Bipolar Theorems for Sets of Non-negative Random Variables*

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Abstract

This paper assumes a robust, in general not dominated, probabilistic framework and provides necessary and sufficient conditions for a bipolar representation of subsets of the set of all quasi-sure equivalence classes of non-negative random variables, without any further conditions on the underlying measure space. This generalizes and unifies existing bipolar theorems proved under stronger assumptions on the robust framework. Applications are in areas of robust financial modeling.
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1 Introduction

The well-known bipolar theorem proved in Brannath and Schachermayer [13] provides necessary and sufficient conditions for the existence of a bipolar representation of a set $\mathcal{C} \subset L^0_{P+}$ by means of elements of L^0_{P+} itself, see Theorem 3.1. Here, $L^0_{P+} := L^0_+(\Omega, \mathcal{F}, P)$ denotes the positive cone of $L^0(\Omega, \mathcal{F}, P)$, which is endowed with the topology induced by convergence in probability, and $(\Omega, \mathcal{F}, P)$ is a probability space. An important application of this result is the dual characterization of solutions to utility maximization problems, see Kramkov and Schachermayer [27, 28].
Brannath and Schachermayer [13] show that $\mathcal{C}$ allows a bipolar representation in L^0_{P+} if and only if $\mathcal{C}$ is convex, solid, and closed in probability. The aim of this paper is to generalize this result to a so-called robust framework, where the probability measure P is replaced by a family of probability measures $\mathcal{P}$ which is not necessarily dominated. Such extensions have already been studied in Bartl and Kupper [6], Gao and Munari [23], and Liebrich et al. [30] where sufficient conditions for a bipolar representation in very particular robust frameworks are given.
In this paper, without further assumptions on the underlying measure space, we provide necessary and sufficient conditions for a bipolar representation of $\mathcal{C} \subset L^0_{c+}$, where c is the upper probability

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induced by the set of probability measures $\mathcal{P}$, and L^0_{c+} denotes the robust counterpart of L^0_{P+}. As a byproduct we obtain a common framework for and unify the already mentioned bipolar results of Bartl and Kupper [6], Brannath and Schachermayer [13], Gao and Munari [23], and Liebrich et al. [30].

Of course, convexity and solidity of $\mathcal{C}$ are necessary conditions for a bipolar representation, also in robust frameworks. A key observation, however, is that any bipolar representation requires a property called *$\mathcal{P}$-sensitivity* of $\mathcal{C}$, see Definition 2.7, a property which is trivially satisfied in the classical dominated case and only reveals itself in a non-dominated robust framework. $\mathcal{P}$-sensitivity as an essential property to handle sets of random variables over robust model space also appeared in Burzoni and Maggis [16] and Maggis et al. [32]. Essentially, a set is $\mathcal{P}$-sensitive if membership to that set is determined by separate evaluations under (all) probability measures which are absolutely continuous to $\mathcal{P}$. This allows a 'localizing' strategy when deriving results for such sets, where localizing means verifying some statement separately under each probability measure. Not only is $\mathcal{P}$-sensitivity necessary for any bipolar representation, but the mentioned localizing strategy allows to *lift* bipolar theorems known within a dominated framework to the robust model space, see Sections 3.2 and 6. Indeed, in this way we derive our main results on bipolar representations for sets $\mathcal{C} \subset L^0_{c+}$ by lifting the mentioned bipolar theorem of Brannath and Schachermayer [13] from L^0_{P+} to L^0_{c+}. In fact, we will pay more attention to lifting a related bipolar theorem given in Kupper and Svindland [29] which has the advantage of the more manageable dual space L^∞_P, see Section 6.

Regarding $\mathcal{P}$-sensitivity, we give a number of conditions that guarantee $\mathcal{P}$-sensitivity of sets in Section 5. In particular, we will show that $\mathcal{P}$-sensitivity is equivalent to the aggregation property known from robust statistics, see, e.g., Torgersen [38], or robust stochastic (financial) models, see, e.g., Soner et al. [37].

Let us revisit the list of necessary properties for a bipolar representation. It is clear that this list must also include some kind of closedness of $\mathcal{C}$. In this respect it turns out that, in contrast to dominated models where closedness with respect to convergence in probability under the dominating probability measure is the canonical choice, in the non-dominated case there are a variety of notions of closedness which offer themselves as necessary and reasonable requirements depending on the point of view on the problem. For a solid set, all of them may be seen as robust generalizations of closedness with respect to convergence in probability. A main contribution of this paper is to relate the underlying notions of convergence on L^0_c, and thus the different types of closedness, to each other, see Section 4. Eventually, we identify sequential order closedness with respect to the quasi-sure order as the appropriate equivalent of a number of notions of closedness for solid sets which are necessary, and in fact sufficient in combination with the other properties mentioned above, for a bipolar representation of $\mathcal{C}$.

In Section 6, we collect versions of the bipolar theorem on L^0_{c+} obtained by the lifting procedure described earlier. The versions differ in the choice of the class of dual elements. Appropriate classes of dual elements turn out to be combinations of probability measures and test functions, or simply the set of finite measures. In Section 7, we collect several applications of the bipolar representations given in Section 6. In particular we show how our results generalize the bipolar theorems of Bartl and Kupper [6], Gao and Munari [23], and Liebrich et al. [30]. Moreover, we sketch applications to mathematical finance which are part of ongoing research, and finally we provide a mass transport type duality adopted from Bartl and Kupper [6].

Related Literature A widely studied source of uncertainty leading to robust models is uncertainty about the volatility of (continuous time) price processes, see, e.g., Cohen [20] and Soner et al. [37]. In that respect Denis et al. [21] investigate capacities and robust function spaces based on sublinear expectations, in particular G-expectation. Bion-Nadal and Kervarec [10] study risk measures under model uncertainty with a focus on dual representations. Robust duality has further been explored by Beissner and Denis [9], who examine general equilibrium theory under Knightian uncertainty. Liebrich and Nendel [31] study robust Orlicz spaces and their duals. As regards the robust bipolar theorems mentioned above, Bartl and Kupper [6] prove a pointwise bipolar theorem within a model-free setting, see Section 7.3. Applications of this result, as provided in Bartl and Kupper [6], include robust hedging in discrete time and the aforementioned mass transport type duality result. In their investigation of surplus-invariant risk measures, Gao and Munari [23] derive another robust bipolar theorem, see Section 7.1. Lastly, Liebrich et al. [30] study model uncertainty from a reverse perspective, trying to understand which conditions the probabilistic model has to satisfy in order to obtain robust analogues of useful properties of the model space known in dominated frameworks. In this regard the bipolar theorem discussed Section 7.2 appears.

Outline The paper is organized as follows. In Section 2, we introduce some notation and preliminary results, including a first discussion of $\mathcal{P}$-sensitivity. Section 3 recalls the bipolar theorems of Brannath and Schachermayer [13] and Kupper and Svindland [29], and provides a reverse study of bipolar representations which already establishes the necessity of $\mathcal{P}$-sensitivity. In Section 4, we discuss the mentioned different concepts of closedness of sets in L^0_c. Then, in Section 5, we give conditions which imply $\mathcal{P}$-sensitivity of sets, and we also provide an equivalent characterization of $\mathcal{P}$-sensitivity in terms of the aggregation property mentioned above. Section 6 collects our main results, which are versions of the bipolar theorem under uncertainty. Lastly, in Section 7, we collect some applications.

2 Preliminaries and Notation

2.1 Basics

Throughout this paper $(\Omega, \mathcal{F})$ denotes an arbitrary measurable space. By ca we denote the real vector space of all countably additive finite variation set functions $\mu\colon \mathcal{F} \to \mathbb{R}$, and by ca_+ its positive elements ($\mu \in ca_+ \Leftrightarrow \forall A \in \mathcal{F}\colon \mu(A) \geq 0$), that is all finite measures on $(\Omega, \mathcal{F})$. Given non-empty subsets $\mathfrak{G}$ and $\mathfrak{I}$ of ca_+, we say that $\mathfrak{I}$ dominates $\mathfrak{G}$ ($\mathfrak{G} \ll \mathfrak{I}$) if for all $N \in \mathcal{F}$ satisfying $\sup_{\nu \in \mathfrak{I}} \nu(N) = 0$, we have $\sup_{\mu \in \mathfrak{G}} \mu(N) = 0$. $\mathfrak{G}$ and $\mathfrak{I}$ are equivalent ($\mathfrak{G} \approx \mathfrak{I}$) if $\mathfrak{G} \ll \mathfrak{I}$ and $\mathfrak{I} \ll \mathfrak{G}$. For the sake of brevity, for $\mu \in ca_+$ we shall write $\mathfrak{G} \ll \mu$, $\mu \ll \mathfrak{I}$, and $\mu \approx \mathfrak{G}$ instead of $\mathfrak{G} \ll \{\mu\}$, $\{\mu\} \ll \mathfrak{I}$, and $\{\mu\} \approx \mathfrak{G}$, respectively.
$\mathfrak{P}(\Omega) \subset ca_+$ denotes the set of probability measures on $(\Omega, \mathcal{F})$ and the letters $\mathcal{P}$ and $\mathcal{Q}$ are used to denote non-empty subsets of $\mathfrak{P}(\Omega)$. Fix such a set $\mathcal{P}$. We then write c for the induced upper probability $c\colon \mathcal{F} \to [0,1]$ defined by

$$c(A) := \sup_{P \in \mathcal{P}} P(A),\ A \in \mathcal{F}.$$

An event $A \in \mathcal{F}$ is called $\mathcal{P}$-polar if $c(A) = 0$. A property holds $\mathcal{P}$-quasi surely (q.s.) if it holds outside a $\mathcal{P}$-polar event. We set $ca_c := \{\mu \in ca \mid \mu \ll \mathcal{P}\}$, $ca_{c+} := ca_+ \cap ca_c$, and $\mathfrak{P}_c(\Omega) := \mathfrak{P}(\Omega) \cap ca_c$.

Consider the $\mathbb{R}$-vector space $\mathcal{L}^0 := \mathcal{L}^0(\Omega, \mathcal{F})$ of all real-valued random variables $f\colon \Omega \to \mathbb{R}$ as well as its subspace $\mathcal{N} := \{f \in \mathcal{L}^0 \mid c(|f| > 0) = 0\}$. The quotient space $L_c^0 := \mathcal{L}^0/\mathcal{N}$ consists of equivalence classes X of random variables up to $\mathcal{P}$-q.s. equality comprising representatives $f \in X$. The equivalence class induced by $f \in \mathcal{L}^0$ in L_c^0 is denoted by $[f]_c$. The space L_c^0 carries the so-called $\mathcal{P}$-quasi-sure order $\preccurlyeq_{\mathcal{P}}$ as a natural vector space order: $X, Y \in L_c^0$ satisfy $X \preccurlyeq_{\mathcal{P}} Y$ if for $f \in X$ and $g \in Y$, $f \le g$ $\mathcal{P}$-q.s., that is $\{f > g\}$ is $\mathcal{P}$-polar. In order to facilitate the notation, we suppress the dependence of $\preccurlyeq_{\mathcal{P}}$ on $\mathcal{P}$ and simply write $\preccurlyeq$ if there is no risk of confusion. $(L_c^0, \preccurlyeq)$ is a vector lattice, and for $X, Y \in L_c^0$, $f \in X$, and $g \in Y$, the minimum $X \wedge Y$ is the equivalence class $[f \wedge g]_c$ generated by the pointwise minimum $f \wedge g$, whereas the maximum $X \vee Y$ is the equivalence class $[f \vee g]_c$ generated by the pointwise maximum $f \vee g$. For an event $A \in \mathcal{F}$, χ_A denotes the indicator of the event (i.e. $\chi_A(\omega) = 1$ if and only if $\omega \in A$, and $\chi_A(\omega) = 0$ otherwise) while $\mathbf{1}_A := [\chi_A]_c$ denotes the generated equivalence class in L_c^0. Throughout the paper, for convenience, we identify the constants $m \in \mathbb{R}$ with the (equivalence classes of) constant random variables they induce, in particular $m = [m]_c = m \cdot \mathbf{1}_\Omega$.
A subspace of L_c^0 which will turn out to be important for our studies is the space L_c^∞ of equivalence classes of $\mathcal{P}$-q.s. bounded random variables, i.e.,

$$L_c^\infty := \{X \in L_c^0 \mid \exists m > 0\colon |X| \preccurlyeq m\}.$$

L_c^∞ is a Banach lattice when endowed with the norm

$$\|X\|_{L_c^\infty} := \inf\{m > 0 \mid |X| \preccurlyeq m\}, \quad X \in L_c^0.$$

L_{c+}^0 and L_{c+}^∞ denote the positive cones of L_c^0 and L_c^∞, respectively. If $\mathcal{P} = \{P\}$ is given by a singleton and thus $c = P$, we write L_P^0, L_P^∞, and $[f]_P$ instead of L_c^0, L_c^∞, and $[f]_c$, and similarly for other expressions where c appears. Also, the $\mathcal{P}$-q.s. order in this case is the P-almost-sure (a.s.) order which we will also denote by $\le_P$ when we are working with both the $\mathcal{P}$-q.s. order $\preccurlyeq$ for some set $\mathcal{P} \subset \mathfrak{P}(\Omega)$ and the P-a.s. order for some $P \in \mathfrak{P}(\Omega)$ (typically $P \ll \mathcal{P}$).
Often we will, as is common practice, identify equivalence classes of random variables with their representatives. However, sometimes it will be helpful to distinguish between them to avoid confusion. Let us clarify this further: X is an equivalence class of random variables if there exists an equivalence relation $\sim$ on $\mathcal{L}^0$ such that $X = \{f \in \mathcal{L}^0 \mid f \sim g\}$ for some $g \in \mathcal{L}^0$. A measure $P \in \mathfrak{P}(\Omega)$ is consistent with the equivalence relation $\sim$ if

$$\forall f, g \in \mathcal{L}^0\colon\ f \sim g \ \Rightarrow\ P(f = g) = 1.$$

In that case we, for instance, write $E_P[X]$ for the expectation of X under P, which actually means $E_P[f]$ for any $f \in X$ provided the latter integral is well-defined. Also, we will write expressions like $P(X = Y)$, where Y is another equivalence class of random variables with respect to the same equivalence relation $\sim$, actually meaning $P(f = g)$ for arbitrary $f \in X$ and $g \in Y$. The difference here to the usual convention of identifying equivalence classes of random variables with their representatives is that the equivalence relation $\sim$ might not be given by P-a.s. equality, but P is only assumed to be consistent with that equivalence relation in the above sense. A typical example is the equivalence relation given by $\mathcal{P}$-q.s. equality of random variables and $P \in \mathfrak{P}_c(\Omega)$.

2.2 Supported Measures and Class (S) Robustness

Supported measures $\mu \in ca_c$ play a key role in handling robustness. This concept is also known in statistics, see Liebrich et al. [30] for a detailed review.

Definition 2.1. Let $\mathcal{P} \subset \mathfrak{P}(\Omega)$ be non-empty.

1. A measure $\mu \in ca_{c+}$ is called supported if there is an event $S(\mu) \in \mathcal{F}$ such that
   (a) $\mu(S(\mu)^c) = 0$;
   (b) whenever $N \in \mathcal{F}$ satisfies $\mu(N \cap S(\mu)) = 0$, then $N \cap S(\mu)$ is $\mathcal{P}$-polar.

   The set $S(\mu)$ is called the (order) support of μ.

2. A signed measure $\mu \in ca_c$ is supported if $|\mu|$ is supported where
$$|\mu|(A) := \sup\{\mu(B) - \mu(A \setminus B) \mid B \in \mathcal{F}, B \subset A\},\ A \in \mathcal{F},$$
   is the total variation of μ.

3. We set
$$sca_c := \{\mu \in ca_c \mid \mu \text{ is supported}\},$$
   the space of all supported signed measures in ca_c, and $sca_{c+} := sca_c \cap ca_{c+}$.

Note that if two sets $S, S' \in \mathcal{F}$ satisfy conditions (a) and (b) in Definition 2.1(1), then $\chi_S = \chi_{S'}$ $\mathcal{P}$-q.s. ($\mathbf{1}_S = \mathbf{1}_{S'}$), i.e., the symmetric difference $S \triangle S'$ is $\mathcal{P}$-polar. The order support $S(\mu)$ is thus usually not unique as an event, but only unique up to $\mathcal{P}$-polar events. In the following $S(\mu)$ therefore denotes an arbitrary version of the order support.
The functional
$$L_c^\infty \ni X \mapsto \int X d\mu \tag{1}$$
is order continuous (with respect to $\preccurlyeq$) if and only if $\mu \in sca_c$. In fact, the space of order continuous linear functionals may be identified with sca_c via (1). In the same way ca_c is identified with the space of all σ-order continuous functionals, and any $\mu \in ca_c \setminus sca_c$ induces a linear σ-order continuous functional which is not order continuous. Note that in robust frameworks $ca_c \setminus sca_c \neq \emptyset$ is often the case. We refer to Liebrich et al. [30] for the latter facts, and, in general, for a concise but comprehensive discussion of the spaces ca_c and sca_c.
Stochastic models, for instance of financial markets, which do not assume a dominating probability measure are often referred to as being robust; see Bouchard and Nutz [12], Burzoni et al. [15], Nutz [34], Soner et al. [37] and the references therein. In Liebrich et al. [30] an important subclass of such robust models, namely the models of class (S) defined next, are discussed:

Definition 2.2. Let $\mathcal{P} \subset \mathfrak{P}(\Omega)$ be non-empty. $\mathcal{P}$ is of class (S) if there exists a set of supported probability measures $\mathcal{Q}$ (i.e. $\mathcal{Q} \subset \mathfrak{P}_c(\Omega) \cap sca_c$) such that $\mathcal{Q} \approx \mathcal{P}$. In that case we call $\mathcal{Q}$ a supported alternative of $\mathcal{P}$.

Let us briefly comment on the significance of the class (S) property: Suppose that $\mathcal{P}$ is of class (S) and let $\mathcal{Q}$ be a supported alternative of $\mathcal{P}$. As $\mathcal{Q} \approx \mathcal{P}$, the $\mathcal{Q}$-q.s. order coincides with the $\mathcal{P}$-q.s. order $\preccurlyeq$. Hence, when arguing by means of the $\mathcal{P}$-q.s. order—think of robust superhedging, for instance—which means to prove some statement for each $P \in \mathcal{P}$, we may indeed switch to $\mathcal{Q}$ and prove the corresponding statement for each $Q \in \mathcal{Q}$. Here we often benefit from Q being supported. In Liebrich et al. [30] it is shown how the class (S) property is important, and indeed necessary, in many situations to handle robustness in non-dominated frameworks.

Definition 2.3. Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega) \cap sca_c$. We say that $\mathcal{Q}$ has disjoint supports if, for all $Q, Q' \in \mathcal{Q}$ such that $Q \neq Q'$, $\mathbf{1}_{S(Q)} \wedge \mathbf{1}_{S(Q')} = 0$, that is $S(Q) \cap S(Q')$ is a $\mathcal{P}$-polar event.

Lemma 2.4 (see Liebrich et al. [30, Lemma 3.7])**.** *Suppose $\mathcal{P}$ is of class (S). Then there exists a supported alternative $\mathcal{Q} \approx \mathcal{P}$ with disjoint supports. $\mathcal{Q}$ will be referred to as a disjoint supported alternative.*

The following Example 2.5 serves as a simple illustration of the preceding discussion. In Example 2.6 below, we collect prominent examples of class (S) models in the literature. The class (S) property of the latter examples is extensively discussed in Liebrich et al. [30].

Example 2.5. Consider the unit interval $\Omega = [0, 1]$ equipped with the Borel-σ-algebra $\mathcal{F} = \mathcal{B}(\Omega)$. Let $\mathcal{P} := \mathfrak{P}(\Omega)$, and let $\mathcal{Q} := \{\delta_\omega \mid \omega \in \Omega\}$ be the set of all Dirac probability measures. Clearly, $\mathcal{Q} \approx \mathcal{P}$. In particular, the upper probability c induced by $\mathcal{P}$ satisfies

$$c(A) = \sup_{P \in \mathcal{P}} P(A) = \sup_{\omega \in \Omega} \delta_\omega(A) = \begin{cases} 1 & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}, \quad A \in \mathcal{F}.$$

Thus, the only $\mathcal{P}$-polar set is $\emptyset$. For each $\omega \in \Omega$, let $S(\delta_\omega) := \{\omega\}$. We have:

(a) $\delta_\omega(\Omega \setminus \{\omega\}) = 0$.

(b) Let $N \in \mathcal{F}$ such that $\delta_\omega(N \cap S(\delta_\omega)) = \delta_\omega(N \cap \{\omega\}) = 0$. Then, $\omega \notin N$ and therefore $N \cap \{\omega\} = \emptyset$ is $\mathcal{P}$-polar.

Hence, for each $\omega \in \Omega$, δ_ω is supported with support $S(\delta_\omega) = \{\omega\}$. Moreover, those supports are obviously disjoint. In sum, we have shown that $\mathcal{Q}$ is a disjoint supported alternative to $\mathcal{P}$. In particular, $\mathcal{P}$ is of class (S).
Also note that in this case $ca_c \neq sca_c$. Indeed consider, for instance, the Lebesgue measure λ on $(\Omega, \mathcal{F})$. For any $S \in \mathcal{F}$ with $\lambda(S^c) = 0$, and for any non-empty countable subset N of S, we have $\lambda(N \cap S) = \lambda(N) = 0$, but $N \cap S = N$ is not $\mathcal{P}$-polar. Hence, λ is not supported, that is $\lambda \in ca_c \setminus sca_c$.

Example 2.6. The underlying robust probabilistic models of the following financial models are all of class (S). We refer to Liebrich et al. [30, Section 3.2] for the details and in particular the proofs of the class (S) property.

1. The financial models on product spaces given in Chau [18] and Chau et al. [19], see Liebrich et al. [30, Section 3.2.1].

2. The volatility uncertainty models discussed in Cohen [20] and Soner et al. [37], see Liebrich et al. [30, Section 3.2.2].

3. A model of innovation considered in Amarante et al. [5], see Liebrich et al. [30, Section 3.2.3].

4. The models applied to study the superhedging problem in Bartl et al. [7], Burzoni et al. [14], Hou and Obłój [25], and Mykland [33], see Liebrich et al. [30, Section 3.2.4].

5. The robust binomial model considered in Blanchard and Carassus [11], see Liebrich et al. [30, Section 3.2.5].

2.3 $\mathcal{P}$-sensitive Sets

Let $\mathcal{P} \subset \mathfrak{P}(\Omega)$. A property that will play a major role in our studies is the so-called $\mathcal{P}$-sensitivity of subsets of L^0_c defined in the following, see also Maggis et al. [32]. To this end, recall that $[f]_c$ denotes the equivalence class in L^0_c generated by $f \in \mathcal{L}^0$, whereas $[f]_Q$ is the equivalence class generated by f in L^0_Q, that is under Q-a.s. equality. The following map identifies any $X, Y \in L^0_c$ which appear to coincide under Q, that is $Q(f = g) = 1$ for $f \in X$ and $g \in Y$:

$$j_Q \colon L^0_c \to L^0_Q, \quad [f]_c \mapsto [f]_Q.$$

Definition 2.7. A set $\mathcal{C} \subset L^0_c$ is called $\mathcal{P}$-sensitive if

$$\mathcal{C} = \bigcap_{Q \in \mathfrak{P}_c(\Omega)} j_Q^{-1} \circ j_Q(\mathcal{C}).$$

$\mathcal{P}$-sensitivity means that the set $\mathcal{C}$ is completely determined by its image under each model $Q \in \mathfrak{P}_c(\Omega)$, so if $X \in L^0_c$ looks like a member of $\mathcal{C}$ under each $Q \in \mathfrak{P}_c(\Omega)$ (i.e. $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathfrak{P}_c(\Omega)$), then in fact $X \in \mathcal{C}$. Note that always $\mathcal{C} \subset \bigcap_{Q \in \mathfrak{P}_c(\Omega)} j_Q^{-1} \circ j_Q(\mathcal{C})$, so the nontrivial inclusion is $\bigcap_{Q \in \mathfrak{P}_c(\Omega)} j_Q^{-1} \circ j_Q(\mathcal{C}) \subset \mathcal{C}$. Trivially, if $\mathcal{P} = \{P\}$, then every set $\mathcal{C} \subset L^0_P$ is P-sensitive. It will sometimes turn out to be useful to know a stronger sensitive representation of $\mathcal{C}$:

Definition 2.8. Let $\mathcal{C} \subset L^0_c$. $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ is called a reduction set for $\mathcal{C}$ if $\mathcal{Q} \neq \emptyset$ and

$$\mathcal{C} = \bigcap_{Q \in \mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{C}). \tag{2}$$

By definition, any $\mathcal{P}$-sensitive set admits the reduction set $\mathfrak{P}_c(\Omega)$. The following lemma relates reduction sets to each other and in particular shows that any set satisfying (2) is indeed $\mathcal{P}$-sensitive.

Lemma 2.9. *Let $\mathcal{C} \subset L^0_c$.*

1. *Consider a reduction set $\mathcal{Q}_1$ for $\mathcal{C}$ and any other set of probability measures $\mathcal{Q}_2 \subset \mathfrak{P}_c(\Omega)$ such that $\mathcal{Q}_1 \subset \mathcal{Q}_2$. Then $\mathcal{Q}_2$ is also a reduction set for $\mathcal{C}$.*

2. *If $\mathcal{C}$ satisfies (2) for some non-empty set $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$, then $\mathcal{C}$ is $\mathcal{P}$-sensitive.*

3. *If $\mathcal{C}$ is $\mathcal{P}$-sensitive and $\tilde{\mathcal{P}} \subset \mathfrak{P}(\Omega)$ dominates $\mathcal{P}$, i.e., $\mathcal{P} \ll \tilde{\mathcal{P}}$, then $\mathcal{C}$ is $\tilde{\mathcal{P}}$-sensitive, where $\tilde{\mathcal{P}}$-sensitive means that*
$$\mathcal{C} = \bigcap_{\{P \in \mathfrak{P}(\Omega) \mid P \ll \tilde{\mathcal{P}}\}} j_P^{-1} \circ j_P(\mathcal{C}).$$

Proof. The first statement follows from

$$\mathcal{C} \subset \bigcap_{Q \in \mathcal{Q}_2} j_Q^{-1} \circ j_Q(\mathcal{C}) \subset \bigcap_{Q \in \mathcal{Q}_1} j_Q^{-1} \circ j_Q(\mathcal{C}) = \mathcal{C}. \tag{3}$$

The second assertion follows from 1. by choosing $\mathcal{Q}_1 = \mathcal{Q}$ and $\mathcal{Q}_2 = \mathfrak{P}_c(\Omega)$. Finally, $\mathcal{P} \ll \tilde{\mathcal{P}}$ implies that $\mathfrak{P}_c(\Omega) \subset \{P \in \mathfrak{P}(\Omega) \mid P \ll \tilde{\mathcal{P}}\}$, so we may argue as in (3). □

The reason for considering other reduction sets than simply $\mathfrak{P}_c(\Omega)$ will become evident throughout the paper. As we will see next, $\mathcal{P}$-sensitive sets are stable under intersection.

Lemma 2.10. *Let I be a non-empty index set and let $\mathcal{C}_\alpha \subset L^0_c$, $\alpha \in I$, be $\mathcal{P}$-sensitive. Then*

$$\mathcal{C} := \bigcap_{\alpha \in I} \mathcal{C}_\alpha$$

is also $\mathcal{P}$-sensitive. If $\mathcal{Q}_\alpha \subset \mathfrak{P}_c(\Omega)$ is a reduction set for $\mathcal{C}_\alpha$ for each $\alpha \in I$, then $\mathcal{Q} := \bigcup_{\alpha \in I} \mathcal{Q}_\alpha$ is a reduction set for $\mathcal{C}$.

Proof. Suppose that $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$. Then in particular $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}_\alpha$ and all $\alpha \in I$. Hence, $X \in \mathcal{C}_\alpha$ for all $\alpha \in I$. □

In the following Example 2.11, we present first simple examples of $\mathcal{P}$-sensitive sets. For a more detailed discussion of $\mathcal{P}$-sensitivity, and further examples, we refer to Section 5.

Example 2.11. For $a \in \mathbb{R}$ consider the sets

$$\mathcal{C}^a_- := \{X \in L^0_c \mid X \preccurlyeq a\} \quad \text{and} \quad \mathcal{C}^a_+ := \{X \in L^0_c \mid a \preccurlyeq X\}.$$

Let us show that $\mathcal{C}^a_-$ and $\mathcal{C}^a_+$ are $\mathcal{P}$-sensitive, and that $\mathcal{P}$ serves as a reduction set in both cases. To this end, fix $X \in L^0_c$ such that $j_P(X) \in j_P(\mathcal{C}^a_-)$ for every $P \in \mathcal{P}$. Then, for each $P \in \mathcal{P}$, there exists $Y^P \in \mathcal{C}^a_-$ such that $j_P(X) = j_P(Y^P)$, meaning that $P(X = Y^P) = 1$. Hence, $P(X \leq a) = P(Y^P \leq a) = 1$ for every $P \in \mathcal{P}$. Therefore, $X \in \mathcal{C}^a_-$. We have thus shown that $\mathcal{C}^a_-$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{P}$. The same reasoning proves $\mathcal{P}$-sensitivity of $\mathcal{C}^a_+$.

3 Bipolar Representations

We recall the well-known bipolar theorem on L^0_{P+} given in Brannath and Schachermayer [13] and used in the seminal study by Kramkov and Schachermayer [27] of the utility maximization problem:

Theorem 3.1 (Brannath and Schachermayer [13, Theorem 1.3])**.** *Let $P \in \mathfrak{P}(\Omega)$ and $\mathcal{C} \subset L^0_{P+}$ be non-empty. Define the polar of $\mathcal{C}$ as*

$$\mathcal{C}^\circ := \{Y \in L^0_{P+} \mid \forall X \in \mathcal{C} \colon E_P[XY] \leq 1\}.$$

Then $\mathcal{C}^\circ$ is a non-empty, P-closed, convex, and solid subset of L^0_{P+}, and the bipolar

$$\mathcal{C}^{\circ\circ} := \{X \in L^0_{P+} \mid \forall Y \in \mathcal{C}^\circ \colon E_P[XY] \leq 1\} \tag{4}$$

of $\mathcal{C}$ is the smallest P-closed, convex, solid set in L^0_{c+} containing $\mathcal{C}$. In particular if $\mathcal{C}$ is P-closed, convex, and solid, then $\mathcal{C} = \mathcal{C}^{\circ\circ}$.

P-closedness in Theorem 3.1 means that the respective set is closed under convergence in probability with respect to P. The definition of solidness is recalled next:

Definition 3.2. Let $\mathcal{C} \subset L^0_c$. $\mathcal{C}$ is called solid in L^0_c if $X \in \mathcal{C}$, $Y \in L^0_c$ and $|Y| \preccurlyeq |X|$ imply $Y \in \mathcal{C}$. $\mathcal{C}$ is solid in L^0_{c+} if $\mathcal{C} \subset L^0_{c+}$ and $X \in \mathcal{C}$, $Y \in L^0_{c+}$ and $Y \preccurlyeq X$ imply $Y \in \mathcal{C}$. We simply say that $\mathcal{C}$ is solid, if $\mathcal{C}$ is either solid in L^0_c or solid in L^0_{c+}.

Note that our usage of the term 'solid' should not cause confusion, because, apart from the empty set, a set which is solid in L^0_{c+} cannot be solid in L^0_c and vice versa. In fact, a non-empty solid set in L^0_{c+} only comprises non-negative elements while a non-empty solid set $\mathcal{C}$ in L^0_c is symmetric in the sense that $X \in \mathcal{C}$ implies $-X \in \mathcal{C}$ (even $YX \in \mathcal{C}$ for all $Y \in L^0_c$ which take values in $\{-1, 1\}$). Indeed, one verifies that the intersection of a solid set in L^0_c with the positive cone L^0_{c+} is a solid set in L^0_{c+}.
In Theorem 3.1 we have $\mathcal{P} = \{P\}$, and the subset $\mathcal{C} \subset L^0_{P+}$ is solid if and only if $X \in \mathcal{C}$, $Y \in L^0_{P+}$, and $Y \leq_P X$ imply $Y \in \mathcal{C}$.
We also like to mention a useful strengthening of Theorem 3.1, still with ambient space L^0_{P+}, given in Kupper and Svindland [29]:

Theorem 3.3 (Kupper and Svindland [29, Corollary 2.7])**.** *Let $P \in \mathfrak{P}(\Omega)$ and $\mathcal{C} \subset L^0_{P+}$ be non-empty. Define the polar of $\mathcal{C}$ as*

$$\mathcal{C}^\circ := \{Y \in L^\infty_{P+} \mid \forall X \in \mathcal{C}\colon E_P[XY] \leq 1\}.$$

Then $\mathcal{C}^\circ$ is a non-empty, $\sigma(L^\infty_P, L^\infty_P)$-closed, convex, solid subset of L^∞_P, and the bipolar

$$\mathcal{C}^{\circ\circ} := \{X \in L^0_{P+} \mid \forall Y \in \mathcal{C}^\circ\colon E_P[XY] \leq 1\} \tag{5}$$

of $\mathcal{C}$ is the smallest P-closed, convex, solid set in L^0_{c+} containing $\mathcal{C}$. In particular if $\mathcal{C}$ is P-closed, convex, and solid, then $\mathcal{C} = \mathcal{C}^{\circ\circ}$.

The important difference between Theorems 3.1 and 3.3 is that the latter replaces the dual cone L^0_{P+} of Theorem 3.1 by L^∞_{P+}. The boundedness of elements in L^∞_{P+} will prove helpful when deriving robust bipolar theorems on L^0_{c+} by *lifting* those on L^0_{P+} for $P \in \mathcal{P}$, see Section 6. Note that by solidness and monotone convergence one directly verifies that the sets in (4) and (5) indeed coincide.

3.1 A Reverse Perspective

In this section we collect some simple observations on necessary conditions for a bipolar representation which will, however, set the direction of our further studies.

Proposition 3.4. *Let $\mathcal{X} \subset L^0_c$ be a non-empty convex subset and suppose that the non-empty set $\mathcal{C} \subset \mathcal{X}$ admits a representation*

$$\mathcal{C} = \{X \in \mathcal{X} \mid \forall h \in \mathcal{K}\colon h(X) \leq 1\} \tag{6}$$

where $\mathcal{K}$ denotes a non-empty set of functions $h : \mathcal{X} \to \mathbb{R} \cup \{-\infty, \infty\}$.

1. *If each $h \in \mathcal{K}$ is dominated by a probability measure $Q \in \mathfrak{P}_c(\Omega)$ in the sense that*

   $$\forall X, Y \in \mathcal{X}\colon\ Q(X = Y) = 1 \Rightarrow h(X) = h(Y),$$

   then $\mathcal{C}$ is $\mathcal{P}$-sensitive. Any set $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ such that every $h \in \mathcal{K}$ is dominated by some $Q \in \mathcal{Q}$ serves as reduction set for $\mathcal{C}$.

2. *If the functions h are convex, then $\mathcal{C}$ is necessarily convex.*

3. *If the functions h are monotone with respect to some partial order $\lhd$ on $\mathcal{X}$, i.e., for all $X, Y \in \mathcal{X}$ we have that $X \lhd Y$ implies $h(X) \leq h(Y)$, then $\mathcal{C}$ is monotone with respect to $\lhd$, i.e., $Y \in \mathcal{C}$, $X \in \mathcal{X}$ and $X \lhd Y$ imply $X \in \mathcal{C}$.*

4. *If the functions h are (sequentially) lower semi-continuous with respect to some topology τ on $\mathcal{X}$, then $\mathcal{C}$ is necessarily (sequentially) τ-closed.*

Proof. 1. Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ be such that every $h \in \mathcal{K}$ is dominated by some $Q \in \mathcal{Q}$. We have to prove that if $X \in \mathcal{X}$ satisfies $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$, then $X \in \mathcal{C}$. To this end, fix such an X and let $h \in \mathcal{K}$ be arbitrary and choose $Q \in \mathcal{Q}$ which dominates h. There is $Y \in \mathcal{C}$ such that $j_Q(Y) = j_Q(X) \in j_Q(\mathcal{C})$. As $Q(X = Y) = 1$, we obtain

$$h(X) = h(Y) \leq 1.$$

Since $h \in \mathcal{K}$ was arbitrarily chosen, we conclude that $X \in \mathcal{C}$.
2., 3. and 4. are easily verified. □

As our focus lies on bipolar representations for subsets of $\mathcal{X} = L^0_{c+}$, let us further refine the implications of Proposition 3.4 in that setting. If $\mathcal{X} = L^0_{c+}$ it seems natural that the functions h appearing in the representation (6) are of type $h(X) = E_P[XZ]$ for some $P \in \mathfrak{P}(\Omega)$ and $Z \in L^0_{c+}$. Under this assumption the following Corollary 3.6 provides more information. However, before we are able to state the corollary we need to introduce some further notation: Let X_n, $n \in \mathbb{N}$, and X be equivalence classes of random variables with respect to the same equivalence relation on $\mathcal{L}^0$, and let $P \in \mathfrak{P}(\Omega)$ be consistent with that equivalence relation, see Section 2.1. We will write $X_n \xrightarrow{P} X$ to indicate that $(X_n)_{n\in\mathbb{N}}$ converges to X in probability with respect to P, that is for any choice $f_n \in X_n$ and $f \in X$ the sequence of random variables $(f_n)_{n\in\mathbb{N}}$ converges to f in probability with respect to P. For a subset $\mathcal{Q}$ of $\mathfrak{P}(\Omega)$ we write $X_n \xrightarrow{\mathcal{Q}} X$ to indicate that every $Q \in \mathcal{Q}$ is consistent with the equivalence relation defining X_n, $n \in \mathbb{N}$, and X, and $X_n \xrightarrow{Q} X$ for all $Q \in \mathcal{Q}$.

Definition 3.5. Let $\mathcal{Q} \subset \mathfrak{P}(\Omega)$ be non-empty. A set $\mathcal{C} \subset L^0_c$ is called $\mathcal{Q}$-closed if $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $X_n \xrightarrow{\mathcal{Q}} X$ for some $X \in L^0_c$ implies that $X \in \mathcal{C}$.

Note that if $\tilde{\mathcal{Q}} \subset \mathcal{Q} \subset \mathfrak{P}(\Omega)$ and if $\mathcal{C}$ is $\tilde{\mathcal{Q}}$-closed, then $\mathcal{C}$ is also $\mathcal{Q}$-closed. In particular, any $\mathcal{Q}$-closed set is $\mathfrak{P}(\Omega)$-closed, and if $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$, any $\mathcal{Q}$-closed set is $\mathfrak{P}_c(\Omega)$-closed.

Corollary 3.6. *Suppose that the non-empty set $\mathcal{C} \subset L^0_{c+}$ admits a bipolar representation of the form*

$$\mathcal{C} = \{X \in L^0_{c+} \mid \forall (P, Z) \in \mathcal{K}\colon E_P[ZX] \leq 1\}$$

where $\mathcal{K} \subset \mathfrak{P}_c(\Omega) \times L^0_{c+}$ is non-empty. Then $\mathcal{C}$ is $\mathcal{P}$-sensitive, convex, solid, and $\mathfrak{P}_c(\Omega)$-closed. Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ denote any set of probabilities such that for all $(P, Z) \in \mathcal{K}$ there is $Q \in \mathcal{Q}$ with $P \ll Q$. Then $\mathcal{Q}$ serves as reduction set for $\mathcal{C}$ and $\mathcal{C}$ is in fact $\mathcal{Q}$-closed.

Proof. Convexity, solidness, and $\mathcal{P}$-sensitivity with reduction set $\mathcal{Q}$ immediately follow from Proposition 3.4. Also $\mathcal{Q}$-closedness is a consequence of Proposition 3.4 since for any $(P, Z) \in \mathcal{K}$ the function $X \ni L^0_{c+} \mapsto E_P[ZX]$ is sequentially lower semi-continuous with respect to $\mathcal{Q}$-convergence. Indeed, consider any $r \in \mathbb{R}$ and let $(X_n)_{n\in\mathbb{N}} \subset L^0_{c+}$ and $X \in L^0_{c+}$ such that $X_n \xrightarrow{\mathcal{Q}} X$ and

$E_P[ZX_n] \leq r$ for all $n \in \mathbb{N}$. As $P \ll Q$ for some $Q \in \mathcal{Q}$ and $X_n \xrightarrow{Q} X$, there is a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$ converging Q-a.s. and thus P-a.s. to X. Hence, by Fatou's lemma

$$E_P[ZX] \leq \liminf_{k\to\infty} E_P[ZX_{n_k}] \leq r.$$

□

Note the relation between the reduction set and the closedness of $\mathcal{C}$ stated in Corollary 3.6.

3.2 Lifting Bipolar Representations

As we have seen above, $\mathcal{P}$-sensitivity is necessary for a bipolar representation. In this section we will see how $\mathcal{P}$-sensitivity can be used to obtain a robust bipolar representation by lifting known bipolar theorems in dominated frameworks to the robust model L^0_c.

Throughout this section let $\mathcal{X}$ be a convex subset of L^0_c, and let $\mathcal{C} \subset \mathcal{X}$ be a non-empty $\mathcal{P}$-sensitive set with reduction set $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$. Further, let $\mathcal{X}_Q := j_Q(\mathcal{X})$ and $\mathcal{C}_Q := j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$. For each $Q \in \mathcal{Q}$, we denote by $\mathcal{Y}_Q$ a non-empty set of mappings $l : \mathcal{X}_Q \to \mathbb{R} \cup \{-\infty, \infty\}$ and let

$$\mathcal{C}_Q^\circ := \{l \in \mathcal{Y}_Q \mid \forall X \in \mathcal{C}_Q : l(X) \leq 1\}$$

and

$$\mathcal{C}_Q^{\circ\circ} := \{X \in \mathcal{X}_Q \mid \forall l \in \mathcal{C}_Q^\circ : l(X) \leq 1\}.$$

Set

$$\mathcal{C}^\circ := \bigcup_{Q\in\mathcal{Q}} \{l \circ j_Q \mid l \in \mathcal{C}_Q^\circ\} \tag{7}$$

and

$$\mathcal{C}^{\circ\circ} := \{X \in \mathcal{X} \mid \forall h \in \mathcal{C}^\circ : h(X) \leq 1\}. \tag{8}$$

Theorem 3.7. *Suppose that $\mathcal{C}_Q = \mathcal{C}_Q^{\circ\circ}$ for all $Q \in \mathcal{Q}$. Then $\mathcal{C} = \mathcal{C}^{\circ\circ}$.*

Proof. Let $X \in \mathcal{C}$, then $j_Q(X) \in \mathcal{C}_Q$ and thus $l(j_Q(X)) \leq 1$ for all $l \in \mathcal{C}_Q^\circ$ and $Q \in \mathcal{Q}$. Hence, $X \in \mathcal{C}^{\circ\circ}$. Now let $X \in \mathcal{C}^{\circ\circ}$. Then for any $Q \in \mathcal{Q}$ we have that $l(j_Q(X)) \leq 1$ for all $l \in \mathcal{C}_Q^\circ$, that is $j_Q(X) \in \mathcal{C}_Q^{\circ\circ}$. Since, by assumption, $\mathcal{C}_Q = \mathcal{C}_Q^{\circ\circ}$ for all $Q \in \mathcal{Q}$, we obtain $j_Q(X) \in \mathcal{C}_Q$ for all $Q \in \mathcal{Q}$. As $\mathcal{Q}$ is a reduction set for $\mathcal{C}$, we conclude that $X \in \mathcal{C}$. □

Clearly, the assumption of Theorem 3.7 that $\mathcal{C}_Q = \mathcal{C}_Q^{\circ\circ}$ holds for all $Q \in \mathcal{Q}$ is rather abstract and does not provide a good bipolar theorem at first sight. However, we will use Theorems 3.1 and 3.3 to conclude that under some conditions on $\mathcal{C} \subset L^0_{c+}$, each $\mathcal{C}_Q \subset L^0_{Q+}$ admits a bipolar representation $\mathcal{C}_Q = \mathcal{C}_Q^{\circ\circ}$. Then Theorem 3.7 allows to lift this bipolar representation to L^0_{c+}. The required conditions on $\mathcal{C}$ will, of course, comprise convexity and solidness with respect to the $\mathcal{P}$-quasi-sure order (Corollary 3.6), and we also need to discuss reasonable closure properties (again Corollary 3.6). The latter is the purpose of the next section.

4 Concepts of Closedness under Uncertainty

Recall the discussion from Section 3.2. If we want to apply Theorem 3.1 or 3.3, we need to ensure that every $j_Q(\mathcal{C})$ is Q-closed. A straightforward way of achieving this is to assume that $\mathcal{C}$ is Q-closed for each $Q \in \mathcal{Q}$. Yet, a still sufficient and indeed also necessary property is the following weaker requirement:

Definition 4.1. Let $\mathcal{C} \subset L^0_c$ and $Q \in \mathfrak{P}_c(\Omega)$. $\mathcal{C}$ is called locally Q-closed if for each sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $X \in L^0_c$ such that $X_n \xrightarrow{Q} X$ there exists $Y \in \mathcal{C}$ such that $j_Q(X) = j_Q(Y)$.

Lemma 4.2. *Let $\mathcal{C} \subset L^0_c$ and $Q \in \mathfrak{P}_c(\Omega)$. $\mathcal{C}$ is locally Q-closed if and only if $j_Q(\mathcal{C})$ is Q-closed.*

Proof. We may assume that $\mathcal{C} \neq \emptyset$. Suppose that $\mathcal{C}$ is locally Q-closed. Let $(X_n^Q)_{n\in\mathbb{N}} \subset j_Q(\mathcal{C})$ and $X^Q \in L^0_Q$ such that $X_n^Q \xrightarrow{Q} X^Q$. Pick $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $j_Q(X_n) = X_n^Q$ and $X \in L^0_c$ such that $j_Q(X) = X^Q$. It follows that $X_n \xrightarrow{Q} X$. As $\mathcal{C}$ is locally Q-closed, there exists $Y \in \mathcal{C}$ such that $j_Q(\mathcal{C}) \ni j_Q(Y) = j_Q(X) = X^Q$. Thus, $\mathcal{C}_Q$ is Q-closed.
Conversely, if $j_Q(\mathcal{C})$ is Q-closed and $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $X \in L^0_Q$ such that $X_n \xrightarrow{Q} X$, then $j_Q(X_n) \xrightarrow{Q} j_Q(X)$ in L^0_Q and thus $j_Q(X) \in j_Q(\mathcal{C})$. Now let $Y \in \mathcal{C}$ such that $j_Q(Y) = j_Q(X)$. $\square$

So far we have encountered two concepts of closedness based on a reduction set $\mathcal{Q}$ for $\mathcal{C}$: $\mathcal{Q}$-closedness appeared as a necessary condition in Corollary 3.6, whereas local Q-closedness for all $Q \in \mathcal{Q}$ enables a lifting of Theorems 3.1 and 3.3. Interestingly, both notions are equivalent for $\mathcal{P}$-sensitive and solid sets:

Proposition 4.3. *Suppose that $\mathcal{C} \subset L^0_c$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$. If $\mathcal{C}$ is locally Q-closed for all $Q \in \mathcal{Q}$, then $\mathcal{C}$ is $\mathcal{Q}$-closed. If additionally $\mathcal{C}$ is solid, then $\mathcal{C}$ is locally Q-closed for all $Q \in \mathcal{Q}$ if and only if $\mathcal{C}$ is $\mathcal{Q}$-closed.*

Proof. Assume that $\mathcal{C} \neq \emptyset$. Suppose $\mathcal{C}$ is locally Q-closed for each $Q \in \mathcal{Q}$. Let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $X \in L^0_c$ such that $X_n \xrightarrow{\mathcal{Q}} X$. By local Q-closedness, for each $Q \in \mathcal{Q}$, there exists $Y_Q \in \mathcal{C}$ such that $j_Q(X) = j_Q(Y_Q) \in j_Q(\mathcal{C})$. Since $\mathcal{Q}$ is a reduction set for $\mathcal{C}$ we obtain $X \in \mathcal{C}$. Hence, $\mathcal{C}$ is $\mathcal{Q}$-closed.
Now suppose that $\mathcal{C}$ is also solid and let $\mathcal{C}$ be $\mathcal{Q}$-closed. Fix $Q \in \mathcal{Q}$ and let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $X_n \xrightarrow{Q} X$ for some $X \in L^0_c$. Then there exists a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$ such that $X_{n_k} \longrightarrow X$ Q-a.s. For an arbitrary choice $g_{n_k} \in X_{n_k}$, $k \in \mathbb{N}$, and $g \in X$ set

$$\{g_{n_k} \to g\} := \{\omega \in \Omega \mid \lim_{k\to\infty} g_{n_k}(\omega) = g(\omega)\}.^1$$

Note that $Q(\{g_{n_k} \to g\}) = 1$ and

$$\forall \omega \in \Omega: \quad g_{n_k}(\omega)\chi_{\{g_{n_k}\to g\}}(\omega) \to g(\omega)\chi_{\{g_{n_k}\to g\}}(\omega).$$

The latter and the fact that every $\tilde{Q} \in \mathcal{Q}$ is consistent with the $\mathcal{P}$-q.s.-order implies

$$X_{n_k}\mathbf{1}_{\{g_{n_k}\to g\}} \xrightarrow{\tilde{Q}} X\mathbf{1}_{\{g_{n_k}\to g\}}$$

[1] At this point, we felt we better drop the convention of identifying equivalence classes of random variables with their representatives for a moment.

for all $\tilde{Q} \in \mathcal{Q}$. By solidness of $\mathcal{C}$ we have $X_{n_k}\mathbf{1}_{\{g_{n_k}\to g\}} \in \mathcal{C}$ for all $k \in \mathbb{N}$, and thus, by $\mathcal{Q}$-closedness, $X\mathbf{1}_{\{g_{n_k}\to g\}} \in \mathcal{C}$. Since $Q(\{g_{n_k} \to g\}) = 1$ we have $j_Q(X) = j_Q(X\mathbf{1}_{\{g_{n_k}\to g\}})$. Therefore, $\mathcal{C}$ is locally Q-closed. □

One of the more commonly used closedness concepts in robust frameworks is order closedness, see for instance Gao and Munari [23] or Liebrich et al. [30].

Definition 4.4. A net $(X_\alpha)_{\alpha\in I} \subset L^0_c$ is order convergent to $X \in L^0_c$, denoted $X_\alpha \xrightarrow[c]{o} X$, if there is another net $(Y_\alpha)_{\alpha\in I} \subset L^0_c$ with the same index set I which is decreasing ($\alpha, \beta \in I$ and $\alpha \le \beta$ imply $Y_\beta \preccurlyeq Y_\alpha$), satisfies $\inf_{\alpha\in I} Y_\alpha = 0$, and for all $\alpha \in I$ it holds that $|X_\alpha - X| \preccurlyeq Y_\alpha$. Here, as usual, $\inf_{\alpha\in I} Y_\alpha$ denotes the largest lower bound of the net $(Y_\alpha)_{\alpha\in I}$.

Note that if $\mathcal{P} = \{P\}$, then $c = P$, and hence order convergence on L^0_P with respect to the P-a.s. order is naturally denoted by $X_\alpha \xrightarrow[P]{o} X$.

Definition 4.5.

1. A set $\mathcal{C} \subset L^0_c$ is order closed if for any net $(X_\alpha)_{\alpha\in I} \subset \mathcal{C}$ and $X \in L^0_c$ such that $X_\alpha \xrightarrow[c]{o} X$ we have $X \in \mathcal{C}$.
2. A set $\mathcal{C} \subset L^0_c$ is sequentially order closed if for any sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $X \in L^0_c$ such that $X_n \xrightarrow[c]{o} X$ we have $X \in \mathcal{C}$.

In the dominated case, for $Q \in \mathfrak{P}(\Omega)$, we know by the super Dedekind completeness of L^0_Q (see Aliprantis and Burkinshaw [3, Definition 1.43]) that $\mathcal{C} \subset L^0_Q$ is order closed if and only if $\mathcal{C}$ is sequentially order closed, and for solid sets this is well-known to be equivalent to Q-closedness:

Lemma 4.6 (see, e.g., Liebrich et al. [30, Lemma 4.1])**.** *Let $Q \in \mathfrak{P}(\Omega)$ and $\mathcal{C} \subset L^0_Q$ be solid. Then the following are equivalent:*

1. *$\mathcal{C}$ is order closed (with respect to the Q-a.s. order).*
2. *$\mathcal{C}$ is sequentially order closed.*
3. *$\mathcal{C}$ is Q-closed.*

Possessing some appealing features, in robust frameworks, authors have tended to focus on order convergence as a generalisation of Q-closedness, see Gao and Munari [23] or Liebrich et al. [30]. However, it turns out that in the non-dominated case order closedness is generally not equivalent to sequential order closedness, see for instance Examples 4.13 and 5.18, and that, in fact, it is the latter notion which is closely related to the other natural robustifications of Q-closedness we have encountered so far, namely $\mathcal{Q}$-closedness or local Q-closedness for all $Q \in \mathcal{Q}$, see Theorem 4.9 below. Before we state Theorem 4.9 we need two auxiliary results:

Lemma 4.7. *Suppose that $\mathcal{C} \subset L^0_c$ is solid. Let $Q \in \mathfrak{P}_c(\Omega)$. Then $j_Q(\mathcal{C})$ is solid.*

Proof. Suppose that $\mathcal{C} \neq \emptyset$ is solid in L^0_c and that $X^Q \in j_Q(\mathcal{C})$ and $Y^Q \in L^0_Q$ satisfy $|Y^Q| \le_Q |X^Q|$. Pick $\tilde{X} \in \mathcal{C}$ such that $j_Q(\tilde{X}) = X^Q$. Further let $f \in \tilde{X}$ and $g \in Y^Q$ and set $X := [f\chi_{\{|f|\ge|g|\}}]_c$ and $Y := [g\chi_{\{|f|\ge|g|\}}]_c$. Note that $Q(|f| \ge |g|) = 1$ and therefore $j_Q(X) = X^Q$. We have $|Y| \preccurlyeq |X| \preccurlyeq |\tilde{X}|$, and thus $Y \in \mathcal{C}$. Since $j_Q(Y) = Y^Q$, we conclude that $Y^Q \in j_Q(\mathcal{C})$, so $j_Q(\mathcal{C})$ is indeed solid with respect to $\le_Q$. A similar argument applies in the case where $\mathcal{C}$ is solid in L^0_{c+}. □

Lemma 4.8. *Suppose that $\emptyset \neq \mathcal{C} \subset L_c^0$ is solid and sequentially order closed, and let $Q \in \mathfrak{P}_c(\Omega)$. Then $j_Q(\mathcal{C})$ is closed with respect to the Q-a.s. order in L_Q^0.*

Proof. As $j_Q(\mathcal{C})$ is solid according to Lemma 4.7, in order to show (sequential) order closedness of $j_Q(\mathcal{C})$ it suffices to consider non-negative increasing sequences $(X_n^Q)_{n\in\mathbb{N}} \subset j_Q(\mathcal{C})$ (that is $0 \leq_Q X_n^Q \leq_Q X_{n+1}^Q$ for all $n \in \mathbb{N}$) such that the supremum $X^Q \in L_Q^0$ of $(X_n^Q)_{n\in\mathbb{N}}$ exists and to show that $X^Q \in j_Q(\mathcal{C})$, see Aliprantis and Burkinshaw [3, Lemma 1.15]. Pick $(\tilde{X}_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $j_Q(\tilde{X}_n) = X_n^Q$ for all $n \in \mathbb{N}$. Let $g \in X^Q$ and $g_n \in \tilde{X}_n$ for all $n \in \mathbb{N}$. Consider the event

$$A := \{\sup_{n\in\mathbb{N}} g_n = g\} \cap \{g_1 \geq 0\} \cap \bigcap_{n\in\mathbb{N}} \{g_n \leq g_{n+1}\}.$$

Note that $Q(A) = 1$. Set $X_n := [g_n\chi_A]_c$ for all $n \in \mathbb{N}$ and $X := [g\chi_A]_c$. Since $X_n \preccurlyeq \tilde{X}_n$ we conclude by solidness of $\mathcal{C}$ that $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$. As for all $\omega \in \Omega$ we have

$$g_n(\omega)\chi_A(\omega) \leq g_{n+1}(\omega)\chi_A(\omega) \leq g(\omega)\chi_A(\omega),\ n \in \mathbb{N},$$

and $g(\omega)\chi_A(\omega) = \sup_{n\in\mathbb{N}} g_n(\omega)\chi_A(\omega)$, we infer that $X_n \preccurlyeq X_{n+1} \preccurlyeq X$, $n \in \mathbb{N}$, and $X = \sup_{n\in\mathbb{N}} X_n$ in $(L_c^0, \preccurlyeq)$. Consequently, $X_n \xrightarrow[c]{o} X$. Hence, by sequential order closedness of $\mathcal{C}$ we obtain $X \in \mathcal{C}$. Now $j_Q(X) = X^Q$ implies $X^Q \in j_Q(\mathcal{C})$. □

Theorem 4.9. *Suppose that $\mathcal{C} \subset L_c^0$ is solid and $\mathcal{P}$-sensitive. Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ be a reduction set for $\mathcal{C}$. Then the following are equivalent:*

1. *$\mathcal{C}$ is sequentially order closed.*
2. *$\mathcal{C}$ is $\mathcal{Q}$-closed.*
3. *$\mathcal{C}$ is locally Q-closed for each $Q \in \mathcal{Q}$.*
4. *$j_Q(\mathcal{C})$ is Q-closed in L_Q^0 for each $Q \in \mathcal{Q}$.*
5. *$j_Q(\mathcal{C})$ is order closed with respect to the Q-a.s. order on L_Q^0 for each $Q \in \mathcal{Q}$.*
6. *$j_Q(\mathcal{C})$ is sequentially order closed with respect to the Q-a.s. order on L_Q^0 for each $Q \in \mathcal{Q}$.*

For the proof of Theorem 4.9 we need another auxiliary lemma:

Lemma 4.10. *Let $(X_n)_{n\in\mathbb{N}} \subset L_c^0$ and $Q \in \mathfrak{P}_c(\Omega)$.*

1. *Suppose that the infimum (supremum) $X = \inf_{n\in\mathbb{N}} X_n$ ($X = \sup_{n\in\mathbb{N}} X_n$) of $(X_n)_{n\in\mathbb{N}}$ in the $\mathcal{P}$-q.s. order exists. Then $j_Q(X) = \inf_{n\in\mathbb{N}} j_Q(X_n)$ ($j_Q(X) = \sup_{n\in\mathbb{N}} j_Q(X_n)$) in L_Q^0, i.e., $j_Q(X)$ is the infimum (supremum) of $(j_Q(X_n))_{n\in\mathbb{N}}$ in the Q-a.s. order.*
2. *Let $Y \in L_c^0$ and suppose that $X_n \xrightarrow[c]{o} Y$ in L_c^0. Then $j_Q(X_n) \xrightarrow[Q]{o} j_Q(Y)$ in L_Q^0.*

Proof. 1. We only prove the case of the infimum. From $Q \ll \mathcal{P}$ it immediately follows that $j_Q(X)$ is a lower bound for $(j_Q(X_n))_{n\in\mathbb{N}}$. Consider another lower bound $Z^Q \in L_Q^0$ of $(j_Q(X_n))_{n\in\mathbb{N}}$. We have to show that $j_Q(X) \geq_Q Z^Q$. For any choice $f_n \in X_n$ and $g \in Z^Q$ we have that $Q(\{f_n \geq g\}) = 1$ and thus also the event

$$A := \bigcap_{n\in\mathbb{N}} \{f_n \geq g\}$$

satisfies $Q(A) = 1$. Let $Z := [g]_c \mathbf{1}_A + X\mathbf{1}_{A^c} \in L^0_c$. Then $Z \preccurlyeq X_n$, $n \in \mathbb{N}$, and hence $Z \preccurlyeq X$ which implies $j_Q(X) \geq_Q j_Q(Z) = Z^Q$.

2. By definition of order convergence, there exists a decreasing sequence $(Y_n)_{n\in\mathbb{N}} \subset L^0_{c+}$ such that $\inf_{n\in\mathbb{N}} Y_n = 0$ in L^0_c and for all $n \in \mathbb{N}$

$$|X_n - X| \preccurlyeq Y_n.$$

Define $X^Q := j_Q(X)$ and $X^Q_n := j_Q(X_n)$, $Y^Q_n := j_Q(Y_n)$, $n \in \mathbb{N}$. As $Q \ll \mathcal{P}$, we have for all $n \in \mathbb{N}$

$$|X^Q_n - X^Q| \leq_Q Y^Q_n \quad \text{and} \quad 0 \leq_Q Y^Q_{n+1} \leq_Q Y^Q_n.$$

According to 1. $\inf_{n\in\mathbb{N}} Y^Q_n = 0$ in L^0_Q. Hence, $X^Q_n \xrightarrow[Q]{o} X^Q$. ☐

Proof of Theorem 4.9. 2. ⇔ 3. ⇔ 4.: see Lemma 4.2 and Proposition 4.3.

4. ⇔ 5. ⇔ 6.: follow from Lemma 4.6.

1. ⇒ 6.: Lemma 4.8.

6. ⇒ 1.: Assume $\mathcal{C} \neq \emptyset$ and let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $X_n \xrightarrow[c]{o} X \in L^0_c$. According to Lemma 4.10, $j_Q(X_n) \xrightarrow[Q]{o} j_Q(X)$. As $j_Q(\mathcal{C})$ is closed in the Q-a.s. order for any $Q \in \mathcal{Q}$ we obtain $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$. Since $\mathcal{Q}$ is a reduction set for $\mathcal{C}$ we infer $X \in \mathcal{C}$. ☐

Interestingly, also in the robust case there are situations in which we may add order closedness to the list in Theorem 4.9. This is closely related to the existence of supports of probability measures as introduced in Section 2.2.

Lemma 4.11. *Let $Q \in \mathfrak{P}_c(\Omega)$.*

1. *Suppose that Q is supported. Let $\mathcal{C} \subset L^0_c$ and suppose that the infimum (supremum) $X := \inf \mathcal{C}$ ($X := \sup \mathcal{C}$) exists in L^0_c. Then $j_Q(X) = \inf j_Q(\mathcal{C})$ ($j_Q(X) = \sup j_Q(\mathcal{C})$) in L^0_Q. In particular, for any net $(X_\alpha)_{\alpha\in I} \subset L^0_c$ and $X \in L^0_c$ we have that $X_\alpha \xrightarrow[c]{o} X$ implies $j_Q(X_\alpha) \xrightarrow[Q]{o} j_Q(X)$.*
2. *Conversely, suppose that for any net $(X_\alpha)_{\alpha\in I} \subset L^0_c$ and $X \in L^0_c$ we have that $X_\alpha \xrightarrow[c]{o} X$ implies $j_Q(X_\alpha) \xrightarrow[Q]{o} j_Q(X)$, then Q is supported.*

Proof. 1. We only prove the case of the infimum. From $Q \ll \mathcal{P}$ it immediately follows that $j_Q(X)$ is a lower bound for $j_Q(\mathcal{C})$. Hence, we only have to show that any lower bound $Y^Q \in L^0_Q$ of $j_Q(\mathcal{C})$ in L^0_Q satisfies $j_Q(X) \geq_Q Y^Q$. Denote by $S(Q)$ a version of the Q-support. Similar to the proof of Lemma 4.10 we pick $f \in Y^Q$ and define $Y := [f]_c \mathbf{1}_{S(Q)} + X1_{S(Q)^c}$. We have that $Y \preccurlyeq Z$ for all $Z \in \mathcal{C}$. Indeed, let $Z \in \mathcal{C}$ and $g \in Z$ (and thus also $g \in j_Q(Z)$). Since $0 = Q(f > g) = Q(S(Q) \cap \{f > g\})$ we infer that $c(S(Q) \cap \{f > g\}) = 0$ (recall Definition 2.1).

Therefore $[f]_c\mathbf{1}_{S(Q)} \preccurlyeq Z\mathbf{1}_{S(Q)}$. X being a lower bound of $\mathcal{C}$ now yields $Y \preccurlyeq Z$. As $Z \in \mathcal{C}$ was arbitrary, and as X is the largest lower bound of $\mathcal{C}$, we conclude that $Y \preccurlyeq X$. This in turn implies that $j_Q(X) \geq_Q j_Q(Y) = Y^Q$ where we have used that $Q(S(Q)) = 1$ for the latter equality. The remaining part of the assertion now follows along similar lines as presented in the proof of Lemma 4.10.

2. Note that by the dominated convergence theorem, for any measure $P \in \mathfrak{P}(\Omega)$, the linear functional

$$l_P : L^\infty_P \ni X \mapsto E_P[X]$$

is always σ-order continuous and thus also order continuous, because L^∞_P is super Dedekind complete. Under the assumption stated in 2. we thus have that

$$L^\infty_c \ni X \mapsto E_Q[X],$$

which we may view as the composition $l_Q \circ j_Q$, is order continuous. Since the order continuous dual of L^∞_c may be identified with sca_c, see Liebrich et al. [30, Proposition B.3], we find that Q must be supported. □

Combining Theorem 4.9 with Lemma 4.11 we obtain:

Theorem 4.12. *Suppose that $\mathcal{C} \subset L^0_c$ is solid and $\mathcal{P}$-sensitive and let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega) \cap sca_c$ be a reduction set for $\mathcal{C}$. Then the following are equivalent:*

1. *$\mathcal{C}$ is order closed.*
2. *$\mathcal{C}$ is sequentially order closed.*
3. *$\mathcal{C}$ is $\mathcal{Q}$-closed.*
4. *$\mathcal{C}$ is locally Q-closed for each $Q \in \mathcal{Q}$.*
5. *$j_Q(\mathcal{C})$ is Q-closed in L^0_Q for each $Q \in \mathcal{Q}$.*
6. *$j_Q(\mathcal{C})$ is order closed with respect to the Q-a.s. order on L^0_Q for each $Q \in \mathcal{Q}$.*
7. *$j_Q(\mathcal{C})$ is sequentially order closed with respect to the Q-a.s. order on L^0_Q for each $Q \in \mathcal{Q}$.*

Proof. In the view of Theorem 4.9 and as obviously 1. $\Rightarrow$ 2., it suffices to prove that 6. $\Rightarrow$ 1. But if $\mathcal{C} \neq \emptyset$ and $(X_\alpha)_{\alpha\in I} \subset \mathcal{C}$ and $X \in L^0_c$ satisfy $X_\alpha \xrightarrow[c]{o} X$, then $j_Q(X_\alpha) \xrightarrow[Q]{o} j_Q(X)$ for all $Q \in \mathcal{Q}$ according to Lemma 4.11. Thus, 6. implies that $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$, and $\mathcal{Q}$ being a reduction set for $\mathcal{C}$ now yields $X \in \mathcal{C}$. □

Note that in Theorem 4.12 it is important that we consider a reduction set $\mathcal{Q}$ for $\mathcal{C}$ which is strictly smaller than $\mathfrak{P}_c(\Omega)$ if $ca_c \neq sca_c$. In fact, $ca_c \neq sca_c$ is often the case according to Liebrich et al. [30, Section 3.3], see also Example 2.5. In the sequel we will encounter more situations in which the existence of a suitable reduction set with further properties than $\mathfrak{P}_c(\Omega)$ is crucial.
The following example shows that the equivalence 1. $\Leftrightarrow$ 2. in Theorem 4.12 generally does not hold if the reduction set is not supported:

Example 4.13. Recall that ca_c and sca_c can be identified with the σ-order and the order continuous dual of L_c^∞, respectively, see, for instance, Liebrich et al. [30]. That means that

$$L_c^\infty \ni X \mapsto \int X d\mu$$

is σ-order continuous, i.e., for every sequence $(X_n)_{n\in\mathbb{N}} \subset L_c^\infty$ such that $X_n \xrightarrow[c]{o} X \in L_c^\infty$ we have $\int X_n d\mu \to \int X d\mu$, whenever $\mu \in ca_c$, and order continuous, i.e., for every net $(X_\alpha)_{\alpha\in I} \subset L_c^\infty$ such that $X_\alpha \xrightarrow[c]{o} X \in L_c^\infty$ we have $\int X_\alpha d\mu \to \int X d\mu$, whenever $\mu \in sca_c$. Let us assume that $sca_c \neq ca_c$ (see Example 2.5) and let $\mu \in ca_{c+} \setminus sca_{c+}$. Consider

$$\mathcal{C}_r := \{X \in L_{c+}^\infty \mid \int X d\mu \leq r\}$$

where $r > 0$. $\mathcal{C}_r$ is obviously convex and solid. Moreover, $\mathcal{C}_r$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} = \{Q\}$ where $Q := \mu(\Omega)^{-1}\mu \in \mathfrak{P}_c(\Omega)$ is not supported. Since $L_c^\infty \ni X \mapsto \int X d\mu$ is not order continuous, there exists a decreasing net $(X_\alpha)_{\alpha\in I} \subset L_{c+}^\infty$ with $\inf_{\alpha\in I} X_\alpha = 0$ such that $\inf_{\alpha\in I} \int X_\alpha d\mu =: b > 0$. Let $\beta \in I$. Then the net $Y_\alpha := X_\beta - X_\alpha$, $\alpha \geq \beta$, is increasing and satisfies $0 \preccurlyeq Y_\alpha$ and $Y_\alpha \xrightarrow[c]{o} X_\beta$. We have $(Y_\alpha)_{\alpha\geq\beta} \subset \mathcal{C}_r$ for $r = \int X_\beta d\mu - b$, but $X_\beta \notin \mathcal{C}_r$. Hence, $\mathcal{C}_r$ is sequentially order closed but not order closed.

5 $\mathcal{P}$-Sensitivity Reloaded

In this section we study necessary and sufficient conditions for $\mathcal{P}$-sensitivity of $\mathcal{C} \subset L_c^0$. We start with some rather evident structural properties.

5.1 $\mathcal{P}$-Sensitivity by Local Defining Conditions

Proposition 5.1. *Let $\emptyset \neq \mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ and fix a quantifier $\dagger \in \{\exists, \forall\}$. Suppose that*

$$\mathcal{C} = \bigcap_{Q\in\mathcal{Q}} \{X \in L_c^0 \mid \dagger\, H \in \mathcal{H}\colon Q(A_Q^H(X)) = 1\},$$

where $\mathcal{H}$ is a non-empty test set and for all $Q \in \mathcal{Q}$ the function $A_Q^H\colon L_c^0 \to \mathcal{F}$ satisfies

$$Q(A_Q^H(X) \triangle A_Q^H(Y)) = 0,$$

whenever $Q(X = Y) = 1$. Then, $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$.

Proof. Assume $\mathcal{C} \neq \emptyset$ and let $X \in L_c^0$ such that $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$. Fix $Q \in \mathcal{Q}$. Then there exists $Y \in \mathcal{C}$ such that $j_Q(X) = j_Q(Y)$, that is $Q(X = Y) = 1$. Hence, depending on the quantifier,

$$\dagger H \in \mathcal{H} \quad Q(A_Q^H(X)) = Q(A_Q^H(Y)) = 1.$$

As $Q \in \mathcal{Q}$ was arbitrary, $X \in \mathcal{C}$. ☐

Example 5.2. Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ be non-empty. To conclude $\mathcal{P}$-sensitivity of the following examples of sets $\mathcal{C}$ we apply Proposition 5.1 in conjunction with the facts that L^0_{c+} is $\mathcal{P}$-sensitive (Example 2.11) and that the intersection of $\mathcal{P}$-sensitive sets remains $\mathcal{P}$-sensitive (Lemma 2.10).

1. (local boundedness condition) Set $\mathcal{H} := \mathbb{N}$ and $A^n_Q(X) := \{\omega \in \Omega \mid f(\omega) \leq n\}$ for some $f \in X$ and $n \in \mathbb{N}$. Then

$$\mathcal{C} := \{X \in L^0_{c+} \mid \forall Q \in \mathcal{Q}\, \exists n \in \mathbb{N} \colon Q(X \leq n) = 1\}$$

is $\mathcal{P}$-sensitive and even convex and solid. However, $\mathcal{C}$ is not sequentially order closed, as we can easily see that $\mathcal{C}$ is not $\mathcal{Q}$-closed.

2. (uniform local boundedness condition) Let $Y_Q \in L^0_{c+}$ for each $Q \in \mathcal{Q}$. Set $\mathcal{H} := \{0\}$ and $A^0_Q(X) := \{\omega \in \Omega \mid f(\omega) \leq g(\omega)\}$ for some $f \in X$ and $g \in Y_Q$. Then

$$\mathcal{C} := \{X \in L^0_{c+} \mid \forall Q \in \mathcal{Q} \colon Q(X \leq Y_Q) = 1\}$$

is $\mathcal{P}$-sensitive, convex, and solid. Clearly, $\mathcal{C}$ is also $\mathcal{Q}$-closed and hence sequentially order closed.

3. (uniform martingale condition) Let $\mathcal{H} := \{(Y, \mathcal{G})\}$ for some sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ and some $Y \in L^0_{c+}$ which admits a $\mathcal{G}$-measurable representative $g \in Y$. The set

$$\mathcal{C} := \{X \in L^0_{c+} \mid \forall Q \in \mathcal{Q}\, \forall f \in E_Q[X \mid \mathcal{G}] \colon f = g \ Q\text{-a.s.}\}$$

is $\mathcal{P}$-sensitive. Here, $E_Q[X \mid \mathcal{G}] \in L^0_c(\Omega, \mathcal{G}, Q)$ denotes the equivalence class of conditional expectations under Q of (any representative of) X given $\mathcal{G}$. Indeed, let

$$A^{(Y,\mathcal{G})}_Q(X) := \{\omega \in \Omega \mid f(\omega) = g(\omega)\}$$

for some arbitrary choice $f \in E_Q[X \mid \mathcal{G}]$. Then

$$\mathcal{C} := \{X \in L^0_{c+} \mid \forall Q \in \mathcal{Q} \colon Q(A^{(Y,\mathcal{G})}_Q(X)) = 1\}$$

4. (uniform supermartingale condition) Again let $\mathcal{H} := \{(Y, \mathcal{G})\}$ for some sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ and some $Y \in L^0_{c+}$ which admits a $\mathcal{G}$-measurable representative $g \in Y$. The set

$$\mathcal{C} := \{X \in L^0_{c+} \mid \forall Q \in \mathcal{Q}\, \forall f \in E_Q[X \mid \mathcal{G}] \colon f \leq g \ Q\text{-a.s.}\}$$

is $\mathcal{P}$-sensitive ($A^{(Y,\mathcal{G})}_Q(X) := \{\omega \in \Omega \mid f(\omega) \leq g(\omega)\}$ for some arbitrary choice $f \in E_Q[X \mid \mathcal{G}]$). Moreover, $\mathcal{C}$ is solid, convex, $\mathcal{Q}$-closed. Hence, by Theorems 4.9 and 4.12, $\mathcal{C}$ is sequentially order closed and even order closed if $\mathcal{Q} \subset sca_c$.

5. Let $Y \in L^0_{c+}$. Then the set

$$\mathcal{C} := \{X \in L^0_{c+} \mid X \preccurlyeq Y\} = \{X \in L^0_{c+} \mid \forall P \in \mathcal{P} \colon P(X \leq Y) = 1\}$$

is convex, solid, and sequentially order closed. $\mathcal{C}$ is also $\mathcal{P}$-sensitive according Proposition 5.1. Indeed, set $\mathcal{H} := \{Y\}$ and $A^Y_P(X) := \{\omega \in \Omega \mid f(\omega) \leq g(\omega)\}$, $P \in \mathcal{P} = \mathcal{Q}$, $X \in L^0_c$, where $f \in X$ and $g \in Y$.

The following lemma is easily verified. We will apply it in our discussion of robust acceptability criteria in Section 7.5.

Lemma 5.3. *Let $\emptyset \neq \mathcal{Q} \subset \mathfrak{P}_c(\Omega)$, and for each $Q \in \mathcal{Q}$ fix a set $\mathcal{C}_Q \in L^0_Q$. Then*

$$\mathcal{C} = \bigcap_{Q\in\mathcal{Q}} j_Q^{-1}(\mathcal{C}_Q)$$

is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$.

Proof. Let $X \in L^0_c$ satisfy $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$. Then, for each $Q \in \mathcal{Q}$, there is $Y^Q \in \mathcal{C}$ such that $Q(Y^Q = X) = 1$. As $Y^Q \in \mathcal{C}$ we have that $j_Q(Y^Q) \in \mathcal{C}_Q$. $Q(Y^Q = X) = 1$ implies that $j_Q(Y^Q) = j_Q(X)$, and therefore $j_Q(X) \in \mathcal{C}_Q$. As $Q \in \mathcal{Q}$ was arbitrary, $\mathcal{P}$-sensitivity of $\mathcal{C}$ with reduction set $\mathcal{Q}$ follows. □

The next result is a slight modification of the observation already made in Corollary 3.6

Lemma 5.4. *Let $\mathcal{M} \subset ca_c \setminus \{0\}$ be non-empty. Further let*

$$\mathcal{C} := \{X \in L^0_c \mid \forall \mu \in \mathcal{M}\colon X \text{ is } \mu\text{-integrable and } \int X d\mu \le a_\mu\},$$

where $a_\mu \in \mathbb{R}$, $\mu \in \mathcal{M}$. Then $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} := \{|\mu|/|\mu|(\Omega) \mid \mu \in \mathcal{M}\}$.

Proof. Let $X \in L^0_c$ such that $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$, and let $\mu \in \mathcal{M}$. For $Q := \frac{|\mu|}{|\mu|(\Omega)} \in \mathcal{Q}$ pick $Y \in \mathcal{C}$ such that $j_Q(X) = j_Q(Y)$. As $Q(X = Y) = 1$, X is μ-integrable, and

$$\int X d\mu = \int Y d\mu \le a_\mu.$$

Since $\mu \in \mathcal{M}$ was arbitrary, we infer that $X \in \mathcal{C}$. □

5.2 $\mathcal{P}$-Sensitivity and Aggregation

In the following we relate $\mathcal{P}$-sensitivity to the concept of aggregation (cf. Hasegawa and Perlman [24], Torgersen [38]).

Definition 5.5. Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$.

1. A family $(X^Q)_{Q\in\mathcal{Q}} \subset L^0_c$ is $\mathcal{Q}$-coherent if there is $X^{\mathcal{Q}} \in L^0_c$ such that

$$\forall Q \in \mathcal{Q}: \quad Q(X^Q = X^{\mathcal{Q}}) = 1.$$

   The equivalence class $X^{\mathcal{Q}}$ is called a $\mathcal{Q}$-aggregator of $(X^Q)_{Q\in\mathcal{Q}}$.

2. A set $\mathcal{C} \subset L^0_c$ is called $\mathcal{Q}$-stable if for any $\mathcal{Q}$-coherent family $(X^Q)_{Q\in\mathcal{Q}} \subset \mathcal{C}$ the set $\mathcal{C}$ contains all $\mathcal{Q}$-aggregators of $(X^Q)_{Q\in\mathcal{Q}}$.

Example 5.6. Recall Example 2.5. For $Q = \delta_\omega$ let $X^Q = \mathbf{1}_{\{\omega\}}$, $\omega \in \Omega$. The family $(\mathbf{1}_{\{\omega\}})_{\omega\in\Omega}$ is $\mathcal{Q}$-coherent with $\mathcal{Q}$-aggregator 1. The set $\mathcal{C} = \{\mathbf{1}_{\{\omega\}} \mid \omega \in \Omega\}$ is not $\mathcal{Q}$-stable since $1 \notin \mathcal{C}$. However, the set $\mathcal{D} = \{X \in L^0_c \mid X \preccurlyeq 1\}$ is $\mathcal{Q}$-stable. Indeed, consider any $\mathcal{Q}$-coherent family $(X^Q)_{Q\in\mathcal{Q}} \subset \mathcal{D}$ and let $X^{\mathcal{Q}} \in L^0_c$ be a $\mathcal{Q}$-aggregator of $(X^Q)_{Q\in\mathcal{Q}}$ (for instance $(\mathbf{1}_{\{\omega\}})_{\omega\in\Omega}$ and 1). Then for each $Q \in \mathcal{Q}$, $Q(X^Q = X^{\mathcal{Q}}) = 1$. It follows that $Q(X^{\mathcal{Q}} \le 1) = Q(X^Q \le 1) = 1$ for every $Q \in \mathcal{Q}$, and thus $X^{\mathcal{Q}} \preccurlyeq 1$ (recall $\mathcal{Q} \approx \mathcal{P}$). Therefore, $X^{\mathcal{Q}} \in \mathcal{D}$, and $\mathcal{D}$ is $\mathcal{Q}$-stable.

Example 5.7. Similarly to the $\mathcal{Q}$-stability of $\mathcal{D}$ in Example 5.6, one verifies that the sets $\mathcal{C}^a_-$ and $\mathcal{C}^a_+$ introduced in Example 2.11 are $\mathcal{P}$-stable.

The $\mathcal{P}$-stable sets $\mathcal{C}^a_-$ and $\mathcal{C}^a_+$ from Example 5.7 are also $\mathcal{P}$-sensitive according to Example 2.11. This is no surprise as the following proposition shows.

Proposition 5.8. *Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$. Then, a non-empty set $\mathcal{C} \subset L^0_c$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$ if and only if $\mathcal{C}$ is $\mathcal{Q}$-stable.*

Proof. Let $\mathcal{C}$ be $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$. Suppose that $(X^Q)_{Q\in\mathcal{Q}} \subset \mathcal{C}$ is $\mathcal{Q}$-coherent and let $X^{\mathcal{Q}} \in L^0_c$ be a $\mathcal{Q}$-aggregator. Then $j_Q(X^{\mathcal{Q}}) = j_Q(X^Q) \in \mathcal{C}_Q$ for all $Q \in \mathcal{Q}$. Hence, as $\mathcal{Q}$ is a reduction set for $\mathcal{C}$, $X^{\mathcal{Q}} \in \mathcal{C}$. Thus $\mathcal{C}$ is $\mathcal{Q}$-stable.
Now suppose that $\mathcal{C}$ is $\mathcal{Q}$-stable. Let $X \in \bigcap_{Q\in\mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{C})$. Then there exist $(X^Q)_{Q\in\mathcal{Q}} \subset \mathcal{C}$ such that $j_Q(X^Q) = j_Q(X)$, that is $Q(X = X^Q) = 1$, for all $Q \in \mathcal{Q}$. Thus, X is a $\mathcal{Q}$-aggregator for $(X^Q)_{Q\in\mathcal{Q}} \subset \mathcal{C}$ and therefore $X \in \mathcal{C}$. Hence, $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$. □

Example 5.9 (Superhedging)**.** Suppose that the (multivariate) process S in continuous or discrete time describes the discounted price evolution of some financial assets. Let $\mathcal{H}$ be a set of investment strategies and denote the portfolio wealth at terminal time $T > 0$ of some $H \in \mathcal{H}$ as $(H \cdot S)_T$, which is a random variable. The latter will typically coincide with a stochastic integral at time T, and $(H \cdot S)_0 = 0$. The set of superhedgeable claims at cost less than 1 is given by

$$\mathcal{C} := \{X \in L^0_{c+} \mid \exists H \in \mathcal{H} \colon X \preccurlyeq 1 + (H \cdot S)_T\}. \tag{9}$$

It is well-known (see e.g. Bartl et al. [8], Kramkov and Schachermayer [27, 28]) that a bipolar representation of $\mathcal{C}$ is closely related to the set of martingale measures, i.e., probability measures under which the discounted price process S is a martingale, see also Section 7.4. Hence, we are interested in criteria which ensure that $\mathcal{C}$ is $\mathcal{P}$-sensitive. Indeed, according to Proposition 5.8, $\mathcal{C}$ is $\mathcal{P}$-sensitive if and only if $\mathcal{C}$ is $\mathcal{Q}$-stable for some $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$. This however requires some aggregation property of the portfolio wealths $(H \cdot S)_T$. For instance, suppose that $\mathcal{P}$ is of class (S) and that L^0_c is Dedekind complete. The latter assumptions are for instance satisfied in the volatility uncertainty framework discussed in Cohen [20] and Soner et al. [37] where Dedekind completeness of L^0_c is achieved by a suitable enlargement of the filtration (Soner et al. [37, Section 5] and Liebrich et al. [30, Example 5.1]). Let $\mathcal{Q}$ be a disjoint supported alternative to $\mathcal{P}$, see Lemma 2.4. Then any family $(X^Q)_{Q\in\mathcal{Q}} \subset \mathcal{C}$ is $\mathcal{Q}$-coherent, see Lemma 5.11 below. Let X be a $\mathcal{Q}$-aggregator of $(X^Q)_{Q\in\mathcal{Q}}$ and let $H^Q \in \mathcal{H}$ be such that $X^Q \preccurlyeq 1 + (H^Q \cdot S)_T$. Consider any $\mathcal{Q}$-aggregator Y of the terminal wealths $((H^Q \cdot S)_T)_{Q\in\mathcal{Q}}$, which exists by Lemma 5.11. Then

$$X \preccurlyeq 1 + Y.$$

A sufficient condition for $\mathcal{P}$-sensitivity is thus that any such $\mathcal{Q}$-aggregator of terminal wealths Y can be replicated, that is, there is $H \in \mathcal{H}$ such that $Y = (H \cdot S)_T$. Indeed, in the case of volatility uncertainty the latter problem is related to finding an aggregator H of the processes H^Q, $Q \in \mathcal{Q}$, in the sense of Soner et al. [37, Definition 3.1]. This is problem is easily solved if the disjoint supports of all probability measures $Q \in \mathcal{Q}$ were $\mathcal{F}_0$-measurable. Then one could simply paste the processes H^Q, $Q \in \mathcal{Q}$, along the supports. More generally, Soner et al. [37, Theorem 5.1 and Theorem 6.5] give sufficient conditions for the existence of such an aggregator.

Remark 5.10. Note that the set of superhedgeable claims $\mathcal{C}$ given in (9) cannot be handled with Proposition 5.1. The reason is that while

$$\mathcal{C} = \{X \in L^0_{c+} \mid \exists H \in \mathcal{H}\, \forall P \in \mathcal{P}\colon P(X \le 1 + (H \cdot S)_T) = 1\},$$

Proposition 5.1 only allows to conclude $\mathcal{P}$-sensitivity of, for instance,

$$\mathcal{D} = \{X \in L^0_{c+} \mid \forall P \in \mathcal{P}\, \exists H \in \mathcal{H}\colon P(X \le 1 + (H \cdot S)_T) = 1\}.$$

The crucial difference is the order of the quantifiers $\exists H \in \mathcal{H}\, \forall P \in \mathcal{P}$ versus $\forall P \in \mathcal{P}\, \exists H \in \mathcal{H}$ in the conditions defining $\mathcal{C}$ and $\mathcal{D}$, respectively. Indeed, a statement like $\forall P \in \mathcal{P}\, \exists H \in \mathcal{H} \ldots$ enables a local argument under each $P \in \mathcal{P}$, that is the principle of $\mathcal{P}$-sensitivity, whereas $\exists H \in \mathcal{H}\, \forall P \in \mathcal{P} \ldots$ imposes a uniform condition over all probability measures $P \in \mathcal{P}$.

Lemma 5.11. *Suppose that $\mathcal{P}$ is of class (S) and L^0_c is Dedekind complete. Let $\mathcal{Q}$ denote a disjoint supported alternative to $\mathcal{P}$ (Lemma 2.4). Then any choice $(X^Q)_{Q \in \mathcal{Q}} \subset L^0_{c+}$ is $\mathcal{Q}$-coherent. Moreover, any $\mathcal{Q}$-aggregator X of $(X^Q)_{Q \in \mathcal{Q}}$ satisfies $X\mathbf{1}_{S(Q)} = X^Q\mathbf{1}_{S(Q)}$ for all $Q \in \mathcal{Q}$.*

Proof. The last assertion follows from $X\mathbf{1}_{S(Q)} = X^Q\mathbf{1}_{S(Q)}$ if and only if $Q(X = X^Q) = 1$. For the first assertion let $(X^Q)_{Q \in \mathcal{Q}} \subset L^0_c$. For $n \in \mathbb{N}$ let $X^n \in L^0_{c+}$ denote the least upper bound of the bounded family $(X^Q \wedge n)\mathbf{1}_{S(Q)}$, $Q \in \mathcal{Q}$. It then follows that $Q(X^n = X^Q \wedge n) = 1$ for all $Q \in \mathcal{Q}$ (see Lemma 4.11) and thus

$$X^n\mathbf{1}_{S(Q)} = (X^Q \wedge n)\mathbf{1}_{S(Q)} \preccurlyeq X^Q\mathbf{1}_{S(Q)}.$$

Therefore, $X^n \preccurlyeq X^{n+1}$ for all $n \in \mathbb{N}$, and the $\mathcal{P}$-quasi sure limit[2] $X := \lim_{n\to\infty} X^n \in L^0_c$ exists and satisfies

$$X\mathbf{1}_{S(Q)} = X^Q\mathbf{1}_{S(Q)}.$$

Hence, X is a $\mathcal{Q}$-aggregator of $(X^Q)_{Q \in \mathcal{Q}}$. □

5.3 $\mathcal{P}$-sensitivity as a Consequence of Weak Closedness

Recall the following classical bipolar theorem for locally convex topologies.

Theorem 5.12 (see, e.g., Aliprantis and Border [2, Theorem 5.103])**.** *Let $\langle \mathcal{X}, \mathcal{Y} \rangle$ be a dual pair, see Aliprantis and Border [2, Definition 5.90], and let $\emptyset \neq \mathcal{C} \subset \mathcal{X}$. Define $\mathcal{C}^\circ := \{Y \in \mathcal{Y} \mid \forall X \in \mathcal{C}\colon\ \langle X, Y \rangle \le 1\}$ and $\mathcal{C}^{\circ\circ} := \{X \in \mathcal{X} \mid \forall Y \in \mathcal{C}^\circ\colon\ \langle X, Y \rangle \le 1\}$. $\mathcal{C} = \mathcal{C}^{\circ\circ}$ if and only if $\mathcal{C}$ is convex, $\sigma(\mathcal{X}, \mathcal{Y})$-closed, and $0 \in \mathcal{C}$.*

The following result shows that if $\mathcal{C} \subset \mathcal{X}$ is convex and $\sigma(\mathcal{X}, \mathcal{Y})$-closed with respect to some dual pair $\langle \mathcal{X}, \mathcal{Y} \rangle$, where $\mathcal{X} \subset L^0_c$ and $\mathcal{Y} \subset \mathrm{ca}_c$, then $\mathcal{C}$ is essentially $\mathcal{P}$-sensitive.

Theorem 5.13. *Let $\mathcal{X} \subset L^0_c$ and $\mathcal{Y} \subset \mathrm{ca}_c$ be subspaces such that $\langle \mathcal{X}, \mathcal{Y} \rangle$ is a dual pair, and $\mathcal{Y}$ satisfies $\mu \in \mathcal{Y} \Rightarrow |\mu| \in \mathcal{Y}$. Suppose that $\mathcal{C} \subset \mathcal{X}$ is non-empty, convex, and $\sigma(\mathcal{X}, \mathcal{Y})$-closed. Then there is $\mathcal{Q} \subset \mathfrak{P}_c(\Omega) \cap \mathcal{Y}$ such that*

$$\mathcal{C} = \bigcap_{Q \in \mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{C}) \cap \mathcal{X}. \tag{10}$$

[2] $(X_n) \subset L^0_c$ is said to converge to $X \in L^0_c$ $\mathcal{P}$-quasi surely if $P(X_n \to X) = 1$ for all $P \in \mathcal{P}$.

The property (10) may be viewed as $\mathcal{C}$ being $\mathcal{P}$*-sensitive in* $\mathcal{X}$ *with reduction set* $\mathcal{Q}$.

Proof. The convex indicator function $f\colon \mathcal{X} \to [0,\infty]$ defined as

$$f(X) := \delta(X \mid \mathcal{C}) = \begin{cases} 0, & X \in \mathcal{C}, \\ \infty, & X \notin \mathcal{C}, \end{cases}$$

is convex and $\sigma(\mathcal{X},\mathcal{Y})$-lower semi-continuous. By the Fenchel-Moreau theorem,

$$f(X) = f^{**}(X) = \sup_{\mu\in\mathcal{Y}} \int X d\mu - f^*(\mu)$$

where $f^* : \mathcal{Y} \to (-\infty, \infty]$ is given by

$$f^*(\mu) = \sup_{X\in\mathcal{X}} \int X d\mu - f(X).$$

We may thus represent $\mathcal{C}$ as

$$\mathcal{C} = \{X \in \mathcal{X} \mid f(X) = 0\} \;=\; \bigcap_{\mu\in \mathrm{dom} f^*\setminus\{0\}} \{X \in \mathcal{X} \mid \int X d\mu - f^*(\mu) \leq 0\},$$

where $\mathrm{dom} f^* := \{\mu \in \mathcal{Y} \mid f^*(\mu) < \infty\}$, and the last equality uses the fact that for $\mu = 0$

$$f^*(\mu) = -\inf_{Y\in\mathcal{X}} f(Y) = 0 = \int X d\mu$$

for all $X \in \mathcal{X}$. Let $\mathcal{Q} := \{\frac{|\mu|}{|\mu|(\Omega)} \mid \mu \in \mathrm{dom} f^* \setminus \{0\}\}$. Then $\mathcal{Q} \subset \mathcal{Y} \cap \mathfrak{P}_c(\Omega)$. Now (10) follows similar to the proof of Lemma 5.4. □

Corollary 5.14. *In the situation of Theorem 5.13 suppose that* $\mathcal{X}$ *is* $\mathcal{P}$*-sensitive with reduction set* $\mathcal{Y} \cap \mathfrak{P}_c(\Omega)$*, then* $\mathcal{C}$ *is* $\mathcal{P}$*-sensitive with reduction set* $\mathcal{Y} \cap \mathfrak{P}_c(\Omega)$.

Proof. This follows from (10) and Lemma 2.10. □

The next simple example shows that even in a dominated framework the $\mathcal{P}$-sensitive sets in L^0_c do not all coincide with weakly closed sets in some locally convex subspace $\mathcal{X}$ of L^0_c.

Example 5.15. Let $\mathcal{P} = \{P\}$ for a non-atomic probability measure $P \in \mathfrak{P}(\Omega)$. In this case, it is well-known that there is no subspace $\mathcal{Y} \subset ca_P \simeq L^1_P$ such that $\langle L^0_P, \mathcal{Y}\rangle$ is a dual pair. Indeed, for any $\mu \in ca_P \setminus \{0\}$ there is $X \in L^0_{P+}$ such that $\int X d\mu$ is not well-defined or infinite. However, $\mathcal{C} := L^0_{P+}$ is convex, solid, and trivially P-sensitive with reduction set $\{P\}$. Also $\mathcal{C}$ admits a bipolar representation with polar set $\mathcal{C}^\circ = \{\mu \in ca_{P+} \mid \forall X \in \mathcal{C}\colon \int X d\mu \leq 1\} = \{0\}$ and $\mathcal{C}^{\circ\circ} = \{X \in L^0_{c+} \mid 0 \leq 1\} = L^0_{c+} = \mathcal{C}$, see Section 6.

5.4 $\mathcal{P}$-Sensitivity as a Consequence of Class (S) and Order Closedness

As mentioned previously, a widely used closedness requirement in robust frameworks is order closedness, see Gao and Munari [23] and Liebrich et al. [30]. Supposing that $\mathcal{P}$ is of class (S), we will in the following show that order closedness combined with convexity and solidness already implies $\mathcal{P}$-sensitivity.

Lemma 5.16. *Suppose that $\mathcal{P}$ is of class (S) and let $\mathcal{Y} \subset sca_c$ be any linear space separating the points of L_c^∞.*[3] *Moreover, let $\mathcal{C} \subset L_c^0$ be convex, solid, and order closed. Then $\mathcal{C} \cap L_c^\infty$ is $\sigma(L_c^\infty, \mathcal{Y})$-closed.*

Proof. $\tau := |\sigma|(L_c^\infty, \mathcal{Y})$ is a locally convex-solid Hausdorff topology with the Lebesgue property[4] since sca_c may be identified with the order continuous dual of L_c^∞, see, e.g., Liebrich et al. [30]. Suppose $\mathcal{C} \neq \emptyset$. Consider the set $\mathcal{D} := \mathcal{C} \cap L_c^\infty$. $\mathcal{D}$ is non-empty (because for each $X \in \mathcal{C}$ and $k \in \mathbb{N}$, $-k \vee X \wedge k \in \mathcal{D}$ by solidness), convex, solid, and order closed. Using Aliprantis and Burkinshaw [3, Lemma 4.2 and Lemma 4.20], we infer that $\mathcal{D}$ is $|\sigma|(L_c^\infty, \mathcal{Y})$-closed. As $|\sigma|(L_c^\infty, \mathcal{Y})$ and $\sigma(L_c^\infty, \mathcal{Y})$ share the same closed convex sets (see Aliprantis and Border [2, Theorem 8.49 and Corollary 5.83]), $\mathcal{D}$ is $\sigma(L_c^\infty, \mathcal{Y})$-closed. □

Corollary 5.17. *Suppose that $\mathcal{P}$ is of class (S) and let $\mathcal{Y} \subset sca_c$ be a subspace separating the points of L_c^∞ such that $\mu \in \mathcal{Y} \Rightarrow |\mu| \in \mathcal{Y}$. Further assume that $\mathcal{C} \subset L_c^0$ is convex, solid, and order closed. Then $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathfrak{P}_c(\Omega) \cap \mathcal{Y}$.*

Note that, when $\mathcal{P}$ is of class (S), sca_c always separates the points of L_c^∞, see Liebrich et al. [30, Proposition B.5].

Proof. $\langle L_c^\infty, \mathcal{Y} \rangle$ is a dual pair. The previous lemma shows that $\mathcal{C} \cap L_c^\infty$ is $\sigma(L_c^\infty, \mathcal{Y})$-closed. According to Theorem 5.13, there is $\mathcal{Q} \subset \mathcal{Y} \cap \mathfrak{P}_c(\Omega)$ such that $\mathcal{C} \cap L_c^\infty$ satisfies (10) where $\mathcal{X} = L_c^\infty$. Let

$$X \in \bigcap_{Q \in \mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{C}).$$

For all $Q \in \mathcal{Q}$ there is $Y \in \mathcal{C}$ such that $j_Q(Y) = j_Q(X)$. As $\mathcal{C}$ is solid, for $n \in \mathbb{N}$, we have $-n \vee Y \wedge n \in \mathcal{C} \cap L_c^\infty$, which in particular implies $j_Q(-n \vee X \wedge n) = j_Q(-n \vee Y \wedge n) \in j_Q(\mathcal{C})$. As $Q \in \mathcal{Q}$ was arbitrary, and by (10), we have that $-n \vee X \wedge n \in \mathcal{C}$ for all $n \in \mathbb{N}$. Finally, order closedness of $\mathcal{C}$ implies $X \in \mathcal{C}$. □

The next example, which can originally be found in Maggis et al. [32], gives us an example of a convex, solid, and sequentially order closed set which is not $\mathcal{P}$-sensitive (and thus not order closed). Moreover, $\mathcal{P}$ will be of class (S) and L_c^0 will be Dedekind complete. However, the example is based on assuming the continuum hypothesis, i.e., there is no set $\mathfrak{X}$ whose cardinality satisfies $|\mathbb{N}| = \aleph_0 < |\mathfrak{X}| < 2^{\aleph_0} = |\mathbb{R}|$.

Example 5.18 (Maggis et al. [32, Example 3.7])**.** Let $(\Omega, \mathcal{F}) = ([0,1], \mathbb{P}([0,1]))$, where $\mathbb{P}([0,1])$ denotes the power set of $[0,1]$. Further, let $\mathcal{P} := \{\delta_\omega \mid \omega \in [0,1]\}$ be the set of all Dirac measures.

[3] that means for any $X, Y \in L_c^\infty$ such that $X \neq Y$ there is $\mu \in \mathcal{Y}$ such that $\int X d\mu \neq \int Y d\mu$.

[4] For the definition of absolute weak topologies $|\sigma|(\mathcal{X}, \mathcal{Y})$, locally convex-solid topologies, and the Lebesgue property, see Aliprantis and Burkinshaw [3].

Every probability measure in $\mathcal{P}$ is supported (Example 2.5), and L^0_c is easily seen to be Dedekind complete.
Consider the set

$$\mathcal{D} := \{\mathbf{1}_A \mid \emptyset \neq A \subset [0,1] \text{ is countable}\}$$

and let $\mathcal{C}$ be the solid hull of $\mathcal{D}$ in L^0_{c+}. $\mathcal{C}$ can be written as

$$\mathcal{C} = \{X \in L^0_{c+} \mid \exists Y \in \mathcal{D}\colon 0 \leq X \leq Y\},$$

which also shows convexity of $\mathcal{C}$. Note that every $X \in \mathcal{C}$ is countably supported. Now let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $X_n \xrightarrow[c]{o} X \in L^0_{c+}$. For each $X_n \in \mathcal{C}$ there exists a countable set $A_n \subset [0,1]$ such that $0 \leq X_n \leq \mathbf{1}_{A_n}$. Set $A := \bigcup_{n\in\mathbb{N}} A_n$. A is still countable and $0 \leq X_n \leq \mathbf{1}_A$ for all $n \in \mathbb{N}$. Hence, $0 \leq X \leq \mathbf{1}_A$ and therefore $X \in \mathcal{C}$. Thus, $\mathcal{C}$ is sequentially order closed. Next we show that $\mathcal{C}$ is not order closed. To this end, set $I := \{A \subset [0,1] \text{ finite}\}$. For $\alpha, \beta \in I$ we let $\alpha \leq \beta$ if $\alpha \subset \beta$ and set $X_\alpha = \mathbf{1}_\alpha$. Then $(X_\alpha)_{\alpha\in I} \subset \mathcal{C}$ converges in order to 1, but $1 \notin \mathcal{C}$. Hence, $\mathcal{C}$ is not order closed. Indeed, so far we have reproduced a well-known text book example of a set which is sequentially order closed, but not order closed, and we have not yet assumed the continuum hypothesis. In order to give an answer to the question whether $\mathcal{C}$ is $\mathcal{P}$-sensitive, we need more structure. From now on assume the continuum hypothesis. Banach and Kuratowski have shown that for any set Λ with the same cardinality as $\mathbb{R}$ there is no measure μ on $(\Lambda, \mathbb{P}(\Lambda))$ such that $\mu(\Lambda) = 1$ and $\mu(\{\omega\}) = 0$ for all $\omega \in \Lambda$, see for instance Dudley [22, Theorem C.1]. It follows that any probability measure μ on $(\Omega, \mathcal{F})$ must be a countable sum of weighted Dirac-measures, i.e., $\mu = \sum_{i=1}^\infty a_i \delta_{\omega_i}$, where $\sum_{i=1}^\infty a_i = 1$, $a_i \geq 0$, and $\omega_i \in \Omega$ for all $i \in \mathbb{N}$. Thus every probability measure has a countable support, and in particular $ca = ca_c = sca = sca_c$. Let $Q \in \mathfrak{P}_c(\Omega) = \mathfrak{P}(\Omega)$. Q has a countable support $S(Q)$, and therefore $\mathbf{1}_{S(Q)} \in \mathcal{C}$. Then $j_Q(1) = j_Q(\mathbf{1}_{S(Q)}) \in j_Q(\mathcal{C})$. As $Q \in \mathfrak{P}_c(\Omega)$ was arbitrary, we have

$$1 \in \bigcap_{Q\in\mathfrak{P}_c(\Omega)} j_Q^{-1} \circ j_Q(\mathcal{C}).$$

However, as before, $1 \notin \mathcal{C}$. Hence, $\mathcal{C}$ is not $\mathcal{P}$-sensitive.

Example 5.18 also implies that there is no proof of the statement that convexity, solidness, and sequential order closedness imply $\mathcal{P}$-sensitivity:

Corollary 5.19. *Let $\mathcal{C} \subset L^0_{c+}$ be convex, solid, and sequentially order closed. Without further assumptions, there exists no proof that the assumed properties of $\mathcal{C}$ imply $\mathcal{P}$-sensitivity.*

Proof. This follows from Example 5.18 and the fact that the continuum hypothesis is consistent with the standard mathematical axioms (ZFC). □

6 Bipolar Theorems on L^0_{c+}

Based on Theorem 3.7 we will now lift Theorem 3.3 to L^0_{c+}.

Theorem 6.1. *Suppose that $\mathcal{C} \subset L^0_{c+}$ and $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ are non-empty . Let*

$$\mathcal{C}^\circ_{\mathcal{Q}} := \{(Q, Z) \in \mathcal{Q} \times L^\infty_{c+} \mid \forall X \in \mathcal{C}\colon E_Q[ZX] \leq 1\}$$

and
$$\mathcal{C}_{\mathcal{Q}}^{\circ\circ} := \{X \in L^0_{c+} \mid \forall (Q,Z) \in \mathcal{C}_{\mathcal{Q}}^{\circ} \colon E_Q[ZX] \leq 1\}.$$
Then $\mathcal{C}_{\mathcal{Q}}^{\circ\circ}$ is convex, solid, sequentially order closed, and $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$, and $\mathcal{C} \subset \mathcal{C}_{\mathcal{Q}}^{\circ\circ}$. Moreover, $\mathcal{C}_{\mathcal{Q}}^{\circ\circ}$ is the smallest such set in the sense that any set $\mathcal{D} \subset L^0_{c+}$ containing $\mathcal{C}$, which is also convex, solid, sequentially order closed, and $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$, satisfies $\mathcal{C}_{\mathcal{Q}}^{\circ\circ} \subset \mathcal{D}$. In particular, $\mathcal{C} = \mathcal{C}_{\mathcal{Q}}^{\circ\circ}$ if and only if $\mathcal{C}$ is convex, solid, sequentially order closed, and $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$.

Proof. Clearly, $\mathcal{C} \subset \mathcal{C}_{\mathcal{Q}}^{\circ\circ}$, and $\mathcal{P}$-sensitivity with reduction set $\mathcal{Q}$, convexity, solidness, and sequentially order closedness of $\mathcal{C}_{\mathcal{Q}}^{\circ\circ}$ follow from Corollary 3.6 and Theorem 4.9.

Now suppose that $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$, convex, solid, and sequentially order closed. Consider any $Q \in \mathcal{Q}$. $j_Q(\mathcal{C})$ is clearly convex in L^0_{Q+} and also solid by Lemma 4.7. Moreover, by Theorem 4.9, $j_Q(\mathcal{C})$ is Q-closed. Hence, according to Theorem 3.3, the requirement of Theorem 3.7 is satisfied, and we have

$$\mathcal{C} = \{X \in L^0_{c+} \mid \forall Q \in \mathcal{Q} \forall Z_Q \in j_Q(\mathcal{C})^{\circ} \colon E_Q[Z_Q j_Q(X)] \leq 1\},$$

where $j_Q(\mathcal{C})^{\circ} \subset L^{\infty}_{Q+}$ is the polar of $j_Q(\mathcal{C})$ given in Theorem 3.3. Fix $Q \in \mathcal{Q}$ and $Z_Q \in j_Q(\mathcal{C})^{\circ}$. Then for any $Z \in j_Q^{-1}(Z_Q) \cap L^{\infty}_{c+}$ and all $X \in L^0_{c+}$ we have $E_Q[Z_Q j_Q(X)] = E_Q[ZX]$. In particular, $E_Q[ZX] = E_Q[Z_Q j_Q(X)] \leq 1$ for all $X \in \mathcal{C}$ because $j_Q(X) \in j_Q(\mathcal{C})$ and $Z_Q \in j_Q(\mathcal{C})^{\circ}$. Hence, $(Q,Z) \in \mathcal{C}_{\mathcal{Q}}^{\circ}$. Therefore, if $X \in L^0_{c+}$ satisfies

$$\forall (Q,Z) \in \mathcal{C}_{\mathcal{Q}}^{\circ} \colon E_Q[ZX] \leq 1,$$

then X satisfies
$$\forall Z_Q \in j_Q(\mathcal{C})^{\circ} \colon E_Q[Z_Q j_Q(X)] \leq 1,$$
and thus $\mathcal{C}_{\mathcal{Q}}^{\circ\circ} \subset \mathcal{C}$. Since always $\mathcal{C} \subset \mathcal{C}_{\mathcal{Q}}^{\circ\circ}$ we have $\mathcal{C} = \mathcal{C}_{\mathcal{Q}}^{\circ\circ}$.

Minimality of $\mathcal{C}_{\mathcal{Q}}^{\circ\circ}$ follows by standard arguments. Indeed, suppose that $\mathcal{D}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$, convex, solid, and sequentially order closed, and that $\mathcal{C} \subset \mathcal{D}$. The latter implies $\mathcal{D}_{\mathcal{Q}}^{\circ} \subset \mathcal{C}_{\mathcal{Q}}^{\circ}$, and therefore $\mathcal{C}_{\mathcal{Q}}^{\circ\circ} \subset \mathcal{D}_{\mathcal{Q}}^{\circ\circ}$. We have shown above that $\mathcal{D}_{\mathcal{Q}}^{\circ\circ} = \mathcal{D}$, that is $\mathcal{C}_{\mathcal{Q}}^{\circ\circ} \subset \mathcal{D}$. ☐

For the generic case $\mathcal{Q} = \mathfrak{P}_c(\Omega)$ we write

$$\mathcal{C}^{\circ} := \mathcal{C}^{\circ}_{\mathfrak{P}_c(\Omega)} \quad \text{and} \quad \mathcal{C}^{\circ\circ} := \mathcal{C}^{\circ\circ}_{\mathfrak{P}_c(\Omega)}. \tag{11}$$

Note that $\mathcal{C}_{\mathcal{Q}}^{\circ} \subset \mathcal{C}^{\circ}$ for any non-empty $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ and thus $\mathcal{C}^{\circ\circ} \subset \mathcal{C}_{\mathcal{Q}}^{\circ\circ}$.

Corollary 6.2. *Suppose that $\mathcal{C} \subset L^0_{c+}$ is non-empty. $\mathcal{C}^{\circ\circ}$ is the smallest convex, solid, sequentially order closed, $\mathcal{P}$-sensitive subset of L^0_{c+} containing $\mathcal{C}$. $\mathcal{C} = \mathcal{C}^{\circ\circ}$ if and only if $\mathcal{C}$ is convex, solid, sequentially order closed, and $\mathcal{P}$-sensitive. If, moreover, $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ is a reduction set for $\mathcal{C}$, then $\mathcal{C} = \mathcal{C}^{\circ\circ} = \mathcal{C}_{\mathcal{Q}}^{\circ\circ}$.*

Proof. This follows from Theorem 6.1 since $\mathfrak{P}_c(\Omega)$ is a (in fact the maximal) reduction set for any $\mathcal{P}$-sensitive set. ☐

Corollary 6.3. *Let $\mathcal{C} \subset L^0_{c+}$ be non-empty. If the non-empty set $\mathcal{Q} \subset sca_c$ has disjoint supports, then*

$$\mathcal{C}^{\circ\circ}_{\mathcal{Q}} = \mathcal{C}^{**}_{\mathcal{Q}} := \{X \in L^0_{c+} \mid \forall Z \in \mathcal{C}^*_{\mathcal{Q}} \colon \sup_{Q \in \mathcal{Q}} E_Q[ZX] \leq 1\}$$

where

$$\mathcal{C}^*_{\mathcal{Q}} := \{Z \in L^\infty_{c+} \mid \forall X \in \mathcal{C} : \sup_{Q \in \mathcal{Q}} E_Q[ZX] \leq 1\}.$$

Proof. As

$$\mathcal{C}^{**}_{\mathcal{Q}} = \{X \in L^0_{c+} \mid \forall Z \in \mathcal{C}^*_{\mathcal{Q}} \; \forall Q \in \mathcal{Q} \colon E_Q[ZX] \leq 1\},$$

Corollary 3.6 and Theorem 4.9 show that $\mathcal{C}^{**}_{\mathcal{Q}}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$, convex, solid, and sequentially order closed. Moreover, $\mathcal{C}^{**}_{\mathcal{Q}}$ contains $\mathcal{C}$. Therefore, by Theorem 6.1 $\mathcal{C}^{\circ\circ}_{\mathcal{Q}} \subset \mathcal{C}^{**}_{\mathcal{Q}}$. It remains to show that $\mathcal{C}^{**}_{\mathcal{Q}} \subset \mathcal{C}^{\circ\circ}_{\mathcal{Q}}$. To this end, let $X \in \mathcal{C}^{**}_{\mathcal{Q}}$. For any $(Q, Z) \in \mathcal{C}^\circ_{\mathcal{Q}}$ we have $Z\mathbf{1}_{S(Q)} \in \mathcal{C}^*_{\mathcal{Q}}$. Indeed, by disjointness of the supports, we obtain

$$\sup_{\tilde{Q} \in \mathcal{Q}} E_{\tilde{Q}}[Z\mathbf{1}_{S(Q)}Y] = E_Q[Z\mathbf{1}_{S(Q)}Y] = E_Q[ZY] \leq 1$$

for all $Y \in \mathcal{C}$. Hence, $Z\mathbf{1}_{S(Q)} \in \mathcal{C}^*_{\mathcal{Q}}$ and thus

$$E_Q[ZX] = \sup_{\tilde{Q} \in \mathcal{Q}} E_{\tilde{Q}}[Z\mathbf{1}_{S(Q)}X] \leq 1.$$

As $(Q, Z) \in \mathcal{C}^\circ_{\mathcal{Q}}$ was arbitrary, this implies $X \in \mathcal{C}^{\circ\circ}_{\mathcal{Q}}$. □

Analogously to the proof of Theorem 6.1 we can obtain a lifting of Theorem 3.1, or we simply conclude it from Theorem 6.1:

Theorem 6.4. *Suppose that $\mathcal{C} \subset L^0_{c+}$ and $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ are non-empty. Let*

$$\mathcal{C}^\diamond_{\mathcal{Q}} := \{(Q, Z) \in \mathcal{Q} \times L^0_{c+} \mid \forall X \in \mathcal{C} \colon E_Q[ZX] \leq 1\}$$

and

$$\mathcal{C}^{\diamond\diamond}_{\mathcal{Q}} := \{X \in L^0_{c+} \mid \forall (Q, Z) \in \mathcal{C}^\diamond_{\mathcal{Q}} \colon E_Q[ZX] \leq 1\}.$$

Then $\mathcal{C}^{\diamond\diamond}_{\mathcal{Q}} = \mathcal{C}^{\circ\circ}_{\mathcal{Q}}$ where $\mathcal{C}^{\circ\circ}_{\mathcal{Q}}$ is given in Theorem 6.1.

Proof. We have $\mathcal{C} \subset \mathcal{C}^{\diamond\diamond}_{\mathcal{Q}} \subset \mathcal{C}^{\circ\circ}_{\mathcal{Q}}$ since $\mathcal{C}^\circ_{\mathcal{Q}} \subset \mathcal{C}^\diamond_{\mathcal{Q}}$. According to Corollary 3.6 and Theorem 4.9, $\mathcal{C}^{\diamond\diamond}_{\mathcal{Q}}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$, convex, solid, and sequentially order closed. Therefore, $\mathcal{C}^{\diamond\diamond}_{\mathcal{Q}} = \mathcal{C}^{\circ\circ}_{\mathcal{Q}}$ by Theorem 6.1. □

Note that the polar in Theorem 6.4 involves unbounded random variables. The advantage of the bipolar representation based on bounded random variables in Theorem 6.1, compared to Theorem 6.4, is that it implies a representation over finite measures:

Corollary 6.5. *Suppose that $\mathcal{C} \subset L^0_{c+}$ is non-empty. Let*

$$\mathcal{C}^{\circ\circ}_{ca} := \{X \in L^0_{c+} \mid \forall \mu \in \mathcal{C}^\circ_{ca} \colon \int X d\mu \leq 1\}$$

where

$$\mathcal{C}_{ca}^{\circ} := \{\mu \in ca_{c+} \mid \forall X \in \mathcal{C}\colon \int X d\mu \leq 1\}.$$

Then $\mathcal{C}_{ca}^{\circ\circ} = \mathcal{C}^{\circ\circ}$ *where* $\mathcal{C}^{\circ\circ}$ *is defined in* (11). *In particular,* $\mathcal{C} = \mathcal{C}_{ca}^{\circ\circ}$ *if and only if* $\mathcal{C}$ *is convex, solid, sequentially order closed, and* $\mathcal{P}$*-sensitive.*
Furthermore, if $\mathcal{C}$ *is convex, solid, sequentially order closed, and* $\mathcal{P}$*-sensitive with reduction set* $\mathcal{Q} \subset sca_c$*, then*

$$\mathcal{C} = \mathcal{C}^{\circ\circ} = \mathcal{C}_{sca}^{\circ\circ} := \{X \in L^0_{c+} \mid \forall \mu \in \mathcal{C}_{sca}^{\circ}\colon \int X d\mu \leq 1\},$$

where

$$\mathcal{C}_{sca}^{\circ} := \{\mu \in sca_{c+} \mid \forall X \in \mathcal{C}\colon \int X d\mu \leq 1\}.$$

Both $\mathcal{C}_{ca}^{\circ}$ *and* $\mathcal{C}_{sca}^{\circ}$ *are convex, solid, and* $\sigma(ca_c, L_c^{\infty})$*-closed or* $\sigma(sca_c, L_c^{\infty})$*-closed, respectively. Here solid means that* $\mu \in \mathcal{C}_{ca}^{\circ}$ *(resp.* $\mu \in \mathcal{C}_{sca}^{\circ}$*) and* $\nu \in ca_{c+}$ *(resp.* $\nu \in sca_{c+}$*) such that* $\nu(A) \leq \mu(A)$ *for all* $A \in \mathcal{F}$ *imply* $\nu \in \mathcal{C}_{ca}^{\circ}$ *(resp.* $\nu \in \mathcal{C}_{sca}^{\circ}$*)*

Proof. Note that any $(Q, Z) \in \mathfrak{P}_c(\Omega) \times L^{\infty}_{c+}$ can be identified with a measure $\mu \in ca_c$ given by $\mu(A) = E_Q[Z\mathbf{1}_A]$, $A \in \mathcal{F}$. Hence, if $(Q, Z) \in \mathcal{C}^{\circ}$ it follows that $\mu \in \mathcal{C}_{ca}^{\circ}$, and therefore $\mathcal{C}_{ca}^{\circ\circ} \subset \mathcal{C}^{\circ\circ}$. $\mathcal{C}_{ca}^{\circ\circ}$ contains $\mathcal{C}$, and $\mathcal{C}_{ca}^{\circ\circ}$ is convex and solid, and also sequentially order closed by the monotone convergence theorem. $\mathcal{P}$-sensitivity of $\mathcal{C}_{ca}^{\circ\circ}$ is shown in Proposition 3.4. Hence, $\mathcal{C}_{ca}^{\circ\circ} = \mathcal{C}^{\circ\circ}$ follows from Corollary 6.2. The latter then also implies that $\mathcal{C} = \mathcal{C}_{ca}^{\circ\circ}$ if and only if $\mathcal{C}$ is convex, solid, sequentially order closed, and $\mathcal{P}$-sensitive.

If $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} \subset sca_c$, then, similar to the previous considerations, we obtain $\mathcal{C} \subset \mathcal{C}_{sca}^{\circ\circ} \subset \mathcal{C}_{\mathcal{Q}}^{\circ\circ}$. If $\mathcal{C}$ is also convex, solid, and sequentially order closed, then $\mathcal{C} = \mathcal{C}^{\circ\circ} = \mathcal{C}_{\mathcal{Q}}^{\circ\circ}$ by Corollary 6.2, so we must have $\mathcal{C} = \mathcal{C}^{\circ\circ} = \mathcal{C}_{sca}^{\circ\circ}$.

Convexity of $\mathcal{C}_{ca}^{\circ}$ and $\mathcal{C}_{sca}^{\circ}$ is easily verified. Regarding solidness, note that if $\nu, \mu \in ca_{c+}$ are such that $\nu(A) \leq \mu(A)$ for all $A \in \mathcal{F}$, then $\int X d\nu \leq \int X d\mu$ for all $X \in L^0_{c+}$. We proceed to prove $\sigma(ca_c, L_c^{\infty})$-closedness of $\mathcal{C}_{ca}^{\circ}$: Consider a net $(\mu_\alpha)_{\alpha \in I} \subset \mathcal{C}_{ca}^{\circ}$ such that $\mu_\alpha \to \mu$ with respect to $\sigma(ca_c, L_c^{\infty})$. For all $X \in \mathcal{C}$ and all $n \in \mathbb{N}$ we have $\int (X \wedge n) d\mu_\alpha \leq \int X d\mu_\alpha \leq 1$ by monotonicity of the integral. Moreover,

$$\int (X \wedge n) d\mu = \lim_{\alpha} \int (X \wedge n) d\mu_\alpha \leq 1$$

since $(X \wedge n) \in L_c^{\infty}$. As $\mu \in ca_{c+}$, the monotone convergence theorem now implies $\int X d\mu \leq 1$. Hence, $\mu \in \mathcal{C}_{ca}^{\circ}$. The same argument shows $\sigma(sca_c, L_c^{\infty})$-closedness of $\mathcal{C}_{sca}^{\circ}$. □

Finally, we give the following standard result on $\mathcal{C}_{ca}^{\circ}$ which will be needed in Section 7.6.

Lemma 6.6. *Let* $\mathcal{M} \subset ca_{c+}$ *be non-empty and define*

$$\mathcal{C} := \{X \in L^0_{c+} \mid \forall \mu \in \mathcal{M}\colon \int X d\mu \leq 1\}.$$

Then $\mathcal{C}_{ca}^{\circ}$ *is the smallest solid, convex, and* $\sigma(ca_c, L_c^{\infty})$*-closed subset of* ca_{c+} *containing* $\mathcal{M}$*.*
The same assertion holds if ca is replaced by sca.

Proof. Clearly, $\mathcal{M} \subset \mathcal{C}^\circ_{ca}$, and solidness, convexity, and $\sigma(ca_c, L^\infty_c)$-closedness is shown in Corollary 6.5. Suppose there is another solid, convex, and $\sigma(ca_c, L^\infty_c)$-closed subset $\mathcal{D}$ of ca_{c+} such that $\mathcal{M} \subset \mathcal{D} \subsetneqq \mathcal{C}^\circ_{ca}$. Let $\mu \in \mathcal{C}^\circ_{ca} \setminus \mathcal{D}$. Then by an appropriate version of the Hahn-Banach separation theorem there is $X \in L^\infty_c$ such that

$$\beta := \sup_{\nu \in \mathcal{D}} \int X d\nu < \int X d\mu.$$

Note that

$$\beta = \sup_{\nu \in \mathcal{D}} \int X^+ d\nu$$

where $X^+ = \max\{X, 0\}$. Indeed, let $A := \{X \geq 0\}$. By solidness of $\mathcal{D}$, for all $\nu \in \mathcal{D}$ we also have $\nu_A \in \mathcal{D}$ where ν_A is given by $\nu_A(\cdot) = \nu(\cdot \cap A)$ ($\nu_A = 0$ in case $\nu(A) = 0$). Clearly,

$$\int X^+ d\nu = \int X d\nu_A \geq \int X d\nu.$$

Since $\int X d\mu \leq \int X^+ d\mu$, we may from now on assume that $X \in L^0_{c+}$. If $\beta = 0$, then $tX \in \mathcal{C}$ for all $t > 0$. However, there is $t > 0$ such that $\int tX d\mu > 1$, so $tX \notin \mathcal{C}^{\circ\circ}_{ca}$. But this contradicts $\mathcal{C} = \mathcal{C}^{\circ\circ}_{ca}$ (Corollary 6.5). Similarly, if $\beta > 0$, then $\frac{X}{\beta} \in \mathcal{C}$, but $\frac{X}{\beta} \notin \mathcal{C}^{\circ\circ}_{ca}$ which again contradicts $\mathcal{C} = \mathcal{C}^{\circ\circ}_{ca}$. Hence, μ cannot exist. □

7 Applications

In Sections 7.1–7.3 we show how the bipolar theorems of Gao and Munari [23], Liebrich et al. [30], and Bartl and Kupper [6] are special cases of our results in Section 6.

7.1 A Robust Bipolar Theorem given in Gao and Munari [23]

Our results imply the following bipolar theorem given in Gao and Munari [23]:

Corollary 7.1 (Gao and Munari [23, Theorem 14])**.** *Assume that $ca^*_c = L^\infty_c$, i.e., the norm dual space of ca_c can be identified with L^∞_c. Let $\mathcal{C} \subset L^0_{c+}$ be non-empty, convex, order closed, and solid in L^0_{c+}. Set*

$$ca^\infty_c := \operatorname{span}\{\mu_{P,Z} \mid P \in \mathcal{P},\ Z \in L^\infty_c\},$$

the linear space spanned by signed measures of type $\mu_{P,Z}(A) := E_P[Z\mathbf{1}_A]$, $A \in \mathcal{F}$. Then we have

$$\mathcal{C} = \mathcal{C}^{**} := \{X \in L^0_{c+} \mid \forall \mu \in \mathcal{C}^* \colon \int X d\mu \leq 1\},$$

where

$$\mathcal{C}^* := \{\mu \in ca^\infty_{c+} \mid \forall X \in \mathcal{C} \colon \int X d\mu \leq 1\}.$$

Proof. The condition $ca^*_c = L^\infty_c$ implies that $\mathcal{P}$ is of class (S) (see Liebrich et al. [30, Lemma 5.15]) and that $sca_c = ca_c$ (see Aliprantis and Burkinshaw [4, Theorem 4.60]). Therefore, in particular, $ca^\infty_c \subset sca_c$. As ca^∞_c is separating the points of L^∞_c, Corollary 5.17 implies that $\mathcal{C}$ is

$\mathcal{P}$-sensitive with reduction set $\mathcal{Q} := ca_{c+}^{\infty} \cap \mathfrak{P}_c(\Omega)$. Let (Q, Z) be an element of the polar $\mathcal{C}_{\mathcal{Q}}^{\circ}$ given in Theorem 6.1. The condition

$$\forall X \in \mathcal{C}\colon E_Q[XZ] \leq 1$$

is equivalent to

$$\forall X \in \mathcal{C}\colon \int X d\mu \leq 1,$$

where $\mu \in ca_c^{\infty}$ is given by $\mu(A) := E_Q[\mathbf{1}_A Z]$, $A \in \mathcal{F}$, and therefore $\mu \in \mathcal{C}^*$. Consequently, $\mathcal{C}^{**} \subset \mathcal{C}_{\mathcal{Q}}^{\circ\circ}$, and as $\mathcal{C} \subset \mathcal{C}^{**}$ and by Theorem 6.1,

$$\mathcal{C} \subset \mathcal{C}^{**} \subset \mathcal{C}_{\mathcal{Q}}^{\circ\circ} = \mathcal{C}.$$

□

7.2 Another Robust Bipolar Theorem provided in Liebrich et al. [30]

Our results also imply the following robust bipolar theorem which can be found in Liebrich et al. [30]:

Theorem 7.2 (Liebrich et al. [30, Theorem 4.2])**.** *Suppose that $\mathcal{P}$ is of class (S). Then for all convex and solid sets $\emptyset \neq \mathcal{C} \subset L_{c+}^0$, order closedness of $\mathcal{C}$ is equivalent to $\mathcal{C} = \mathcal{C}_{sca}^{\circ\circ}$ where $\mathcal{C}_{sca}^{\circ\circ}$ is given in Corollary 6.5.*

Proof. Clearly, $\mathcal{C}_{sca}^{\circ\circ}$ is order closed (and convex and solid). If the convex and solid set $\mathcal{C}$ is also order closed, then, according to Corollary 5.17, $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathfrak{P}_c(\Omega) \cap sca_c$. Thus, we can apply Corollary 6.5 to obtain the result. □

7.3 Yet another Robust Bipolar Theorem given in Bartl and Kupper [6]

Consider the case $\mathcal{P} = (\delta_\omega)_{\omega \in \Omega}$, so that $\preceq$ coincides with the pointwise order and $L_c^0 = \mathcal{L}^0$ and $ca_c = ca$. In Bartl and Kupper [6] the following pointwise bipolar theorem is proved:

Theorem 7.3. *(Bartl and Kupper [6, Theorem 1]) Let $\mathcal{C}$ be a non-empty solid regular subset of $\mathcal{L}_+^0$. Then $\mathcal{C} = \mathcal{C}_{ca}^{\circ\circ}$ (where $\mathcal{C}_{ca}^{\circ\circ}$ is given in Corollary 6.5) if and only if $\mathcal{C}$ is convex and closed under* lim inf*.*

In Bartl and Kupper [6], $\mathcal{C}$ is called regular if

$$\forall \mu \in ca_+\colon \sup_{h \in \mathcal{C} \cap U_b} \int h d\mu = \sup_{h \in \mathcal{C} \cap C_b} \int h d\mu \tag{12}$$

where C_b and U_b denote the spaces of bounded functions $f \in \mathcal{L}^0$ which are in addition continuous or upper semi-continuous, respectively. Involving continuity properties of course requires that Ω carries a topology, and in fact, Bartl and Kupper [6] assume Ω to be a σ-compact metric space and $\mathcal{F}$ to be the corresponding Borel σ-algebra.
$\mathcal{C}$ is said to be closed under lim inf whenever $\liminf_{n\to\infty} h_n \in \mathcal{C}$ for any sequence $(h_n)_{n\in\mathbb{N}} \subset \mathcal{C}$. One verifies that, for solid sets, being closed under lim inf is equivalent to sequential order closedness. In view of Theorem 6.1, we observe that the rather technical assumption of regularity (12) simply implies $\mathcal{P}$-sensitivity of $\mathcal{C}$. Indeed, recall that by Corollary 6.5 $\mathcal{C} = \mathcal{C}_{ca}^{\circ\circ}$ if and only if $\mathcal{C}$ is $\mathcal{P}$-sensitive,

convex, solid, and closed under lim inf (sequentially order closed). Since, given a set $\mathcal{C}$ which is convex, solid, and closed under lim inf, regularity of $\mathcal{C}$ implies $\mathcal{C} = \mathcal{C}_{ca}^{\circ\circ}$ according to Theorem 7.3, regularity must in this case imply $\mathcal{P}$-sensitivity.
In the following we illustrate that the necessary and sufficient requirements for $\mathcal{C} = \mathcal{C}_{ca}^{\circ\circ}$ we provide in Theorem 6.1, namely that $\mathcal{C}$ be $\mathcal{P}$-sensitive, convex, solid, and sequentially order closed, are weaker than replacing $\mathcal{P}$-sensitivity by regularity in the latter list of properties. To this end, we give an example of a set $\mathcal{C}$ which is $\mathcal{P}$-sensitive, convex, solid, and sequentially order closed, but not regular.

Example 7.4. Suppose that $\Omega = [0, 1]$. Let

$$\mathcal{C} := \{X \in \mathcal{L}^0_+ \mid X \preccurlyeq \mathbf{1}_{[\frac{1}{2},1]}\}.$$

Note that $\mathcal{C}$ is $\mathcal{P}$-sensitive (see Example 5.2 (5.)), convex, solid, and sequentially order closed. However, $\mathcal{C}$ is not regular, because $x \mapsto \mathbf{1}_{[\frac{1}{2},1]}(x)$ is upper semi-continuous and for $\mu = \delta_{\frac{1}{2}}$ we have

$$\sup_{X \in \mathcal{C} \cap U_b} \int X d\mu = 1 > 0 = \sup_{X \in \mathcal{C} \cap C_b} \int X d\mu.$$

7.4 Superhedging and Martingale Measures

Recall Example 5.9 and the set of superhedgeable claims at cost less than 1

$$\mathcal{C} = \{X \in L^0_{c+} \mid \exists H \in \mathcal{H} \colon X \preccurlyeq 1 + (H \cdot S)_T\}. \tag{13}$$

Clearly, $\mathcal{C}$ is non-empty, convex, and solid. Suppose that $\mathcal{C}$ is also $\mathcal{P}$-sensitive, see Example 5.9, and sequentially order closed. Then, according to Corollary 6.5, $\mathcal{C} = \mathcal{C}_{ca}^{\circ\circ}$. We recall that under some conditions on S and $\mathcal{H}$ the set $\mathcal{C}_{ca}^{\circ} \cap \mathfrak{P}_c(\Omega)$ is well-known to be closely related to the set of (local) martingale measures for S (see e.g. Kramkov and Schachermayer [27] for the dominated case): For illustration, suppose that S is a one-dimensional continuous process adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$, and let $\mathcal{P}_{b,sem} \subset \mathfrak{P}_c(\Omega)$ be the set of probability measures such that S is a bounded semi-martingale under each $Q \in \mathcal{P}_{b,sem}$. Let $\mathcal{H}$ be (a subset of) the set of all processes such that the stochastic integral $(H \cdot S) = {}^Q(H \cdot S)$ is defined for all $Q \in \mathcal{P}_{b,sem}$ in the usual semi-martingale sense. Further, let $\mathcal{P} \subset \mathcal{P}_{b,sem}$. The condition $X \preccurlyeq 1 + (H \cdot S)_T$ can then be interpreted as $P(X \leq 1 + {}^P(H \cdot S)_T) = 1$ for all $P \in \mathcal{P}$. One can go further and require that there is a universal process $(H \cdot S)$ which coincides Q-a.s. with the stochastic integrals ${}^Q(H \cdot S)$ for all $Q \in \mathcal{P}_{b,sem}$. This can be achieved for cadlag integrands $H \in \mathcal{H}$ according to Karandikar [26]. The probability model typically chosen here is the Wiener space with S being the canonical process, and $(\mathcal{F}_t)_{t\geq 0}$ being the canonical filtration (or completions of that, appropriate for various purposes), see, for instance, Soner et al. [37] or Bartl et al. [8]. Let us verify that any $Q \in \mathcal{C}_{ca}^{\circ} \cap \mathfrak{P}_c(\Omega)$ is a martingale measure for S: Note that for simple processes of type $H_a^{A,t}(s, \omega) := a1_A(\omega)1_{(t,T]}(s)$, where $A \in \mathcal{F}_t$, $a > 0$, and $t \in [0, T]$, the (universal) process

$$(H_a^{A,t} \cdot S)_u = a1_A(S_{T\wedge u} - S_{t\wedge u})$$

satisfies $(H_a^{A,t} \cdot S) = {}^P(H_a^{A,t} \cdot S)$ P-a.s. for all $P \in \mathcal{P}_{b,sem}$. By boundedness of S, we find $a > 0$ such that

$$-1 \preccurlyeq (H_a^{A,t} \cdot S)_T = a1_A(S_T - S_t) \preccurlyeq 1.$$

Hence, $1 + (H_a^{A,t} \cdot S)_T \in \mathcal{C}$ and $1 - (H_a^{A,t} \cdot S)_T \in \mathcal{C}$. It follows that for all $Q \in \mathcal{C}_{ca}^\circ \cap \mathfrak{P}_c(\Omega)$

$$E_Q[(H_a^{A,t} \cdot S)_T] = 0.$$

That implies $E_Q[1_A(S_T - S_t)] = 0$ and hence the martingale property of S under Q.
Conversely, let us show that under mild conditions $\mathcal{C}_{ca}^\circ \cap \mathfrak{P}_c(\Omega)$ includes all martingale measures which are dominated by some probability measure in $\mathcal{P}$. To this end, assume that the stochastic integral $(H \cdot S)$ is universally defined in the above sense, and that $\mathcal{H}$ is further restricted to those processes such that $(H \cdot S) = {}^P(H \cdot S)$ is P-a.s. bounded from below for any $P \in \mathcal{P}$, where the bound may depend on H and P. Consider a martingale measure $Q \in \mathfrak{P}_c(\Omega)$ for S such that $Q \ll P$ for some $P \in \mathcal{P}$. The stochastic integrals $(H \cdot S) = {}^Q(H \cdot S)$ are local martingales under Q and the required P-a.s. lower bound for $(H \cdot S)$ is also a Q-a.s. lower bound for $(H \cdot S)$. Thus, the processes $(H \cdot S)$, $H \in \mathcal{H}$, are in fact Q-supermartingales. Hence, for all $H \in \mathcal{H}$,

$$E_Q[(H \cdot S)_T] \leq (H \cdot S)_0 = 0.$$

Consequently, for any $X \in \mathcal{C}$, it follows that $E_Q[X] \leq 1$, so $Q \in \mathcal{C}_{ca}^\circ \cap \mathfrak{P}_c(\Omega)$.
A unifying study of robust fundamental theorems of asset pricing such as discussed in Acciaio et al. [1], Blanchard and Carassus [11], Bouchard and Nutz [12], Burzoni et al. [14], Burzoni and Maggis [16], Burzoni et al. [17], Chau [18], Obłój and Wiesel [35], and Riedel [36] as well as a dual theory for robust utility maximization, adopting the considerations made in Kramkov and Schachermayer [27], is work in progress.

7.5 $\mathcal{P}$-sensitivity and Acceptability Criteria for Random Costs/Losses

In this example we show that $\mathcal{P}$-sensitivity is a natural property of acceptance sets in risk assessment. To this end, identify L_{c+}^0 with random costs/losses. Consider a non-empty set $\mathcal{A} \subset L_{c+}^0$ of acceptable random costs. Assuming that $\mathcal{A}$ is solid means that if some costs are acceptable, then less costs are too. Convexity means that cost diversification is not penalized, and sequential order closedness implies that for an order convergent increasing sequence of acceptable losses, the limit remains acceptable. Finally, consider $\mathcal{P}$-sensitivity: Suppose that $\mathcal{A}$ is also $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$. Then Theorem 6.1 provides a dual characterisation of acceptability

$$X \in \mathcal{A} \quad \Leftrightarrow \quad X \in L_{c+}^0 \wedge \sup_{(Q,Z) \in \mathcal{A}_\mathcal{Q}^\circ} E_Q[ZX] \leq 1.$$

The interpretation of $\mathcal{A}_\mathcal{Q}^\circ$ is clear: Whether a loss X is acceptable depends on a number of probability models $\mathcal{Q}$ under which X is (stress) tested. Under each $Q \in \mathcal{Q}$, X has to meet the requirement that $E_Q[ZX] \leq 1$ for the model specific test functions $T_Q := \{Z \mid (Z, Q) \in \mathcal{A}_\mathcal{Q}^\circ\}$. Defining the local acceptance sets $\mathcal{A}_Q := \{X \in L_{Q+}^0 \mid \forall Z \in T_Q \colon E_Q[X j_Q(Z)] \leq 1\}$, $Q \in \mathcal{Q}$, we have that

$$\mathcal{A} = L_{c+}^0 \cap \bigcap_{Q \in \mathcal{Q}} j_Q^{-1}(\mathcal{A}_Q). \tag{14}$$

Conversely, a natural approach to robust risk assessment is to fix a set of probability measures $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ and to evaluate the risk of a loss X under each model $Q \in \mathcal{Q}$, for instance, by verifying whether $j_Q(X) \in \mathcal{A}_Q$ for a set of Q-acceptable losses $\mathcal{A}_Q \subset L^0_Q$. In that case the acceptable losses, these are those $X \in L^0_{c+}$ which are acceptable under each $Q \in \mathcal{Q}$, satisfy

$$X \in \mathcal{A} = L^0_{c+} \cap \bigcap_{Q \in \mathcal{Q}} j_Q^{-1}(\mathcal{A}_Q).$$

Hence, the acceptance set $\mathcal{A}$ is by construction of type (14), and therefore $\mathcal{P}$-sensitive according to Lemmas 5.3 and 2.10.
This illustrates that $\mathcal{P}$-sensitivity is indeed a quite natural requirement for robust risk assessment, since it corresponds to acceptability criteria of type (14), which are based on evaluating the risk under different possible probability models and then taking a worst-case approach.

7.6 A Mass Transport Type Duality

This application is inspired by Bartl and Kupper [6] and a straightforward generalisation of Bartl and Kupper [6, Section 4]. Consider two measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$. Let $\Omega := \Omega_1 \times \Omega_2$ and $\mathcal{F} := \mathcal{F}_1 \otimes \mathcal{F}_2$ denote the product space. Consider probability measures P_1 on $(\Omega_1, \mathcal{F}_1)$ and P_2 on $(\Omega_2, \mathcal{F}_2)$ and the set of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{F})$ consisting of all $P \in \mathfrak{P}(\Omega)$ with marginals $P(\cdot \times \Omega_2) = P_1$ and $P(\Omega_1 \times \cdot) = P_2$. Any $f \in \mathcal{L}^0_+(\Omega)$, which serves as a goal function, gives rise to the optimal mass transport (or Monge-Kantorovich) problem

$$\int f dP \to \max \quad \text{subject to} \quad P \in \mathcal{P}.$$

In fact, as we have been practising so far, we may identify f with the equivalence class $X = [f]_c$ generated by f in $L^0_c(\Omega)$ and write

$$\int X dP \to \max \quad \text{subject to} \quad P \in \mathcal{P} \tag{15}$$

where $c(A) = \sup_{P \in \mathcal{P}} P(A)$, $A \in \mathcal{F}$, is the upper probability corresponding to $\mathcal{P}$ on the product space $(\Omega, \mathcal{F})$, and $L^0_c(\Omega)$ is the space of equivalence classes of $\mathcal{P}$-q.s. equal random variables on $(\Omega, \mathcal{F})$.
A robustification of this problem is obtained by replacing the marginals P_1 and P_2 with sets of marginals $\mathcal{P}_1 \subset \mathfrak{P}(\Omega_1)$ and $\mathcal{P}_2 \subset \mathfrak{P}(\Omega_2)$. Now $\mathcal{P}$ is the set of all probability measures $\mathcal{P}$ on $(\Omega, \mathcal{F})$ such that $P(\cdot \times \Omega_2) \in \mathcal{P}_1$ and $P(\Omega_1 \times \cdot) \in \mathcal{P}_2$. We thus obtain the upper probabilities

$$c_1(A) = \sup_{P \in \mathcal{P}_1} P(A),\ A \in \mathcal{F}_1, \quad c_2(A) = \sup_{P \in \mathcal{P}_2} P(A),\ A \in \mathcal{F}_2, \quad \text{and} \quad c(A) = \sup_{P \in \mathcal{P}} P(A),\ A \in \mathcal{F},$$

and the corresponding spaces $L^0_{c_1}(\Omega_1)$, $L^0_{c_2}(\Omega_2)$, and $L^0_c(\Omega)$. For $X_1 \in L^0_{c_1}(\Omega_1)$ and $X_2 \in L^0_{c_2}(\Omega_2)$ we write $X_1 \oplus X_2$ for the $\mathcal{P}$-q.s. equivalence class in $L^0_c(\Omega)$ given by $f_1 \oplus f_2(\omega) := f_1(\omega_1) + f_2(\omega_2)$, $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, where $f_1 \in X_1$ and $f_2 \in X_2$.
Unfortunately, before we can state our duality result, we have to relax the mass transport problem as follows: Let $\mathcal{M}_i \subset ca_{c_i+}(\Omega_i)$ be a set such that $\mathcal{M}_i = \mathcal{C}^\circ_{i,ca}$ for some non-empty, convex, solid, $\mathcal{P}_i$-sensitive, and sequentially order closed sets $\mathcal{C}_i \subset L^0_{c_i+}(\Omega_i)$, i=1,2. This assumption is needed to apply Theorem 6.1 in the version of Corollary 6.5 in the proof of Theorem 7.5 below. Then, for $X \in L^0_{c+}$, we consider the problem

$$\int X d\mu \to \max \quad \text{subject to} \quad \mu \in \mathcal{M} \tag{16}$$

where $\mathcal{M} \subset ca_{c+}(\Omega)$ is the set of finite measures μ on $(\Omega, \mathcal{F})$ such that the marginals satisfy $\mu(\cdot \times \Omega_2) \in \mathcal{M}_1$ and $\mu(\Omega_1 \times \cdot) \in \mathcal{M}_2$. The dual problem to (16) is given by

$$\sup_{\mu_1 \in \mathcal{M}_1} \int X_1 d\mu_1 + \sup_{\mu_2 \in \mathcal{M}_2} \int X_2 d\mu_2 \to \min \quad \text{subject to} \quad (X_1, X_2) \in \Psi_X \tag{17}$$

where

$$\Psi_X := \{(X_1, X_2) \in L^0_{c_1+}(\Omega_1) \times L^0_{c_2+}(\Omega_2) \mid X \preccurlyeq_{\mathcal{P}} X_1 \oplus X_2\}.$$

Suppose that the problem (16) is non-trivial in the sense that $\sup_{\mu \in \mathcal{M}} \int X d\mu > 0$. Further suppose that (16) is well-posed in the sense that $\sup_{\mu \in \mathcal{M}} \int X d\mu < \infty$. Then, after a suitable normalisation, we may assume that $\sup_{\mu \in \mathcal{M}} \int X d\mu = 1$. Hence, X is an element of the following set

$$\mathcal{D} := \{Y \in L^0_{c+}(\Omega) \mid \sup_{\mu \in \mathcal{M}} \int Y d\mu \leq 1\}.$$

Consider the set

$$\mathcal{C} := \{Y \in L^0_{c+}(\Omega) \mid \exists (Y_1, Y_2) \in \Psi_Y \colon \sup_{\mu_1 \in \mathcal{M}_1} \int Y_1 d\mu_1 + \sup_{\mu_2 \in \mathcal{M}_2} \int Y_2 d\mu_2 \leq 1\}. \tag{18}$$

By monotonicity of the integral, we have that $\mathcal{C} \subset \mathcal{D}$. If we are able to show that $\mathcal{C} = \mathcal{D}$, then there is $(X_1, X_2) \in \Psi_X$ such that

$$1 \geq \sup_{\mu_1 \in \mathcal{M}_1} \int X_1 d\mu_1 + \sup_{\mu_2 \in \mathcal{M}_2} \int X_2 d\mu_2 \geq \sup_{\mu \in \mathcal{M}} \int X d\mu = 1.$$

In other words, the dual problem (17) admits a solution (X_1, X_2) and there is no duality gap, i.e.,

$$\min_{(X_1, X_2) \in \Psi_X} \sup_{\mu_1 \in \mathcal{M}_1} \int X_1 d\mu_1 + \sup_{\mu_2 \in \mathcal{M}_2} \int X_2 d\mu_2 = \sup_{\mu \in \mathcal{M}} \int X d\mu.$$

Theorem 7.5. *$\mathcal{C}^{\circ\circ}_{ca} = \mathcal{D}^{\circ\circ}_{ca} = \mathcal{D}$. $\mathcal{C} = \mathcal{D}$ if and only if $\mathcal{C}$ is $\mathcal{P}$-sensitive and sequentially order closed.*

Before we prove Theorem 7.5, consider the following auxiliary lemma.

Lemma 7.6. *For $\mu \in ca_{c+}(\Omega)$ we denote by $\mu_1(\cdot) = \mu(\cdot \times \Omega_2) \in ca_{c_1+}(\Omega_1)$ and $\mu_2(\cdot) = \mu(\Omega_1 \times \cdot) \in ca_{c_2+}(\Omega_2)$ the corresponding marginal distributions. Then*

$$\sup_{X \in \mathcal{C}} \int X d\mu = \max_{i \in \{1,2\}} \sup_{X_i \in \mathcal{C}_i} \int X_i d\mu_i. \tag{19}$$

Consequently,

$$\mathcal{C}^{\circ}_{ca} = \{\mu \in ca_{c+}(\Omega) \mid \mu_i \in \mathcal{M}_i, i \in \{1, 2\}\} = \mathcal{M}.$$

Proof. Consider $X \in \mathcal{C}$ and let $(X_1, X_2) \in \Psi_X$ such that

$$\sup_{\nu_1 \in \mathcal{M}_1} \int X_1 d\nu_1 + \sup_{\nu_2 \in \mathcal{M}_2} \int X_2 d\nu_2 \leq 1.$$

Suppose that $\sup_{\nu_i\in\mathcal{M}_i}\int X_i d\nu_i > 0$, $i = 1,2$, then

$$\begin{aligned}
\int X d\mu &\leq \int X_1 \oplus X_2 d\mu \\
&= \int X_1 d\mu_1 + \int X_2 d\mu_2 \\
&= \sup_{\nu_1\in\mathcal{M}_1}\int X_1 d\nu_1 \int \frac{X_1}{\sup\limits_{\nu_1\in\mathcal{M}_1}\int X_1 d\nu_1} d\mu_1 + \sup_{\nu_2\in\mathcal{M}_2}\int X_2 d\nu_2 \int \frac{X_2}{\sup\limits_{\nu_2\in\mathcal{M}_2}\int X_2 d\nu_2} d\mu_2 \\
&\leq \sup_{\nu_1\in\mathcal{M}_1}\int X_1 d\nu_1 \sup_{Y_1\in\mathcal{C}_1}\int Y_1 d\mu_1 + \sup_{\nu_2\in\mathcal{M}_2}\int X_2 d\nu_2 \sup_{Y_2\in\mathcal{C}_2}\int Y_2 d\mu_2 \\
&\leq \max_{i\in\{1,2\}}\sup_{Y_i\in\mathcal{C}_i}\int Y_i d\mu_i
\end{aligned}$$

where, for the second inequality, we used Corollary 6.5 to infer that

$$\frac{X_i}{\sup\limits_{\nu_i\in\mathcal{M}_i}\int X_i d\nu_i} \in \mathcal{C}_{i,ca}^{\circ\circ} = \mathcal{C}_i \ , i = 1,2.$$

Again by Corollary 6.5, if $\sup_{\nu_i\in\mathcal{M}_i}\int X_i d\nu_i = 0$, then $X_i \in \mathcal{C}_i$, and additionally, for all $t > 0$, $X_i/t \in \mathcal{C}_i$. Without loss of generality assume now that $\sup_{\nu_1\in\mathcal{M}_1}\int X_1 d\nu_1 = 0$. We then have that $\sup_{\nu_2\in\mathcal{M}_2}\int X_2 d\nu_2 \leq 1$ and therefore $X_2 \in \mathcal{C}_{2,ca}^{\circ\circ} = \mathcal{C}_2$ (see Corollary 6.5). Thus, for all $t > 0$,

$$\begin{aligned}
\int X d\mu &\leq \int X_1 d\mu_1 + \int X_2 d\mu_2 \\
&= t\int \frac{1}{t} X_1 d\mu_1 + \int X_2 d\mu_2 \\
&\leq t \sup_{Y_1\in\mathcal{C}_1}\int Y_1 d\mu_1 + \sup_{Y_2\in\mathcal{C}_2}\int Y_2 d\mu_2 \\
&\leq (1+t)\max_{i\in\{1,2\}}\sup_{Y_i\in\mathcal{C}_i}\int Y_i d\mu_i.
\end{aligned}$$

Letting $t \to 0$ shows that indeed $\int X d\mu \leq \max_{i\in\{1,2\}}\sup_{Y_i\in\mathcal{C}_i}\int Y_i d\mu_i$. Hence, we obtain

$$\sup_{X\in\mathcal{C}}\int X d\mu \leq \max_{i\in\{1,2\}}\sup_{X_i\in\mathcal{C}_i}\int X_i d\mu_i.$$

In order to show the reverse inequality, for $X_1 \in \mathcal{C}_1$ let $X := X_1 \oplus 0$ and for $X_2 \in \mathcal{C}_2$ let $\tilde{X} = 0 \oplus X_2$. Then, using the fact that $C_i = C_{i,ca}^{\circ\circ}$ by Corollaries 6.2 and 6.5 another time, we infer that $X, \tilde{X} \in \mathcal{C}$. Moreover,

$$\int X_1 d\mu_1 = \int X d\mu \leq \sup_{Y\in\mathcal{C}}\int Y d\mu \quad \text{and} \quad \int X_2 d\mu_2 = \int \tilde{X} d\mu \leq \sup_{Y\in\mathcal{C}}\int Y d\mu.$$

It follows that

$$\max_{i\in\{1,2\}}\sup_{X_i\in\mathcal{C}_i}\int X_i d\mu_i \leq \sup_{X\in\mathcal{C}}\int X d\mu,$$

and thus (19) is proved. Finally,

$$\begin{aligned}
\mathcal{C}_{ca}^{\circ} &= \{\mu \in ca_{c+}(\Omega) \mid \forall X \in \mathcal{C}\colon \int X d\mu \leq 1\} \\
&= \{\mu \in ca_{c+}(\Omega) \mid \sup_{X\in\mathcal{C}} \int X d\mu \leq 1\} \\
&= \{\mu \in ca_{c+}(\Omega) \mid \max_{i\in\{1,2\}} \sup_{X_i\in\mathcal{C}_i} \int X_i d\mu_i \leq 1\} \\
&= \{\mu \in ca_{c+}(\Omega) \mid \mu_i \in \mathcal{C}_{i,ca}^{\circ}, i \in \{1,2\}\} = \mathcal{M}.
\end{aligned}$$

□

Corollary 7.7. $\mathcal{C}_{ca}^{\circ} = \mathcal{M} = \mathcal{D}_{ca}^{\circ}$.

Proof. $\mathcal{C} \subset \mathcal{D}$, Lemma 6.6, and the definition of $\mathcal{D}$ imply that $\mathcal{M} \subset \mathcal{D}_{ca}^{\circ} \subset \mathcal{C}_{ca}^{\circ}$. According to Lemma 7.6, $\mathcal{M} = \mathcal{C}_{ca}^{\circ}$. □

Proof of Theorem 7.5. $\mathcal{D}$ is non-empty, convex, solid, $\mathcal{P}$-sensitive, and sequentially order closed by definition (see also Proposition 3.4). By Corollary 6.5, it thus holds that $\mathcal{D} = \mathcal{D}_{ca}^{\circ\circ}$. $\mathcal{C}_{ca}^{\circ\circ} = \mathcal{D}_{ca}^{\circ\circ} = \mathcal{D}$ then follows from Corollary 7.7.
Clearly, if $\mathcal{C} = \mathcal{D}$, then $\mathcal{C}$ inherits the properties $\mathcal{P}$-sensitivity and sequential order closedness from $\mathcal{D}$. Now suppose that $\mathcal{C}$ is $\mathcal{P}$-sensitive and sequentially order closed. It is clear that $\mathcal{C}$ is also non-empty, convex, and solid. Hence, by Corollary 6.5, we have $\mathcal{C} = \mathcal{C}_{ca}^{\circ\circ}$. □

The following examples illustrate Theorem 7.5. By definition of the set $\mathcal{C}$ it is clear that, in general, $\mathcal{P}$-sensitivity and sequential order closedness may be challenging to verify. In Example 7.8 finiteness of one of the spaces Ω_i will make this possible. In Example 7.9, we give an example where $\mathcal{C} = \mathcal{D}$ is easily directly verified, while showing that $\mathcal{C}$ in its representation (18) is sequentially order closed and $\mathcal{P}$-sensitive seems more challenging.
However, in any case Theorem 7.5 is interesting from a structural point of view. It shows that if things behave nicely—that is, the dual problem admits solutions for all $X \in L_{c+}^0$ such that $\sup_{\mu\in\mathcal{M}} \int X d\mu \leq 1$ and there is no duality gap—then this requires $\mathcal{P}$-sensitivity of $\mathcal{C}$ (and sequential order closedness), again highlighting $\mathcal{P}$-sensitivity as a structural condition which makes robust models manageable.

Example 7.8. Let

$$\mathcal{C}_1 := \{X \in L_{c_1+}^0(\Omega_1) \mid X\mathbf{1}_{A_1} = 0\},$$

where $A_1 \in \mathcal{F}_1$ satisfies $c_1(A_1) > 0$ and $c_1(A_1^c) > 0$, and let

$$\mathcal{C}_2 := \{X \in L_{c_2+}^0(\Omega_2) \mid X \preccurlyeq_{\mathcal{P}_2} 1\}.$$

Clearly, another representation of $\mathcal{C}_1$ is $\mathcal{C}_1 = \{X\mathbf{1}_{A_1^c} \mid X \in L_{c_1+}^0(\Omega_1)\}$. Since $\mathcal{C}_1$ is a cone, one verifies that

$$\mathcal{M}_1 = \mathcal{C}_{1,ca}^{\circ} = \{\mu_1 \in ca_{c_1+}(\Omega_1) \mid \mu_1(A_1^c) = 0\}.$$

Regarding $\mathcal{M}_2$, we have

$$\mathcal{M}_2 = \mathcal{C}_{2,ca}^{\circ} = \{\mu_2 \in ca_{c_2+}(\Omega_2) \mid \mu_2(\Omega_2) \leq 1\}.$$

Let us, from now on, assume that Ω_2 is finite and that $\mathcal{F}_2$ is the power set of Ω_2. Moreover, we assume that $\mathcal{P}$ is of class (S). We will show that $\mathcal{C}$ is order closed, and thus deduce $\mathcal{P}$-sensitivity by Corollary 5.17. To this end, let the net $(Y_\alpha)_{\alpha\in I}\subset\mathcal{C}$ satisfy $Y_\alpha \xrightarrow[c]{o} Y$ where $Y\in L^0_c$. Obviously, $0\preccurlyeq_{\mathcal{P}} Y$. Let $(Y_{1,\alpha},Y_{2,\alpha})\in\Psi_{Y_\alpha}$ such that

$$\sup_{\mu_1\in\mathcal{M}_1}\int Y_{1,\alpha}d\mu_1+\sup_{\mu_2\in\mathcal{M}_2}\int Y_{2,\alpha}d\mu_2\le 1. \tag{20}$$

Since $\mathcal{M}_1$ is a cone, the latter is satisfied only if $Y_{1,\alpha}\in\mathcal{C}_1$, and thus $\sup_{\mu_1\in\mathcal{M}_1}\int Y_{1,\alpha}d\mu_1=0$. This implies that $Y_{2,\alpha}\in\mathcal{C}_2$. Conversely, any pair $(Y_1,Y_2)\in\mathcal{C}_1\times\mathcal{C}_2$ satisfies (20). Define (identifying equivalence classes and their representatives)

$$Y_1(\cdot):=\max_{\omega_2\in\Omega_2}Y(\cdot,\omega_2)\mathbf{1}_{A_1^c}\quad\text{and}\quad Y_2:=\sup_{\alpha\in I}Y_{\alpha,2}$$

in $L^0_{c_1}(\Omega_1)$ and $L^0_{c_2}(\Omega_2)$, respectively. Then $(Y_1,Y_2)\in\mathcal{C}_1\times\mathcal{C}_2$, and thus (20) is satisfied. Moreover,

$$Y\mathbf{1}_{A_1^c\times\Omega_2}\preccurlyeq_{\mathcal{P}} Y_1\oplus 0\preccurlyeq_{\mathcal{P}}(Y_1\oplus Y_2)\mathbf{1}_{A_1^c\times\Omega_2},$$

and, for $\alpha\in I$,

$$Y_\alpha\mathbf{1}_{A_1\times\Omega_2}\preccurlyeq_{\mathcal{P}}(Y_{1,\alpha}\oplus Y_{2,\alpha})\mathbf{1}_{A_1\times\Omega_2}=(0\oplus Y_{2,\alpha})\mathbf{1}_{A_1\times\Omega_2}\preccurlyeq_{\mathcal{P}}(0\otimes Y_2)\mathbf{1}_{A_1\times\Omega_2}=(Y_1\otimes Y_2)\mathbf{1}_{A_1\times\Omega_2},$$

where we used that $Y_1\mathbf{1}_{A_1}=0$ and $Y_{1,\alpha}\mathbf{1}_{A_1}=0$ for all $\alpha\in I$. We conclude that $Y\preccurlyeq_{\mathcal{P}}Y_1\oplus Y_2$, and therefore $Y\in\mathcal{C}$. Hence, $\mathcal{C}$ is order closed and Theorem 7.5 yields $\mathcal{D}=\mathcal{C}$.

Example 7.9. Let

$$\mathcal{C}_i:=\{X\in L^0_{c_i+}(\Omega_i)\mid X\preccurlyeq_{\mathcal{P}_i}1\},\ i=1,2.$$

One verifies that

$$\mathcal{M}_i=\mathcal{C}^\circ_{i,ca}=\{\mu_i\in ca_{c_i+}(\Omega_i)\mid \mu_i(\Omega)\le 1\},\ i=1,2.$$

Hence,

$$\mathcal{M}=\{\mu\in ca_{c+}(\Omega)\mid\mu(\Omega)\le 1\},$$

and

$$\mathcal{D}=\{X\in L^0_{c+}(\Omega)\mid X\preccurlyeq_{\mathcal{P}}1\}.$$

Now consider $\mathcal{C}$. Let $Y\in L^0_{c+}$ such that $Y\preccurlyeq_{\mathcal{P}}1$. Then $Y_i\in L^0_{c_i+}$ given by $Y_i\equiv 1/2$ trivially satisfy $Y\preccurlyeq_{\mathcal{P}}Y_1\otimes Y_2$, that is $(Y_1,Y_2)\in\Psi_Y$, and

$$\sup_{\mu_1\in\mathcal{M}_1}\int Y_1d\mu_1+\sup_{\mu_2\in\mathcal{M}_2}\int Y_2d\mu_2=1.$$

Hence, $\mathcal{D}\subset\mathcal{C}$, and recalling that $\mathcal{C}\subset\mathcal{D}$, we have $\mathcal{D}=\mathcal{C}$. However, concluding this from Theorem 7.5, that is verifying sequential order closedness and $\mathcal{P}$-sensitivity of $\mathcal{C}$ in the representation (18) does not seem trivial.

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