we describe the underlying market model. In Section 3 we review European contracts and establish a direct characterization of market-consistent prices in Theorem 3.9. In Section 4 we shift the focus to American options. The main result on market-consistent prices is Theorem 4.16, which builds on separate characterizations for seller and buyer prices. In Section 5 we introduce market-consistent exercise strategies and describe the first and last time to exercise in a market-consistent way in Theorems 5.3 and 5.6. In Section 6 we establish a variety of dual representations for prices and exercise strategies by means of equivalent martingale measures. The final appendix includes a simple conditional version of the intermediate value theorem used in the dual representation of option prices.

2 The market model

We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and consider a discrete-time financial market with dates $0, \dots, T$ with $T \geq 2$. The terminal states of the economy at date T are represented by the elements of Ω. The particular choice of $\mathbb{P}$ is important only up to equivalence, i.e., agents may hold different beliefs about the probability of occurrence of events as long as they agree on which events have strictly positive probability. We denote by $\mathcal{X}$ the (real) vector space of equivalence classes of random variables, i.e., of Borel measurable functions $X : \Omega \to \mathbb{R}$, with respect to almost-sure equality under $\mathbb{P}$. In line with standard practice, we do not distinguish explicitly between an element of $\mathcal{X}$ and any of its representatives. In particular, the elements of $\mathbb{R}$ are identified with the almost-surely constant random variables. Equalities and inequalities between random variables are always understood in the almost-sure sense. In the same vein, inclusions between events are understood to hold up to sets of zero probability. For convenience, for any $X, Y \in \mathcal{X}$ we write $X \gneq Y$ when $X \geq Y$ but $X \neq Y$. A set $\mathcal{S} \subset \mathcal{X}$ is called upward, respectively downward, directed if for all $X, Y \in \mathcal{S}$ there exists $Z \in \mathcal{S}$ such that $\max\{X, Y\} \leq Z$, respectively $\min\{X, Y\} \geq Z$. The essential supremum and infimum of $\mathcal{S}$ are denoted by $\operatorname{ess\,sup} \mathcal{S}$ and $\operatorname{ess\,inf} \mathcal{S}$, respectively. If $\mathcal{S}$ is upward directed and $\operatorname{ess\,sup} \mathcal{S} \in \mathcal{X}$, respectively downward directed and $\operatorname{ess\,inf} \mathcal{S} \in \mathcal{X}$, then there exists a sequence $(X_n) \subset \mathcal{S}$ such that $X_n \uparrow \operatorname{ess\,sup} \mathcal{S}$, respectively $X_n \downarrow \operatorname{ess\,inf} \mathcal{S}$, almost surely.

The flow of information in the economy is modelled by a filtration $\mathcal{F} = (\mathcal{F}_t)$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{A}$. For every $t \in \{0, \dots, T\}$ we denote by $\mathcal{X}_t$ the space of $\mathcal{F}_t$-measurable random variables and by $\mathcal{X}_{0:T}$ the space of $\mathcal{F}$-adapted stochastic processes. The *truncation* of $X \in \mathcal{X}_{0:T}$ at date $t \in \{0, \dots, T\}$ is the process $X^t := (0, \dots, 0, X_t, \dots, X_T)$.

Throughout the paper we assume the existence of a frictionless market in which a finite number N of assets are traded at dates $0, \dots, T-1$ and pay off at date T. As customary, these *basic securities* are represented by their price processes $S^i \in \mathcal{X}_{0:T}$ for $i \in \{1, \dots, N\}$, which are assumed to be exogenously given. We work under the following assumptions:

(BS1) $S^i_t \geq 0$ for all $i \in \{1, \dots, N\}$ and $t \in \{0, \dots, T\}$;

(BS2) $S^1_t = 1$ for every $t \in \{0, \dots, T\}$.

We refer to the first security as the *numéraire security*.[2] By trading in the market, agents can

[2] We could also have required that $S^0_t := \sum_{i=1}^N S^i_t > 0$ for every $t \in \{0, \dots, T\}$. One can then introduce an additional artificial security with price process S^0 and use this security as the numéraire, i.e., express all security prices in terms of this artificial security. The new security then satisfies (BS2).

set up portfolios of basic securities. A *trading strategy* with starting date $t \in \{0, \dots, T-1\}$ is an N-dimensional process of the form

$$\alpha = (\alpha_t, \dots, \alpha_{T-1}) \in \mathcal{X}_t^N \times \cdots \times \mathcal{X}_{T-1}^N.$$

For every $s \in \{t, \dots, T-1\}$, the N-dimensional random vector $\alpha_s = (\alpha_s^1, \dots, \alpha_s^N)$ describes the number of physical units of the various basic securities held at date s. The set of trading strategies with starting date t is denoted by $\mathcal{S}_t$. The *acquisition value* of α at date $s \in \{t, \dots, T-1\}$ and the *liquidation value* of α at date $s \in \{t+1, \dots, T\}$ are respectively defined by

$$V_s^{acq}(\alpha) := \sum_{i=1}^N \alpha_s^i S_s^i, \qquad V_s^{liq}(\alpha) := \sum_{i=1}^N \alpha_{s-1}^i S_s^i.$$

We say that α is *self-financing* if $V_s^{acq}(\alpha) = V_s^{liq}(\alpha)$ for every $s \in \{t+1, \dots, T-1\}$, i.e., at any date $t \in \{1, \dots, T-1\}$, the acquisition of portfolio α_{t+1} can be fully financed by the liquidation of portfolio α_t. In this case, we may define the *value* of α at date $s \in \{t, \dots, T\}$ by setting

$$V_s(\alpha) := \begin{cases} V_t^{acq}(\alpha) & \text{if } s = t, \\ V_s^{acq}(\alpha) = V_s^{liq}(\alpha) & \text{if } s \in \{t+1, \dots, T-1\}, \\ V_T^{liq}(\alpha) & \text{if } s = T. \end{cases}$$

The set of self-financing trading strategies with starting date t is denoted by $\mathcal{S}_t^0$. For backward induction arguments that involve trading strategies it is convenient to view a random vector $\alpha \in \mathcal{X}_T^N$ as a trading strategy starting at date T and set

$$V_T(\alpha) := V_T^{acq}(\alpha) := V_T^{liq}(\alpha) := \sum_{i=1}^N \alpha^i S_T^i.$$

The set of trading strategies starting at date T is denoted by $\mathcal{S}_T$ or $\mathcal{S}_T^0$. We will often use the "static" self-financing strategy $\eta \in \mathcal{S}_0^0$ corresponding to holding one unit of the numéraire security from the initial date until maturity, i.e.,

$$\eta_0 := \cdots := \eta_{T-1} := (1, 0, \dots, 0) \in \mathbb{R}^N. \tag{2.1}$$

Finally, for each $t \in \{0, \dots, T-1\}$ and $u \in \{t+1, \dots, T\}$, we say that a self-financing strategy $\alpha \in \mathcal{S}_t^0$ is an *arbitrage opportunity* (between t and u) if $V_t(\alpha) \leq 0$ and $V_u(\alpha) \gneq 0$. The remainder of this paper is concerned with pricing payoffs in a market with given equilibrium prices. Therefore, we assume that **the market is arbitrage-free**, i.e., no arbitrage opportunities exist between any two dates.

3 Market-consistent prices for European contracts

In preparation for our treatment of American options, this section examines the range of prices at which two rational agents might consider transacting a European contract off-market. A *(European) contract* entitles the holder to receiving either a single payment at a given maturity, the contract's *payoff*, or a stream of payments through time, the contract's *payoff stream*. As usual in the context of no-arbitrage pricing, we make no assumptions about agents' monetary preferences other than that they prefer more to less. It is also worth emphasizing that the question we ask is not at what price agents *should* transact, but rather at what prices they *should not*, because they could otherwise do better by trading directly in the market. The complement of the set of these prices is then the set of prices at which they *might consider* transacting, i.e., the set of prices at which it would not be foolish to transact. Whether they *actually* transact depends on whether they find common ground given their individual preferences, but that is beyond the scope of arbitrage pricing theory. Therefore, arbitrage pricing theory examines whether the existence of a market accessible to all agents imposes preference-free "rationality" constraints on off-market transactions between these agents; see also Remark 3.4.

3.1 European payoffs

In this subsection **we fix a European payoff** $X \in \mathcal{X}_u$ maturing at date $u \in \{1, \dots, T\}$ and such that $X \geq 0$. A superreplication strategy is a trading strategy that generates a market payoff that is more favorable than X from the perspective of a potential buyer. Similarly, a subreplication strategy is a trading strategy that generates a market payoff that is more favorable than X from the perspective of a potential seller. The cost of implementing sub- and superreplication strategies place natural limits on the price at which a seller or a buyer might reasonably consider transacting.

Definition 3.1. Let $t \in \{0, \dots, u\}$ and $E \in \mathcal{F}_t$. A self-financing trading strategy $\alpha \in \mathcal{S}_t^0$ is a *subreplication strategy* for X on E if $V_u(\alpha) \leq X$ on E and a *superreplication strategy* for X on E if $V_u(\alpha) \geq X$ on E. If $\mathbb{P}(E) = 1$ we omit the reference to E. The *subreplication price* and *superreplication price*, of X at date t are defined, respectively, by

$$\pi_{t,u}^-(X) := \operatorname{ess\,sup}\{V_t(\alpha)\,;\ \alpha \in \mathcal{S}_t^0 \text{ is a subreplication strategy for } X\},$$
$$\pi_{t,u}^+(X) := \operatorname{ess\,inf}\{V_t(\alpha)\,;\ \alpha \in \mathcal{S}_t^0 \text{ is a superreplication strategy for } X\}.$$

In particular, note that $\pi_{u,u}^-(X) = \pi_{u,u}^+(X) = X$.

In the above definition, we could have restricted our attention to trading strategies terminating at the maturity date u as the behavior of a sub- and superreplication strategy after the payoff's maturity is irrelevant. Nevertheless, for ease of notation, we will consistently work with trading strategies that are liquidated at the terminal date T.

Remark 3.2. It is clear how to define local versions $\pi_{t,u}^\pm(X|E)$ of the sub/superreplication prices restricted to events $E \in \mathcal{F}_t$. However, this is unnecessary since these local versions can always be expressed using the global definitions. Specifically,

$$\pi_{t,u}^\pm(X|E) = \pi_{t,u}^\pm(1_E X) = 1_E \pi_{t,u}^\pm(X).$$

This relationship between local and global pricing operators would be consistently repeated throughout the subsequent concepts we introduce, allowing us to work primarily with the global definitions without risk of confusion. So while we will define price-related concepts also locally, we will refrain from providing local versions of price operators.

As already mentioned, the concept of a market-consistent price is based on the principle that if two rational parties transact X at that price, neither the buyer nor the seller should be able to achieve a better outcome by trading directly in the market. A "localized" version of this concept, which is useful when considering intermediate dates, is formalized in the following definition.

Definition 3.3. Let $t \in \{0, \dots, u\}$ and $E \in \mathcal{F}_t$. We say that $P \in \mathcal{X}_t$ is a *market-consistent price* for X at date t on E if for all self-financing trading strategies $\alpha \in \mathcal{S}_t^0$ and $A \in \mathcal{F}_t$ with $A \subset E$:

(1) If $V_u(\alpha) \gneq X$ on A, then $V_t(\alpha) \gneq P$ on A.

(2) If $V_u(\alpha) \lneq X$ on A, then $V_t(\alpha) \lneq P$ on A.

We say that P is a *market-consistent buyer price* at date t on E if (1) holds and a *market-consistent seller price* at date t on E if (2) holds. If $\mathbb{P}(E) = 1$ we omit the reference to E. The set of market-consistent prices is denoted by $MCP_{t,u}(X)$ and the sets of market-consistent buyer and seller prices by $MCP^b_{t,u}(X)$ and $MCP^s_{t,u}(X)$, respectively.

Note that, since X is assumed to be nonnegative, the set of subreplication strategies is always nonempty as it contains at least the zero strategy. As a result, $\pi^-_{t,u}(X)$ belongs to $\mathcal{X}_t$ and is nonnegative. By contrast, superreplication strategies may not always exist. When this is the case, the buyer cannot directly compare X with any market alternative and the market remains silent about the prices at which the buyer may consider buying X. In other words, *every* price is a market-consistent buyer price. Therefore, to avoid keeping track of this trivial case, **we assume that X admits a superreplication strategy starting at date** 0. This assumption, along with the absence of arbitrage opportunities, implies that both $\pi^-_{t,u}(X)$ and $\pi^+_{t,u}(X)$ belong to $\mathcal{X}_t$ and satisfy $0 \le \pi^-_{t,u}(X) \le \pi^+_{t,u}(X)$ for every $t \in \{0, \dots, u\}$.

Remark 3.4. (i) Unlike most literature, we consider prices at any date before the maturity of X, not just at the initial date. This helps clarify many of the concepts related to American options and allows us to provide better economic interpretation of key results.

(ii) Recall that $P \in \mathbb{R}$ is an *arbitrage-free price* at date 0 for X if there exists a positive process $S^{N+1} \in \mathcal{X}_{0:T}$ satisfying $S_0^{N+1} = P$ and $S_T^{N+1} = X$ such that the *extended market* with basic securities $S^1, \dots, S^{N+1}$ is arbitrage-free; see, e.g., [6, Definition 5.28]. We will later establish the non-trivial equivalence of arbitrage-free prices and market-consistent prices. At this point we stress that, as observed in the introduction, there is no compelling economic reason for requiring that a price at which X is transacted should be based on arbitrage considerations:

- When new securities enter a market, prices of existing securities typically adjust — a market with unchanged prices after introducing new securities is unrealistic.
- Arbitrage considerations are irrelevant for off-market transactions between parties, as there is no market mechanism to eliminate arbitrage opportunities. However, rational parties should not transact if better opportunities exist in the market.
- Finally, it is unclear why the existence of an hypothetical future price process for X should have any bearing on a transaction happening today. While pricing X today requires considering dynamic trading strategies involving future market prices of the basic securities, it should not depend on the theoretical ability to transact X itself at all future dates. A transaction's economic validity should be determined by current market conditions and available trading opportunities, independent of whether X could hypothetically be traded again in the future.

For these reasons, we prefer market-consistent prices as the primitive notion of the theory.

It is evident from the definition of a market-consistent price that every market-consistent price is contained between the sub- and superreplication prices. The next proposition is the key technical result needed to show that the sub- and superreplication prices are sharp bounds for the set of market-consistent prices, and to provide a complete characterization of this set. As a byproduct, the proposition establishes a useful "time consistency" formula for sub- and superreplication prices.

Proposition 3.5. *For every $t \in \{0, \dots, u\}$ the following statements hold:*

(i) $\pi^+_{t,u}(X) = V_t(\alpha)$ *for some superreplication strategy* $\alpha \in \mathcal{S}^0_t$ *for* X. *Moreover, for* $t < u$,

$$\pi^+_{t,u}(X) = \pi^+_{t,t+1}(\pi^+_{t+1,u}(X)).$$

(ii) $\pi^-_{t,u}(X) = V_t(\alpha)$ *for some subreplication strategy* $\alpha \in \mathcal{S}^0_t$ *for* X. *Moreover, for* $t < u$,

$$\pi^-_{t,u}(X) = \pi^-_{t,t+1}(\pi^-_{t+1,u}(X)).$$

Proof. We only prove (i). The proof of (ii) is similar and omitted. As a preliminary step, let $s \in \{0, \dots, T-1\}$ and take any payoff $Z \in \mathcal{X}_{s+1}$ admitting a superreplication strategy starting at date s. We claim that

$$\pi^+_{s,s+1}(Z) = V_s(\alpha) \text{ for some superreplication strategy } \alpha \in \mathcal{S}^0_s \text{ for } Z, \tag{3.1}$$

The set of payoffs maturing at time $s+1$ that can be superreplicated at date s at zero cost is

$$\mathcal{C}_{s+1} := \left\{ Y \in \mathcal{X}_{s+1} \,;\; \exists \alpha \in \mathcal{S}^0_s \,:\, V_s(\alpha) = 0,\; V_{s+1}(\alpha) \geq Y \right\}.$$

Setting $\mathcal{P}_s(Z) := (Z - \mathcal{C}_{s+1}) \cap \mathcal{X}_s$, a straightforward verification shows that

$$\pi^+_{s,s+1}(Z) = \operatorname{ess\,inf}\{P \in \mathcal{X}_s \,;\; Z - P \in \mathcal{C}_{s+1}\} = \operatorname{ess\,inf} \mathcal{P}_s(Z).$$

By [6, Lemma 1.68], the set $\mathcal{C}_{s+1}$ is closed with respect to convergence in probability. Note that $\mathcal{P}_s(Z)$ is downward directed. Indeed, take $P_1, P_2 \in \mathcal{P}_s(Z)$ and $\alpha^1, \alpha^2 \in \mathcal{S}^0_s$ such that $V_s(\alpha^1) = V_s(\alpha^2) = 0$ and $V_{s+1}(\alpha^1) \geq Z - P_1$ and $V_{s+1}(\alpha^2) \geq Z - P_2$. Set $E = \{P_1 \leq P_2\}$ and define $\alpha \in \mathcal{S}^0_s$ by setting $\alpha_u = 1_E \alpha^1_u + 1_{E^c} \alpha^2_u$ for $u \in \{s, \dots, T-1\}$. Then, $V_s(\alpha) = 0$ and $V_{s+1}(\alpha) \geq Z - \min\{P_1, P_2\}$, proving that $\mathcal{P}_s(Z)$ is downward directed. Hence, $\pi^+_{s,s+1}(Z)$ is the almost-sure limit, and thus the limit in probability, of a sequence in $\mathcal{P}_s(Z)$. The closedness of $\mathcal{C}_{s+1}$ implies that $\pi^+_{s,s+1}(Z) \in \mathcal{P}_s(Z)$. As a result, there exists $\beta \in \mathcal{S}^0_s$ such that $V_s(\beta) = 0$ and $V_{s+1}(\beta) \geq Z - \pi^+_{s,s+1}(Z)$.
To prove (3.1), define $\alpha \in \mathcal{S}^0_s$ by setting $\alpha_u = \pi^+_{s,s+1}(Z)\eta_u + \beta_u$ for $u \in \{s, \dots, T-1\}$ and observe that

$$V_s(\alpha) = \pi^+_{s,s+1}(Z) + V_s(\beta) = \pi^+_{s,s+1}(Z), \qquad V_{s+1}(\alpha) = \pi^+_{s,s+1}(Z) + V_{s+1}(\beta) \geq Z.$$

To show (i), we prove by backward induction that for every $s \in \{t, \dots, u-1\}$

$$\pi^+_{s,u}(X) = \pi^+_{s,s+1}(\pi^+_{s+1,u}(X)) = V_s(\alpha) \text{ for some superreplication strategy } \alpha \in \mathcal{S}^0_s \text{ for } X.$$

For $s = u-1$, the claim follows directly from (3.1). Assume the claim holds for some $s \in \{t+1, \dots, u-1\}$. Again by (3.1), there exists $\beta \in \mathcal{S}^0_{s-1}$ such that $V_s(\beta) \geq \pi^+_{s,u}(X) = V_s(\alpha)$ and $V_{s-1}(\beta) = \pi^+_{s-1,s}(\pi^+_{s,u}(X))$. Define the trading strategy $\gamma \in \mathcal{S}^0_{s-1}$ by

$$\gamma_{s-1} = \beta_{s-1}, \qquad \gamma_r = \alpha_r + (V_s(\beta) - V_s(\alpha))\eta_r, \quad r \in \{s, \dots, T-1\}.$$

Since $V_s(\beta) \geq V_s(\alpha)$, we have $V_u(\gamma) \geq V_u(\alpha) \geq X$. In particular,

$$\pi^+_{s-1,u}(X) \leq V_{s-1}(\gamma) = V_{s-1}(\beta) = \pi^+_{s-1,s}(\pi^+_{s,u}(X)).$$

Since the inequality $\pi^+_{s-1,s}(\pi^+_{s,u}(X)) \leq \pi^+_{s-1,u}(X)$ is clear, we obtain

$$V_{s-1}(\gamma) = \pi^+_{s-1,u}(X) = \pi^+_{s-1,s}(\pi^+_{s,u}(X)).$$

This concludes the induction argument. $\square$

Remark 3.6. What makes the proof of Proposition 3.5 work is the closedness result in [6, Lemma 1.68] which, not surprisingly, is also implicitly or explicitly the key ingredient in all proofs of the Fundamental Theorem of Asset Pricing. There, it ensures that a separation of closed convex sets can be performed. Here, we use it to ensure the attainability of sub- and superreplication prices.

A natural notion in the study of market-consistent prices is that of a replicable payoff, where a self-financing strategy achieves both sub- and superreplication simultaneously, thus exactly matching the payoff. When discussing replicability on an event, we must specify the starting date of the replication strategy, since the event might also belong to the information set at earlier dates, yet a replication strategy beginning at those earlier dates might not exist.

Definition 3.7. Let $t \in \{0, \ldots, u\}$ and $E \in \mathcal{F}_t$. We say that X is *replicable* on E at date t if there exists a self-financing trading strategy $\alpha \in \mathcal{S}^0_t$ that is both a sub- and a superreplication strategy for X on E. In this case, $X = V_u(\alpha)$ on E and we say that α is a *replication strategy* for X on E. If $\mathbb{P}(E) = 1$ we omit the reference to E. The largest event on which X is replicable at date t is called the *maximum domain of replicability* of X at date t.

The maximum domain of replicability of a payoff corresponds to the event where its sub- and superreplication prices coincide. On this event, both these prices trivially coincide with the value of any replication strategy and can be therefore interpreted as the cost that has to be paid to replicate the payoff. This is the second key result to obtain a description of market-consistent prices in terms of sub- and superreplication prices.

Proposition 3.8. *For every $t \in \{0, \ldots, u\}$ the set $\{\pi^-_{t,u}(X) = \pi^+_{t,u}(X)\}$ is the maximum domain of replicability of X at date t.*

Proof. Take a subreplication strategy $\alpha \in \mathcal{S}^0_t$ for X with $V_t(\alpha) = \pi^-_{t,u}(X)$ and a superreplication strategy $\beta \in \mathcal{S}^0_t$ for X with $V_t(\beta) = \pi^+_{t,u}(X)$, which exist by Proposition 3.5. The absence of arbitrage opportunities implies $V_u(\alpha) = V_u(\beta) = X$ on $\{\pi^-_{t,u}(X) = \pi^+_{t,u}(X)\}$, showing that X is replicable on $\{\pi^-_{t,u}(X) = \pi^+_{t,u}(X)\}$ at date t. To conclude, assume that X is replicable on $E \in \mathcal{F}_t$ at date t and note that this implies that $\pi^+_{t,u}(X) = \pi^-_{t,u}(X)$ on E. $\square$

Following our earlier results, we can now provide the announced characterization of market-consistent prices showing that the set of market-consistent prices is interval-like and its bounds are given by the sub- and superreplication prices. The theorem also precisely characterizes under which conditions these bounds are attained. This result effectively summarizes all there is to say about the structure of the set of market-consistent prices for European payoffs.

Theorem 3.9. *For every $t \in \{0, \ldots, u\}$ the sets $MCP^b_{t,u}(X)$ and $MCP^s_{t,u}(X)$ are nonempty and for every $P \in \mathcal{X}_t$ the following statements hold:*

(i) $P \in \mathcal{X}_t$ *is a market-consistent buyer price for* X *at date* t *if and only if:*

(1) $P < \pi^+_{t,u}(X)$ *on* $\{\pi^-_{t,u}(X) < \pi^+_{t,u}(X)\}$*;*

(2) $P \leq \pi^+_{t,u}(X)$ *on* $\{\pi^-_{t,u}(X) = \pi^+_{t,u}(X)\}$*.*

(ii) $P \in \mathcal{X}_t$ *is a market-consistent seller price for* X *at date* t *if and only if:*

(1) $P > \pi^-_{t,u}(X)$ *on* $\{\pi^-_{t,u}(X) < \pi^+_{t,u}(X)\}$*;*

(2) $P \geq \pi^-_{t,u}(X)$ *on* $\{\pi^-_{t,u}(X) = \pi^+_{t,u}(X)\}$*.*

(iii) $P \in \mathcal{X}_t$ *is a market-consistent price for* X *at date* t *if and only if:*

(1) $\pi^-_{t,u}(X) < P < \pi^+_{t,u}(X)$ *on* $\{\pi^-_{t,u}(X) < \pi^+_{t,u}(X)\}$*;*

(2) $P = \pi^-_{t,u}(X) = \pi^+_{t,u}(X)$ *on* $\{\pi^-_{t,u}(X) = \pi^+_{t,u}(X)\}$*.*

(iv) $\pi^+_{t,u}(X) = \operatorname{ess\,sup} MCP_{t,u}(X) = \operatorname{ess\,sup} MCP^b_{t,u}(X)$*.*

(v) $\pi^-_{t,u}(X) = \operatorname{ess\,inf} MCP_{t,u}(X) = \operatorname{ess\,inf} MCP^s_{t,u}(X)$*.*

Proof. We only prove (i). The proof of (ii) is similar and omitted while statements (iii) to (v) follow from (i) and (ii). If P is a market-consistent buyer price for X at date t, then it is clear that (2) holds. Furthermore, take any superreplication strategy $\alpha \in \mathcal{S}^0_t$ for X with $V_t(\alpha) = \pi^+_{t,u}(X)$, which exists by Proposition 3.5. It follows from Proposition 3.8 that, on the event $A = \{P \geq \pi^+_{t,u}(X) > \pi^-_{t,u}(X)\}$, we must have $V_u(\alpha) \gneq X$ and, hence, $\pi^+_{t,u}(X) = V_t(\alpha) \gneq P \geq \pi^+_{t,u}(X)$ by market consistency. This forces $\mathbb{P}(A) = 0$ and yields (1). Conversely, assume that P satisfies (1) and (2) and take any $A \in \mathcal{F}_t$ and any self-financing strategy $\alpha \in \mathcal{S}^0_t$ such that $V_u(\alpha) \gneq X$ on A. Note that $V_t(\alpha) \geq \pi^+_{t,u}(X) \geq P$ on A by assumption. Now, take a subreplication strategy $\beta \in \mathcal{S}^0_t$ for X with $V_t(\beta) = \pi^-_{t,u}(X)$, which exists again by Proposition 3.5. If $V_t(\alpha) = P$ on A, then $A \subset \{\pi^-_{t,u}(X) = \pi^+_{t,u}(X)\}$ by (1) and we would have an arbitrage opportunity because $V_t(\alpha) = \pi^+_{t,u}(X) = \pi^-_{t,u}(X) = V_t(\beta)$ on A whereas $V_u(\alpha) \gneq X \geq V_u(\beta)$ on A. As a result, $V_t(\alpha) \gneq P$ on A, showing that P is a market-consistent buyer price. □

Remark 3.10. A corresponding result exists for arbitrage-free prices; see [6, Theorem 5.29]. The identical mathematical characterization of the sets of market-consistent prices and arbitrage-free prices establishes their equivalence, at least at date $t = 0$ (for which the results in [6] are established). However, the result for arbitrage-free prices requires the use of duality theory in the form of equivalent martingale measures. In Section 6, we will revisit this equivalence from the perspective of dual representations. By contrast, our characterization is established directly in our "primal" setting, requiring no duality theory, which underscores the mathematical simplicity of our approach.

3.2 European payoff streams

The problem of pricing a payoff stream in a market-consistent way can be reduced to pricing a single payoff. Throughout this subsection **we fix a payoff stream** $X \in \mathcal{X}_{0:T}$ with $X_0 = 0$ and $X_t \geq 0$ for each $t \in \{1, \dots, T\}$. When we say "X is transacted at date t", we specifically mean that X^t is transacted (rather than X^{t+1}). We begin by defining sub- and superreplication strategies. Since payoff streams involve intermediate payoffs, we must use trading strategies that are generally not self-financing. For notational convenience, for a trading strategy $\alpha \in \mathcal{S}_t$ we set $V^{liq}_t(\alpha) := 0$ and $V^{acq}_T(\alpha) := 0$.

Definition 3.11. Let $t \in \{0, \dots, T-1\}$ and $E \in \mathcal{F}_t$. A trading strategy $\alpha \in \mathcal{S}_t$ is a *subreplication strategy* for X on E if $V_s^{liq}(\alpha) - V_s^{acq}(\alpha) \le X_s$ on E for every $s \in \{t+1, \dots, T\}$ and a *superreplication strategy* for X on E if $V_s^{liq}(\alpha) - V_s^{acq}(\alpha) \ge X_s$ on E for every $s \in \{t+1, \dots, T\}$. If $\mathbb{P}(E) = 1$ we omit the reference to E. The *subreplication price*, respectively *superreplication price*, of X at date t are defined by

$$\pi_t^-(X) := \operatorname{ess\,sup}\{V_t^{acq}(\alpha)\,;\ \alpha \in \mathcal{S}_t \text{ is a subreplication strategy for } X\},$$
$$\pi_t^+(X) := \operatorname{ess\,inf}\{V_t^{acq}(\alpha)\,;\ \alpha \in \mathcal{S}_t \text{ is a superreplication strategy for } X\}.$$

For convenience, we also set $\pi_T^-(X) := \pi_T^+(X) := X_T$.

The following definition formalizes, in the context of payoff streams, the intuition that when a payoff stream is transacted at a rational price, neither the seller nor the buyer should be able to do better by trading directly in the market.

Definition 3.12. Let $t \in \{0, \dots, T-1\}$ and $E \in \mathcal{F}_t$. We say that $P \in \mathcal{X}_t$ is a *market-consistent price* for X at date t on E if for every trading strategy $\alpha \in \mathcal{S}_t$ and every $A \in \mathcal{F}_t$ such that $A \subset E$:

(1) If $V_s^{liq}(\alpha) - V_s^{acq}(\alpha) \ge X_s$ on A for every $s \in \{t+1, \dots, T\}$ and some inequality is not an equality, then $V_t^{acq}(\alpha) \gneq P$ on A.

(2) If $V_s^{liq}(\alpha) - V_s^{acq}(\alpha) \le X_s$ on A for every $s \in \{t+1, \dots, T\}$ and some inequality is not an equality, then $V_t^{acq}(\alpha) \lneq P$ on A.

The set of market-consistent prices is denoted by $MCP_t(X)$. We say that P is a *market-consistent buyer price* if (1) holds and a *market-consistent seller price* if (2) holds. The sets of market-consistent buyer and seller prices are denoted by $MCP_t^b(X)$ and $MCP_t^s(X)$, respectively.

As in the case of single payoffs, to ensure that the problem of determining market-consistent buyer prices is not trivial we need to assume that X **admits a superreplication strategy starting at date** 0. Together with the postulated absence of arbitrage opportunities, this assumption implies that $\pi_t^-(X)$ and $\pi_t^+(X)$ both belong to $\mathcal{X}_t$ and satisfy $0 \le \pi_t^-(X) \le \pi_t^+(X)$ for every $t \in \{0, \dots, T\}$. For $t \in \{0, \dots, T\}$ we define the *terminal payoff equivalent* of X at date t by

$$Z_t(X) := \sum_{u=t}^{T} \frac{X_u}{S_u^1} S_T^1 = \sum_{u=t}^{T} X_u.$$

The terminal payoff equivalent is thus the terminal payoff of a trading strategy that, at each date $u \in \{t, \dots, T\}$, invests the amount X_u in the numéraire security. This special payoff enables us to translate results for single payoffs into results for payoff streams as recorded next.

Proposition 3.13. *Let $t \in \{0, \dots, T-1\}$. The following statements hold:*

(i) $MCP_t^b(X) = MCP_{t,T}^b(Z_t(X))$.

(ii) $MCP_t^s(X) = MCP_{t,T}^s(Z_t(X))$.

(iii) $MCP_t(X) = MCP_{t,T}(Z_t(X))$.

Proof. We only sketch the proof of (i). If $\alpha \in \mathcal{S}_t$ is a superreplication strategy for X, then there exists a superreplication strategy $\beta \in \mathcal{S}_t^0$ for $Z_t(X)$ such that $V_t(\beta) = V_t^{acq}(\alpha) + X_t$. Conversely, if $\beta \in \mathcal{S}_t^0$ is a superreplication strategy for $Z_t(X)$, then there exists a superreplication strategy $\alpha \in \mathcal{S}_t$ for X such that $V_t^{acq}(\alpha) = V_t(\beta) - X_t$. In both cases, if some inequality featuring in the superreplication condition for the given trading strategy is not an equality, then this carries over to the derived trading strategy. This delivers at once the desired statement. $\square$

Remark 3.14. Let $t \in \{0, \dots, T-1\}$. Replicability of the payoff stream X is defined similarly to replicability of a single payoff, i.e., X is *replicable* at date t if there exists a strategy $\alpha \in \mathcal{S}_t$ that is both a sub- and a superreplication strategy for X. It is easy to verify that X is replicable at date t precisely when $Z_t(X)$ is. It is also clear that, if X_u is replicable at date t for every $u \in \{t, \dots, T\}$, then X is also replicable at date t. The converse is, however, not true. To see this, take any $Y \in \mathcal{X}_u$ for some $u \in \{t+1, \dots, T-1\}$ that is not replicable at date t and consider the payoff stream X defined by setting $X_s = 0$ for $s \in \{0, \dots, u-1\} \cup \{u+1, \dots, T-1\}$ as well as $X_u = Y$ and $X_T = S_T^1 - Y$. As $Z_t(X) = S_T^1$, the payoff stream X is replicable at date t.

4 Market-consistent prices for American options

An *American option* is specified by a stream of non-negative payoffs with the following special contractual *optionality* feature: At any date during the life of the contract, the holder can either *exercise* the option to receive the payment at that date, foregoing any future payment, or retain the option for later exercise, foregoing the payment. Throughout this section, **we fix an American option** $X \in \mathcal{X}_{0:T}$ with $X_0 = 0$ and $X_t \geq 0$ for each $t \in \{1, \dots, T\}$. As is customary, exercise strategies are modelled by stopping times.

Definition 4.1. Let $t \in \{0, \dots, T\}$. An *exercise strategy* for X (starting at date t) is a random variable $\tau \in \mathcal{X}_T$ such that $\mathbb{P}(\tau \in \{t, \dots, T\}) = 1$ and $\{\tau = u\} \in \mathcal{F}_u$ for every $u \in \{t, \dots, T\}$. The set of exercise strategies starting at date t is denoted by $\mathcal{T}_t$.

For any date $t \in \{0, \dots, T\}$, exercising X according to a strategy $\tau \in \mathcal{T}_t$ entitles the holder of the option to receive the (European) payoff stream given by

$$X^\tau := (0, \dots, 0, 1_{\{\tau=t\}} X_t, \dots, 1_{\{\tau=T\}} X_T).$$

As we know from the previous section, from the perspective of market-consistent prices, this payoff stream can be identified with its terminal payoff equivalent that we denote by X_τ, i.e.,

$$X_\tau := Z_t(X^\tau) = \sum_{u=t}^{T} 1_{\{\tau=u\}} X_u.$$

From now on we then work with the payoff X_τ instead of with the payoff stream X^τ and loosely refer to X_τ as the *payoff* obtained when exercising X according to τ, even though what the holder obtains is in fact the payoff stream X^τ. The key observation for our analysis of American options is that buying the American option at a given date is economically equivalent to acquiring the right to select one, and only one, item from a basket containing the (European) payoffs X_τ with τ starting at that date. As a result, the seller of the option needs to be prepared to deliver any of the payoffs in the basket, ignoring which one it will be until the owner actually exercises. This interpretation leads to a natural definition of sub- and superreplication strategies as well as of market-consistent prices for American options. As our previous discussion implies, in contrast to the European case, sub- and superreplication strategies for American options exhibit a fundamental asymmetry. Consequently, we examine each concept in its own dedicated section. We introduce two special classes of exercise strategies that, to the best of our knowledge, have not been considered in the literature and that at this point will play a crucial technical role in our analysis. Their financial interpretation will be discussed in the next sections.

Definition 4.2. Let $t \in \{0, \dots, T\}$. We denote by $\mathcal{T}_t^-$, respectively $\mathcal{T}_t^+$, the set of *pricing-lowerbound exercise strategies*, respectively *pricing-upperbound exercise strategies*, for X at date t, i.e., $\sigma \in \mathcal{T}_t$ such that, respectively,

$$\pi_{t,T}^-(X_\sigma) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \pi_{t,T}^-(X_\tau), \qquad \pi_{t,T}^+(X_\sigma) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \pi_{t,T}^+(X_\tau).$$

In addition, we denote by $\mathcal{T}_t^s$, respectively $\mathcal{T}_t^b$, the set of *seller*, respectively *buyer*, *pricing exercise strategies* for X at date t, i.e., $\sigma \in \mathcal{T}_t$ such that, respectively,

$$MCP_{t,T}^s(X_\sigma) = \bigcap_{\tau \in \mathcal{T}_t} MCP_{t,T}^s(X_\tau), \qquad MCP_{t,T}^b(X_\sigma) = \bigcup_{\tau \in \mathcal{T}_t} MCP_{t,T}^b(X_\tau).$$

Note that, by Theorem 3.9, we always have $\mathcal{T}_t^s \subset \mathcal{T}_t^-$ and $\mathcal{T}_t^b \subset \mathcal{T}_t^+$.

4.1 Market-consistent seller prices

Since the seller of an American option is committed to delivering any of the European payoffs in the basket, he or she would rationally prefer a less onerous commitment — specifically, one arising from a short position in a self-financing strategy that subreplicates one of these payoffs. Indeed, we can think of selling the American option as borrowing funds where repayment is effectively driven by the exercise strategy the buyer decides to adopt. From this perspective, if possible, the seller would rationally prefer to borrow the same amount by shorting a trading strategy that is dominated by one of these potential repayment schedules. This leads to the following notion of a subreplication strategy and later to that of market-consistent seller prices.

Definition 4.3. Let $t \in \{0, \dots, T\}$ and $E \in \mathcal{F}_t$. We say that $\alpha \in \mathcal{S}_t^0$ is a *subreplication strategy* for X on E if it is a subreplication strategy for X_τ on E for some $\tau \in \mathcal{T}_t$. If $\mathbb{P}(E) = 1$ we omit the reference to E. The *subreplication price* for X at date t is

$$\pi_t^-(X) := \operatorname{ess\,sup}\{V_t(\alpha)\,;\ \alpha \in \mathcal{S}_t^0 \text{ is a subreplication strategy for } X\}.$$

In particular, $\pi_T^-(X) = X_T$.

Being a stream of positive payments, X always admits a subreplication strategy. Along with the no-arbitrage assumption, this readily implies that $\pi_t^-(X)$ belongs to $\mathcal{X}_t$ and satisfies $\pi_t^-(X) \geq 0$. The following result provides a variety of equivalent formulations of and a recursion formula for the subreplication price, showing in particular that pricing-lowerbound pricing strategies always exist. The recursion formula formalizes the economic intuition that the subreplication price of an American option equals the maximum between its immediate exercise value and the subreplication price of the residual option should exercise be deferred.

Proposition 4.4. *For every $t \in \{0, \ldots, T\}$ the following statements hold:*

(i) $\pi_t^-(X) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \pi_{t,T}^-(X_\tau)$.

(ii) $\pi_t^-(X) = \pi_{t,T}^-(X_\tau)$ *for some* $\tau \in \mathcal{T}_t$.

(iii) $\pi_t^-(X) = \max\{X_t, \pi_{t,t+1}^-(\pi_{t+1}^-(X))\} = \max\{X_t, \pi_t^-(X^{t+1})\}$ *for* $t < T$.

(iv) $\pi_t^-(X) = V_t(\alpha)$ *for some subreplication strategy* $\alpha \in \mathcal{S}_t^0$ *for* X.

In particular, the set $\mathcal{T}_t^-$ *is nonempty.*

Proof. Note that assertion (i) is immediate from the definition of subreplication strategies. Note also that the second equality in assertion (iii) follows from the first equality in (iii) by applying it to X^{t+1} because we easily see that $X_t^{t+1} = 0$ and, hence, $\pi_t^-(X^{t+1}) = \pi_{t,t+1}^-(\pi_{t+1}^-(X^{t+1}))$. We establish (ii) by a backward induction argument that also delivers the first equality in (iii). Start by observing that (ii) is clearly true for $t = T$. Assume the assertion is satisfied by $\tau \in \mathcal{T}_{t+1}$ for $t+1$ for some $t \in \{0, \ldots, T-1\}$. Set $E := \{\pi_t^-(X) = X_t\}$ and $\sigma := t1_E + \tau 1_{E^c} \in \mathcal{T}_t$. On E we clearly have

$$X_t = \pi_t^-(X) \geq \pi_{t,T}^-(X_\sigma) = X_t, \quad \pi_t^-(X) \geq \pi_{t,T}^-(X_\tau) = \pi_{t,t+1}^-(\pi_{t+1,T}^-(X_\tau)) = \pi_{t,t+1}^-(\pi_{t+1}^-(X)).$$

Similarly, on the complement event E^c we have

$$\begin{aligned}\pi_{t,T}^-(X_\sigma) &= \pi_{t,t+1}^-(\pi_{t+1,T}^-(X_\sigma)) = \pi_{t,t+1}^-(\pi_{t+1,T}^-(X_\tau)) = \pi_{t,t+1}^-(\pi_{t+1}^-(X)) \\ &\geq \operatorname*{ess\,sup}_{\rho \in \mathcal{T}_{t+1}} \pi_{t,t+1}^-(\pi_{t+1,T}^-(X_\rho)) = \operatorname*{ess\,sup}_{\rho \in \mathcal{T}_{t+1}} \pi_{t,T}^-(X_\rho) = \pi_t^-(X) \geq \pi_{t,T}^-(X_\sigma),\end{aligned}$$

Combining the two shows (ii) and the first equality in (iii) for date t and concludes the induction argument. Finally, (iv) follows by taking any $\tau \in \mathcal{T}_t^-$, which exists by (ii), and any subreplication strategy $\alpha \in \mathcal{S}_t^0$ for X_τ such that $V_t(\alpha) = \pi_{t,T}^-(X_\tau)$, which exists by Proposition 3.5. □

Thinking of an American option as a basket, it is clear that the seller will not contemplate selling the option for any price that is not a market-consistent seller price for every item in the basket. We formalize this in the following definition.

Definition 4.5. Let $t \in \{0, \ldots, T\}$ and $E \in \mathcal{F}_t$. We say that $P \in \mathcal{X}_t$ is a *market-consistent seller price* for X at date t on E if it is a market-consistent seller price for X_τ at date t on E for every $\tau \in \mathcal{T}_t$. If $\mathbb{P}(E) = 1$ we omit the reference to E. The set of market-consistent seller prices is denoted by $MCP_t^s(X)$.

Market-consistent seller prices for American options can be determined by focusing only on payoffs in the basket with the highest subreplication price, i.e., on payoffs X_τ corresponding to exercise strategies $\tau \in \mathcal{T}_t^-$. In fact, we prove that amongst these payoffs there always exists at least one that is "most expensive" in the sense that it has the highest superreplication price. In other words there exists an exercise strategy $\sigma \in \mathcal{T}_t^-$ such that

$$\pi_{t,T}^+(X_\sigma) = \widetilde{\pi}_t(X) := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t^-} \pi_{t,T}^+(X_\tau). \tag{4.1}$$

Importantly, the market-consistent prices for X_σ coincide with the market-consistent prices of the American option. Moreover, every such exercise strategy turns out to be a "seller pricing strategy" in the sense of Definition 4.2, providing an ex-post justification for the adopted terminology. Note also that being the "most expensive" among the payoffs X_τ with $\tau \in \mathcal{T}_t^-$ implies that if X_σ is replicable, then every X_τ with $\tau \in \mathcal{T}_t^-$ is also replicable. The next result formalizes the above discussion by establishing a characterization of seller pricing strategies and by showing that a seller pricing strategy always exists.

Proposition 4.6. *For every $t \in \{0, \dots, T\}$ there exists $\sigma \in \mathcal{T}_t$ such that*

$$MCP_t^s(X) = \bigcap_{\tau \in \mathcal{T}_t} MCP_{t,T}^s(X_\tau) = \bigcap_{\tau \in \mathcal{T}_t^-} MCP_{t,T}^s(X_\tau) = MCP_{t,T}^s(X_\sigma).$$

In particular, $\mathcal{T}_t^s$ is nonempty and for every $\sigma \in \mathcal{T}_t$ the following statements are equivalent:

(a) $\sigma \in \mathcal{T}_t^-$ and $\{\pi_{t,T}^-(X_\sigma) = \pi_{t,T}^+(X_\sigma)\} = \bigcap_{\tau \in \mathcal{T}_t^-}\{\pi_{t,T}^-(X_\tau) = \pi_{t,T}^+(X_\tau)\}$.

(b) $\sigma \in \mathcal{T}_t^-$ and $\{\pi_{t,T}^-(X_\sigma) = \pi_{t,T}^+(X_\sigma)\} = \{\pi_t^-(X) = \widetilde{\pi}_t(X)\}$.

(c) $MCP_{t,T}^s(X_\sigma) = MCP_t^s(X)$.

(d) $\sigma \in \mathcal{T}_t^s$.

Proof. To establish the proposition it suffices to prove the equivalence between the four assertions and the existence of an exercise strategy satisfying any of the equivalent assertions. The equivalence between (a) and (b) is obvious and that between (c) and (d) is a direct consequence of the definition of a market-consistent price. To conclude the proof of equivalence, we establish that (a) and (d) are equivalent. First, assume that (d) holds and take any $\tau \in \mathcal{T}_t^-$. Recall that $\mathcal{T}_t^s \subset \mathcal{T}_t^-$, which yields $\sigma \in \mathcal{T}_t^-$ and the inclusion "$\supset$" in (a). As $\pi_{t,T}^-(X_\sigma) = \pi_{t,T}^-(X_\tau)$, we must have $\pi_{t,T}^-(X_\tau) = \pi_{t,T}^+(X_\tau)$ on the event $\{\pi_{t,T}^-(X_\sigma) = \pi_{t,T}^+(X_\sigma)\}$, for otherwise $\pi_{t,T}^-(X_\tau)$ would belong to $MCP_{t,T}^s(X_\sigma)$ but not to $MCP_{t,T}^s(X_\tau)$ by Theorem 3.9, which is impossible because $MCP_{t,T}^s(X_\sigma) \subset MCP_{t,T}^s(X_\tau)$ by assumption on σ. This shows the inclusion "$\subset$" in (a). Conversely, assume (a) holds and take any $\tau \in \mathcal{T}_t$. Note that $\pi_{t,T}^-(X_\sigma) \geq \pi_{t,T}^-(X_\tau)$. Set $E := \{\pi_{t,T}^-(X_\tau) = \pi_{t,T}^-(X_\sigma)\}$ and define $\rho := 1_E\tau + 1_{E^c}\sigma \in \mathcal{T}_t^-$. By assumption, $\pi_{t,T}^-(X_\tau) = \pi_{t,T}^-(X_\rho) = \pi_{t,T}^+(X_\rho) = \pi_{t,T}^+(X_\tau)$ on $E \cap \{\pi_{t,T}^-(X_\sigma) = \pi_{t,T}^+(X_\sigma)\}$. This implies that $MCP_{t,T}^s(X_\sigma) \subset MCP_{t,T}^s(X_\tau)$, delivering (d) and concluding the proof of the equivalence.

We conclude by proving that there exists an exercise strategy satisfying (b). To this effect, note that the collection $\{\pi_{t,T}^+(X_\tau)\,;\ \tau \in \mathcal{T}_t^-\}$ is upward directed. Indeed, for all $\tau_1, \tau_2 \in \mathcal{T}_t^-$ set $E = \{\pi_{t,T}^+(X_{\tau_1}) \geq \pi_{t,T}^+(X_{\tau_2})\}$ and note that $\tau = 1_E\tau_1 + 1_{E^c}\tau_2 \in \mathcal{T}_t^-$ satisfies $\pi_{t,T}^+(X_\tau) = \max\{\pi_{t,T}^+(X_{\tau_1}), \pi_{t,T}^+(X_{\tau_2})\}$. Hence, we find a sequence $(\tau_n) \subset \mathcal{T}_t^-$ such that $\pi_{t,T}^+(X_{\tau_n}) \uparrow \widetilde{\pi}_t(X)$ almost surely. For $n \in \mathbb{N}$ define recursively

$$A_1 := \{\pi_{t,T}^+(X_{\tau_1}) = \widetilde{\pi}_t(X)\}, \quad A_{n+1} := \{\pi_{t,T}^+(X_{\tau_{n+1}}) > \pi_{t,T}^+(X_{\tau_n})\} \setminus \bigcup_{k=1}^n A_k \in \mathcal{F}_t.$$

The events A_n's are pairwise disjoint and $\bigcup_{n\in\mathbb{N}} A_n = \Omega$. Now, define the exercise strategy

$$\sigma := \sum_{n\in\mathbb{N}} \tau_n 1_{A_n} \in \mathcal{T}_t^-.$$

For every $n \in \mathbb{N}$ we have $\pi^-_{t,T}(X_\sigma) = \pi^-_{t,T}(X_{\tau_n}) \leq \pi^+_{t,T}(X_{\tau_n}) < \pi^+_{t,T}(X_{\tau_{n+1}}) = \pi^+_{t,T}(X_\sigma)$ on A_{n+1}. In particular, $\{\pi^-_{t,T}(X_\sigma) = \pi^+_{t,T}(X_\sigma)\} \subset A_1$, implying that on $\{\pi^-_{t,T}(X_\sigma) = \pi^+_{t,T}(X_\sigma)\}$

$$\pi^-_t(X) = \pi^-_{t,T}(X_\sigma) = \pi^+_{t,T}(X_\sigma) = \pi^+_{t,T}(X_{\tau_1}) = \widetilde{\pi}_t(X).$$

This implies that σ satisfies (b), concluding the proof. □

The market-consistent seller prices of an American option were shown to coincide with the market-consistent seller prices of the European contract that is obtained by exercising the option according to a seller pricing strategy. As a direct consequence of the results for European contracts we therefore derive a characterization of the market-consistent seller prices for an American option.

Theorem 4.7. *For every $t \in \{0, \dots, T\}$ the set $MCP^s_t(X)$ is nonempty and $P \in \mathcal{X}_t$ is a market-consistent seller price for X at date t if and only if the following conditions hold:*

(1) $P > \pi^-_t(X)$ on $\{\pi^-_t(X) < \widetilde{\pi}_t(X)\}$;

(2) $P \geq \pi^-_t(X)$ on $\{\pi^-_t(X) = \widetilde{\pi}_t(X)\}$.

In particular, the following statements are equivalent:

(a) $\pi^-_t(X)$ is a market-consistent seller price for X at date t.

(b) $\pi^-_t(X) = \widetilde{\pi}_t(X)$.

(c) X_τ is replicable at date t for every $\tau \in \mathcal{T}^s_t$.

(d) X_τ is replicable at date t for some $\tau \in \mathcal{T}^s_t$.

(e) X_τ is replicable at date t for every $\tau \in \mathcal{T}^-_t$.

Remark 4.8 (The seller does not have to charge for optionality). At first glance, one might expect that the optionality embedded in an American option must be valuable and that the seller should therefore charge a "market premium" for it. However, as implied by Proposition 4.6, this is not the case. Indeed, regardless of the optionality, the market-consistent seller prices for the American option simply correspond to the market-consistent seller prices of a single European payoff, which corresponds to an item in the basket that is "most expensive" in the eyes of the seller.

4.2 Market-consistent buyer prices

Since the buyer of an American option has the right to choose any of the European payoffs in the basket, he or she will always prefer a self-financing strategy dominating all of these payoffs compared to holding the option itself.

Definition 4.9. Let $t \in \{0, \dots, T\}$ and $E \in \mathcal{F}_t$. We say that $\alpha \in \mathcal{S}^0_t$ is a *superreplication strategy* for X on E if it is a superreplication strategy for X_τ on E for every $\tau \in \mathcal{T}_t$. If $\mathbb{P}(E) = 1$ we omit the reference to E. The *superreplication price* for X at date t is

$$\pi^+_t(X) := \operatorname{ess\,inf}\{V_t(\alpha)\,;\ \alpha \in \mathcal{S}^0_t \text{ is a superreplication strategy for } X\}.$$

In particular, $\pi^+_T(X) = X_T$.

As in the European case, we exclude the case that every price is a market-consistent buyer price by assuming that X **admits a superreplication strategy starting at date** 0. This assumption implies that $\pi_t^+(X)$ belongs to $\mathcal{X}_t$ and satisfies $0 \leq \pi_t^-(X) \leq \pi_t^+(X)$ for every $t \in \{0, \dots, T\}$. The following result establishes a variety of equivalent formulations of the superreplication price in terms of the superreplication prices of the European payoffs in the basket and of superreplication strategies.

Proposition 4.10. *For every $t \in \{0, \dots, T\}$ the following statements hold:*

(i) $\pi_t^+(X) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \pi_{t,T}^+(X_\tau)$.

(ii) $\pi_t^+(X) = \pi_{t,T}^+(X_\tau)$ *for some* $\tau \in \mathcal{T}_t$.

(iii) $\pi_t^+(X) = \max\{X_t, \pi_{t,t+1}^+(\pi_{t+1}^+(X))\} = \max\{X_t, \pi_t^+(X^{t+1})\}$ *for* $t < T$.

(iv) $\pi_t^+(X) = V_t(\alpha)$ *for some superreplication strategy* $\alpha \in \mathcal{S}_t^0$ *for* X.

In particular, the set $\mathcal{T}_t^+$ *is nonempty.*

Proof. The inequality "$\geq$" in (i) follows directly from the definition of superreplication strategy and the second equality in (iii) follows from the first equality in (iii) by applying it to X^{t+1} because $X_t^{t+1} = 0$ and, hence, $\pi_t^+(X^{t+1}) = \pi_{t,t+1}^+(\pi_{t+1}^+(X^{t+1}))$. We show (ii) and (iv) using a backward recursion argument that will also deliver the first equality in (iii). The inequality "$\leq$" in (i) will be a consequence of (ii). Observe first that (ii) and (iv) are clear for $t = T$, so assume they hold for $t+1$ for some $t \in \{0, \dots, T-1\}$ so that we find an exercise strategy $\sigma \in \mathcal{T}_{t+1}$ and a superreplication strategy $\alpha \in \mathcal{S}_{t+1}^0$ for X satisfying

$$\pi_{t+1}^+(X) = \pi_{t+1,T}^+(X_\sigma) = V_{t+1}(\alpha).$$

By Proposition 3.5 there exists a superreplication strategy $\beta \in \mathcal{S}_t^0$ for $\pi_{t+1}^+(X)$ such that

$$V_t(\beta) = \pi_{t,t+1}^+(\pi_{t+1}^+(X)).$$

Let $E = \{X_t \geq \pi_{t,t+1}^+(\pi_{t+1}^+(X))\}$ and consider the combined trading strategy $\gamma \in \mathcal{S}_t^0$ given by

$$\gamma_u := \begin{cases} 1_E X_t \eta_t + 1_{E^c} \beta_t & \text{if } u = t, \\ [1_E X_t + V_{t+1}(\beta) - V_{t+1}(\alpha)]\eta_u + \alpha_u & \text{if } u > t. \end{cases}$$

As $V_u(\gamma) \geq X_u$ for every $u \in \{t, \dots, T\}$, we see that γ is a superreplication strategy for X. In addition, define the exercise strategy $\rho := t 1_E + \sigma 1_{E^c} \in \mathcal{T}_t$. We obtain

$$\begin{aligned} \pi_t^+(X) \leq V_t(\gamma) &= \max\{X_t, \pi_{t,t+1}^+(\pi_{t+1}^+(X))\} = \max\{X_t, \pi_{t,t+1}^+(\pi_{t+1,T}^+(X_\sigma))\} \\ &= \max\{X_t, \pi_{t,T}^+(X_\sigma)\} = \pi_{t,T}^+(X_\rho) \leq \pi_t^+(X), \end{aligned}$$

where we used time consistency in the third equality and the inequality "$\geq$" in (i) to get the last inequality. As a result, we must have equality throughout, concluding the induction argument. $\square$

Thinking again of an American option as a basket of European payoffs from which the buyer can choose his or her favorite item, the buyer should never buy the option for a price that is not a market-consistent buyer price for at least one item in the basket. We formalize this in the following definition.

Definition 4.11. Let $t \in \{0, \dots, T\}$ and $E \in \mathcal{F}_t$. We say that $P \in \mathcal{X}_t$ is a *market-consistent buyer price* for X at date t on E if it is a market-consistent buyer price for X_τ at date t on E for some $\tau \in \mathcal{T}_t$. We omit the reference to E when $\mathbb{P}(E) = 1$. The set of market-consistent buyer prices is denoted by $MCP_t^b(X)$.

Similarly to the seller's case, but requiring a different proof strategy, we can establish the existence of exercise strategies whose associated payoff streams have precisely the same market-consistent buyer prices as the American option. These exercise strategies turn out to be "buyer pricing strategies" introduced in Definition 4.2, which validates the terminology. From the buyer's perspective, such payoff streams are the "most expensive" in the basket. A key characteristic of these exercise strategies is that they have the largest possible domain of replicability, namely the event on which the sub- and superreplication prices for the option coincide.

Proposition 4.12. *For every $t \in \{0, \dots, T\}$ there exists $\sigma \in \mathcal{T}_t$ such that*

$$MCP_t^b(X) = \bigcup_{\tau \in \mathcal{T}_t} MCP_{t,T}^b(X_\tau) = \bigcup_{\tau \in \mathcal{T}_t^+} MCP_{t,T}^b(X_\tau) = MCP_{t,T}^b(X_\sigma).$$

In particular, $\mathcal{T}_t^b$ is nonempty and for every $\sigma \in \mathcal{T}_t$ the following statements are equivalent:

(a) $\sigma \in \mathcal{T}_t^+$ *and* $\{\pi_{t,T}^-(X_\sigma) = \pi_{t,T}^+(X_\sigma)\} = \bigcup_{\tau \in \mathcal{T}_t^+} \{\pi_{t,T}^-(X_\tau) = \pi_{t,T}^+(X_\tau)\}$.

(b) $\sigma \in \mathcal{T}_t^+$ *and* $\{\pi_{t,T}^-(X_\sigma) = \pi_{t,T}^+(X_\sigma)\} = \{\pi_t^-(X) = \pi_t^+(X)\}$.

(c) $MCP_{t,T}^b(X_\sigma) = MCP_t^b(X)$.

(d) $\sigma \in \mathcal{T}_t^b$.

Proof. To establish the proposition it is enough to prove the equivalence between the four assertions and show the existence of an exercise strategy satisfying (b). Clearly, (b) implies (a) and the equivalence between (c) and (d) follows from the definition of a market-consistent buyer price. To show that (d) implies (b), assume that $\sigma \in \mathcal{T}_t^b$. As $\mathcal{T}_t^b \subset \mathcal{T}_t^+$ and the inclusion "$\subset$" is therefore clear, it suffices to show that $\{\pi_t^-(X) = \pi_t^+(X)\} \subset \{\pi_{t,T}^-(X_\sigma) = \pi_{t,T}^+(X_\sigma)\}$. To this end, set $E := \{\pi_t^-(X) = \pi_t^+(X)\}$ and take any $\tau \in \mathcal{T}_t^-$. Note that $\pi_{t,T}^-(X_\tau) = \pi_t^-(X) = \pi_t^+(X) \geq \pi_{t,T}^+(X_\tau)$ on E, implying that X_τ is replicable on E at date t. As $MCP_{t,T}^b(X_\tau) \subset MCP_{t,T}^b(X_\sigma)$ and $\pi_{t,T}^+(X_\sigma) = \pi_{t,T}^+(X_\tau)$ on E by assumption on σ, we deduce that X_σ is also replicable on E at date t. Finally, let $\sigma \in \mathcal{T}_t$ satisfy (a). Take an arbitrary $\tau \in \mathcal{T}_t$ and note that $\pi_{t,T}^+(X_\sigma) \geq \pi_{t,T}^+(X_\tau)$. Set $E := \{\pi_{t,T}^-(X_\tau) = \pi_{t,T}^+(X_\tau) = \pi_{t,T}^+(X_\sigma)\}$ and define $\rho := 1_E\tau + 1_{E^c}\sigma \in \mathcal{T}_t^+$. On the event E we have $\pi_{t,T}^-(X_\rho) = \pi_{t,T}^+(X_\rho)$ and thus $\pi_{t,T}^-(X_\sigma) = \pi_{t,T}^+(X_\sigma)$ by assumption on σ. But then $MCP_{t,T}^b(X_\tau) \subset MCP_{t,T}^b(X_\sigma)$, delivering (d) and concluding the proof of equivalence.

It remains to prove that there exists an exercise strategy satisfying (b). To this effect, take any $\sigma_1 \in \mathcal{T}_t^-$ and $\sigma_2 \in \mathcal{T}_t^+$ and define $E := \{\pi_t^-(X) = \pi_t^+(X)\}$ and $\sigma := 1_E\sigma_1 + 1_{E^c}\sigma_2$. On the event E we have $\pi_{t,T}^+(X_{\sigma_1}) \leq \pi_t^+(X) = \pi_t^-(X) = \pi_{t,T}^-(X_{\sigma_1})$, showing that $\pi_{t,T}^-(X_\sigma) = \pi_t^+(X) = \pi_{t,T}^+(X_\sigma)$. This implies that $\sigma \in \mathcal{T}_t^+$ and $E = \{\pi_{t,T}^-(X_\sigma) = \pi_{t,T}^+(X_\sigma)\}$. Hence, σ satisfies (b). $\square$

The previous proposition established that market-consistent buyer prices for American options equal those of the payoff streams obtained when exercising according to a buyer pricing strategies. Together with our characterization for market-consistent buyer prices for European payoffs, this directly yields our desired result for market-consistent buyer prices for American options.

Theorem 4.13. *For every $t \in \{0, \dots, T\}$ the set $MCP_t^b(X)$ is nonempty and $P \in \mathcal{X}_t$ is a market-consistent buyer price for X at date t if and only if the following conditions hold:*

(1) $P < \pi_t^+(X)$ on $\{\pi_t^-(X) < \pi_t^+(X)\}$;

(2) $P \leq \pi_t^+(X)$ on $\{\pi_t^-(X) = \pi_t^+(X)\}$.

In particular, the following statements are equivalent:

(a) $\pi_t^+(X)$ is a market-consistent buyer price for X at date t.

(b) $\pi_t^-(X) = \pi_t^+(X)$.

(c) X_τ is replicable at date t for every $\tau \in \mathcal{T}_t^b$.

(d) X_τ is replicable at date t for some $\tau \in \mathcal{T}_t^b$.

(e) X_τ is replicable at date t for some $\tau \in \mathcal{T}_t^+$.

Remark 4.14 (The buyer does not need to pay for optionality). Similar to the seller's perspective, the optionality inherent in an American option does not justify that the buyer pays a "market premium" . According to Proposition 4.12, the market-consistent buyer prices for the entire American option are identical to the market-consistent buyer prices for a single European payoff — specifically, the payoff that results from exercising the option according to a buyer pricing strategy. Therefore, despite the flexibility provided by optionality, the market-consistent buyer prices for an American option simply correspond to the market-consistent buyer prices of a single payoff stream in the basket that the buyer considers "most expensive".

4.3 Market-consistent prices

It is clear how to define market-consistent prices for an American option that admits a superreplication strategy at the initial date.

Definition 4.15. Let $t \in \{0, \dots, T\}$ and $E \in \mathcal{F}_t$. We say that $P \in \mathcal{X}_t$ is a *market-consistent price* for X at date t on E if it is both a market-consistent seller and a market-consistent buyer price for X at date t on E. If $\mathbb{P}(E) = 1$ we omit the reference to E. The set of market-consistent prices is denoted by $MCP_t(X)$.

From Theorem 4.7 and Theorem 4.13 we immediately obtain the following characterization of market-consistent prices. Note that $\pi_t^-(X) \leq \widetilde{\pi}_t(X) \leq \pi_t^+(X)$ for every $t \in \{0, \dots, T\}$. To the best of our knowledge, a complete characterization of the price interval for an American option and, notably, of its price bounds was still missing in the literature. In addition, the existing results in this direction make heavy use of duality theory in the form of equivalent martingale measures; see, e.g., [6, Chapter 7]. As in the European case, our approach is direct and requires no duality theory.

Theorem 4.16. *For every $t \in \{0, \dots, T\}$ the set $MCP_t(X)$ is nonempty and $P \in \mathcal{X}_t$ is a market-consistent price for X at date t if and only if the following three conditions hold:*

(1) $\pi_t^-(X) < P < \pi_t^+(X)$ *on* $\{\pi_t^-(X) < \widetilde{\pi}_t(X)\}$;

(2) $\pi_t^-(X) \le P < \pi_t^+(X)$ *on* $\{\pi_t^-(X) = \widetilde{\pi}_t(X)\} \cap \{\pi_t^-(X) < \pi_t^+(X)\}$;

(3) $P = \pi_t^+(X) = \pi_t^-(X)$ *on* $\{\pi_t^-(X) = \pi_t^+(X)\}$.

In particular, the following statements hold:

(i) $\pi_t^+(X) = \operatorname{ess\,sup} MCP_t(X) = \operatorname{ess\,sup} MCP_t^b(X)$.

(ii) $\pi_t^-(X) = \operatorname{ess\,inf} MCP_t(X) = \operatorname{ess\,inf} MCP_t^s(X)$.

4.4 Replicable American options

Replicability of an American option is defined similarly to replicablity for European payoffs.

Definition 4.17. Let $t \in \{0, \dots, T\}$ and $E \in \mathcal{F}_t$. We say that X is *replicable* on E at date t if there exists a self-financing trading strategy $\alpha \in \mathcal{S}_t^0$ that is both a sub- and a superreplication strategy for X on E. In this case, we say that α is a *replication strategy* for X on E. If $\mathbb{P}(E) = 1$ we omit the reference to E. The largest event on which X is replicable at date t is called the *maximum domain of replicability* of X at date t.

The maximum domain of replicability of an American option can be easily determined and, as in European case, consists of the set where the sub- and superreplication prices coincide. Additional equivalent conditions for replicability in terms of the replicability of European payoffs in the basket follow from Theorem 4.13.

Proposition 4.18. *For every* $t \in \{0, \dots, T\}$ *the set* $\{\pi_t^-(X) = \pi_t^+(X)\}$ *is the maximum domain of replicability for* X*. In addition, the following statements are equivalent:*

(a) X *is replicable at date* t.

(b) $\pi_t^-(X) = \pi_t^+(X)$.

(c) $\pi_t^+(X) \in MCP_t(X)$.

Proof. Take a subreplication strategy $\alpha \in \mathcal{S}_t^0$ for X with $V_t(\alpha) = \pi_t^-(X)$ and a superreplication strategy $\beta \in \mathcal{S}_t^0$ for X with $V_t(\beta) = \pi_t^+(X)$, which exist by Propositions 4.4 and 4.10. Clearly, $V_T(\alpha) \le X_\tau \le V_T(\beta)$ for some $\tau \in \mathcal{T}_t$. Then, the absence of arbitrage opportunities implies $V_T(\alpha) = X_\tau = V_T(\beta)$ on $\{\pi_t^-(X) = \pi_t^+(X)\}$, showing that X is replicable on $\{\pi_{t,u}^-(X) = \pi_{t,u}^+(X)\}$ at date t. Now, assume that X is replicable on $E \in \mathcal{F}_t$ at date t and note that this implies that $\pi_t^+(X) = \pi_t^-(X)$ on E. This characterizes the maximum domain of replicability.
The equivalence between (a) and (b) follows from the characterization of the maximum domain of replicability and that between (b) and (c) follows from Theorem 4.13. $\square$

As we have just shown, the superreplication price of the option is a market-consistent price for the option if and only if the option is replicable. The next example, the simplest that comes to mind, illustrates that this is not true for the subreplication price, which can be a market-consistent price even when the option is not replicable; see also [6, Example 6.32].

Example 4.19. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and assume each scenario has a strictly positive probability. Consider a two-period market with $\mathcal{F}_1 = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$ and $\mathcal{F}_2 = 2^\Omega$. The market consists of a single basic security defined by $S_0 = S_1 = S_2 = 1$. Finally, consider an American option X given by

$$X_0 = 0, \quad X_1 = 1, \quad X_2(\omega_1) = X_2(\omega_3) = 2, \quad X_2(\omega_2) = X_2(\omega_4) = 0.$$

By the recursion formula we have

$$\pi_0^+(X) = \pi_{0,1}^+(\max\{X_1, \underbrace{\pi_{1,2}^+(X_2)}_{=2}\}) = \pi_{0,1}^+(2) = 2,$$
$$\pi_0^-(X) = \pi_{0,1}^-(\max\{X_1, \underbrace{\pi_{1,2}^-(X_2)}_{=0}\}) = \pi_{0,1}^+(1) = 1.$$

Clearly, $\mathcal{T}_0^- = \{1\}$, which implies that $\widetilde{\pi}_0(X) = \pi_{0,2}^+(X_1) = 1$. It then follows from Theorem 4.7 that $\pi_0^-(X)$ is a market-consistent price for X at date 0 and from Proposition 4.18 that X is not replicable at date 0.

Remark 4.20 (Never buy a replicable American option). If an American option is replicable, it possesses a unique market-consistent price at which it can be rationally transacted. While this price is fully in line with market prices, we claim that it is not reasonable to buy the option at this price. Indeed, according to Proposition 4.10, this price exactly matches the acquisition value of a replication strategy for the option. Consequently, a potential buyer would clearly benefit more by directly acquiring this replication strategy from the market at the same price. The strategy offers superior flexibility because it is, in particular, a superreplication strategy — liquidating it according to any given exercise strategy yields results at least as favorable as exercising the option under the identical strategy. This shows that replicable American options provide no additional advantages beyond those attainable through direct implementation of self-financing trading strategies in the market. In particular, this shows that American options do not enrich a complete market, because in such a market every American option is replicable.

5 Market-consistent exercise strategies

In this section, we address when the holder of an American option might rationally consider exercising, based on the market-consistent pricing theory developed above. To this effect, we introduce market-consistent exercise strategies, which, to our knowledge, have not been previously studied in the literature. The analysis is based on two key insights:

1. *Exercise as a sale*: Exercising the option is equivalent to selling the residual option (the right to exercise later) in exchange for the immediate exercise value. Therefore, the holder should *never* exercise when the exercise value is not a market-consistent seller price for the residual option.

2. *Deferred exercise as a purchase*: Deferring exercise is equivalent to buying the residual option at a price equal to the current exercise value. Therefore, the holder should *always* exercise when the exercise value is not a market-consistent buyer price for the residual option. It follows that, if exercise is deferred, then the exercise value must be a market-consistent

buyer price for the residual option. Moreover, when analyzing a specific exercise strategy, we must ensure that the exercise value is a market-consistent buyer price specifically for the payoff stream associated with that particular strategy — not just any potential strategy. This ensures that deferral is justified based on a rational future exercise plan rather than an arbitrary one.

These complementary conditions lead us to define a "market-consistent exercise strategy" as a strategy according to which it is not foolish for the holder to exercise.

Definition 5.1. Let $t \in \{0, \dots, T-1\}$. We say that $\tau \in \mathcal{T}_t$ is a *market-consistent exercise strategy* for X at date t if it satisfies for every $u \in \{t, \dots, T-1\}$:

(1) X_u is a market-consistent seller price for X^{u+1} at date u on $\{\tau = u\}$;

(2) X_u is a market-consistent buyer price for X_τ at date u on $\{\tau > u\}$.

The set of market-consistent exercise strategies at date t is denoted by $\mathcal{T}_t^{MC}$.

5.1 First exercise time

The first property in the definition of a market-consistent exercise strategy leads to considering the exercise strategy that prescribes exercising as soon as the exercise price is a market-consistent seller price for the residual option, which can be described using Theorem 4.7.

Definition 5.2. Let $t \in \{0, \dots, T\}$. The *first exercise time* from date t is the exercise strategy $\tau_t^- \in \mathcal{T}_t$ defined by $\tau_t^- := T$ if $t = T$ and otherwise

$$\tau_t^- := \min\{u \in \{t, \dots, T-1\}\, ;\ X_u > \pi_u^-(X^{u+1}) \text{ or } X_u = \pi_u^-(X^{u+1}) = \widetilde{\pi}_u(X^{u+1})\} \wedge T.$$

The next result collects the main properties of the first exercise time and shows, among other things, that the first exercise time is the *earliest* market-consistent exercise strategy.

Theorem 5.3. *For every $t \in \{0, \dots, T-1\}$ the following statements hold:*

(i) $\tau_t^- \in \mathcal{T}_t^{MC}$ and $\tau_t^- = \operatorname{ess\,inf} \mathcal{T}_t^{MC}$.

(ii) $\tau_t^- \in \mathcal{T}_t^s$.

Proof. We start by establishing (ii). To this effect, we first prove by backward recursion that $\tau_t^- \in \mathcal{T}_t^-$ for every $t \in \{0, \dots, T\}$. The statement is clearly true if $t = T$. Now, assume it holds for $t+1$ for some $t \in \{0, \dots, T-1\}$ so that $\pi_{t+1}^-(X) = \pi_{t+1,T}^-(X_{\tau_{t+1}^-})$. Using Proposition 4.4, we see that on $\{\tau_t^- = t\}$

$$\pi_{t,T}^-(X_{\tau_t^-}) = X_t = \max\{X_t, \pi_t^-(X^{t+1})\} = \pi_t^-(X),$$

and similarly on the complement event $\{\tau_t^- > t\}$

$$\begin{aligned}\pi_{t,T}^-(X_{\tau_t^-})) &= \pi_{t,t+1}^-(\pi_{t+1,T}^-(X_{\tau_t^-})) = \pi_{t,t+1}^-(\pi_{t+1,T}^-(X_{\tau_{t+1}^-})) \\ &= \pi_{t,t+1}^-(\pi_{t+1}^-(X^{t+1})) = \pi_t^-(X^{t+1}) = \max\{X_t, \pi_t^-(X^{t+1})\} = \pi_t^-(X),\end{aligned}$$

which concludes the recursion argument. By Proposition 4.6, to conclude the proof of (ii) it then suffices to show that, for every $\tau \in \mathcal{T}_t^-$,

$$\pi_{t,T}^-(X_\tau) = \pi_{t,T}^+(X_\tau) \text{ on } \{\pi_{t,T}^-(X_{\tau_t^-}) = \pi_{t,T}^+(X_{\tau_t^-})\}. \tag{5.1}$$

We proceed by backward recursion. The statement is clearly true for $t = T$. Now, assume it holds for $t+1$ for some $t \in \{0, \dots, T-1\}$ and fix $\tau \in \mathcal{T}_t^-$. Set $E := \{\pi_{t,T}^-(X_{\tau_t^-}) = \pi_{t,T}^+(X_{\tau_t^-})\}$ and define

$$A := \{\tau = t\} \cap E, \quad B := \{\tau > t\} \cap \{\tau_t^- = t\} \cap E, \quad C := \{\tau > t\} \cap \{\tau_t^- > t\} \cap E.$$

On A assertion (5.1) clearly holds. On B we have

$$X_t = \pi_{t,T}^-(X_{\tau_t^-}) = \pi_{t,T}^-(X_\tau) \leq \pi_t^-(X^{t+1}).$$

By definition of τ_t^-, we must thus have $\pi_t^-(X^{t+1}) = \widetilde{\pi}_t(X^{t+1}) = X_t$, which implies that the inequality above is, in fact, an equality and yields

$$\pi_{t,T}^-(X_\tau) = \pi_t^-(X^{t+1}) = \widetilde{\pi}_t(X^{t+1}) \geq \pi_{t,T}^+(X_\tau).$$

As a result, (5.1) holds on B. Finally, on C, note first that $\tau_t^- = \tau_{t+1}^-$ and $\pi_{t+1,T}^-(X_\tau) \leq \pi_{t+1,T}^-(X_{\tau_t^-})$. As the right-hand side is replicable at date t,

$$\begin{aligned}\pi_t^-(X) &= \pi_{t,t+1}^-(\pi_{t+1,T}^-(X_\tau)) \leq \pi_{t,t+1}^+(\pi_{t+1,T}^-(X_\tau)) \\ &\leq \pi_{t,t+1}^+(\pi_{t+1,T}^-(X_{\tau_t^-})) = \pi_{t,t+1}^-(\pi_{t+1,T}^-(X_{\tau_t^-})) = \pi_t^-(X).\end{aligned}$$

It follows that $\pi_{t+1,T}^-(X_\tau)$ is also replicable at date t and has the same subreplication price of $\pi_{t+1,T}^-(X_{\tau_t^-})$ at date t. To avoid arbitrage opportunities, $\pi_{t+1,T}^-(X_\tau) = \pi_{t+1,T}^-(X_{\tau_t^-}) = \pi_{t+1}^-(X)$ must hold. Using the induction hypothesis we find

$$\begin{aligned}\pi_{t,T}^+(X_\tau) &= \pi_{t,t+1}^+(\pi_{t+1,T}^+(X_\tau)) \leq \pi_{t,t+1}^+(\pi_{t+1,T}^-(X_{\tau_{t+1}^-})) \\ &= \pi_{t,t+1}^+(\pi_{t+1,T}^+(X_{\tau_t^-})) = \pi_{t,T}^+(X_{\tau_t^-}) = \pi_{t,T}^-(X_{\tau_t^-}) = \pi_{t,T}^-(X_\tau),\end{aligned}$$

which delivers $\pi_{t,T}^-(X_\tau) = \pi_{t,T}^+(X_\tau)$. This proves (5.1) on C and ends the recursion argument.

As a final step, we establish (i). It follows directly from Theorem 4.7 that $\tau_t^- \leq \tau$ for every $\tau \in \mathcal{T}_t^{MC}$. Fix $u \in \{0, \dots, T-1\}$. The same result implies that X_u is a market-consistent seller price for X^{u+1} at date u on $\{\tau_t^- = u\}$. On $\{\tau_t^- > u\}$ we have two cases. First, if $X_u < \pi_u^-(X^{u+1})$, then we deduce from Proposition 4.4 and from (ii) that

$$X_u < \pi_u^-(X) = \pi_u^-(X_{\tau_u^-}) = \pi_u^-(X_{\tau_t^-}).$$

Second, if $X_u = \pi_u^-(X^{u+1}) < \widetilde{\pi}_u(X^{u+1})$, then $\pi_u^-(X) = X_u < \widetilde{\pi}_u(X)$ again by Proposition 4.4 and we deduce from (ii) that

$$X_u = \pi_u^-(X) = \pi_u^-(X_{\tau_u^-}) < \pi_u^+(X_{\tau_u^-}) = \pi_u^+(X_{\tau_t^-}).$$

In either case we see that X_u is a market-consistent buyer price for $X_{\tau_t^-}$ at date u on $\{\tau_t^- > u\}$ by virtue of Theorem 4.7. This concludes the proof. $\square$

Remark 5.4. A key exercise strategy in the treatment of [6] is defined for $t \in \{0, \ldots, T\}$ by

$$\tau_t^{--} := \min\{u \in \{t, \ldots, T\}\,;\ X_u = \pi_u^-(X)\};$$

see [6, Theorem 6.47]. This exercise strategy is used in the context of lower Snell envelopes in connection with the problem of analyzing the lower bound of the set of arbitrage-free prices of an American option. It is immediate to verify that τ_t^{--} always prescribes exercising not later than the first exercise time, i.e., $\tau_t^{--} \le \tau_t^-$, but it turns out that the two exercise strategies do not generally coincide. As a result, τ_t^{--} fails to be a market-consistent exercise strategy in general, which explains why it cannot help yield a characterization of when the subreplication price is market consistent or, equivalently, arbitrage free. To see this, consider the setting of Example 4.19 and the American option X defined by

$$X_0 = 0, \quad X_1 = 1, \quad X_2(\omega_1) = X_2(\omega_3) = 2, \quad X_2(\omega_2) = X_2(\omega_4) = 1.$$

As $\pi_1^-(X) = \max\{X_1, \pi_{1,2}^-(X_2)\} = 1 = X_1$, we infer that $\tau_0^{--} = 1$. However, $\tau_0^- = 2$ because

$$\widetilde{\pi}_1(X^2) = \pi_{1,2}^+(X_\tau) = 2 > 1 = X_1 = \pi_{1,2}^-(X_2) = \pi_1^-(X^2).$$

5.2 Last exercise time

The second property in the definition of a market-consistent exercise strategy suggests considering the strategy that prescribes exercising as soon as the exercise price is no longer a market-consistent buyer price for the residual option. This exercise strategy can be described formally using Theorem 4.13.

Definition 5.5. Let $t \in \{0, \ldots, T\}$. The *last exercise time* from date t is the exercise strategy $\tau_t^+ \in \mathcal{T}_t$ defined by $\tau_t^+ := T$ if $t = T$ and otherwise

$$\tau_t^+ := \min\{u \in \{t, \ldots, T-1\}\,;\ X_u > \pi_u^+(X^{u+1}) \text{ or } X_u = \pi_u^+(X^{u+1}) > \pi_u^-(X^{u+1})\} \wedge T.$$

As we show next, the last exercise time is the latest market-consistent exercise strategy.

Theorem 5.6. *For every $t \in \{0, \ldots, T-1\}$ the following statements hold:*

(i) $\tau_t^+ \in \mathcal{T}_t^{MC}$ *and* $\tau_t^+ = \operatorname{ess\,sup} \mathcal{T}_t^{MC}$.

(ii) $\tau_t^+ \in \mathcal{T}_t^b$.

Proof. We start by (ii). As a first step, we establish that $\tau_t^+ \in \mathcal{T}_t^+$. The statement is clear for $t = T$. We proceed by backward induction assuming the statement is true for $t+1$ for some $t \in \{0, \ldots, T-1\}$. Set $E := \{\tau_t^+ = t\}$. Note that on E

$$\pi_t^+(X) = \max\{X_t, \pi_t^+(X^{t+1})\} = X_t = \pi_{t,T}^+(X_{\tau_t^+}).$$

Moreover, using the induction hypothesis, we have on E^c

$$\begin{aligned}\pi_t^+(X) &= \max\{X_t, \pi_t^+(X^{t+1})\} = \pi_t^+(X^{t+1}) = \pi_{t,t+1}^+(\pi_{t+1}^+(X^{t+1})) \\ &= \pi_{t,t+1}^+(\pi_{t+1,T}^+(X_{\tau_{t+1}^+})) = \pi_{t,t+1}^+(\pi_{t+1,T}^+(X_{\tau_t^+})) = \pi_{t,T}^+(X_{\tau_t^+}).\end{aligned}$$

This concludes the recursion argument. By Proposition 4.12, to establish (ii) it suffices to show

$$\{\pi^-_{t,T}(X_{\tau^+_t}) = \pi^+_{t,T}(X_{\tau^+_t})\} = \{\pi^-_t(X) = \pi^+_t(X)\}.$$

The assertion clearly holds for $t = T$. We proceed by backward induction and assume it holds for $t+1$ for some $t \in \{0, \dots, T-1\}$. For convenience, set $E := \{\tau^+_t = t\}$ and define $F := \{\pi^-_{t,T}(X_{\tau^+_t}) = \pi^+_{t,T}(X_{\tau^+_t})\}$ and $G := \{\pi^-_t(X) = \pi^+_t(X)\}$. Note that we always have $E \subset F \subset G$. Hence, it is enough to show that $G \cap E^c \subset F$. On $G \cap E^c$

$$\pi^+_t(X^{t+1}) = \max\{X_t, \pi^+_t(X^{t+1})\} = \pi^+_t(X) = \pi^-_t(X) = \max\{X_t, \pi^-_t(X^{t+1})\} = \pi^-_t(X^{t+1}).$$

Hence, $\pi^-_{t,t+1}(\pi^-_{t+1}(X)) = \pi^+_{t,t+1}(\pi^+_{t+1}(X))$. Since $\pi^-_{t+1}(X) \leq \pi^+_{t+1}(X)$, the absence of arbitrage opportunities implies that $\pi^-_{t+1}(X) = \pi^+_{t+1}(X)$. By the induction hypothesis we find

$$\begin{aligned}\pi^-_{t,T}(X_{\tau^+_t}) &= \pi^-_{t,t+1}(\pi^-_{t+1}(X_{\tau^+_{t+1}})) = \pi^-_{t,t+1}(\pi^+_{t+1}(X_{\tau^+_{t+1}})) = \pi^-_{t,t+1}(\pi^+_{t+1}(X)) \\ &= \pi^-_{t,t+1}(\pi^-_{t+1}(X)) = \pi^-_t(X) = \pi^+_t(X) = \pi^+_{t,T}(X_{\tau^+_t}).\end{aligned}$$

This yields $G \cap E^c \subset F$ and concludes the recursion argument.

As a final step, we establish (i). It follows directly from Theorem 4.13 that $\tau^+_t \geq \tau$ for every $\tau \in \mathcal{T}^{MC}_t$. Fix $u \in \{0, \dots, T-1\}$. The same result implies that X_u is a market-consistent seller price for X^{u+1} at date u on $\{\tau^+_t = u\}$. On $\{\tau^+_t > u\}$ we have two cases. First, if $X_u < \pi^+_u(X^{u+1})$, then we deduce from Proposition 4.10 and from (ii) that

$$X_u < \pi^+_u(X) = \pi^+_u(X_{\tau^+_u}) = \pi^+_u(X_{\tau^+_t}).$$

Second, if $X_u = \pi^+_u(X^{u+1}) = \pi^-_u(X^{u+1})$, then $X_u = \pi^+_u(X) = \pi^-_u(X)$ again by Proposition 4.10 and we deduce from (ii) that

$$\pi^+_u(X_{\tau^+_t}) = \pi^+_u(X_{\tau^+_u}) = X_u = \pi^-_u(X_{\tau^+_u}) = \pi^-_u(X_{\tau^+_t}).$$

In either case we see that X_u is a market-consistent buyer price for $X_{\tau^+_t}$ at date u on $\{\tau^-_t > u\}$ due to Theorem 4.13. This concludes the proof. $\square$

5.3 A more optimal last time to exercise

Even though it is not irrational to exercise according to the last exercise time τ^+_t, there is a different exercise strategy that could lay a claim on being a more rational last time to exercise. Indeed, by Remark 4.20, holding a replicable American option is suboptimal — it is more advantageous to sell it at its unique market-consistent price and instead purchase a superreplication strategy which provides greater flexibility in terms of exercise timing. Therefore, it is arguably more rational to exercise as soon as the option becomes replicable, which may very well occur earlier than the technically correct last exercise time τ^+_t.

Definition 5.7. Let $t \in \{0, \dots, T\}$. The *earliest replication time* from date t is the exercise strategy $\tau^{rep}_t \in \mathcal{T}_t$ defined by

$$\tau^{rep}_t := \min\{u \in \{t, \dots, T\}\,;\; X_u = \pi^+_u(X)\}.$$

The earliest replication time is a market-consistent exercise strategy as well as a buyer pricing strategy. The proof follows the same lines of that of Theorem 5.6 and is therefore omitted.

Proposition 5.8. *For every $t \in \{0, \dots, T-1\}$ we have $\tau_t^{rep} \in \mathcal{T}_t^{MC} \cap \mathcal{T}_t^b$.*

As illustrated by the following example, the earliest replication time does not coincide with the last exercise time in general. This happens when the option remains replicable through time, in which case waiting longer than the earliest replication time does not violate market consistency.

Example 5.9. In the simple market model of Example 4.19 consider the American option X given by $X_0 = 0$ and $X_1 = X_2 = 1$. It is immediate to see that $\pi_1^+(X) = \pi_1^+(X^2) = \pi_1^-(X^2) = 1$ so that $\tau_0^{rep} = 1$ but $\tau_0^+ = 2$.

6 Representation theorems

In this final section, we show that market-consistent prices can be described in terms of expectations with respect to equivalent martingale measures. While we employ the terminology of martingales, we do not rely on advanced martingale theory.

For each $t \in \{0, \dots, T-1\}$ we denote by $\mathcal{M}_t$ the set of probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F}_{t+1})$ satisfying the following properties:

(EMM1) S_t^i and S_{t+1}^i are integrable with respect to $\mathbb{Q}$ for every $i \in \{1, \dots, N\}$;

(EMM2) $\mathbb{E}_{\mathbb{Q}}[S_{t+1}^i|\mathcal{F}_t] = S_t^i$ for every $i \in \{1, \dots, N\}$;

(EMM3) $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_{t+1}$.

A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ is called an *equivalent martingale measure* if it belongs to $\mathcal{M}_t$ for every $t \in \{0, \dots, T-1\}$. The set of equivalent martingale measures is denoted by $\mathcal{M}$. It is well known that, under an equivalent martingale measure, the martingale property (EMM2) holds not only for the price process of the basic securities but more generally for the value process of any self-financing strategy with non-negative terminal value. Moreover, the Fundamental Theorem of Asset Pricing establishes the equivalence of the absence of arbitrage opportunities and the existence of equivalent martingale measures.

The next result collects some simple and useful facts about equivalent martingale measures.

Lemma 6.1. *Let $t \in \{0, \dots, T-1\}$. The following statements hold:*

(i) For all $\mathbb{Q}_1 \in \mathcal{M}_t$ and $\mathbb{Q}_2 \in \mathcal{M}$ there exists $\mathbb{Q} \in \mathcal{M}$ such that

$$\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}_1}[\mathbb{E}_{\mathbb{Q}_2}[X|\mathcal{F}_{t+1}]|\mathcal{F}_t]$$

for every non-negative $X \in \mathcal{X}_T$ admitting a superreplication strategy at date t.

(ii) For all $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{M}$ and $\lambda \in \mathcal{X}_t$ satisfying $0 \leq \lambda \leq 1$ there exists $\mathbb{Q} \in \mathcal{M}$ such that

$$\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] = \lambda \mathbb{E}_{\mathbb{Q}_1}[X|\mathcal{F}_t] + (1-\lambda)\mathbb{E}_{\mathbb{Q}_2}[X|\mathcal{F}_t]$$

for every non-negative $X \in \mathcal{X}_T$ admitting a superreplication strategy at date t.

(iii) For all sequences $(\mathbb{Q}_n) \subset \mathcal{M}$ and $(A_n) \subset \mathcal{F}_t$ such that the events A_n are pairwise disjoint and their union is Ω there exists $\mathbb{Q} \in \mathcal{M}$ such that

$$\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] = \sum_{n \in \mathbb{N}} 1_{A_n} \mathbb{E}_{\mathbb{Q}_n}[X|\mathcal{F}_t]$$

for every non-negative $X \in \mathcal{X}_T$ admitting a superreplication strategy at date t.

Proof. A simple verification shows that the desired equivalent martingale measures can be respectively defined through the densities

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}_1}{d\mathbb{P}}\middle|\mathcal{F}_t\right]^{-1}\frac{d\mathbb{Q}_1}{d\mathbb{P}}\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}_2}{d\mathbb{P}}\middle|\mathcal{F}_t\right]\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}_2}{d\mathbb{P}}\middle|\mathcal{F}_{t+1}\right]^{-1}\frac{d\mathbb{Q}_2}{d\mathbb{P}},$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \lambda\frac{d\mathbb{Q}_1}{d\mathbb{P}} + (1-\lambda)\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}_1}{d\mathbb{P}}\middle|\mathcal{F}_t\right]\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}_2}{d\mathbb{P}}\middle|\mathcal{F}_t\right]^{-1}\frac{d\mathbb{Q}_2}{d\mathbb{P}},$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \sum_{n\in\mathbb{N}}\mathbb{1}_{A_n}\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}_1}{d\mathbb{P}}\middle|\mathcal{F}_t\right]\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}_n}{d\mathbb{P}}|\mathcal{F}_t\right]^{-1}\frac{d\mathbb{Q}_n}{d\mathbb{P}}.$$

□

6.1 European payoffs

As in Section 3, we fix a nonnegative European payoff X with maturity $u \in \{1,\dots,T\}$ that admits a superreplication strategy at date 0. The following result shows that the set of market-consistent prices for X coincides with the set of conditional expectations of X taken with respect to all possible equivalent martingale measures.

Theorem 6.2. *For every $t \in \{0,\dots,u-1\}$ the following statements hold:*

(i) $MCP_{t,u}(X) = \{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]\,;\ \mathbb{Q}\in\mathcal{M}\}$.

(ii) $\pi^-_{t,u}(X) = \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$.

(iii) $\pi^+_{t,u}(X) = \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$.

Proof. Note that (ii) and (iii) follow immediately from (i). To prove (i), note that for every $\mathbb{Q}\in\mathcal{M}$ and for all $\alpha\in\mathcal{S}^0_t$ and $A\in\mathcal{F}_t$ such that $V_u(\alpha)\gneq X$ on A,

$$V_t(\alpha) = \mathbb{E}_{\mathbb{Q}}[V_u(\alpha)|\mathcal{F}_t] \gneq \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \quad \text{on } A.$$

This shows that $\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$ is a market-consistent buyer price for X at date t. In the same way one shows that $\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$ is a market-consistent seller price for X at date t, establishing the inclusion "$\supset$" in (i).

To prove the converse inclusion, we proceed by backward recursion. we first show that, for any $s\in\{0,\dots,T-1\}$ and any $P\in MCP_{s,s+1}(Z)$ such that $Z\in\mathcal{X}_{s+1}$ satisfies $Z\geq 0$ and admits a superreplication strategy at date s, there exists $\mathbb{Q}\in\mathcal{M}_s$ such that

$$P = \mathbb{E}_{\mathbb{Q}}[Z|\mathcal{F}_s]. \tag{6.1}$$

Note that $P\geq 0$ and take any process $S^{N+1}\in\mathcal{X}_{0:T}$ such that $S^{N+1}_t\geq 0$ for each $t\in\{0,\dots,T\}$ as well as $S^{N+1}_s = P$ and $S^{N+1}_{s+1} = Z$. We claim that the market for the $N+1$ basic securities $S^1,\dots,S^{N+1}$ is arbitrage free between s and $s+1$. To show this, take a self-financing strategy α in the larger market and assume that

$$\overline{V}_s = \sum_{i=1}^{N+1}\alpha^i_s S^i_s \leq 0, \quad \overline{V}_{s+1} = \sum_{i=1}^{N+1}\alpha^i_s S^i_{s+1} \gneq 0. \tag{6.2}$$

Define $A = \{\alpha_s^{N+1} \neq 0\}$ and note that $\mathbb{P}(A) > 0$ and $\overline{V}_{s+1} = 0$ on A^c for otherwise there would be arbitrage opportunities between s and $s+1$ in the market for the basic securities $S^1, \dots, S^N$. As a result, $\overline{V}_{s+1} \neq 0$ on A. Define $\beta \in \mathcal{S}_s^0$ by

$$\beta_r = \left(-\frac{\alpha_s^1}{\alpha_s^{N+1}} 1_A, \dots, -\frac{\alpha_s^N}{\alpha_s^{N+1}} 1_A\right), \quad r \in \{s, \dots, T-1\}.$$

Set $B = \{\alpha_s^{N+1} > 0\}$ and suppose that $\mathbb{P}(B) > 0$. By rearranging the terms in (6.2) we obtain that $V_s(\beta) \geq P$ on B and $V_{s+1}(\beta) \lneq Z$ on B, which is not possible because P is a market-consistent seller price for Z. If $\mathbb{P}(B) = 0$, then $\mathbb{P}(A \setminus B) > 0$ and it follows from (6.2) that $V_s(\beta) \leq P$ on $A \setminus B$ and $V_{s+1}(\beta) \gneq Z$ on $A \setminus B$, which is not possible because P is a market-consistent buyer price for Z. It follows that the market for the $N+1$ basic securities $S^1, \dots, S^{N+1}$ is arbitrage free between s and $s+1$. By [6, Theorem 1.55], we find $\mathbb{Q} \in \mathcal{M}_s$ such that $P = S_s^{N+1} = \mathbb{E}_{\mathbb{Q}}[S_{s+1}^{N+1}|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[Z|\mathcal{F}_s]$, concluding the proof.

We now establish the inclusion "$\subset$" in (i). The inclusion is clearly true for $t = u$. Assume it is true for $t+1$ for some $t \in \{0, \dots, u-1\}$. Note that, by virtue of point (ii) in Lemma 6.1, the set $\{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_{t+1}]\,;\ \mathbb{Q} \in \mathcal{M}\}$ is upward directed, i.e., is closed with respect to pointwise maxima. Then, there exists a sequence $(\mathbb{Q}_n) \subset \mathcal{M}$ such that

$$\mathbb{E}_{\mathbb{Q}_n}[X|\mathcal{F}_{t+1}] \uparrow \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_{t+1}] = \pi_{t+1,u}^+(X) \tag{6.3}$$

in the almost-sure sense. Let $P \in MCP_{t,u}(X)$ and set $E = \{\pi_{t,u}^-(X) < \pi_{t,u}^+(X)\}$. On E we have $P < \pi_{t,u}^+(X) = \pi_{t,t+1}^+(\pi_{t+1,u}^+(X))$. It follows that P is a market-consistent buyer for $\pi_{t+1,u}^+(X)$ on E at date t. As a result of (6.1), we find $\widetilde{\mathbb{Q}} \in \mathcal{M}_t$ such that $P < \mathbb{E}_{\widetilde{\mathbb{Q}}}[\pi_{t+1,u}^+(X)|\mathcal{F}_t]$ on E. By monotone convergence and (6.3) we find $L \in \mathcal{X}_t$ such that

$$P < L \uparrow \mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathbb{E}_{\mathbb{Q}_n}[X|\mathcal{F}_{t+1}]|\mathcal{F}_t] \quad \text{on } E.$$

By point (i) in Lemma 6.1 there exists $(\widetilde{\mathbb{Q}}_n) \subset \mathcal{M}$ such that $\mathbb{E}_{\widetilde{\mathbb{Q}}_n}[X|\mathcal{F}_t] \uparrow L$ on E. Set $A_1 = \{\mathbb{E}_{\widetilde{\mathbb{Q}}_1}[X|\mathcal{F}_t] > P\} \cap E$ and for every $n \in \mathbb{N}$ define $A_{n+1} = (\{\mathbb{E}_{\widetilde{\mathbb{Q}}_{n+1}}[X|\mathcal{F}_t] > P\} \cap E) \setminus A_n$. It follows from point (iii) in Lemma 6.1 that we find $\overline{\mathbb{Q}} \in \mathcal{M}$ such that $\mathbb{E}_{\overline{\mathbb{Q}}}[X|\mathcal{F}_t] = \mathbb{E}_{\widetilde{\mathbb{Q}}_n}[X|\mathcal{F}_t]$ on A_n for every $n \in \mathbb{N}$, whence $P \leq \mathbb{E}_{\overline{\mathbb{Q}}}[X|\mathcal{F}_t]$ on E. By a similar argument, we find $\underline{\mathbb{Q}} \in \mathcal{M}$ such that $P \geq \mathbb{E}_{\underline{\mathbb{Q}}}[X|\mathcal{F}_t]$ on E. We can now again apply point (ii) in Lemma 6.1 to obtain $\mathbb{Q} \in \mathcal{M}$ such that $P = \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$ on E. We conclude the recursion argument by observing that $P = \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$ on E^c for any $\mathbb{Q} \in \mathcal{M}$ due to the inclusion "$\supset$" in (i). □

Remark 6.3. It is worth noting that in the proof of Theorem 6.2 we extended the market by introducing a new security at a market-consistent price, leaving the prices of the original securities unchanged, and then demonstrated that this extended market remains arbitrage-free. While this approach might initially appear circular — given our claim that such extended markets are not economically meaningful — it serves only as a technical device to build on established results about equivalent martingale measures in single-period arbitrage-free markets. The representation is established directly from market consistency, without invoking the disputed multi-period characterization of arbitrage-free prices. Thus, there is no circularity in the argument.

6.2 American options

We now fix an American option X with $X_0 = 0$ and $X_t \geq 0$ for each $t \in \{1, \dots, T\}$. As in Section 4, we assume that X admits a superreplication strategy at date 0.

Market-consistent prices

To streamline the proof of the dual characterization of market-consistent prices and, later, of market-consistent exercise strategies, it is useful to start by recalling a classical notion from the theory of optimal stopping.

Definition 6.4. Let $t \in \{0, \dots, T-1\}$ and $\mathbb{Q} \in \mathcal{M}$. We say that $\sigma \in \mathcal{T}_t$ is $\mathbb{Q}$-*optimal* if

$$\mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_t] = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_t].$$

The set of such exercise strategies is denoted by $\mathcal{T}_t^{\mathbb{Q}}$.

It is well known, e.g., [6, Theorem 6.18, Proposition 6.20, Theorem 6.21], or can be immediately derived by mirroring the proof that τ_t^- and τ_t^+ are lower/upperbound pricing strategies in Theorem 5.3 and Theorem 5.6, that there exist optimal exercise strategies under any equivalent martingale measure.

Proposition 6.5. *Let $t \in \{0, \dots, T-1\}$ and $\mathbb{Q} \in \mathcal{M}$. Then, $\tau_t^{\mathbb{Q},-}, \tau_t^{\mathbb{Q},+} \in \mathcal{T}_t^{\mathbb{Q}}$ where*

$$\tau_t^{\mathbb{Q},-} := \min\left\{u \in \{t, \dots, T\}\,;\ X_u = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_u} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_u]\right\},$$

$$\tau_t^{\mathbb{Q},+} := \min\left\{u \in \{t, \dots, T-1\}\,;\ X_u > \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{u+1}} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_u]\right\} \wedge T.$$

In addition, $\tau_t^{\mathbb{Q},-} = \operatorname{ess\,inf} \mathcal{T}_t^{\mathbb{Q}}$ and $\tau_t^{\mathbb{Q},+} = \operatorname{ess\,sup} \mathcal{T}_t^{\mathbb{Q}}$.

The next result records the desired characterization of market-consistent prices in terms of equivalent martingale measures.

Theorem 6.6. *For every $t \in \{0, \dots, T-1\}$ the following statements hold:*

(i) $MCP_t(X) = \{\operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_t]\,;\ \mathbb{Q} \in \mathcal{M}\}$.

(ii) $\pi_t^+(X) = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{M}} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_t]$.

(iii) $\pi_t^-(X) = \operatorname{ess\,inf}_{\mathbb{Q} \in \mathcal{M}} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_t]$.

Proof. Clearly, (ii) and (iii) follow from (i). To show the inclusion "$\supset$" in (i), take any $\mathbb{Q} \in \mathcal{M}$ and set $P = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_t]$. Take $\alpha \in \mathcal{S}_t^0$ and $A \in \mathcal{F}_t$ such that $V_T(\alpha) \lneq X_\sigma$ on A for some $\sigma \in \mathcal{T}_t$ and observe that

$$V_t(\alpha) = \mathbb{E}_{\mathbb{Q}}[V_T(\alpha)|\mathcal{F}_t] \lneq \mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_t] \leq P \quad \text{on } A,$$

showing that P is a market-consistent seller price for X at date t. Assume now that $V_T(\alpha) \gneq X_{\tau_t^{\mathbb{Q}}}$ on A. In the same vein we obtain

$$V_t(\alpha) = \mathbb{E}_{\mathbb{Q}}[V_T(\alpha)|\mathcal{F}_t] \gneq \mathbb{E}_{\mathbb{Q}}[X_{\tau_t^{\mathbb{Q}}}|\mathcal{F}_t] = P \quad \text{on } A,$$

showing that P is a market-consistent buyer price for X at date t.

We turn to establishing the inclusion "$\subset$" in (i). Take an arbitrary $P \in MCP_t(X)$. Since $P \in MCP^b_{t,T}(X_\sigma)$ for some $\sigma \in \mathcal{T}_t$, it follows from Theorem 6.2 that there is $\overline{\mathbb{Q}} \in \mathcal{M}$ such that

$$P \leq \mathbb{E}_{\overline{\mathbb{Q}}}[X_\sigma|\mathcal{F}_t] \leq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_{\overline{\mathbb{Q}}}[X_\tau|\mathcal{F}_t]. \tag{6.4}$$

Since $P \in MCP^s_{t,T}(X_{\tau_t^-})$, it follows again from Theorem 6.2 that there is $\widetilde{\mathbb{Q}} \in \mathcal{M}$ such that

$$P \geq \mathbb{E}_{\widetilde{\mathbb{Q}}}[X_{\tau_t^-}|\mathcal{F}_t]. \tag{6.5}$$

To show that the right-hand side in (6.5) can be bounded from below by an essential supremum as in (6.4), we show by backward recursion that for each $u \in \{t, \dots, T\}$ and for each $\mathbb{Q} \in \mathcal{M}$ there exists $\mathbb{Q}^* \in \mathcal{M}$ such that

$$\mathbb{E}_{\mathbb{Q}}[X_{\tau_u^-}|\mathcal{F}_u] \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_u} \mathbb{E}_{\mathbb{Q}^*}[X_\tau|\mathcal{F}_u]. \tag{6.6}$$

The statement clearly holds if $u = T$. So assume it holds for $u+1$ for some $u \in \{t, \dots, T-1\}$ and set $A = \{\tau_u^- = u\}$. By definition of τ_u^-, we see that X_u is a market-consistent seller price for X^{u+1}, hence for $X_{\tau_{u+1}^-}$, on A at date u. It then follows from Theorem 6.2 that there is $\mathbb{Q}_1 \in \mathcal{M}$ such that

$$\mathbb{E}_{\mathbb{Q}}[X_{\tau_u^-}|\mathcal{F}_u] = X_u \geq \mathbb{E}_{\mathbb{Q}_1}[X_{\tau_{u+1}^-}|\mathcal{F}_u] = \mathbb{E}_{\mathbb{Q}_1}[\mathbb{E}_{\mathbb{Q}_1}[X_{\tau_{u+1}^-}|\mathcal{F}_{u+1}]|\mathcal{F}_u] \quad \text{on } A.$$

Combining the induction hypothesis and point (i) in Lemma 6.1 yields $\mathbb{Q}_2, \mathbb{Q}_A \in \mathcal{M}$ such that

$$\mathbb{E}_{\mathbb{Q}}[X_{\tau_u^-}|\mathcal{F}_u] \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{u+1}} \mathbb{E}_{\mathbb{Q}_1}[\mathbb{E}_{\mathbb{Q}_2}[X_\tau|\mathcal{F}_{u+1}]|\mathcal{F}_u] = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{u+1}} \mathbb{E}_{\mathbb{Q}_A}[X_\tau|\mathcal{F}_u] \quad \text{on } A.$$

Bearing in mind that $X_{\tau_u^-} = X_u$ on A, we conclude that

$$\mathbb{E}_{\mathbb{Q}}[X_{\tau_u^-}|\mathcal{F}_u] \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_u} \mathbb{E}_{\mathbb{Q}_A}[X_\tau|\mathcal{F}_u] \quad \text{on } A. \tag{6.7}$$

Note that $\tau_u^- = \tau_{u+1}^-$ on A^c. Then,

$$\mathbb{E}_{\mathbb{Q}}[X_{\tau_u^-}|\mathcal{F}_u] = \mathbb{E}_{\mathbb{Q}}[X_{\tau_{u+1}^-}|\mathcal{F}_u] = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[X_{\tau_{u+1}^-}|\mathcal{F}_{u+1}]|\mathcal{F}_u] \quad \text{on } A^c.$$

Combining the induction hypothesis and point (i) in Lemma 6.1 yields $\mathbb{Q}_3, \mathbb{Q}_{A^c} \in \mathcal{M}$ such that

$$\mathbb{E}_{\mathbb{Q}}[X_{\tau_u^-}|\mathcal{F}_u] \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{u+1}} \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}_3}[X_\tau|\mathcal{F}_{u+1}]|\mathcal{F}_u] = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{u+1}} \mathbb{E}_{\mathbb{Q}_{A^c}}[X_\tau|\mathcal{F}_u] \quad \text{on } A^c.$$

In particular, using that $\tau_u^- \in \mathcal{T}_u^-$, it follows from Theorem 6.2 that

$$\mathbb{E}_{\mathbb{Q}}[X_{\tau_u^-}|\mathcal{F}_u] \geq \mathbb{E}_{\mathbb{Q}_{A^c}}[X_{\tau_{u+1}^-}|\mathcal{F}_u] = \mathbb{E}_{\mathbb{Q}_{A^c}}[X_{\tau_u^-}|\mathcal{F}_u] \geq \pi^-_{u,T}(X_{\tau_u^-}) = \pi_u^-(X) \geq X_u \quad \text{on } A^c.$$

As a consequence, we conclude that

$$\mathbb{E}_{\mathbb{Q}}[X_{\tau_u^-}|\mathcal{F}_u] \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_u} \mathbb{E}_{\mathbb{Q}_{A^c}}[X_\tau|\mathcal{F}_u] \quad \text{on } A^c. \tag{6.8}$$

Combining (6.7) and (6.8) with point (ii) in Lemma 6.1 yields (6.6) and concludes the recursion argument. We can now derive from (6.5) and (6.6) that there is $\underline{\mathbb{Q}} \in \mathcal{M}$ such that

$$P \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_{\underline{\mathbb{Q}}}[X_\tau|\mathcal{F}_t]. \tag{6.9}$$

Set $\Lambda_t = \{\lambda \in \mathcal{X}_t\,;\ 0 \le \lambda \le 1\}$ and define a map $F : \Lambda_t \to \mathcal{X}_t$ by

$$F(\lambda) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t}\{(1-\lambda)\mathbb{E}_{\underline{\mathbb{Q}}}[X_\tau|\mathcal{F}_t] + \lambda\mathbb{E}_{\overline{\mathbb{Q}}}[X_\tau|\mathcal{F}_t]\}.$$

Note that $F(0) \le P \le F(1)$ by (6.4) and (6.9). In view of point (ii) in Lemma 6.1, to conclude the proof of (i) we verify that F satisfies the conditions of the Intermediate Value Theorem 6.9. Conditionality is immediate. To show sequential monotone continuity, take a sequence $(\lambda_n) \subset \Lambda_t$ converging to some $\lambda \in \Lambda_t$ almost surely from above or below. There is a sequence $(\mu_n) \subset \Lambda_t$ that converges to 0 almost surely such that $\lambda_n = \mu_n\lambda_1 + (1-\mu_n)\lambda$ for every $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\liminf_{n\to\infty} F(\lambda_n) &\ge \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \liminf_{n\to\infty} \big((1-\lambda_n)\mathbb{E}_{\underline{\mathbb{Q}}}[X_\tau|\mathcal{F}_t] + \lambda_n\mathbb{E}_{\overline{\mathbb{Q}}}[X_\tau|\mathcal{F}_t]\big) \\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \big((1-\lambda)\mathbb{E}_{\underline{\mathbb{Q}}}[X_\tau|\mathcal{F}_t] + \lambda\mathbb{E}_{\overline{\mathbb{Q}}}[X_\tau|\mathcal{F}_t]\big) = F(\lambda) \\
&= \limsup_{n\to\infty} \big(\mu_n F(\lambda_n) + (1-\mu_n)F(\lambda)\big) \\
&\ge \limsup_{n\to\infty} F(\lambda_n).
\end{aligned}$$

It follows that $F(\lambda_n) \to F(\lambda)$ almost surely as desired. □

Market-consistent exercise strategies

We conclude the paper with a representation result for market-consistent exercise strategies showing that such stopping strategies correspond precisely to optimal stopping times under equivalent martingale measures. This also yields a dual representation of the first and last exercise times.

Theorem 6.7. *Let $t \in \{0,\dots,T-1\}$. For every $\tau \in \mathcal{T}_t$ the following statements are equivalent:*

(a) $\tau \in \mathcal{T}_t^{MC}$.

(b) $\tau \in \mathcal{T}_t^{\mathbb{Q}}$ for some $\mathbb{Q} \in \mathcal{M}$.

In addition, the following statements hold:

(i) $\tau_t^- = \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{M}} \tau_t^{\mathbb{Q},-}$.

(ii) $\tau_t^+ = \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}} \tau_t^{\mathbb{Q},+}$.

Proof. Assertions (i) and (ii) follow by combining Theorem 5.3 with the equivalence between (a) and (b). To show that (a) implies (b), take $\sigma \in \mathcal{T}_t^{MC}$. It follows from market consistency that, for every $u \in \{t,\dots,T-1\}$, there exist $\mathbb{Q}_u^1, \mathbb{Q}_u^2 \in \mathcal{M}$ such that

$$X_u = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_u} \mathbb{E}_{\mathbb{Q}_u^1}[X_\tau|\mathcal{F}_u] \text{ on } \{\sigma = u\}, \quad X_u \le \mathbb{E}_{\mathbb{Q}_u^2}[X_\sigma|\mathcal{F}_u] \text{ on } \{\sigma \ge u+1\}.$$

For every $u \in \{t,\dots,T-1\}$ we can apply point (ii) in Lemma 6.1 to obtain $\mathbb{Q}_u \in \mathcal{M}$ such that

$$X_u = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_u} \mathbb{E}_{\mathbb{Q}_u}[X_\tau|\mathcal{F}_u] \text{ on } \{\sigma = u\}, \quad X_u \le \mathbb{E}_{\mathbb{Q}_u}[X_\sigma|\mathcal{F}_u] \text{ on } \{\sigma \ge u+1\}.$$

By point (i) in Lemma 6.1, we find $\mathbb{Q} \in \mathcal{M}$ such that, for every $\tau \in \mathcal{T}_t$,

$$\mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}_t}[\mathbb{E}_{\mathbb{Q}_{t+1}}[\cdots\mathbb{E}_{\mathbb{Q}_{T-1}}[X_\tau|\mathcal{F}_{T-1}]\cdots|\mathcal{F}_{t+1}|\mathcal{F}_t].$$

As a result, for every $u \in \{t, \dots, T-1\}$ we get

$$X_u = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_u} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_u] \text{ on } \{\sigma = u\}, \quad X_u \leq \mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_u] \text{ on } \{\sigma \geq u+1\}. \tag{6.10}$$

We show by backward induction that for every $u \in \{t, \dots, T\}$

$$\mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_u] = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_u} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_u] \quad \text{on } \{\sigma \geq u\}. \tag{6.11}$$

The statement holds if $u = T$. If it holds for $u+1$ for some $u \in \{t, \dots, T-1\}$, then on $\{\sigma \geq u\}$

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_u] &= 1_{\{\sigma=u\}} X_u + 1_{\{\sigma \geq u+1\}} \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_{u+1}]|\mathcal{F}_u] \\ &\geq 1_{\{\sigma=u\}} X_u + 1_{\{\sigma \geq u+1\}} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{u+1}} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_u] = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_u} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_u]. \end{aligned}$$

The inequality follows from the induction hypothesis and the last equality from (6.10). This concludes the induction argument and establishes (6.11) and, hence, $\sigma \in \mathcal{T}_t^{\mathbb{Q}}$.
To show that (b) yields (a), take $\sigma \in \mathcal{T}_t^{\mathbb{Q}}$ for some $\mathbb{Q} \in \mathcal{M}$. We claim that for each $u \in \{t, \dots, T-1\}$

$$\mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_u] = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_u} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_u] \quad \text{on } \{\sigma \geq u\}. \tag{6.12}$$

To the contrary, let $u \in \{t, \dots, T-1\}$ and $\tau \in \mathcal{T}_u$ be such that the event $A = \{\mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_u] > \mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_u]\} \cap \{\sigma \geq u\}$ has strictly positive probability. Define $\rho = 1_A \tau + 1_{A^c} \sigma \in \mathcal{T}_t$ and observe that $\mathbb{E}_{\mathbb{Q}}[X_\rho|\mathcal{F}_u] \gneq \mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_u]$. As a consequence, $\mathbb{E}_{\mathbb{Q}}[X_\rho|\mathcal{F}_t] \gneq \mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_t]$ as well, which contradicts the $\mathbb{Q}$-optimality of σ. It then follows from (6.12) that, for every $u \in \{t, \dots, T-1\}$,

$$X_u = \mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_u] = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_u} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_u] \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{u+1}} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_u] \quad \text{on } \{\sigma = u\},$$

implying that X_u is a market-consistent seller price for X^{u+1} on $\{\sigma = u\}$, as well as

$$X_u = \mathbb{E}_{\mathbb{Q}}[X_u|\mathcal{F}_u] \leq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_u} \mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_u] = \mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_u] \quad \text{on } \{\sigma > u\},$$

implying that X_u is a market-consistent buyer price for X_σ on $\{\sigma > u\}$. This gives $\tau \in \mathcal{T}_t^{MC}$. □

Remark 6.8. The key to characterize when the subreplication price of an American option is an arbitrage-free price in [1] was to work with the stopping time defined for $t \in \{0, \dots, T-1\}$ by

$$\widehat{\tau}_t := \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{M}} \tau_t^{\mathbb{Q},-} = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{M}} \mathcal{T}_t^{\mathbb{Q}}.$$

By Theorem 6.7, this exercise strategy is always market consistent and coincides with the first exercise time τ_t^-. In line with our approach, starting from τ_t^- highlights the economic rationale and the identity $\tau_t^- = \widehat{\tau}_t$ establishes a dual representation of the first exercise time in the absence of which $\widehat{\tau}_t$ would not carry any compelling economic interpretation.

A final remark about representation theorems

We conclude by emphasizing a natural way to think about the above representation theorems that highlights the similarities between representation theorems for market-consistent strategies and market consistent prices. If P is a market-consistent price for the American option X at date t, our representation theorem for market-consistent prices tells us that

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t}\mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_t]$$

for some equivalent martingale measure $\mathbb{Q}$. This allows us to interpret P as the unique price that X would have in a hypothetical "complete market", let us call it the $\mathbb{Q}$-*market*, where every payoff is priced as an expectation with respect to $\mathbb{Q}$, i.e., P is the most expensive payoff stream in the basket when priced with respect to $\mathbb{Q}$. We use quotation marks because, unless the state space was finite, this market would not qualify as one of the markets studied here because it would feature infinitely many basic securities. Similarly, if τ is a market-consistent exercise strategy starting at t, then there would be an equivalent martingale measure $\mathbb{Q}$ such that τ is $\mathbb{Q}$-optimal, meaning

$$\mathbb{E}_{\mathbb{Q}}[X_\sigma|\mathcal{F}_t]=\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t}\mathbb{E}_{\mathbb{Q}}[X_\tau|\mathcal{F}_t].$$

This allows us to interpret market-consistent exercise strategies as strategies that are optimal in the hypothetical $\mathbb{Q}$-market described above.

Appendix: A conditional Intermediate Value Theorem

In this appendix we prove a simple conditional version of the Intermediate Value Theorem needed for the representation of market-consistent prices of American options. Other conditional versions of the same result can be found in [5] and [7]. For each $t\in\{0,\dots,T-1\}$ we set $\Lambda_t:=\{\lambda\in X_t\,;\ \ 0\le\lambda\le 1\}$.

Theorem 6.9 (Conditional Intermediate Value Theorem)**.** *Let $t\in\{0,\dots,T-1\}$ and assume a map $F:\Lambda_t\to\mathcal{X}_t$ satisfies the following conditions:*

(1) Sequential monotone continuity*: $F(\lambda_n)\to F(\lambda)$ almost surely for every sequence $(\lambda_n)\subset\Lambda_t$ that converges from above or below to $\lambda\in\Lambda_t$ almost surely.*

(2) Conditionality*: $F(\lambda_1 1_A+\lambda_2 1_{A^c})=F(\lambda_1)1_A+F(\lambda_2)1_{A^c}$ for all $A\in\mathcal{F}_t$ and $\lambda_1,\lambda_2\in\Lambda_t$.*

Then, for every $P\in\mathcal{X}_t$ such that $F(0)\le P\le F(1)$ there exists $\lambda\in\Lambda_t$ such that $F(\lambda)=P$.

Proof. Set $a_1:=0$, $b_1:=1$ and $\mu_1:=\frac{a_1+b_1}{2}=\frac{1}{2}$. We recursively define for $n\in\mathbb{N}$

$$A_n:=\{F(\mu_n)>P\},\quad a_{n+1}:=a_n 1_{A_n}+\mu_n 1_{A_n^c},\quad b_{n+1}:=\mu_n 1_{A_n}+b_n 1_{A_n^c},\quad \mu_{n+1}:=\frac{a_{n+1}+b_{n+1}}{2}.$$

The sequences (a_n) and (b_n) are contained in Λ_t and are increasing, respectively decreasing. It follows that there exists $a,b\in\Lambda_t$ such that $a_n\uparrow a$ and $b_n\downarrow b$ almost surely. Since $b_{n+1}-a_{n+1}=\frac{b_n-a_n}{2}=\frac{1}{2^n}$ for every $n\in\mathbb{N}$, we conclude that $a=b$. Note also that $F(a_n)\le P\le F(b_n)$ for every $n\in\mathbb{N}$ by conditionality. As a result of sequential monotone continuity,

$$F(a)=\lim_{n\to\infty}F(a_n)\le P\le\lim_{n\to\infty}F(b_n)=F(b)=F(a),$$

which yields $P=F(a)$ and concludes the proof. $\square$

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