Sufficient Convexity and Best Approximation

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Abstract

Working constructively throughout, we introduce the notion of sufficient convexity for functions and sets and study its implications on the existence of best approximations of points in sets and of sets mutually.
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The framework of this paper is Bishop-style constructive mathematics (**BISH**), which, for all practical purposes, can be viewed as mathematics developed using intuitionistic logic and based on an appropriate foundation such as CZF [1], Martin-Löf type theory [9, 10], or constructive Morse set theory [5]. For more on BISH see [6]. Thus all our proofs embody algorithms that can be extracted for computer implementation (see, for example, [8, 11, 12]).

We call a mapping f of a metric space X into $\mathbf{R}$ *sufficiently convex* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in X$ with $\rho(x, x') > \varepsilon$, there exists $z \in X$ such that $f(z) + \delta < \max\{f(x), f(x')\}$. Here ρ denotes the metric on X.

Proposition 1 *The following are equivalent conditions on a mapping f of a metric space X into $\mathbf{R}$, such that $\mu \equiv \inf f$ exists.*

(i) f *is sufficiently convex.*

(ii) for each $\varepsilon > 0$ there exists $\tilde{\delta} > 0$ such that if $x, x' \in X$, $f(x) < \mu + \tilde{\delta}$, and $f(x') < \mu + \tilde{\delta}$, then $\rho(x, x') < \varepsilon$.

Proof. First suppose that f is sufficiently convex. Given $\varepsilon > 0$, pick $\delta > 0$ such that if $x, x' \in X$ and $\rho(x, x') > \varepsilon/2$, then $f(z) + \delta < \max\{f(x), f(x')\}$ for some $z \in X$. Let $\tilde{\delta} := \delta$ and consider $x, x' \in X$ such that $f(x) < \mu + \delta$, and $f(x') < \mu + \delta$. If $\rho(x, x') > \varepsilon/2$, then there exists $z \in X$ such that

$$f(z) + \delta < \max\{f(x), f(x')\} < \mu + \delta$$

and therefore $f(z) < \mu$, which is absurd. Hence $\rho(x, x') \leq \varepsilon/2 < \varepsilon$.

Conversely, suppose that f satisfies condition (ii). Given $\varepsilon > 0$, choose $\tilde{\delta}$ as in that condition. If $x, x' \in X$ and $\rho(x, x') > \varepsilon$, then $\max\{f(x), f(x')\} \geq \mu + \tilde{\delta}$. By the definition of μ, there exists $z \in X$ such that

$$f(z) < \mu + \frac{\tilde{\delta}}{2}$$

and hence

$$f(z) + \frac{\tilde{\delta}}{2} < \mu + \tilde{\delta} \leq \max\{f(x), f(x')\}.$$

Therefore we may set $\delta := \frac{\tilde{\delta}}{2}$. □

The following result is was communicated to us by Peter Aczel many years ago.

Proposition 2 *Let* X *be a complete metric space, and let* f *be a sequentially continuous, sufficiently convex mapping of* X *into* $\mathbf{R}$ *such that* $\mu \equiv \inf f$ *exists. Then there exists* $\xi \in X$ *such that* $f(\xi) = \mu$. *Moreover, if* $x \in X$ *and* $x \neq \xi$, *then* $f(x) > \mu$.

Proof. In view of Proposition 1, we can construct a strictly decreasing sequence $(\delta_n)_{n \geqslant 1}$ of positive numbers such that for each n, if $x, x' \in X$, $f(x) < \mu + \delta_n$, and $f(x') < \mu + \delta_n$, then $\rho(x, x') < 2^{-n}$. For each n, pick $x_n \in X$ such that $f(x_n) < \mu + \delta_n$. Then $\rho(x_m, x_n) < 2^{-n}$ for all $m \geqslant n$, so $(x_n)_{n \geqslant 1}$ is a Cauchy sequence in X. Since X is complete, $\xi \equiv \lim_{n \to \infty} x_n$ exists in X. By the sequential continuity of f, $\mu \leq f(\xi) \leq \mu$, so $f(\xi) = \mu$. Moreover, if $x \in X$ and $\rho(x, \xi) > 0$, then, with $\varepsilon := \frac{1}{2}\rho(x, \xi)$ and $\delta > 0$ as in the definition of 'sufficiently convex', there exists $z \in X$ such that

$$\mu < \mu + \delta \leq f(z) + \delta < \max\{f(\xi), f(x)\} = \max\{\mu, f(x)\} = f(x).$$

□

A subset K of a metric space X is *sufficiently convex given* $x \in X$ if K is inhabited, and if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y, y' \in K$ with $\rho(y, y') > \varepsilon$, there exists $z \in K$ such that

$$\rho(x, z) + \delta < \max\{\rho(x, y), \rho(x, y')\}.$$

In other words, K is sufficiently convex given $x \in X$ if $f(y) \equiv \rho(x, y)$ defines a sufficiently convex function on K. We call K *sufficiently convex* if K is sufficiently convex given any $x \in X$. The following theorem on best approximation of points is an immediate consequence of Proposition 2.

Theorem 3 *Let* K *be a complete subset of a metric space* X *that is sufficienctly convex given* $x \in X$. *Further suppose that* $\mu = \inf\{\rho(x, y) \mid y \in K\}$ *exists. Then there exists* $\xi \in K$ *such that* $\rho(x, \xi) = \mu$. *Moreover, if* $y \in K$ *and* $y \neq \xi$, *then* $\rho(x, y) > \mu$.

A normed space X is *uniformly convex* if for each $\varepsilon > 0$ there exists δ with $0 < \delta < 1$ such that if x, y are elements of X with $\|x\| = 1 = \|y\|$ and $\|x - y\| \geq \varepsilon$, then $\left\|\frac{1}{2}(x + y)\right\| \leq \delta$. Hilbert spaces, and L_p spaces with $p > 1$, are uniformly convex [4, page 322, Corollary (3.22)].

Lemma 4 *Let X be a uniformly convex normed space. Then for all $\tilde{\varepsilon} > 0$ and $M > 0$ there exists $\tilde{\delta} > 0$ such that if x, y are elements of X with $\|x\| = \|y\| \leq M$ and $\|x - y\| \geq \tilde{\varepsilon}$, then $\|\frac{1}{2}(x + y)\| + \tilde{\delta} \leq \|x\|$.*

Proof. Let $\tilde{\varepsilon} > 0$ and consider any $x, y \in X$ such that $\|x\| = \|y\| \leq M$ and $\|x - y\| \geq \tilde{\varepsilon}$. As $\tilde{\varepsilon} \leq \|x - y\| \leq 2\|x\|$, we deduce that $\|x\| = \|y\| \geq \tilde{\varepsilon}/2 > 0$. Set $\varepsilon := \frac{\tilde{\varepsilon}}{M}$ and compute $\delta \in (0, 1)$ as in the definition of uniform convexity. As $x/\|x\|$ and $y/\|y\|$ are unit vectors with

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \frac{1}{\|x\|}\|x - y\| \geq \frac{\tilde{\varepsilon}}{M} = \varepsilon,$$

we obtain

$$\frac{1}{\|x\|}\left\|\frac{1}{2}(x + y)\right\| \leq \delta.$$

Hence, since $\|x\| \geq \tilde{\varepsilon}/2$,

$$\left\|\frac{1}{2}(x + y)\right\| \leq \delta\|x\| \leq \|x\| - (1 - \delta)\|x\| \leq \|x\| - (1 - \delta)\frac{\tilde{\varepsilon}}{2}.$$

It remains to take $\tilde{\delta} := (1 - \delta)\frac{\tilde{\varepsilon}}{2}$. □

Lemma 5 *Let X be a uniformly convex normed space, and let $K \subset X$ be inhabited, convex, and norm bounded. Then for each $x \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|\frac{1}{2}(y + y') - x\| + \delta < \max\{\|y - x\|, \|y' - x\|\}$$

whenever $y, y' \in K$ satisfy $\|y - y'\| > \varepsilon$. In particular, K is sufficiently convex.

Proof. Let $x \in X$ and $f(y) = \|y - x\|$ $(y \in X)$. In addition pick $M > 0$ such that $\|y - x\| \leq M$ for all $y \in K$ (recall that K is bounded). Let $\varepsilon > 0$. For $\tilde{\varepsilon} := \varepsilon/2$ and M compute $\tilde{\delta} > 0$ as in Lemma 4. Choose $\delta > 0$ with $\delta < \min\{\varepsilon/4, \tilde{\delta}/2\}$, and consider $y, y' \in K$ with $\|y - y'\| > \varepsilon$. Either $\left|f(y) - f(y')\right| > \delta$ or $\left|f(y) - f(y')\right| < 2\delta$. In the first case either $f(y) < f(y') - \delta$ or $f(y') < f(y) - \delta$, and hence by the triangle inequality,

$$\begin{aligned} f(\frac{1}{2}(y + y')) &= \|\frac{1}{2}(y + y') - x\| \\ &\leq \frac{1}{2}(\|y - x\| + \|y' - x\|) \\ &= \frac{1}{2}(f(y) + f(y')) \\ &< \max\{f(y), f(y')\} - \frac{\delta}{2}. \end{aligned}$$

Now assume the second case. Then by the triangle inequality,

$$\varepsilon < \|y - y'\| \leq \|y - x\| + \|y' - x\| = f(y) + f(y') < 2(f(y) + \delta),$$

so $f(y) > \varepsilon/4$. Likewise $f(y') > \varepsilon/4$; whence $\min\{f(y), f(y')\} > \varepsilon/4 > 0$. Letting

$$z := \frac{\|y - x\|}{\|y' - x\|}(y' - x),$$

note that

$$\|z - (y' - x)\| = \big|\|y - x\| - \|y' - x\|\big| = \big|f(y) - f(y')\big| < 2\delta,$$

$$\|z\| = \|y - x\| = f(y) \leq M,$$

and

$$\|(y - x) - z\| \geq \|y - y'\| - \|(y' - x) - z\| > \varepsilon - 2\delta > \frac{\varepsilon}{2} = \tilde{\varepsilon}.$$

By our choice of $\tilde{\delta}$,

$$\begin{aligned}
f(y) = \|y - x\| &\geq \frac{1}{2}\|(y - x) + z\| + \tilde{\delta} \\
&= \frac{1}{2}\|(y - x) + (y' - x) - (y' - x) + z\| + \tilde{\delta} \\
&\geq \|\frac{1}{2}(y + y') - x\| - \frac{1}{2}\|z - (y' - x)\| + \tilde{\delta} \\
&> f(\frac{1}{2}(y + y')) - \delta + \tilde{\delta} \\
&> f(\frac{1}{2}(y + y')) + \delta.
\end{aligned}$$

As $f(y) \leq \max\{f(y), f(y')\}$ and as $\frac{1}{2}(y + y') \in K$, the lemma is proved.

□

Lemma 6 *Let X be a uniformly convex normed space, and let $K \subset X$ be inhabited and convex. Then K is sufficiently convex.*

Proof. Let $x \in X$ and $f(y) = \|y - x\|$ $(y \in X)$. We have to prove that f is sufficiently convex on K. To this end, let $y_0 \in K$ and $M > 2(\|y_0 - x\| + 1) = 2(f(y_0) + 1)$. Note that

$$K' = \{y - x : y \in K, \|y - x\| \leq M\}$$

is convex, norm bounded, and inhabited (since $y_0 - x \in K'$). Therefore, by Lemma 5, for $\varepsilon > 0$ there exists δ with $0 < \delta < 1$, such that if $y, y' \in K$, $\|y - y'\| > \varepsilon$, and $(y - x), (y' - x) \in K'$, then

$$f(\frac{1}{2}(y + y')) + \delta = \|\frac{1}{2}(y + y') - x\| + \delta < \max\{\|y - x\|, \|y' - x\|\}.$$

If $y, y' \in K$ are such that $f(y) = \|y - x\| > M/2$ or $f(y') = \|y' - x\| > M/2$, then

$$f(y_0) + \delta < \frac{M}{2} - 1 + 1 < \max\{f(y), f(y')\}.$$

$\square$

Lemma 6 and Theorem 3 lead to

Theorem 7 *Let X be a uniformly convex normed space, and let $K \subset X$ be an inhabited, complete, and convex set. Moreover, let $x \in X$ and assume that*

$$\mu := \inf\{\|y - x\| : y \in K\}$$

exists. Then there exists $\xi \in K$ such that $\|\xi - x\| = \mu$. If $y' \in K$ such that $y' \neq \xi$, then $\|y' - x\| > \mu$.

An immediate consequence of Theorem 7 is the proof of [4, Problem 11, p. 391] which corresponds to Corollary 8 below. To this end, we recall that a subset L of a metric space X is *located* if L is inhabited and for all $x \in X$ the distance

$$\rho(y, L) := \inf\{\rho(x, y) \mid x \in L\}$$

exists.

Corollary 8 *Let B be a uniformly convex Banach space, and let $K \subset B$ be an inhabited, closed, located, and convex set. Then each $x \in B$ has a unique closest point $\xi \in K$—that is, $\|x - \xi\| = \rho(x, K)$—and if $y \in K$ is such that $y \neq \xi$, then $\|x - y\| > \rho(x, K)$.*

A subset K of a normed space X is *uniformly rotund* if it is inhabited, convex, and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, x' \in K$ and $\|x - x'\| \geq \varepsilon$, then $\frac{1}{2}(x + x') + z \in K$ for all $z \in X$ with $\|z\| \leq \delta$.

Proposition 9 *A normed linear space X is uniformly convex if and only if its closed unit ball B is uniformly rotund.*

Proof. Suppose that X is uniformly convex, and let $\varepsilon > 0$. Compute $\delta > 0$ for ε, $x = 0$, and $K = B$ as in Lemma 5. Then for all $y, y' \in B$ such that $\|y - y'\| \geq \varepsilon$, and any $z \in X$ with $\|z\| \leq \delta$, it follows that

$$\left\|\frac{1}{2}(y + y') + z\right\| \leq \left\|\frac{1}{2}(y + y')\right\| + \delta \leq \max\{\|y\|, \|y'\|\} \leq 1.$$

Hence, $\frac{1}{2}(y + y') + z \in B$, so B is uniformly rotund.

Conversely, suppose that B is uniformly rotund, let $\varepsilon > 0$, and choose $\delta < 1$ as in the definition of uniformly rotund. If x, y are unit vectors of X with $\|x - y\| \geq \varepsilon$, then $\left\|\frac{1}{2}\delta(x + y)\right\| \leq \delta$, so

$$(1 + \delta)\left\|\tfrac{1}{2}(x + y)\right\| = \left\|\tfrac{1}{2}(x + y) + \tfrac{1}{2}\delta(x + y)\right\| \leq 1$$

and therefore $\left\|\frac{1}{2}(x + y)\right\| \leq (1 + \delta)^{-1} < 1$. $\square$

Lemma 10 *A uniformly rotund subset of a normed space is sufficiently convex.*

Proof. Let K be a uniformly rotund subset of a normed space X and let $x \in X$. In addition let $\varepsilon > 0$ and $y, y' \in K$ be such that $\|y - y'\| > \varepsilon$. Pick $\delta > 0$ as in the definition of uniformly rotund and note that we may assume that $\delta < \varepsilon/2$. Let $v := \frac{1}{2}(y + y') - x$. Either $\|v\| > 0$ or $\|v\| < \delta/2$. In the first case

$$\|\delta \frac{v}{\|v\|}\| \leq \delta$$

and therefore $z := \frac{1}{2}(y + y') - \delta \frac{v}{\|v\|} \in K$. We have

$$\begin{aligned}\|z - x\| = \|v\| - \delta &\leq \frac{1}{2}(\|y - x\| + \|y' - x\|) - \delta \\ &\leq \max\{\|y - x\|, \|y' - x\|\} - \delta,\end{aligned}$$

and thus $\|z - x\| + \delta/2 < \max\{\|y - x\|, \|y' - x\|\}$. Next suppose that $\|v\| < \delta/2$. As $\|y - x\| + \|y' - x\| \geq \|y - y'\| > \varepsilon$, we must have $\max\{\|y - x\|, \|y' - x\|\} \geq \varepsilon/2$. Hence, in this case, letting $z := \frac{1}{2}(y + y') \in K$ we have

$$\|z - x\| + \delta/2 = \|v\| + \delta/2 < \delta < \varepsilon/2 \leq \max\{\|y - x\|, \|y' - x\|\}.$$

$\square$

Now Lemma 10 and Theorem 3 imply:

Theorem 11 *Let K be a complete, uniformly rotund subset of a normed space X. Moreover, let $x \in X$ and assume that*

$$\mu := \inf\{\|y - x\| : y \in K\}$$

exists. Then there exists $\xi \in K$ such that $\|\xi - x\| = \mu$. If $y' \in K$ such that $y' \neq \xi$, then $\|y' - x\| > \mu$.

So far we have considered the best approximation of a point in a set. Now we move on to mutual best approximations of sets. Let K and L be subsets of a metric space X such that L is inhabited. We call K *sufficiently convex relative to* L if K is inhabited, and for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y, y' \in K$ with $\rho(y, y') > \varepsilon$, there exists $z \in K$ such that for all $x, x' \in L$ there is $u \in L$ with

$$\rho(u, z) + \delta < \max\{\rho(x, y), \rho(x', y')\}.$$

Note that K is sufficiently convex given $x \in X$ if and only if K is sufficiently convex relative to $\{x\}$.

Lemma 12 *Let K and L be subsets of a metric space X such that L is located. Then K is sufficiently convex relative to L if and only if $f(y) \equiv \rho(y, L)$ defines a sufficiently convex function on K.*

Proof. Suppose that K is sufficiently convex relative to L. For $\varepsilon > 0$ let $\delta > 0$ be as in the definition of *sufficiently convex relative to* L. Consider $y, y' \in \mathsf{K}$ such that $\rho(y, y') > \varepsilon$. Then there is $z \in \mathsf{K}$ such that for all $x, x' \in \mathsf{L}$ there is $u \in \mathsf{L}$ with

$$\rho(u, z) + \delta < \max\{\rho(x, y), \rho(x', y')\}.$$

Let $x, x' \in \mathsf{L}$ be such that $\rho(y, \mathsf{L}) > \rho(x, y) - \delta/2$ and $\rho(y', \mathsf{L}) > \rho(x', y') - \delta/2$. Then with $u \in \mathsf{L}$ as above, we have

$$\begin{aligned}\rho(z, \mathsf{L}) + \delta &\leq \rho(u, z) + \delta \\ &< \max\{\rho(x, y), \rho(x', y')\} \\ &< \max\{\rho(y, \mathsf{L}), \rho(y', \mathsf{L})\} + \delta/2.\end{aligned}$$

Hence,

$$\rho(z, \mathsf{L}) + \delta/2 < \max\{\rho(y, \mathsf{L}), \rho(y', \mathsf{L})\}.$$

Conversely, if $f(y) \equiv \rho(y, \mathsf{L})$ defines a sufficiently convex function on K, then for every $\varepsilon > 0$ there is $\delta > 0$ such that if $y, y' \in \mathsf{K}$ satisfy $\rho(y, y') > \varepsilon$ there is $z \in \mathsf{K}$ such that

$$\rho(z, \mathsf{L}) + \delta < \max\{\rho(y, \mathsf{L}), \rho(y', \mathsf{L})\}.$$

Let $u \in \mathsf{L}$ be such that $\rho(z, \mathsf{L}) > \rho(u, z) - \delta/2$. Then for all $x, x' \in \mathsf{L}$ we have

$$\rho(u, z) + \delta/2 < \rho(z, \mathsf{L}) + \delta < \max\{\rho(y, \mathsf{L}), \rho(y', \mathsf{L})\} \leq \max\{\rho(x, y), \rho(x', y')\}.$$

□

Lemma 13 *Let* L *be a located, convex subset of a normed space* X. *Then for all* x, x' *in* X *and* $t \in [0, 1]$,

$$\rho(tx + (1 - t)x', \mathsf{L}) \leq t\rho(x, \mathsf{L}) + (1 - t)\rho(x', \mathsf{L}).$$

Proof. Given $x, x' \in \mathsf{X}$, $t \in [0, 1]$, and $\varepsilon > 0$, pick $y, y' \in \mathsf{L}$ such that

$$\|x - y\| < \rho(x, \mathsf{L}) + \varepsilon \text{ and } \|x' - y'\| < \rho(x', \mathsf{L}) + \varepsilon.$$

Then

$$\begin{aligned}\rho(tx + (1 - t)x', \mathsf{L}) &\leq \|tx + (1 - t)x' - ty - (1 - ty')\| \\ &\leq t\,\|x - y\| + (1 - t)\,\|x' - y'\| \\ &\leq t\rho(x, \mathsf{L}) + (1 - t)\rho(x', \mathsf{L}) + t\varepsilon + (1 - t)\varepsilon \\ &\leq t\rho(x, \mathsf{L}) + (1 - t)\rho(x', \mathsf{L}) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

Proposition 14 *Let* K *be an inhabited, uniformly rotund subset of a normed space* X, *and* L *a located, convex subset of* X *that is disjoint from* K. *Then* K *is sufficiently convex relative to* L.

Proof. By Lemma 12 it will suffice to show that $f(x) \equiv \rho(x, L)$ defines a sufficiently convex function on K. For $\varepsilon > 0$ let $\delta > 0$ be as in the definition of *uniform rotundity* for K. Consider $x, x' \in K$ such that $\|x - x'\| > \varepsilon$. Let $u := \frac{1}{2}(x + x')$ and fix $v \in L$ such that $\|v - u\| < \rho(u, L) + \delta/2$. Note that $\|v-u\| \geq \delta$, because by choice of δ, if we had $\|v-u\| < \delta$, then $v = u+(v-u) \in K$ which is absurd since K and L are disjoint. Let

$$z := u + \frac{\delta}{\|v - u\|}(v - u).$$

Then $\|z - u\| = \delta$, and therefore $z = u + (z - u) \in K$. Recalling both our choice of v and Lemma 13, we have

$$\begin{aligned} f(z) + \delta/2 &\leq \|v - z\| + \delta/2 \\ &= \left(1 - \frac{\delta}{\|v - u\|}\right) \|v - u\| + \delta/2 \\ &= \|v - u\| - \delta/2 \\ &< f(u) \\ &\leq \max\{f(x), f(x')\}. \end{aligned}$$

□

To see that in Proposition 14 we cannot replace uniform rotundity by mere convexity (in a uniformly convex space), take X to be the Euclidean plane $\mathbf{R}^2$,

$$K = \{(a, b) \in \mathbf{R}^2 : a \leq 0\},$$

and

$$L = \{(a, b) \in \mathbf{R}^2 : a \geq 1\}.$$

We have

$$\inf_{x \in K} \rho(x, L) = 1 = \|(0, b) - (1, b)\|$$

for all $b \in \mathbf{R}$, so, in view of Proposition 2, $x \mapsto \rho(x, L)$ is not sufficiently convex on K.

Recall here *Bishop's Lemma* [7, Proposition 3.1.1]:

> Let Y be a complete, located subset of a metric space X. Then for each $x \in X$ there exists $y \in Y$ such that if $x \neq y$, then $\rho(x, Y) > 0$.

Theorem 15 *Let K and L be subsets of a metric space X such that K is complete, L is located, and K is sufficiently convex relative to L. Suppose also that $d \equiv \inf_{y \in K} \rho(y, L)$ exists. Then there exists $\xi \in K$ such that* (i) $\rho(\xi, L) = d$ *and* (ii) $\rho(y, L) > d$ *for all $y \in K$ with $y \neq \xi$. If, in addition, L is complete, then there exists $x \in L$ such that if $\xi \neq x$, then $d > 0$.*

Proof. Since $f(y) \equiv \rho(y, L)$ defines a sufficiently convex function on K, and since K is complete and d exists, Proposition 2 produces $\xi \in K$ with properties (i) and (ii). If also L is complete, then we complete the proof by invoking Bishop's Lemma. □

Lemma 16 *Let Y be an inhabited, convex subset of a Hilbert space H, and a a point of H such that $d = \rho(a, Y)$ exists. Then there exists $b \in \overline{Y}$ such that $\|a - b\| = d$. Moreover,*

(i) $\|a - y\| > d$ *whenever* $y \in \overline{Y}$ *and* $y \neq b$*;*

(ii) $\langle a - b, b - y \rangle \geq 0$*, and therefore* $\langle a - b, a - y \rangle \geq d^2$*, for all* $y \in Y$.

Proof. This is a well-known result on Hilbert space. For instance Lemma 1 in [2] proves the existence of $b \in \overline{Y}$ such that $\|a - b\| = d$ and (ii) holds. Conclusion (i) follows from (ii) since for all $y \in Y$

$$\begin{aligned}\|a - y\|^2 &= \|a - b + b - y\|^2 \\ &= \|a - b\|^2 + \|b - y\|^2 + 2\langle a - b, b - y \rangle \\ &\geq d^2 + \|b - y\|^2.\end{aligned}$$

□

Theorem 17 *Let K and L be closed subsets of a Hilbert space H such that L is convex and located, and K is sufficiently convex relative to L. Suppose also that $d \equiv \inf_{x \in K} \rho(x, L)$ exists. Then there exist $x_\infty \in K$ and $y_\infty \in L$ such that $\|x_\infty - y_\infty\| = d$. Moreover,*

(i) $\|x - y\| > d$ *whenever* $x \in K$ *and* $y \in L$ *and either* $x \neq x_\infty$ *or* $y \neq y_\infty$*;*

(ii) $\langle x_\infty - y_\infty, y_\infty - y \rangle \geq 0$*, and therefore* $\langle x_\infty - y_\infty, x_\infty - y \rangle \geq d^2$*, for all* $y \in L$.

Proof. By Theorem 15, there exists $x_\infty \in K$ such that $d = \rho(x_\infty, L)$. By Lemma 16 there exists $y_\infty \in L$ such that $\|x_\infty - y_\infty\| = \rho(x_\infty, L)$ and properties (i) and (ii) hold. □

Note that also in Theorem 17 we cannot replace sufficient convexity by mere convexity: Let $H = \mathbf{R}^2$,

$$K = \{(a, b) \in \mathbf{R}^2 : b \geq e^a + 1\},$$

and

$$L = \{(a, b) \in \mathbf{R}^2 : b \leq -e^a - 1\}.$$

Then $d = 2$, but there are no $x \in K$ and $y \in L$ such that $\|x - y\| = 2$.

Theorem 17 leads us to a new constructive separation theorem where the separating linear functional is constructed as the difference of the points of closest distance.

Theorem 18 *Let* K *and* L *be closed, convex, and located subsets of a Hilbert space* H*, such that* K *is sufficiently convex relative to* L*. Suppose that* $d \equiv \inf_{x \in K} \rho(x, L)$ *exists and is positive, let* $x_\infty \in K$ *and* $y_\infty \in L$ *be as in Theorem 17, and let* $p = x_\infty - y_\infty$*. Then*

$$\langle p, x - y \rangle \geq d^2 \ (x \in K, y \in L).$$

Moreover, if $u(x) = \langle d^{-1}p, x \rangle \ (x \in H)$*, then*

(a) *u is a normed real linear functional,* $\|u\| = 1$*, and* $u(x) \geq u(y) + d$ *for all* $x \in K$ *and* $y \in L$*.*

(b) $u(x_\infty) \leq u(x)$ *for all* $x \in K$*, and* $u(y_\infty) \geq u(y)$ *for all* $y \in L$*.*

(c) *If also* K *is uniformly rotund, then* $u(x_\infty) < u(x)$ *whenever* $x \neq x_\infty$*.*

Proof. By Theorem 17, for all $y \in L$ we have

$$\langle p, x_\infty - y \rangle = \langle x_\infty - y_\infty, x_\infty - y \rangle \geq d^2.$$

On the other hand, since K is located, convex, and closed, Lemma 16 provides a unique $b \in K$ such that $\rho(y_\infty, K) = \|y_\infty - b\|$. As $\rho(y_\infty, K) = d = \|y_\infty - x_\infty\|$, it follows that $b = x_\infty$ and thus, by Lemma 16, that

$$\langle y_\infty - x_\infty, x_\infty - x \rangle \geq 0 \tag{1}$$

for all $x \in K$. Hence, for $x \in K$ and $y \in L$,

$$\begin{aligned} \langle p, x - y \rangle &= \langle p, x_\infty - y \rangle + \langle p, x - x_\infty \rangle \\ &\geq d^2 + \langle x_\infty - y_\infty, x - x_\infty \rangle \\ &= d^2 + \langle y_\infty - x_\infty, x_\infty - x \rangle \geq d^2. \end{aligned}$$

It is straightforward to prove (a) and, using (1) and noting Theorem 17 (ii), to prove (b). To prove (c), suppose also that K is also uniformly rotund, and let $x \in K$ be such that $x \neq x_\infty$. Choose $\delta > 0$ such that $\frac{1}{2}(x_\infty + x) + z \in K$ for all $z \in H$ with $\|z\| \leq \delta$. Let $z := -\frac{\delta}{d}p$. Then $\|z\| = \delta$ and therefore $\frac{1}{2}(x_\infty + x) + z \in K$. It follows that $u(\frac{1}{2}(x_\infty + x) + z) \geq u(x_\infty)$, and thus $u(x) + 2u(z) \geq u(x_\infty)$. As $u(z) = -\frac{\delta}{d^2}\langle p, p \rangle = -\delta < 0$, we conclude that $u(x) > u(x_\infty)$. □

By Theorem 18 we may construct supporting hyperplanes $P_K := \{x \in H : u(x) = u(x_\infty)\}$ of K and $P_L := \{x \in H : u(x) = u(y_\infty)\}$ of L, respectively, where P_K intersects with K in the point x_∞, and P_L intersects with L in y_∞. If K is uniformly rotund, then the intersection point x_∞ of P_K and K is strongly unique in the sense that any point $x \in K$ distinct from x_∞ is bounded away from P_K since $u(x) > u(x_\infty)$.

In trying to apply the foregoing theorems, it is natural to think of a uniformly rotund set K which is compact. In that case, if K is nontrivial, Corollary 20 below shows that H is finite-dimensional.

Proposition 19 *Let* X *be a normed space, and* S *be a uniformly rotund subset of* X *that contains two distinct points. Then* S *contains an open ball of positive radius.*

Proof. Let a, b be two distinct points of S. There exists $\delta > 0$ such that if $x, y \in S$ and $\|x - y\| \geq \|a - b\|$, then $\frac{1}{2}(x + y) + z \in S$ for all $z \in X$ with $\|z\| \leq \delta$. Consider the open ball $B(\frac{1}{2}(a+b), \delta)$ of radius δ with centre $\frac{1}{2}(a+b)$. If $z \in B(\frac{1}{2}(a + b), \delta)$, then $\|z - \frac{1}{2}(a + b)\| < \delta$, so

$$z = \tfrac{1}{2}(a + b) + \left(z - \tfrac{1}{2}(a + b)\right) \in S,$$

and therefore $B(\frac{1}{2}(a + b), \delta)$ is the required ball. □

Corollary 20 *A normed space that has a totally bounded, uniformly rotund subset which contains two distinct points is finite-dimensional.*

Proof. This follows from the Proposition 19 and [7, Proposition 4.1.13]. □

References

[1] P. Aczel and M. Rathjen, *Notes on Constructive Set Theory*, `http://www1.maths.leeds.ac.uk/~rathjen/book.pdf`.

[2] J. Berger and G. Svindland, A separating hyperplane theorem, the fundamental theorem of asset pricing, and Markov's principle. *Ann. Pure and Applied Logic* **167** (2016), 1161–1171

[3] E. Bishop, *Foundations of Constructive Analysis.* McGraw-Hill, New York, 1967

[4] E. Bishop and D. S. Bridges, *Constructive Analysis.* Grundlehren der math. Wiss. 279, Springer-Verlag, Heidelberg, 1985

[5] D. S. Bridges, Morse set theory as a foundation for constructive mathematics. *Theoretical Computer Science* **928** (2022), 115–135

[6] D. S. Bridges, H. Ishihara, M. J. Rathjen, and H. Schwichtenberg (editors), *Handbook of Constructive Mathematics*, Encyclopedia [sic] of Mathematics and Its Applications 185, Cambridge University Press, 2023

[7] D. S. Bridges and L. S. Vîţă, *Techniques of Constructive Analysis.* Springer, New York, 2006

[8] R. L. Constable et al., *Implementing Mathematics with the Nuprl Proof Development System.* Prentice-Hall, Englewood Cliffs, New Jersey, 1986

[9] P. Martin-Löf, *Intuitionistic type theory.* Notes by Giovanni Sambin of a series of lectures given in Padua, June 1980, Bibliopolis, Napoli, 1984

[10] G. Sambin and J. Smith (editors), *Twenty-five Years of Constructive Type Theory.* Oxford Logic Guides 36, The Clarendon Press, Oxford, 1998

[11] H. Schwichtenberg, Computational Aspects of Bishop's Constructive Mathematics. In: *Handbook of Constructive Mathematics*, Encyclopedia [sic] of Mathematics and Its Applications 185, pp. 715–748, Cambridge University Press, 2023

[12] H. Schwichtenberg and S. S. Wainer, Proofs and Computations. In *Perspectives in Logic*. Association for Symbolic Logic and Cambridge University Press, 2012