Uniform Rotundity and Separation

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Abstract

Working constructively throughout, we prove that if K is an inhabited, complete, uniformly rotund subset of a normed space X, L is a located convex subset of X containing at least two distinct points, and $d \equiv \inf_{x \in K} \rho(x, L)$ exists, then there exists a strongly unique point $x_\infty \in K$ such that $\rho(x_\infty, L) = d$. To do so, we introduce the notion of sufficient convexity for real-valued functions on a metric space, and discuss the attainment of the infimum of such a function when that infimum exists. Our main theorem leads to new constructive versions of the separation theorem in a Hilbert space.
Keywords: sufficiently convex functions, uniform rotundity, separation theorem for convex sets

The framework of this paper is Bishop-style constructive mathematics (**BISH**), which, for all practical purposes, can be viewed as mathematics developed using intuitionistic logic and based on an appropriate foundation such as CZF [1], Martin-Löf type theory [9, 10], or constructive Morse set theory [6]. Thus all our proofs embody algorithms that can be extracted for computer implementation (see, for example, [8, 11, 12]).

We call a mapping f of a metric space X into $\mathbf{R}$ *sufficiently convex* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in X$ with $\rho(x, x') > \varepsilon$, there exists $z \in X$ such that $f(z) + \delta < \max\{f(x), f(x')\}$. Here ρ denotes the metric on X.

Proposition 1 *The following are equivalent conditions on a mapping f of a metric space X into $\mathbf{R}$, such that $\mu \equiv \inf f$ exists.*

(i) f *is sufficiently convex.*

(ii) for each $\varepsilon > 0$ there exists $\tilde{\delta} > 0$ such that if $x, x' \in X$, $f(x) < \mu + \tilde{\delta}$, and $f(x') < \mu + \tilde{\delta}$, then $\rho(x, x') < \varepsilon$.

Proof. First suppose that f is sufficiently convex. Given $\varepsilon > 0$, pick $\delta > 0$ such that if $x, x' \in X$ and $\rho(x, x') > \varepsilon/2$, then $f(z) + \delta < \max\{f(x), f(x')\}$ for some $z \in X$. Let $\tilde{\delta} := \delta$ and consider $x, x' \in X$ such that $f(x) < \mu + \delta$, and $f(x') < \mu + \delta$. If $\rho(x, x') > \varepsilon/2$, then there exists $z \in X$ such that

$$f(z) + \delta < \max\{f(x), f(x')\} < \mu + \delta$$

and therefore $f(z) < \mu$, which is absurd. Hence $\rho(x, x') \leq \varepsilon/2 < \varepsilon$.

Conversely, suppose that f satisfies condition (ii). Given $\varepsilon > 0$, choose $\tilde{\delta}$ as in that condition. If $x, x' \in X$ and $\rho(x, x') > \varepsilon$, then $\max\{f(x), f(x')\} \geq \mu + \tilde{\delta}$. By the definition of μ, there exists $z \in X$ such that

$$f(z) < \mu + \frac{\tilde{\delta}}{2}$$

and hence

$$f(z) + \frac{\tilde{\delta}}{2} < \mu + \tilde{\delta} \leq \max\{f(x), f(x')\}.$$

Therefore, we may set $\delta := \frac{\tilde{\delta}}{2}$. □

The following result is was communicated to us by Peter Aczel many years ago.

Proposition 2 *Let X be a complete metric space, and let f be a sequentially continuous, sufficiently convex mapping of X into $\mathbf{R}$ such that $\mu \equiv \inf f$ exists. Then there exists $\xi \in X$ such that $f(\xi) = \mu$. Moreover, if $x \in X$ and $x \neq \xi$, then $f(x) > \mu$.*

Proof. In view of Proposition 1, we can construct a strictly decreasing sequence $(\delta_n)_{n \geqslant 1}$ of positive numbers such that for each n, if $x, x' \in X$, $f(x) < \mu + \delta_n$, and $f(x') < \mu + \delta_n$, then $\rho(x, x') < 2^{-n}$. For each n, pick $x_n \in X$ such that $f(x_n) < \mu + \delta_n$. Then $\rho(x_m, x_n) < 2^{-n}$ for all $m \geqslant n$, so $(x_n)_{n \geqslant 1}$ is a Cauchy sequence in X. Since X is complete, $\xi \equiv \lim_{n \to \infty} x_n$ exists in X. By the sequential continuity of f, $\mu \leqslant f(\xi) \leqslant \mu$, so $f(\xi) = \mu$. Moreover, if $x \in X$ and $\rho(x, \xi) > 0$, then, with $\varepsilon := \frac{1}{2}\rho(x, \xi)$ and $\delta > 0$ as in the definition of 'sufficiently convex', there exists $z \in X$ such that

$$\mu < \mu + \delta \leq f(z) + \delta < \max\{f(\xi), f(x)\} = \max\{\mu, f(x)\} = f(x).$$

□

A subset L of a metric space is *located* if for all $x \in X$ the distance

$$\rho(x, L) := \inf\{\rho(x, y) \mid y \in L\}$$

exists.

Lemma 3 *Let L be an inhabited, located, convex subset of a normed space X. Then for all x, x' in X and $t \in [0, 1]$,*

$$\rho(tx + (1-t)x', L) \leq t\rho(x, L) + (1-t)\rho(x', L).$$

Proof. Given $x, x' \in X$, $t \in [0,1]$, and $\varepsilon > 0$, pick $y, y' \in L$ such that

$$\|x - y\| < \rho(x, L) + \varepsilon \text{ and } \|x' - y'\| < \rho(x', L) + \varepsilon.$$

Then

$$\begin{aligned}
\rho(tx + (1-t)x', L) &\le \|tx + (1-t)x' - ty - (1-ty')\| \\
&\le t\,\|x - y\| + (1-t)\,\|x' - y'\| \\
&\le t\rho(x, L) + (1-t)\rho(x', L) + t\varepsilon + (1-t)\varepsilon \\
&\le t\rho(x, L) + (1-t)\rho(x', L) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

A normed space X is *uniformly convex* if for each $\varepsilon > 0$ there exists δ with $0 < \delta < 1$ such that if x, y are elements of X with $\|x\| = 1 = \|y\|$ and $\|x - y\| \ge \varepsilon$, then $\left\|\frac{1}{2}(x+y)\right\| \le \delta$. Hilbert spaces, and L_p spaces with $p > 1$, are uniformly convex [4, page 322, Corollary (3.22)].

Lemma 4 *Let X be a uniformly convex normed space. Then for all $\tilde{\varepsilon} > 0$ and $M > 0$ there exists $\tilde{\delta} > 0$ such that if x, y are elements of X with $\|x\| = \|y\| \le M$ and $\|x - y\| \ge \tilde{\varepsilon}$, then $\|\frac{1}{2}(x+y)\| + \tilde{\delta} \le \|x\|$.*

Proof. Let $\tilde{\varepsilon} > 0$ and consider any $x, y \in X$ such that $\|x\| = \|y\| \le M$ and $\|x - y\| \ge \tilde{\varepsilon}$. As $\tilde{\varepsilon} \le \|x - y\| \le 2\|x\|$, we infer $\|x\| = \|y\| \ge \tilde{\varepsilon}/2 > 0$. Set $\varepsilon := \frac{\tilde{\varepsilon}}{M}$ and compute $\delta \in (0,1)$ as in the definition of uniform convexity. As $x/\|x\|$ and $y/\|y\|$ are unit vectors with

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \frac{1}{\|x\|}\|x - y\| \ge \frac{\tilde{\varepsilon}}{M} = \varepsilon,$$

we obtain

$$\frac{1}{\|x\|}\left\|\frac{1}{2}(x+y)\right\| \le \delta.$$

Hence, using that $\|x\| \ge \tilde{\varepsilon}/2$,

$$\left\|\frac{1}{2}(x+y)\right\| \le \delta\|x\| \le \|x\| - (1-\delta)\|x\| \le \|x\| - (1-\delta)\frac{\tilde{\varepsilon}}{2}.$$

Set $\tilde{\delta} := (1-\delta)\frac{\tilde{\varepsilon}}{2}$. □

Lemma 5 *Let X be a uniformly convex normed space, and let $K \subset X$ be an inhabited, convex, and norm bounded set. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in K$ with $\|x - x'\| \ge \varepsilon$ we have $\left\|\frac{1}{2}(x+x')\right\| + \delta \le \max\{\|x\|, \|x'\|\}$. In particular $f(x) = \|x\|$, $x \in K$, defines a sufficiently convex function.*

Proof. Let $\varepsilon > 0$ and let $M > 0$ be a norm bound for K. For $\tilde{\varepsilon} := \varepsilon/2$ and M compute $\tilde{\delta} > 0$ as in Lemma 4. Choose $\delta > 0$ with $\delta < \min\{\varepsilon/4, \tilde{\delta}/2\}$ and consider $x, x' \in K$ with $\|x - x'\| \geq \varepsilon$. Either $\big|\|x\| - \|x'\|\big| > \delta$ or $\big|\|x\| - \|x'\|\big| < 2\delta$. In the first case note that $\min\{\|x\|, \|x'\|\} < \max\{\|x\|, \|x'\|\} - \delta$ and thus

$$\left\|\frac{1}{2}(x + x')\right\| \leq \frac{1}{2}(\max\{\|x\|, \|x'\|\} + \min\{\|x\|, \|x'\|\}) < \max\{\|x\|, \|x'\|\} - \frac{\delta}{2}.$$

Now assume the second case. Then by the triangle inequality,

$$\varepsilon \leq \|x - x'\| \leq 2(\|x\| + \delta) \quad \text{and} \quad \varepsilon \leq \|x - x'\| \leq 2(\|x'\| + \delta)$$

implying that $\min\{\|x\|, \|x'\|\} > 0$. Consider $y := \frac{\|x\|}{\|x'\|} x'$, and note that

$$\|x' - y\| = \big|\|x'\| - \|x\|\big| < 2\delta, \quad \|y\| = \|x\| \leq M,$$

and

$$\|x - y\| \geq \|x - x'\| - \|x' - y\| > \varepsilon - 2\delta > \frac{\varepsilon}{2} = \tilde{\varepsilon}.$$

By choice of $\tilde{\delta}$ we have

$$\begin{aligned} \|x\| &\geq \frac{1}{2}\|x + y\| + \tilde{\delta} \;\geq\; \frac{1}{2}(\|x + x'\| - \|x' - y\|) + \tilde{\delta} \\ &> \frac{1}{2}\|x + x'\| - \delta + \tilde{\delta} \;>\; \frac{1}{2}\|x + x'\| + \delta. \end{aligned}$$

As $\|x\| \leq \max\{\|x\|, \|x'\|\}$, the lemma is proved. □

Proposition 6 *Let X be a uniformly convex normed space, and let $K \subset X$ be an inhabited complete convex set. Moreover, let $y \in X$ and assume that*

$$\mu := \inf\{\|y - x\| : x \in K\}$$

exists. Then there exists $x_0 \in K$ such that $\|y - x_0\| = \mu$. If $x' \in K$ such that $x' \neq x_0$, then $\|y - x'\| > \mu$.

Proof. As the set $K - \{y\}$ inherits all properties from K, we may assume that $y = 0$. Pick $z \in K$. Then

$$\mu = \inf\{\|x\| : x \in K, \|x\| \leq M\}$$

where $M > 0$ satisfies $M > \|z\|$. The set $\tilde{K} := \{x \in K : \|x\| \leq M\}$ is inhabited, convex, bounded, and complete. Therefore, $\tilde{K} \ni x \mapsto \|x\|$ is sufficiently convex by Lemma 5 and has a unique minimum point $x_0 \in \tilde{K}$ by Proposition 2. □

An immediate consequence of Proposition 6 is the proof of [4, Problem 11, p. 391], namely:

Corollary 7 *Let* B *be a uniformly convex Banach space, and let* $\mathsf{K} \subset \mathsf{B}$ *be a closed located convex set. Then each* $y \in \mathsf{B}$ *has a unique closest point* $x_0 \in \mathsf{K}$*, i.e.* $\|y - x_0\| = \rho(y, \mathsf{K})$*, and if* $x' \in \mathsf{K}$ *is such that* $x' \neq x_0$*, then* $\|y - x'\| > \rho(y, \mathsf{K})$*.*

A subset C of a normed space X is *uniformly rotund* if it is convex and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, x' \in \mathsf{C}$ and $\|x - x'\| \geq \varepsilon$, then $\frac{1}{2}(x + x') + z \in \mathsf{C}$ for all $z \in \mathsf{X}$ with $\|z\| \leqslant \delta$.

Proposition 8 *A normed linear space* X *is uniformly convex if and only if its closed unit ball* B *is uniformly rotund.*

Proof. Suppose that X is uniformly convex, and let $\varepsilon > 0$. Compute $\delta > 0$ for ε and $\mathsf{K} = \mathsf{B}$ as in Lemma 5. Then for all $x, x' \in \mathsf{B}$ such that $\|x - x'\| \geq \varepsilon$ and any $z \in \mathsf{X}$ with $\|z\| \leq \delta$ it follows that

$$\left\| \frac{1}{2}(x + x') + z \right\| \leq \left\| \frac{1}{2}(x + x') \right\| + \delta \leq \max\{\|x\|, \|x'\|\} \leq 1.$$

Hence, $\frac{1}{2}(x + x') + z \in \mathsf{B}$, so B is uniformly rotund.
Conversely, suppose that B is uniformly rotund, let $\varepsilon > 0$, and choose $\delta < 1$ as in the definition of uniformly rotund. If x, y are unit vectors of X with $\|x - y\| \geq \varepsilon$, then $\left\| \frac{1}{2}\delta(x + y) \right\| \leq \delta$, so

$$(1 + \delta) \left\| \tfrac{1}{2}(x + y) \right\| = \left\| \tfrac{1}{2}(x + y) + \tfrac{1}{2}\delta(x + y) \right\| \leq 1$$

and therefore $\left\| \frac{1}{2}(x + y) \right\| \leq (1 + \delta)^{-1} < 1$. □

Let C be a convex subset of a normed space X. We say that a mapping $f : \mathsf{C} \to \mathbf{R}$ is *quasiuniformly convex* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, x' \in \mathsf{X}$ and $\|x - x'\| > \varepsilon$, then there exists $z \in \mathsf{X}$ such that $\frac{1}{2}(x + x') + z \in \mathsf{C}$ and

$$f(\tfrac{1}{2}(x + x') + z) + \delta < \tfrac{1}{2}\left(f(x) + f(x')\right).$$

Clearly, such a function is sufficiently convex on C.

Proposition 9 *Let* K *be an inhabited, complete, uniformly rotund subset of a normed space* X*, and* L *a located convex subset of* X *that is disjoint from* K *and contains at least two distinct points. Then* $f(x) \equiv \rho(x, \mathsf{L})$ *defines a quasiuniformly convex function on* K*.*

Proof. With $\varepsilon > 0$ and $\delta < 1$ as in the definition of 'uniform rotundity' for K, let $x, x' \in \mathsf{K}$ and $\|x - x'\| > \varepsilon$. Choose $y, y' \in \mathsf{L}$ such that $y \neq y'$ and

$$\left\| \tfrac{1}{2}(x + x') - y \right\| < \rho(\tfrac{1}{2}(x + x'), \mathsf{L}) + \frac{\delta}{8}.$$

Let

$$0 < t < \min\left\{1, \frac{\delta}{8\,\|y - y'\|}\right\}$$

and let

$$\eta = (1-t)y + ty'.$$

Then $\eta \in L$ by convexity, and since $y \neq \eta$, either $\frac{1}{2}(x+x') \neq y$ or $\frac{1}{2}(x+x') \neq \eta$. In the latter case,

$$\begin{aligned}
\left\|\tfrac{1}{2}(x+x') - \eta\right\| &\leq \left\|\tfrac{1}{2}(x+x') - y\right\| + \|y-\eta\| \\
&< \rho(\tfrac{1}{2}(x+x'), L) + \frac{\delta}{8} + t\,\|y-y'\| \\
&< \rho(\tfrac{1}{2}(x+x'), L) + \frac{\delta}{4}.
\end{aligned}$$

Thus in either case there exists $p \in L$ such that $p \neq \frac{1}{2}(x+x')$ and

$$0 < \left\|\tfrac{1}{2}(x+x') - p\right\| < \rho(\tfrac{1}{2}(x+x'), L) + \frac{\delta}{2}.$$

Then

$$0 \neq \nu := \tfrac{1}{2}(x+x') - p.$$

Either $\|\nu\| > \frac{3\delta}{4}$ or $\|\nu\| < \delta$. In the latter case $p = \frac{1}{2}(x+x') - \nu \in K$ which is absurd since K and L are disjoint. Hence, $\|\nu\| > \frac{3\delta}{4}$. Now, with

$$z := \frac{3\delta}{4\,\|\nu\|}\nu,$$

we have $\|z\| < \delta$, so

$$\tfrac{1}{2}(x+x') - z \in K.$$

Hence,

$$\begin{aligned}
\left\|\tfrac{1}{2}(x+x') - p - z\right\| &= \left\|\nu - \tfrac{3\delta}{4\|\nu\|}\nu\right\| \\
&= \left(1 - \tfrac{3\delta}{4\|\nu\|}\right)\|\nu\| \\
&= \|\nu\| - \frac{3\delta}{4} \\
&< \rho(\tfrac{1}{2}(x+x'), L) + \frac{\delta}{2} - \frac{3\delta}{4} \\
&= \rho(\tfrac{1}{2}(x+x'), L) - \frac{\delta}{4} \\
&\leqslant \tfrac{1}{2}(\rho(x, L) + \rho(x', L)) - \frac{\delta}{4},
\end{aligned}$$

the last step using Lemma 3. Thus

$$\begin{aligned}
f\left(\tfrac{1}{2}(x+x') - z\right) + \frac{\delta}{4} &= \rho\left(\tfrac{1}{2}(x+x') - z, L\right) + \frac{\delta}{4} \\
&\leqslant \left\|\tfrac{1}{2}(x+x') - z - p\right\| + \frac{\delta}{4} \\
&< \tfrac{1}{2}(\rho(x, L) + \rho(x', L)) - \frac{\delta}{4} + \frac{\delta}{4} \\
&= \tfrac{1}{2}(f(x) + f(x')).
\end{aligned}$$

□

To see that in Proposition 9 we cannot replace uniform rotundity by mere convexity, take X to be the Euclidean plane $\mathbf{R}^2$, $K = \{(x, y) \in \mathbf{R}^2 : x \le 0\}$, and $L = \{(x, y) \in \mathbf{R}^2 : x \ge 1\}$; we have

$$\inf_{x \in K} \rho(x, L) = 1 = \|(0, y) - (1, y)\|$$

for all $y \in \mathbf{R}$, so, in view of Proposition 2, $x \mapsto \rho(x, L)$ is not sufficiently convex on K.

Recall here *Bishop's Lemma* [7, Proposition 3.1.1]:

> Let Y be an inhabited, complete, located subset of a metric space X. Then for each $x \in X$ there exists $y \in Y$ such that if $x \ne y$, then $\rho(x, Y) > 0$.

From Proposition 9, Proposition 2, and Bishop's Lemma we obtain:

Theorem 10 *Let K be an inhabited, complete, uniformly rotund subset of a normed space X, and L a located convex subset of X that is disjoint from K and contains at least two distinct points. Suppose also that $d \equiv \inf_{x \in K} \rho(x, L)$ exists. Then there exists $\xi \in K$ such that* (i) $\rho(\xi, L) = d$ *and (ii)* $\rho(x, L) > d$ *for all $x \in K$ with $x \ne \xi$. If, in addition, L is complete, then there exists $y \in L$ such that if $\xi \ne y$, then $d > 0$.*

Proof. By Proposition 9, $f(x) \equiv \rho(x, L)$ defines a quasiuniformly convex, and hence sufficiently convex, function on K. Since K is complete and d exists, Proposition 2 produces $\xi \in K$ with properties (i) and (ii). If also L is complete, then we complete the proof by invoking Bishop's Lemma. □

Lemma 11 *Let Y be an inhabited convex subset of a Hilbert space H, and a a point of H such that $d = \rho(a, Y)$ exists. Then there exists $b \in \overline{Y}$ such that $\|a - b\| = d$. Moreover,*

(i) $\|a - y\| > d$ *whenever $y \in \overline{Y}$ and $y \ne b$;*

(ii) $\langle a - b, b - y \rangle \ge 0$*, and therefore $\langle a - b, a - y \rangle \ge d^2$, for all $y \in Y$.*

Proof. This is a well-known result on Hilbert space. For instance Lemma 1 in [2] proves the existence of $b \in \overline{Y}$ such that $\|a - b\| = d$ and (ii) holds. Conclusion (i) follows from (ii) since for all $y \in Y$

$$\|a-y\|^2 = \|a-b+b-y\|^2 = \|a-b\|^2+\|b-y\|^2+2\langle a - b, b - y\rangle \ge d^2+\|b-y\|^2.$$

□

Corollary 12 *Let* K *be an inhabited, closed, uniformly rotund subset of a Hilbert space* H*, and* L *a closed, located, convex subset of* H *that is disjoint from* K *and contains at least two distinct points. Suppose also that* $d \equiv \inf_{x \in K} \rho(x, L)$ *exists. Then there exist* $x_\infty \in K$ *and* $y_\infty \in L$ *such that* $\|x_\infty - y_\infty\| = d$*. Moreover,*

(i) $\|x - y\| > d$ *whenever* $x \in K$ *and* $y \in L$ *and either* $x \neq x_\infty$ *or* $y \neq y_\infty$*;*

(ii) $\langle x_\infty - y_\infty, y_\infty - y \rangle \geq 0$*, and therefore* $\langle x_\infty - y_\infty, x_\infty - y \rangle \geq d^2$*, for all* $y \in L$.

Proof. By Theorem 10, there exists $x_\infty \in K$ such that $d = \rho(x_\infty, L)$. By Lemma 11 there exists $y_\infty \in Y$ such that $\|x_\infty - y_\infty\| = \rho(x_\infty, L)$ and properties (i) and (ii) hold. □

This leads us to a new constructive separation theorem.

Theorem 13 *Let* K *be an inhabited, closed, located, uniformly rotund subset of a Hilbert space* H*, and* L *a closed, located, convex subset of* H *containing at least two distinct points. Suppose that* $d \equiv \inf_{x \in K} \rho(x, L)$ *exists and is positive, let* x_∞, y_∞ *be as in* Corollary 12*, and let* $p = x_\infty - y_\infty$*. Then*

$$\langle p, x - y \rangle \geq d^2 \quad \textit{for all } x \in K \textit{ and } y \in L.$$

Proof. Construct $x_\infty \in K$ and $y_\infty \in L$ as in Corollary 12, and let

$$p = x_\infty - y_\infty.$$

Then, by Corollary 12, for all $y \in Y$ we have

$$\langle p, x_\infty - y \rangle = \langle x_\infty - y_\infty, x_\infty - y \rangle \geq d^2.$$

On the other hand, since K is located Lemma 11 provides the existence of a unique $b \in K$ such that $\rho(y_\infty, K) = \|y_\infty - b\|$. As $\rho(y_\infty, K) = d = \|y_\infty - x_\infty\|$ it follows that indeed $b = x_\infty$ and thus by Lemma 11 that

$$\langle y_\infty - x_\infty, x_\infty - x \rangle \geq 0$$

for all $x \in K$. Hence, for $x \in K$ and $y \in L$,

$$\begin{aligned} \langle p, x - y \rangle &= \langle p, x_\infty - y \rangle + \langle p, x - x_\infty \rangle \\ &\geq d^2 + \langle x_\infty - y_\infty, x - x_\infty \rangle \\ &= d^2 + \langle y_\infty - x_\infty, x_\infty - x \rangle \geq d^2. \end{aligned}$$

□

Corollary 14 *Let* K *be an inhabited, closed, located, uniformly rotund subset of a Hilbert space* H*, and* L *a closed, located, convex subset of* H *containing at least two distinct points. Suppose that* $d \equiv \inf_{x \in K} \rho(x, L)$ *exists and is positive. Then there exists a normed linear functional* u *on* H *with* $\|u\| = 1$*, such that* $u(x) \geq u(y) + d$ *for all* $x \in X$ *and* $y \in Y$.

Proof. In the notation of Theorem 13, let

$$u(x) = \left\langle x, d^{-1}p \right\rangle \qquad (x \in H).$$

Then u is a bounded linear functional with norm 1. Also, for $x \in X$ and $y \in Y$,

$$u(x-y) = d^{-1} \langle p, x-y \rangle \geq d$$

and therefore $u(x) \geq u(y) + d$. □

In trying to apply the foregoing theorems, it is natural to think of the case where the uniformly rotund set K is compact. In that case, if K is nontrivial, x is finite-dimensional.

Proposition 15 *A normed space that has a totally bounded and uniformly rotund subset which contains two distinct points is finite-dimensional.*

Proof. Let S be a totally bounded, uniformly rotund subset of a normed space X. Assume that S contains two distinct points a, b. There exists $\delta > 0$ such that $\left\| \frac{1}{2}(x+y) + z \right\| \in S$ whenever $x, y \in S$, $\|x-y\| > \frac{1}{2}\|a-b\|$, $z \in X$, and $\|z\| \leq \delta$. Since S is totally bounded, there exists r such that $0 < r \leq \delta$ and the set $S \cap \overline{B}(\frac{1}{2}(a+b), r)$, which contains $\frac{1}{2}(a+b)$, is totally bounded [7, Theorem 2.2.13]. If $z \in \overline{B}(\frac{1}{2}(a+b), r)$, then $\left\| z - \frac{1}{2}(a+b) \right\| \leq \delta$, so

$$z = \tfrac{1}{2}(a+b) + \left(z - \tfrac{1}{2}(a+b)\right) \in S.$$

Thus $\overline{B}(\frac{1}{2}(a+b), r) = S \cap \overline{B}(\frac{1}{2}(a+b), r)$ and so is a totally bounded ball of positive radius. It follows from [7, Proposition 4.1.13] that X is finite-dimensional. □

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