Bipolar Theorems for Sets of Non-negative Random Variables*

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Abstract

This paper assumes a robust, in general not dominated, probabilistic framework and provides necessary and sufficient conditions for a bipolar representation of subsets of the set of all quasi-sure equivalence classes of non-negative random variables without any further conditions on the underlying measure space. This generalises and unifies existing bipolar theorems proved under stronger assumptions on the robust framework. Moreover, we sketch applications to issues in mathematical finance and a mass transport type duality.
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1 Introduction

The well-known bipolar theorem provided by Brannath and Schachermayer in [5] provides necessary and sufficient conditions for the existence of a bipolar representation of a set $\mathcal{C} \subset L^0_{P+}$ where $L^0_{P+} := L^0_+(\Omega, \mathcal{F}, P)$ is endowed with the topology induced by convergence in probability. Important applications of this result are, for instance, dual characterisations of solutions to utility maximisation problems in financial economics. [5] shows that $\mathcal{C}$ allows a bipolar representation in L^0_{P+} if and only if $\mathcal{C}$ is convex, solid, and closed in probability. The aim of this paper is to generalise this result to a non-dominated—so-called robust—framework, where the probability measure P is replaced by a family of probability measures $\mathcal{P}$ which is not necessarily dominated. Such extensions have already been studied in, e.g., [4, 10, 13] where sufficient conditions for the existence of a bipolar representation in very particular robust frameworks are given. In this paper, without further assumptions, we provide necessary and sufficient conditions for a bipolar representation of $\mathcal{C} \subset L^0_{c+}$ where c is the upper probability induced by the set of probability measures $\mathcal{P}$ and L^0_{c+} denotes the robust counterpart of L^0_{P+}. As a byproduct we obtain a common framework for and unify the bipolar results of [4, 5, 10, 13].

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Of course, convexity and solidity of $\mathcal{C}$ are necessary for a bipolar representation also in robust frameworks. A key point in our findings, however, is the necessity of $\mathcal{P}$-sensitivity of $\mathcal{C}$, see Definition 2.5 and [6, 14]. This property, which is trivially satisfied in the classical dominated case, allows to *lift* bipolar theorems known within a dominated framework to the robust model space, see Sections 3.1, 3.2, and 6. We will show that $\mathcal{P}$-sensitivity is equivalent to the aggregation-property known from robust statistics, see, e.g., [16], or robust stochastic modeling, see, e.g., [15]. The list of necessary conditions for a bipolar representation should obviously also include some kind of closedness of $\mathcal{C}$. In this respect it turns out that, in contrast to dominated models where closedness in probability under the dominating probability measure is the canonical choice, in the non-dominated case there are a variety of notions of closedness which offer themselves as necessary and reasonable requirements depending on the point of view on the problem. All of them may be seen as robust generalisations of closedness in probability in case of a solid set $\mathcal{C}$. A main contribution of this paper is to relate the underlying notions of convergence on L^0_c and thus the different closedness properties to each other, see Section 4. Eventually we identify sequential order closedness with respect to the quasi-sure order as the appropriate equivalent of a number of notions of closedness for solid sets which are necessary, and in fact sufficient in combination with the mentioned properties above, for a bipolar representation of $\mathcal{C}$. Versions of the bipolar theorem for different dual sets are then provided in Section 6. The different dual sets comprise combinations of probability measures and test functions or simply the set of finite measures. In Section 7 we provide several applications of the bipolar representations given in Section 6. In particular we show how our results generalise the bipolar theorems of [4, 10, 13]. Also we sketch applications in mathematical finance and a mass transport type duality.

2 Preliminaries and Notation

2.1 Basics

Throughout this paper $(\Omega, \mathcal{F})$ denotes an arbitrary measurable space. By ca we denote the real vector space of all countably additive finite variation set functions $\mu\colon \mathcal{F} \to \mathbb{R}$, and by ca_+ its positive elements ($\mu \in ca_+ \Leftrightarrow \forall\, A \in \mathcal{F}\colon \mu(A) \geq 0$), that is all finite measures on $(\Omega, \mathcal{F})$. Given non-empty subsets $\mathfrak{G}$ and $\mathfrak{I}$ of ca_+, we say that $\mathfrak{I}$ dominates $\mathfrak{G}$ ($\mathfrak{G} \ll \mathfrak{I}$) if for all $N \in \mathcal{F}$ satisfying $\sup_{\nu\in\mathfrak{I}} \nu(N) = 0$, we have $\sup_{\mu\in\mathfrak{G}} \mu(N) = 0$. $\mathfrak{G}$ and $\mathfrak{I}$ are equivalent ($\mathfrak{G} \approx \mathfrak{I}$) if $\mathfrak{G} \ll \mathfrak{I}$ and $\mathfrak{I} \ll \mathfrak{G}$. For the sake of brevity, for $\mu \in ca_+$ we shall write $\mathfrak{G} \ll \mu$, $\mu \ll \mathfrak{I}$, and $\mu \approx \mathfrak{G}$ instead of $\mathfrak{G} \ll \{\mu\}$, $\{\mu\} \ll \mathfrak{I}$, and $\{\mu\} \approx \mathfrak{G}$, respectively.

$\mathfrak{P}(\Omega) \subset ca_+$ denotes the set of probability measures on $(\Omega, \mathcal{F})$ and the letters $\mathcal{P}$ and $\mathcal{Q}$ are used to denote non-empty subsets of $\mathfrak{P}(\Omega)$. Fix such a set $\mathcal{P}$. We then write c for the induced upper probability $c\colon \mathcal{F} \to [0, 1]$ defined by

$$c(A) = \sup_{P\in\mathcal{P}} P(A).$$

An event $A \in \mathcal{F}$ is called $\mathcal{P}$-polar if $c(A) = 0$. A property holds $\mathcal{P}$-quasi surely (q.s.) if it holds outside a $\mathcal{P}$-polar event. We set $ca_c := \{\mu \in ca \mid \mu \ll \mathcal{P}\}$, $ca_{c+} := ca_+ \cap ca_c$, and $\mathfrak{P}_c(\Omega) := \mathfrak{P}(\Omega) \cap ca_c$.

Consider the $\mathbb{R}$-vector space $\mathcal{L}^0 := \mathcal{L}^0(\Omega, \mathcal{F})$ of all real-valued random variables $f\colon \Omega \to \mathbb{R}$ as well as its subspace $\mathcal{N} := \{f \in \mathcal{L}^0 \mid c(|f| > 0) = 0\}$. The quotient space $L^0_c := \mathcal{L}^0/\mathcal{N}$ consists of

equivalence classes X of random variables up to $\mathcal{P}$-q.s. equality comprising representatives $f \in X$. The equivalence class induced by $f \in \mathcal{L}^0$ in L_c^0 is denoted by $[f]_c$. The space L_c^0 carries the so-called $\mathcal{P}$-quasi-sure order $\preccurlyeq_{\mathcal{P}}$ as a natural vector space order: $X, Y \in L_c^0$ satisfy $X \preccurlyeq_{\mathcal{P}} Y$ if for $f \in X$ and $g \in Y$, $f \leq g$ $\mathcal{P}$-q.s., that is $\{f > g\}$ is $\mathcal{P}$-polar. In order to facilitate the notation, we suppress the dependence of $\preccurlyeq_{\mathcal{P}}$ on $\mathcal{P}$ and simply write $\preccurlyeq$ if there is no risk of confusion. $(L_c^0, \preccurlyeq)$ is a vector lattice, and for $X, Y \in L_c^0$, $f \in X$, and $g \in Y$, the minimum $X \wedge Y$ is the equivalence class $[f \wedge g]_c$ generated by the pointwise minimum $f \wedge g$, whereas the maximum $X \vee Y$ is the equivalence class $[f \vee g]_c$ generated by the pointwise maximum $f \vee g$. For an event $A \in \mathcal{F}$, χ_A denotes the indicator of the event (i.e. $\chi_A(\omega) = 1$ if and only if $\omega \in A$, and $\chi_A(\omega) = 0$ otherwise) while $\mathbf{1}_A := [\chi_A]_c$ denotes the generated equivalence class in L_c^0.
A subspace of L_c^0 which will turn out to be important for our studies is the space L_c^∞ of equivalence classes of $\mathcal{P}$-q.s. bounded random variables, i.e.,

$$L_c^\infty := \{X \in L_c^0 \mid \exists m > 0 \colon |X| \preccurlyeq m\}.$$

L_c^∞ is a Banach lattice when endowed with the norm

$$\|X\|_{L_c^\infty} := \inf\{m > 0 \mid |X| \preccurlyeq m\}, \quad X \in L_c^0.$$

L_{c+}^0 and L_{c+}^∞ denote the positive cones of L_c^0 and L_c^∞, respectively. If $\mathcal{P} = \{P\}$ is given by a singleton and thus $c = P$, we write L_P^0, L_P^∞, and $[f]_P$ instead of L_c^0, L_c^∞, and $[f]_c$, and similarly for other expressions where c appears. Also, the $\mathcal{P}$-q.s. order in this case is the P-almost-sure (a.s.) order which we will also denote by $\leq_P$ when we are working with both the $\mathcal{P}$-q.s. order $\preccurlyeq$ for some set $\mathcal{P} \subset \mathfrak{P}(\Omega)$ and the P-a.s. order for some $P \in \mathfrak{P}(\Omega)$ (typically $P \ll \mathcal{P}$).
Often we will, as is common practice, identify equivalence classes of random variables with their representatives. However, sometimes it will be useful to distinguish between them to avoid confusion. Let us clarify this further: We say that X is an equivalence class of random variables if there exists an equivalence relation $\sim$ on $\mathcal{L}^0$ such that $X = \{f \in \mathcal{L}^0 \mid f \sim g\}$ for some $g \in \mathcal{L}^0$. A measure $P \in \mathfrak{P}(\Omega)$ is consistent with the equivalence relation $\sim$ if

$$\forall f, g \in \mathcal{L}^0 \colon\ f \sim g \ \Rightarrow\ P(f = g) = 1.$$

In that case we, for instance, write $E_P[X]$ for the expectation of X under P which actually means $E_P[f]$ for any $f \in X$ provided the latter integral is well-defined. Also we will write expressions like $P(X = Y)$, where Y is another equivalence class of random variables with respect to the same equivalence relation $\sim$, actually meaning $P(f = g)$ for arbitrary $f \in X$ and $g \in Y$. The difference here to the usual convention of identifying equivalence classes of random variables with their representatives is that the equivalence relation $\sim$ might not be given by P-a.s. equality, but P is only assumed to be consistent with that equivalence relation in the above sense. A typical example is the equivalence relation given by $\mathcal{P}$-q.s. equality of random variables and $P \in \mathfrak{P}_c(\Omega)$.

2.2 Supported Measures and Class (S) Robustness

Supported measures $\mu \in ca_c$ play a key role in handling robustness. This concept is also known in statistics, see [13] for a detailed review.

Definition 2.1. Let $\mathcal{P} \subset \mathfrak{P}(\Omega)$ be non-empty.

1. A measure $\mu \in ca_{c+}$ is called supported if there is an event $S(\mu) \in \mathcal{F}$ such that
   (a) $\mu(S(\mu)^c) = 0$;
   (b) whenever $N \in \mathcal{F}$ satisfies $\mu(N \cap S(\mu)) = 0$, then $N \cap S(\mu)$ is $\mathcal{P}$-polar.

   The set $S(\mu)$ is called the (order) support of μ.

2. A signed measure $\mu \in ca_c$ is supported if $|\mu|$ is supported where

$$|\mu|(A) := \sup\{\mu(B) - \mu(A \setminus B) \mid B \in \mathcal{F}, B \subset A\}, \ A \in \mathcal{F},$$

   is the total variation of μ.

3. We set

$$sca_c := \{\mu \in ca_c \mid \mu \text{ supported}\},$$

   the space of all supported signed measures in ca_c, and $sca_{c+} := sca_c \cap ca_{c+}$.

Note that if two sets $S, S' \in \mathcal{F}$ satisfy conditions (a) and (b) in Definition 2.1(1), then $\chi_S = \chi_{S'}$ $\mathcal{P}$-q.s. ($\mathbf{1}_S = \mathbf{1}_{S'}$), i.e., the symmetric difference $S \triangle S'$ is $\mathcal{P}$-polar. The order support $S(\mu)$ is thus usually not unique as an event, but only unique up to $\mathcal{P}$-polar events. In the following $S(\mu)$ therefore denotes an arbitrary version of the order support. Note that the functional

$$L_c^\infty \ni X \mapsto \int X d\mu \tag{1}$$

is order continuous (with respect to $\preccurlyeq$) if and only if $\mu \in sca_c$. In fact, the space of order continuous linear functionals may be identified with sca_c via (1). In the same way ca_c is identified with the space of all σ-order continuous functionals, and any $\mu \in ca_c \setminus sca_c$ induces a linear σ-order continuous functional which is not order continuous. Note that in robust frameworks $ca_c \setminus sca_c \neq \emptyset$ is often the case. We refer to [13] for a concise but comprehensive discussion of the spaces ca_c and sca_c.

Definition 2.2. Let $\mathcal{P} \subset \mathfrak{P}(\Omega)$ be non-empty. $\mathcal{P}$ is of class (S) if there exists a set of supported probability measures $\mathcal{Q}$ (i.e. $\mathcal{Q} \subset \mathfrak{P}_c(\Omega) \cap sca_c$) such that $\mathcal{Q} \approx \mathcal{P}$. In that case we call $\mathcal{Q}$ a supported alternative of $\mathcal{P}$.

Suppose that $\mathcal{P}$ is of class (S) and let $\mathcal{Q}$ be a supported alternative of $\mathcal{P}$. As $\mathcal{Q} \approx \mathcal{P}$, the $\mathcal{Q}$-q.s. order coincides with the $\mathcal{P}$-q.s. order $\preccurlyeq$. In [13] it is shown how the class (S) property is important, and indeed necessary, in many situations to handle robustness in non-dominated frameworks.

Definition 2.3. Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega) \cap sca_c$. We say that $\mathcal{Q}$ has disjoint supports if, for all $Q, Q' \in \mathcal{Q}$ such that $Q \neq Q'$, $\mathbf{1}_{S(Q)} \wedge \mathbf{1}_{S(Q')} = 0$, that is $S(Q) \cap S(Q')$ is a $\mathcal{P}$-polar event.

Lemma 2.4 (see [13, Lemma 3.7])**.** *Suppose $\mathcal{P}$ is of class (S). Then there exists a supported alternative $\mathcal{Q} \approx \mathcal{P}$ with disjoint supports. $\mathcal{Q}$ will be referred to as a disjoint supported alternative.*

2.3 $\mathcal{P}$-sensitive Sets

Let $\mathcal{P} \subset \mathfrak{P}(\Omega)$. A property that will be key in our studies is the so-called $\mathcal{P}$-sensitivity of subsets of L^0_c defined in the following, see also [14]. To this end, recall that $[f]_c$ denotes the equivalence class in L^0_c generated by $f \in \mathcal{L}^0$, whereas $[f]_Q$ is the equivalence class generated by f in L^0_Q, that is under Q-a.s. equality. The following map identifies any $X, Y \in L^0_c$ which appear to coincide under Q, that is $Q(f = g) = 1$ for $f \in X$ and $g \in Y$.

$$j_Q \colon L^0_c \to L^0_Q, \quad [f]_c \mapsto [f]_Q.$$

Definition 2.5. A set $\mathcal{C} \subset L^0_c$ is called $\mathcal{P}$-sensitive if

$$\mathcal{C} = \bigcap_{Q \in \mathfrak{P}_c(\Omega)} j_Q^{-1} \circ j_Q(\mathcal{C}).$$

$\mathcal{P}$-sensitivity means that the set $\mathcal{C}$ is completely determined by its image under each model $Q \in \mathfrak{P}_c(\Omega)$, so if $X \in L^0_c$ looks like a member of $\mathcal{C}$ under each $Q \in \mathfrak{P}_c(\Omega)$ (i.e. $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathfrak{P}_c(\Omega)$) then in fact $X \in \mathcal{C}$. Note that always $\mathcal{C} \subset \bigcap_{Q \in \mathfrak{P}_c(\Omega)} j_Q^{-1} \circ j_Q(\mathcal{C})$, so the nontrivial inclusion is $\bigcap_{Q \in \mathfrak{P}_c(\Omega)} j_Q^{-1} \circ j_Q(\mathcal{C}) \subset \mathcal{C}$. Trivially, if $\mathcal{P} = \{P\}$, then every set $\mathcal{C} \subset L^0_P$ is P-sensitive. It will sometimes turn out to be useful to know a stronger sensitive representation of $\mathcal{C}$:

Definition 2.6. Let $\mathcal{C} \subset L^0_c$. $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ is called a reduction set for $\mathcal{C}$ if $\mathcal{Q} \neq \emptyset$ and

$$\mathcal{C} = \bigcap_{Q \in \mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{C}). \tag{2}$$

Clearly, any $\mathcal{P}$-sensitive set admits the reduction set $\mathfrak{P}_c(\Omega)$. The following lemma relates reduction sets to each other and in particular shows that any set satisfying (2) is indeed $\mathcal{P}$-sensitive.

Lemma 2.7. *Let $\mathcal{C} \subset L^0_c$.*

1. *Consider a reduction set $\mathcal{Q}_1$ for $\mathcal{C}$ and any other set of probability measures $\mathcal{Q}_2 \subset \mathfrak{P}_c(\Omega)$ such that $\mathcal{Q}_1 \subset \mathcal{Q}_2$. Then $\mathcal{Q}_2$ is also a reduction set for $\mathcal{C}$.*

2. *If $\mathcal{C}$ satisfies* (2) *for some non-empty set $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$, then $\mathcal{C}$ is $\mathcal{P}$-sensitive.*

3. *If $\tilde{\mathcal{P}} \subset \mathfrak{P}(\Omega)$ dominates $\mathcal{P}$, i.e. $\mathcal{P} \ll \tilde{\mathcal{P}}$, then $\mathcal{C}$ is $\tilde{\mathcal{P}}$-sensitive.*

Proof. The first statement follows from

$$\mathcal{C} \subset \bigcap_{Q \in \mathcal{Q}_2} j_Q^{-1} \circ j_Q(\mathcal{C}) \subset \bigcap_{Q \in \mathcal{Q}_1} j_Q^{-1} \circ j_Q(\mathcal{C}) = \mathcal{C}. \tag{3}$$

The second assertion follows from 1. by choosing $\mathcal{Q}_1 = \mathcal{Q}$ and $\mathcal{Q}_2 = \mathfrak{P}_c(\Omega)$. Finally, $\mathcal{P} \ll \tilde{\mathcal{P}}$ implies that $\mathfrak{P}_c(\Omega) \subset \{P \in \mathfrak{P}(\Omega) \mid P \ll \tilde{\mathcal{P}}\}$, so we may argue as in (3). $\square$

The reason for considering other reduction sets than simply $\mathfrak{P}_c(\Omega)$ will become evident throughout the paper. As we will see next, $\mathcal{P}$-sensitive sets are stable under intersection.

Lemma 2.8. *Let I be a non-empty index set and let $\mathcal{C}_\alpha \subset L^0_c$, $\alpha \in I$, be $\mathcal{P}$-sensitive. Then*

$$\mathcal{C} := \bigcap_{\alpha\in I} \mathcal{C}_\alpha$$

is also $\mathcal{P}$-sensitive. If $\mathcal{Q}_\alpha \subset \mathfrak{P}_c(\Omega)$ is a reduction set for $\mathcal{C}_\alpha$ for each $\alpha \in I$, then $\mathcal{Q} := \bigcup_{\alpha\in I} \mathcal{Q}_\alpha$ is a reduction set for $\mathcal{C}$.

Proof. Suppose that $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$. Then in particular $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}_\alpha$ and all $\alpha \in I$. Hence, $X \in \mathcal{C}_\alpha$ for all $\alpha \in I$. □

3 Bipolar Representations

Our focus will be on extensions to L^0_{c+} of the well-known bipolar theorem on L^0_{P+} given in [5]:

Theorem 3.1 ([5, Theorem 1.3])**.** *Let $P \in \mathfrak{P}(\Omega)$ and $\mathcal{C} \subset L^0_{P+}$ be non-empty. Define the polar of $\mathcal{C}$ as*

$$\mathcal{C}^\circ := \{Y \in L^0_{P+} \mid \forall X \in \mathcal{C} \colon E_P[XY] \le 1\}.$$

Then $\mathcal{C}^\circ$ is a non-empty, P-closed, convex, and solid subset of L^0_{P+}, and the bipolar

$$\mathcal{C}^{\circ\circ} := \{X \in L^0_{P+} \mid \forall Y \in \mathcal{C}^\circ \colon E_P[XY] \le 1\} \tag{4}$$

of $\mathcal{C}$ is the smallest P-closed, convex, solid set in L^0_{c+} containing $\mathcal{C}$. In particular if $\mathcal{C}$ is P-closed, convex, and solid, then $\mathcal{C} = \mathcal{C}^{\circ\circ}$.

P-closedness in Theorem 3.1 means that the respective set is closed under convergence in probability with respect to P. The definition of solidness is recalled next:

Definition 3.2. Let $\mathcal{C} \subset L^0_c$. $\mathcal{C}$ is called solid in L^0_c if $X \in \mathcal{C}$, $Y \in L^0_c$ and $|Y| \preccurlyeq |X|$ imply $Y \in \mathcal{C}$. $\mathcal{C}$ is solid in L^0_{c+} if $\mathcal{C} \subset L^0_{c+}$ and $X \in \mathcal{C}$, $Y \in L^0_{c+}$ and $Y \preccurlyeq X$ imply $Y \in \mathcal{C}$. We simply say that $\mathcal{C}$ is solid, if $\mathcal{C}$ is either solid in L^0_c or solid in L^0_{c+}.

Note that a set which is solid in L^0_{c+} cannot be solid in L^0_c and vice versa. In Theorem 3.1 we have $\mathcal{P} = \{P\}$, and the subset $\mathcal{C} \subset L^0_{P+}$ is solid if and only if $X \in \mathcal{C}$, $Y \in L^0_{P+}$, and $Y \le_P X$ imply $Y \in \mathcal{C}$.

We also like to mention a useful strengthening of Theorem 3.1, still with ambient space L^0_{P+}, given in [12]:

Theorem 3.3 ([12, Colorllary 2.7])**.** *Let $P \in \mathfrak{P}(\Omega)$ and $\mathcal{C} \subset L^0_{P+}$ be non-empty. Define the polar of $\mathcal{C}$ as*

$$\mathcal{C}^\circ := \{Y \in L^\infty_{P+} \mid \forall X \in \mathcal{C} \colon E_P[XY] \le 1\}.$$

Then $\mathcal{C}^\circ$ is a non-empty, $\sigma(L^\infty_P, L^\infty_P)$-closed, convex, solid subset of L^∞_P, and the bipolar

$$\mathcal{C}^{\circ\circ} := \{X \in L^0_{P+} \mid \forall Y \in \mathcal{C}^\circ \colon E_P[XY] \le 1\} \tag{5}$$

of $\mathcal{C}$ is the smallest P-closed, convex, solid set in L^0_{c+} containing $\mathcal{C}$. In particular if $\mathcal{C}$ is P-closed, convex, and solid, then $\mathcal{C} = \mathcal{C}^{\circ\circ}$.

The important difference between Theorems 3.1 and 3.3 is that the latter replaces the dual cone L^0_{P+} of Theorem 3.1 by L^∞_{P+}. The boundedness of elements in L^∞_{P+} will prove helpful when deriving robust bipolar theorems on L^0_{c+} by *lifting* those on L^0_{P+} for $P \in \mathcal{P}$, see Section 6. Note that by solidness of $\mathcal{C}$ and by monotone convergence one directly verifies that the sets in (4) and (5) indeed coincide.

3.1 A Reverse Perspective

In this section we collect some simple observations on necessary conditions for a bipolar representation which will, however, set the direction of our further studies.

Proposition 3.4. *Let $\mathcal{X} \subset L_c^0$ be a non-empty convex subset and suppose that the non-empty set $\mathcal{C} \subset \mathcal{X}$ admits a bipolar representation*

$$\mathcal{C} = \{X \in \mathcal{X} \mid \forall h \in \mathcal{K} \colon h(X) \leq 1\} \tag{6}$$

where $\mathcal{K}$ denotes a non-empty set of functions $h : \mathcal{X} \to \mathbb{R} \cup \{-\infty, \infty\}$.

1. *If each $h \in \mathcal{K}$ is dominated by a probability measure $Q \in \mathfrak{P}_c(\Omega)$ in the sense that*

$$\forall X, Y \in \mathcal{X} \colon \ Q(X = Y) = 1 \Rightarrow h(X) = h(Y),$$

   then $\mathcal{C}$ is $\mathcal{P}$-sensitive. Any set $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ such that every $h \in \mathcal{K}$ is dominated by some $Q \in \mathcal{Q}$ serves as reduction set for $\mathcal{C}$.

2. *If the functions h are convex, then $\mathcal{C}$ is necessarily convex.*

3. *If the functions h are monotone with respect to some partial order $\lhd$ on $\mathcal{X}$, i.e. for all $X, Y \in \mathcal{X}$ we have that $X \lhd Y$ implies $h(X) \leq h(Y)$, then $\mathcal{C}$ is monotone with respect to $\lhd$, i.e. $Y \in \mathcal{C}$, $X \in \mathcal{X}$ and $X \lhd Y$ imply $X \in \mathcal{C}$.*

4. *If the functions h are (sequentially) lower semi-continuous with respect to some topology τ on $\mathcal{X}$, then $\mathcal{C}$ is necessarily (sequentially) τ-closed.*

Proof. 1. Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ be such that every $h \in \mathcal{K}$ is dominated by some $Q \in \mathcal{Q}$. We have to prove that if $X \in \mathcal{X}$ satisfies $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$, then $X \in \mathcal{C}$. To this end, fix such an X and let $h \in \mathcal{K}$ be arbitrary and choose $Q \in \mathcal{Q}$ which dominates h. There is $Y \in \mathcal{C}$ such that $j_Q(Y) = j_Q(X) \in j_Q(\mathcal{C})$. As $Q(X = Y) = Q(j_Q(X) = j_Q(Y)) = 1$, we obtain

$$h(X) = h(Y) \leq 1.$$

Since $h \in \mathcal{K}$ was arbitrarily chosen, we conclude that $X \in \mathcal{C}$.
2., 3. and 4. are easily verified. □

As our focus lies on bipolar representations for subsets of $\mathcal{X} = L_{c+}^0$, let us further refine the implications of Proposition 3.4 in that setting. If $\mathcal{X} = L_{c+}^0$ it seems natural that the functions h appearing in the bipolar representation (6) are of type $h(X) = E_P[XZ]$ for some $P \in \mathfrak{P}(\Omega)$ and $Z \in L_{c+}^0$. Under this assumption the following Corollary 3.6 provides more information. However, before we are able to state the corollary we need to introduce some further notation: Let X_n, $n \in \mathbb{N}$, and X be equivalence classes of random variables with respect to the same equivalence relation on $\mathcal{L}^0$, and let $P \in \mathfrak{P}(\Omega)$ be consistent with that equivalence relation, see Section 2.1. We will write $X_n \xrightarrow{P} X$ to indicate that $(X_n)_{n \in \mathbb{N}}$ converges to X in probability with respect to P, that is for any choice $f_n \in X_n$ and $f \in X$ the sequence of random variables $(f_n)_{n \in \mathbb{N}}$ converges to f in probability with respect to P. For a subset $\mathcal{Q}$ of $\mathfrak{P}(\Omega)$ we write $X_n \xrightarrow{\mathcal{Q}} X$ to indicate that every $Q \in \mathcal{Q}$ is consistent with the equivalence relation defining X_n, $n \in \mathbb{N}$, and X, and $X_n \xrightarrow{Q} X$ for all $Q \in \mathcal{Q}$.

Definition 3.5. Let $\mathcal{Q} \subset \mathfrak{P}(\Omega)$ be non-empty. A set $\mathcal{C} \subset L_c^0$ is called $\mathcal{Q}$-closed if $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $X_n \xrightarrow{\mathcal{Q}} X$ to some $X \in L_c^0$ implies that $X \in \mathcal{C}$.

Note that if $\tilde{\mathcal{Q}} \subset \mathcal{Q} \subset \mathfrak{P}(\Omega)$ and if $\mathcal{C}$ is $\tilde{\mathcal{Q}}$-closed, then $\mathcal{C}$ is also $\mathcal{Q}$-closed. In particular, any $\mathcal{Q}$-closed set is $\mathfrak{P}(\Omega)$-closed, and if $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$, any $\mathcal{Q}$-closed set is $\mathfrak{P}_c(\Omega)$-closed.

Corollary 3.6. *Suppose that the non-empty set $\mathcal{C} \subset L_{c+}^0$ admits a bipolar representation of the form*

$$\mathcal{C} = \{X \in L_{c+}^0 \mid \forall (P,Z) \in \mathcal{K} \colon E_P[ZX] \leq 1\}$$

where $\mathcal{K} \subset \mathfrak{P}_c(\Omega) \times L_{c+}^0$ is non-empty. Then $\mathcal{C}$ is $\mathcal{P}$-sensitive, convex, solid, and $\mathfrak{P}_c(\Omega)$-closed. Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ denote any set of probabilities such that for all $(P,Z) \in \mathcal{K}$ there is $Q \in \mathcal{Q}$ with $P \ll Q$. Then $\mathcal{Q}$ serves as reduction set for $\mathcal{C}$ and $\mathcal{C}$ is in fact $\mathcal{Q}$-closed.

Proof. Convexity, solidness, and $\mathcal{P}$-sensitivity with reduction set $\mathcal{Q}$ immediately follow from Proposition 3.4. Also $\mathcal{Q}$-closedness is a consequence of Proposition 3.4 since for any $(P,Z) \in \mathcal{K}$ the function $X \ni L_{c+}^0 \mapsto E_P[ZX]$ is sequentially lower semi-continuous with respect to $\mathcal{Q}$-convergence. Indeed, consider any $r \in \mathbb{R}$ and let $(X_n)_{n\in\mathbb{N}} \subset L_{c+}^0$ and $X \in L_{c+}^0$ such that $X_n \xrightarrow{\mathcal{Q}} X$ and $E_P[ZX_n] \leq r$ for all $n \in \mathbb{N}$. As $P \ll Q$ for some $Q \in \mathcal{Q}$ and $X_n \xrightarrow{Q} X$, there is a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$ converging Q-a.s. and thus P-a.s. to X. Hence, by Fatou's lemma

$$E_P[ZX] \leq \liminf_{k\to\infty} E_P[ZX_{n_k}] \leq r.$$

□

Note the relation between the reduction set and the closedness of $\mathcal{C}$ stated in Corollary 3.6.

3.2 Lifting Bipolar Representations

As we have seen above, $\mathcal{P}$-sensitivity arises naturally in the context of sets with a bipolar representation. Conversely, in this section we will see how $\mathcal{P}$-sensitivity can be used to obtain a robust bipolar representation by lifting known bipolar theorems in dominated frameworks to the robust model L_c^0.

Throughout this section let $\mathcal{X}$ be a convex subset of L_c^0, and let $\mathcal{C} \subset \mathcal{X}$ be a non-empty $\mathcal{P}$-sensitive set with reduction set $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$. Further let $\mathcal{X}_Q := j_Q(\mathcal{X})$ and $\mathcal{C}_Q := j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$. For each $Q \in \mathcal{Q}$ we denote by $\mathcal{Y}_Q$ a non-empty set of maps $l : \mathcal{X}_Q \to \mathbb{R} \cup \{-\infty, \infty\}$ and let

$$\mathcal{C}_Q^\circ := \{l \in \mathcal{Y}_Q \mid \forall X \in \mathcal{C}_Q \colon l(X) \leq 1\}$$

and

$$\mathcal{C}_Q^{\circ\circ} := \{X \in \mathcal{X}_Q \mid \forall X \in \mathcal{C}_Q^\circ \colon l(X) \leq 1\}.$$

Set

$$\mathcal{C}^\circ := \bigcup_{Q\in\mathcal{Q}} \{l \circ j_Q \mid l \in \mathcal{C}_Q^\circ\} \tag{7}$$

and

$$\mathcal{C}^{\circ\circ} := \{X \in \mathcal{X} \mid \forall h \in \mathcal{C}^\circ \colon h(X) \leq 1\}. \tag{8}$$

Theorem 3.7. *Suppose that $\mathcal{C}_Q = \mathcal{C}_Q^{\circ\circ}$ for all $Q \in \mathcal{Q}$. Then $\mathcal{C} = \mathcal{C}^{\circ\circ}$.*

Proof. Let $X \in \mathcal{C}$, then $j_Q(X) \in \mathcal{C}_Q$ and thus $l(j_Q(X)) \leq 1$ for all $l \in \mathcal{C}_Q^\circ$ and $Q \in \mathcal{Q}$. Hence, $X \in \mathcal{C}^{\circ\circ}$. Now let $X \in \mathcal{C}^{\circ\circ}$. Then for any $Q \in \mathcal{Q}$ we have that $l(j_Q(X)) \leq 1$ for all $l \in \mathcal{C}_Q^\circ$. Since $\mathcal{C}_Q = \mathcal{C}_Q^{\circ\circ}$ holds by assumption for all $Q \in \mathcal{Q}$, we obtain $j_Q(X) \in \mathcal{C}_Q$ for all $Q \in \mathcal{Q}$. As $\mathcal{Q}$ is a reduction set for $\mathcal{C}$, we conclude that $X \in \mathcal{C}$. □

Clearly, supposing that $\mathcal{C}_Q = \mathcal{C}_Q^{\circ\circ}$ holds for all $Q \in \mathcal{Q}$ is a rather abstract assumption. As we focus on $\mathcal{X} = L^0_{c+}$, we will use Theorems 3.1 and 3.3 to conclude that under some conditions on $\mathcal{C}$ each $\mathcal{C}_Q$ admits a bipolar representation $\mathcal{C}_Q = \mathcal{C}_Q^{\circ\circ}$. Then we may lift this bipolar representation with Theorem 3.7. The conditions on $\mathcal{C}$ will, of course, comprise convexity and solidness with respect to the $\mathcal{P}$-quasi-sure order, and these requirements are easily seen to imply convexity and, respectively, solidness with respect to the Q-a.s. order of any $\mathcal{C}_Q$. However, we also need to discuss reasonable closure properties. This is the purpose of the next section.

4 Concepts of Closedness under Uncertainty

Recall the discussion from the previous Section 3.2. If we want to apply Theorem 3.1 or 3.3, we need to ensure that every $j_Q(\mathcal{C})$ is Q-closed. A straight forward way of achieving this is to assume that $\mathcal{C}$ is Q-closed for each $Q \in \mathcal{Q}$. Yet, a still sufficient and indeed also necessary property is the following weaker requirement:

Definition 4.1. Let $\mathcal{C} \subset L^0_c$ and $Q \in \mathfrak{P}_c(\Omega)$. $\mathcal{C}$ is called locally Q-closed if for each sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $X \in L^0_c$ such that $X_n \xrightarrow{Q} X$ there exists $Y \in \mathcal{C}$ such that $j_Q(X) = j_Q(Y)$.

Lemma 4.2. *Let $\mathcal{C} \subset L^0_c$ and $Q \in \mathfrak{P}_c(\Omega)$. $\mathcal{C}$ is locally Q-closed if and only if $j_Q(\mathcal{C})$ is Q-closed.*

Proof. We may assume that $\mathcal{C} \neq \emptyset$. Suppose that $\mathcal{C}$ is locally Q-closed. Let $(X_n^Q)_{n\in\mathbb{N}} \subset j_Q(\mathcal{C})$ and $X^Q \in L^0_Q$ such that $X_n^Q \xrightarrow{Q} X^Q$. Pick $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $j_Q(X_n) = X_n^Q$ and $X \in L^0_c$ such that $j_Q(X) = X^Q$. It follows that $X_n \xrightarrow{Q} X$. As $\mathcal{C}$ is locally Q-closed, there exists $Y \in \mathcal{C}$ such that $j_Q(\mathcal{C}) \ni j_Q(Y) = j_Q(X) = X^Q$. Thus, $\mathcal{C}_Q$ is Q-closed.
Conversely, if $j_Q(\mathcal{C})$ is Q-closed and $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $X \in L^0_Q$ such that $X_n \xrightarrow{Q} X$, then $j_Q(X_n) \xrightarrow{Q} j_Q(X)$ in L^0_Q and thus $j_Q(X) \in j_Q(\mathcal{C})$. Now let $Y \in \mathcal{C}$ such that $j_Q(Y) = j_Q(X)$. □

So far we have encountered two concepts of closedness which arise naturally in our studies: $\mathcal{Q}$-closedness appeared as a necessary condition in Corollary 3.6 whereas local Q-closedness for all $Q \in \mathcal{Q}$ is equivalent to Q-closedness of $\mathcal{C}_Q$ for all $Q \in \mathcal{Q}$ and thus enables a lifting of Theorems 3.1 and 3.3. Interestingly, both notions are equivalent for $\mathcal{P}$-sensitive and solid sets:

Proposition 4.3. *Suppose that $\mathcal{C} \subset L^0_c$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$. If $\mathcal{C}$ is locally Q-closed for all $Q \in \mathcal{Q}$, then $\mathcal{C}$ is $\mathcal{Q}$-closed. If additionally $\mathcal{C}$ is solid, then $\mathcal{C}$ is locally Q-closed for all $Q \in \mathcal{Q}$ if and only if $\mathcal{C}$ is $\mathcal{Q}$-closed.*

Proof. Assume that $\mathcal{C} \neq \emptyset$. Suppose $\mathcal{C}$ is locally Q-closed for each $Q \in \mathcal{Q}$. Let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $X \in L^0_c$ such that $X_n \xrightarrow{\mathcal{Q}} X$. By assumption there exists $Y_Q \in \mathcal{C}$ for each $Q \in \mathcal{Q}$ such that $j_Q(X) = j_Q(Y_Q) \in j_Q(\mathcal{C})$. Since $\mathcal{Q}$ is a reduction set for $\mathcal{C}$ we obtain $X \in \mathcal{C}$. Hence, $\mathcal{C}$ is $\mathcal{Q}$-closed.

Now suppose that $\mathcal{C}$ is also solid and let $\mathcal{C}$ be $\mathcal{Q}$-closed. Fix $Q \in \mathcal{Q}$ and let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $X_n \xrightarrow{Q} X$ for some $X \in L^0_c$. Then there exists a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$ such that $X_{n_k} \to X$ Q-a.s. For an arbitrary choice $g_{n_k} \in X_{n_k}$, $k \in \mathbb{N}$, and $g \in X$ set

$$\{g_{n_k} \to g\} := \{\omega \in \Omega \mid g_{n_k}(\omega) \to g(\omega)\}.^1$$

Note that $Q(\{g_{n_k} \to g\}) = 1$ and

$$\forall \omega \in \Omega \quad g_{n_k}(\omega)\chi_{\{g_{n_k}\to g\}}(\omega) \to g(\omega)\chi_{\{g_{n_k}\to g\}}(\omega).$$

The latter and the fact that every $\tilde{Q} \in \mathcal{Q}$ is consistent with the $\mathcal{P}$-q.s.-order implies

$$X_{n_k}\mathbf{1}_{\{g_{n_k}\to g\}} \xrightarrow{\tilde{Q}} X\mathbf{1}_{\{g_{n_k}\to g\}}$$

for all $\tilde{Q} \in \mathcal{Q}$. By solidness of $\mathcal{C}$ we have $X_{n_k}\mathbf{1}_{\{g_{n_k}\to g\}} \in \mathcal{C}$ for all $k \in \mathbb{N}$, and thus, by $\mathcal{Q}$-closedness, $X\mathbf{1}_{\{g_{n_k}\to g\}} \in \mathcal{C}$. Since $Q(\{g_{n_k} \to g\}) = 1$ we have $j_Q(X) = j_Q(X\mathbf{1}_{\{g_{n_k}\to g\}}) \in j_Q(\mathcal{C})$. Therefore, $\mathcal{C}$ is locally Q-closed. □

One of the more commonly used closedness concepts in robust frameworks is order closedness, see for instance [10] or [13].

Definition 4.4. A net $(X_\alpha)_{\alpha\in I} \subset L^0_c$ is order convergent to $X \in L^0_c$, denoted $X_\alpha \xrightarrow[c]{o} X$, if there is another net $(Y_\alpha)_{\alpha\in I} \subset L^0_c$ with the same index set I which is decreasing ($\alpha, \beta \in I$ and $\alpha \leq \beta$ imply $Y_\beta \preccurlyeq Y_\alpha$), satisfies $\inf_{\alpha\in I} Y_\alpha = 0$, and for all $\alpha \in I$ it holds that $|X_\alpha - X| \preccurlyeq Y_\alpha$. Here, as usual, $\inf_{\alpha\in I} Y_\alpha$ denotes the largest lower bound of the net $(Y_\alpha)_{\alpha\in I}$.

Note that if $\mathcal{P} = \{P\}$, then $c = P$, and hence order convergence on L^0_P with respect to the P-a.s. order is naturally denoted by $X_\alpha \xrightarrow[P]{o} X$.

Definition 4.5.

1. A set $\mathcal{C} \subset L^0_c$ is order closed if for any net $(X_\alpha)_{\alpha\in I} \subset \mathcal{C}$ and $X \in L^0_c$ such that $X_\alpha \xrightarrow[c]{o} X$ it holds that $X \in \mathcal{C}$.

2. A set $\mathcal{C} \subset L^0_c$ is sequentially order closed if for any sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $X \in L^0_c$ such that $X_n \xrightarrow[c]{o} X$ it holds that $X \in \mathcal{C}$.

In the dominated case, for $Q \in \mathfrak{P}(\Omega)$, we know by the super Dedekind completeness of L^0_Q (see [2, Definition 1.43]) that $\mathcal{C} \subset L^0_Q$ is order closed if and only if $\mathcal{C}$ is sequentially order closed, and for solid sets this is well-known to be equivalent to Q-closedness:

Lemma 4.6 (see e.g. [13, Lemma 4.1])**.** *Let $Q \in \mathfrak{P}(\Omega)$ and $\mathcal{C} \subset L^0_Q$ be solid. Then the following are equivalent:*

1. *$\mathcal{C}$ is order closed (with respect to the Q-a.s. order).*

2. *$\mathcal{C}$ is sequentially order closed.*

[1] At this point, we felt we better drop the convention of identifying equivalence classes of random variables with their representatives for a moment.

3. $\mathcal{C}$ *is* Q*-closed.*

Having some other appealing features, in robust frameworks, authors have tended to focus on order convergence as a generalisation of Q-closedness, see [10] or [13]. However, it turns out that in the non-dominated case order closedness is generally not equivalent to sequential order closedness, see for instance Examples 4.13 and 5.12, and that in fact it is the latter notion which is closely related to the other natural robustifications of Q-closedness we have encountered so far, namely $\mathcal{Q}$-closedness or local Q-closedness for all $Q \in \mathcal{Q}$, see Theorem 4.9 below. Before we state Theorem 4.9 we need two auxiliary results:

Lemma 4.7. *Suppose that* $\mathcal{C} \subset L^0_c$ *is solid. Let* $Q \in \mathfrak{P}_c(\Omega)$*. Then* $j_Q(\mathcal{C})$ *is solid.*

Proof. Suppose that $\mathcal{C} \neq \emptyset$ is solid in L^0_c and that $X^Q \in j_Q(\mathcal{C})$ and $Y^Q \in L^0_Q$ satisfy $|Y^Q| \leq_Q |X^Q|$. Pick $\tilde{X} \in \mathcal{C}$ such that $j_Q(\tilde{X}) = X^Q$. Further let $f \in \tilde{X}$ and $g \in Y^Q$ and set $X := [f\chi_{\{|f|\geq|g|\}}]_c$ and $Y := [g\chi_{\{|f|\geq|g|\}}]_c$. Note that $Q(|f| \geq |g|) = 1$ and therefore $j_Q(X) = X^Q$. We have $|Y| \preccurlyeq |X| \preccurlyeq |\tilde{X}|$, and thus $Y \in \mathcal{C}$. Since $j_Q(Y) = Y^Q$ we conclude that $Y^Q \in j_Q(\mathcal{C})$, so $j_Q(\mathcal{C})$ is indeed solid with respect to $\leq_Q$. The assertion in case that $\mathcal{C}$ is solid in L^0_{c+} follows similarly. □

Lemma 4.8. *Suppose that* $\emptyset \neq \mathcal{C} \subset L^0_c$ *is solid and sequentially order closed, and let* $Q \in \mathfrak{P}_c(\Omega)$*. Then* $j_Q(\mathcal{C})$ *is closed with respect to the* Q*-a.s. order in* L^0_Q*.*

Proof. As $j_Q(\mathcal{C})$ is solid according to Lemma 4.7, in oder to show (sequential) order closedness it suffices to consider non-negative increasing sequences $(X^Q_n)_{n\in\mathbb{N}} \subset \mathcal{C}_Q$ (that is $0 \leq_Q X^Q_n \leq_Q X^Q_{n+1}$ for all $n \in \mathbb{N}$) such that the supremum $X^Q \in L^0_Q$ of $(X^Q_n)_{n\in\mathbb{N}}$ exists and to show that $X^Q \in j_Q(\mathcal{C})$, see [2, Lemma 1.15]. Pick $(\tilde{X}_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $j_Q(\tilde{X}_n) = X^Q_n$ for all $n \in \mathbb{N}$. Let $g \in X^Q$ and $g_n \in \tilde{X}_n$ for all $n \in \mathbb{N}$. Consider the event

$$A := \{\sup_{n\in\mathbb{N}} g_n = g\} \cap \bigcap_{n\in\mathbb{N}} \{g_n \leq g_{n+1}\}.$$

Note that $Q(A) = 1$. Set $X_n := [g_n\chi_A]_c$ for all $n \in \mathbb{N}$ and $X := [g\chi_A]_c$. Since $X_n \preccurlyeq \tilde{X}_n$ we conclude by solidness of $\mathcal{C}$ that $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$. One verifies that indeed $X = \sup_{n\in\mathbb{N}} X_n$ in $(L^0_c, \preccurlyeq)$ and $X_n \xrightarrow[c]{o} X$. Hence, by sequential order closedness of $\mathcal{C}$ we obtain $X \in \mathcal{C}$. As $j_Q(X) = X^Q$ we infer that $X^Q \in \mathcal{C}_Q$. □

Theorem 4.9. *Suppose that* $\mathcal{C} \subset L^0_c$ *is solid and* $\mathcal{P}$*-sensitive. Let* $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ *be a reduction set for* $\mathcal{C}$*. Then the following are equivalent:*

1. $\mathcal{C}$ *is sequentially order closed.*
2. $\mathcal{C}$ *is* $\mathcal{Q}$*-closed.*
3. $\mathcal{C}$ *is locally* Q*-closed for each* $Q \in \mathcal{Q}$*.*
4. $j_Q(\mathcal{C})$ *is* Q*-closed in* L^0_Q *for each* $Q \in \mathcal{Q}$*.*
5. $j_Q(\mathcal{C})$ *is order closed with respect to the* Q*-a.s. order on* L^0_Q *for each* $Q \in \mathcal{Q}$*.*
6. $j_Q(\mathcal{C})$ *is sequentially order closed with respect to the* Q*-a.s. order on* L^0_Q *for each* $Q \in \mathcal{Q}$*.*

For the proof of Theorem 4.9 we need an auxiliary lemma:

Lemma 4.10. *Let $(X_n)_{n\in\mathbb{N}} \subset L^0_c$ and $Q \in \mathfrak{P}_c(\Omega)$.*

1. *Suppose that the infimum (supremum) $X = \inf_{n\in\mathbb{N}} X_n$ ($X = \sup_{n\in\mathbb{N}} X_n$) of $(X_n)_{n\in\mathbb{N}}$ in the $\mathcal{P}$-q.s. order exists. Then $j_Q(X) = \inf_{n\in\mathbb{N}} j_Q(X_n)$ ($j_Q(X) = \sup_{n\in\mathbb{N}} j_Q(X_n)$) in L^0_Q, i.e. $j_Q(X)$ is the infimum (supremum) of $(j_Q(X_n))_{n\in\mathbb{N}}$ in the Q-a.s. order.*
2. *Let $Y \in L^0_c$ and suppose that $X_n \xrightarrow[c]{o} Y$ in L^0_c. Then $j_Q(X_n) \xrightarrow[Q]{o} j_Q(Y)$ in L^0_Q.*

Proof. (1.): We only prove the case of the infimum. From $Q \ll \mathcal{P}$ it immediately follows that $j_Q(X)$ is a lower bound for $(j_Q(X_n))_{n\in\mathbb{N}}$. Consider another lower bound $Z^Q \in L^0_Q$ of $(j_Q(X_n))_{n\in\mathbb{N}}$. We have to show that $j_Q(X) \geq_Q Z^Q$. For any choice $f_n \in X_n$ and $g \in Z^Q$ we have that $Q(\{f_n \geq g\}) = 1$ and thus also the event

$$A := \bigcap_{n\in\mathbb{N}} \{f_n \geq g\}$$

satisfies $Q(A) = 1$. Let $Z := [g]_c \mathbf{1}_A + X\mathbf{1}_{A^c} \in L^0_c$. Then $Z \preccurlyeq X_n$, and hence $Z \preccurlyeq X$ which implies $j_Q(X) \geq_Q j_Q(Z) = Z^Q$.
(2.): By definition of order convergence, there exists a decreasing sequence $(Y_n)_{n\in\mathbb{N}} \subset L^0_{c+}$ such that $\inf_{n\in\mathbb{N}} Y_n = 0$ in L^0_c and for all $n \in \mathbb{N}$

$$|X_n - X| \preccurlyeq Y_n.$$

Define $X^Q := j_Q(X)$ and $X^Q_n := j_Q(X_n)$, $Y^Q_n := j_Q(Y_n)$, $n \in \mathbb{N}$. As $Q \ll \mathcal{P}$, we have for all $n \in \mathbb{N}$

$$|X^Q_n - X^Q| \leq_Q Y^Q_n \quad \text{and} \quad 0 \leq_Q Y^Q_{n+1} \leq_Q Y^Q_n.$$

According to 1. $\inf_{n\in\mathbb{N}} Y^Q_n = 0$ in L^0_Q. Hence, $X^Q_n \xrightarrow[Q]{o} X^Q$. □

Proof of Theorem 4.9. (2.) $\Leftrightarrow$ (3.) $\Leftrightarrow$ (4.): see Lemma 4.2 and Proposition 4.3.
(4.) $\Leftrightarrow$ (5.) $\Leftrightarrow$ (6.): follow from Lemma 4.6.
(1.) $\Rightarrow$ (6.): Lemma 4.8.
(6.) $\Rightarrow$ (1.): Assume $\mathcal{C} \neq \emptyset$ and let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $X_n \xrightarrow[c]{o} X \in L^0_c$. According to Lemma 4.10, $j_Q(X_n) \xrightarrow[Q]{o} j_Q(X)$. As $j_Q(\mathcal{C})$ is closed in the Q-a.s. order for any $Q \in \mathcal{Q}$ we obtain $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$. Since $\mathcal{Q}$ is a reduction set for $\mathcal{C}$ we infer $X \in \mathcal{C}$. □

Interestingly, also in the robust case there are situations in which we may add order closedness to the list in Theorem 4.9. This is closely related to the existence of supports of probability measures as introduced in Section 2.2.

Lemma 4.11. *Let $Q \in \mathfrak{P}_c(\Omega)$.*

1. *Suppose that Q is supported. Let $\mathcal{C} \subset L^0_c$ and suppose that the infimum (supremum) $X := \inf \mathcal{C}$ ($X := \sup \mathcal{C}$) exists in L^0_c. Then $j_Q(X) = \inf j_Q(\mathcal{C})$ ($j_Q(X) = \sup j_Q(\mathcal{C})$) in L^0_Q. In particular, for any net $(X_\alpha)_{\alpha\in I} \subset L^0_c$ and $X \in L^0_c$ we have that $X_\alpha \xrightarrow[c]{o} X$ implies $j_Q(X_\alpha) \xrightarrow[Q]{o} j_Q(X)$.*

2. *Conversely, suppose that for any net* $(X_\alpha)_{\alpha\in I} \subset L^0_c$ *and* $X \in L^0_c$ *we have that* $X_\alpha \xrightarrow[c]{o} X$ *implies* $j_Q(X_\alpha) \xrightarrow[Q]{o} j_Q(X)$*, then* Q *is supported.*

Proof. (1.): We only prove the case of the infimum. Recalling the observations already made in the proof of Lemma 4.10, we only have to show that any lower bound $Y^Q \in L^0_Q$ of $j_Q(\mathcal{C})$ in L^0_Q satisfies $j_Q(X) \geq_Q Y^Q$. Denote by $S(Q)$ a version of the Q-support. Similar to the proof of Lemma 4.10 we pick $f \in Y^Q$ and define $Y := [f]_c\mathbf{1}_{S(Q)} + X1_{S(Q)^c}$. We have that $Y \preccurlyeq Z$ for all $Z \in \mathcal{C}$. Indeed, let $Z \in \mathcal{C}$ and $g \in Z$ (and thus also $g \in j_Q(Z)$). Since $0 = Q(f > g) = Q(S(Q) \cap \{f > g\})$ we infer that $c(S(Q) \cap \{f > g\}) = 0$ (recall Definition 2.1). Therefore $Y\mathbf{1}_{S(Q)} \preccurlyeq Z\mathbf{1}_{S(Q)}$. X being a lower bound of $\mathcal{C}$ now yields $Y \preccurlyeq Z$. As $Z \in \mathcal{C}$ was arbitrary and as X is the largest lower bound of $\mathcal{C}$ we conclude that $Y \preccurlyeq X$. This in turn implies that $j_Q(X) \geq_Q j_Q(Y) = Y^Q$ where we have used that $Q(S(Q)) = 1$ for the latter equality. The remaining part of the assertion now follows along similar lines as presented in the proof of Lemma 4.10.
(2.): Note that by the dominated convergence theorem, for any measure $P \in \mathfrak{P}(\Omega)$, the linear functional

$$l_P:\ L^\infty_P \ni X \mapsto E_P[X]$$

is always σ-order continuous and thus also order continuous, because L^∞_P is super Dedekind complete. Under the assumption stated in (2.) we thus have that

$$L^\infty_c \ni X \mapsto E_Q[X],$$

which we may view as the composition $l_Q \circ j_Q$, is order continuous. Since the order continuous dual of L^∞_c may be identified with sca_c, see [13, Proposition B.3], we find that Q must be supported. □

Combining Theorem 4.9 with Lemma 4.11 we obtain:

Theorem 4.12. *Suppose that* $\mathcal{C} \subset L^0_c$ *is solid and* $\mathcal{P}$*-sensitive and let* $\mathcal{Q} \subset \mathfrak{P}_c(\Omega) \cap sca_c$ *be a reduction set for* $\mathcal{C}$*. Then the following are equivalent:*

1. $\mathcal{C}$ *is order closed.*
2. $\mathcal{C}$ *is sequentially order closed.*
3. $\mathcal{C}$ *is* $\mathcal{Q}$*-closed.*
4. $\mathcal{C}$ *is locally* Q*-closed for each* $Q \in \mathcal{Q}$*.*
5. $j_Q(\mathcal{C})$ *is* Q*-closed in* L^0_Q *for each* $Q \in \mathcal{Q}$*.*
6. $j_Q(\mathcal{C})$ *is order closed with respect to the* Q*-a.s. order on* L^0_Q *for each* $Q \in \mathcal{Q}$*.*
7. $j_Q(\mathcal{C})$ *is sequentially order closed with respect to the* Q*-a.s. order on* L^0_Q *for each* $Q \in \mathcal{Q}$*.*

Proof. In the view of Theorem 4.9 and as obviously (1.) $\Rightarrow$ (2.), it suffices to prove that (6.) $\Rightarrow$ (1.). But if $\mathcal{C} \neq \emptyset$ and $(X_\alpha)_{\alpha\in I} \subset \mathcal{C}$ and $X \in L^0_c$ satisfy $X_\alpha \xrightarrow[c]{o} X$, then $j_Q(X_\alpha) \xrightarrow[Q]{o} j_Q(X)$ according to Lemma 4.11. Thus (6.) implies that $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$, and $\mathcal{Q}$ being a reduction set for $\mathcal{C}$ now yields $X \in \mathcal{C}$. □

Note that in Theorem 4.12 it is important that we consider a reduction set $\mathcal{Q}$ for $\mathcal{C}$ which is strictly smaller than $\mathfrak{P}_c(\Omega)$ if $ca_c \neq sca_c$. In fact, $ca_c \neq sca_c$ is often the case according to [13, Section 3.3]. In the sequel we will encounter more situations in which the existence of a suitable reduction set with further properties than $\mathfrak{P}_c(\Omega)$ is crucial.
The following example shows that the equivalence (1.) $\Leftrightarrow$ (2.) in Theorem 4.12 generally does not hold if the reduction set is not supported:

Example 4.13. Recall that ca_c and sca_c can be identified with the σ-order and the order continuous dual of L_c^∞, respectively, see, for instance, [13]. That means that

$$L_c^\infty \ni X \mapsto \int X d\mu$$

is σ-order continuous, i.e. for every sequence $(X_n)_{n\in\mathbb{N}} \subset L_c^\infty$ such that $X_n \xrightarrow[c]{o} X \in L_c^\infty$ we have $\int X_n d\mu \to \int X d\mu$, whenever $\mu \in ca_c$, and order continuous, i.e. for every net $(X_\alpha)_{\alpha\in I} \subset L_c^\infty$ such that $X_\alpha \xrightarrow[c]{o} X \in L_c^\infty$ we have $\int X_\alpha d\mu \to \int X d\mu$, whenever $\mu \in sca_c$. [13, Section 3.3] shows that $sca_c \neq ca_c$ is often the case. Hence, let us assume that $sca_c \neq ca_c$ and let $\mu \in ca_{c+} \setminus sca_{c+}$ and consider

$$\mathcal{C}_r := \{X \in L_{c+}^\infty \mid \int X d\mu \leq r\}$$

where $r > 0$. $\mathcal{C}_r$ is obviously convex and solid. Moreover, $\mathcal{C}_r$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} = \{Q\}$ where $Q := \mu(\Omega)^{-1}\mu \in \mathfrak{P}_c(\Omega)$. As $L_c^\infty \ni X \mapsto \int X d\mu$ is not order continuous there exists a decreasing net $(X_\alpha)_{\alpha\in I} \subset L_{c+}^\infty$ with $\inf_{\alpha\in I} X_\alpha = 0$ such that $\inf_{\alpha\in I} \int X_\alpha d\mu =: b > 0$. Let $\beta \in I$. Then the net $Y_\alpha := X_\beta - X_\alpha$, $\alpha \geq \beta$, is increasing and satisfies $0 \preccurlyeq Y_\alpha$ and $Y_\alpha \xrightarrow[c]{o} X_\beta$. However, $(Y_\alpha)_{\alpha\geq\beta} \subset \mathcal{C}_r$ for $r = \int X_\beta d\mu - b$, but $X_\beta \notin \mathcal{C}_r$. Hence, $\mathcal{C}_r$ is sequentially order closed but not order closed.

5 $\mathcal{P}$-Sensitivity Reloaded

In this section we study necessary and sufficient conditions for ensuring $\mathcal{P}$-sensitivity of $\mathcal{C} \subset L_c^0$. We start with some rather evident structural properties.

5.1 $\mathcal{P}$-Sensitivity by Local Defining Conditions

Proposition 5.1. *Let $\emptyset \neq \mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ and suppose that*

$$\mathcal{C} = \bigcap_{Q\in\mathcal{Q}} \{X \in L_c^0 \mid \dagger\, H \in \mathcal{H} \colon Q(A_Q^H(X)) = 1\},$$

where $\dagger \in \{\exists, \forall\}$, $\mathcal{H}$ is a non-empty test set, and for all $Q \in \mathcal{Q}$ the function $A_Q^H \colon L_c^0 \longrightarrow \mathcal{F}$ satisfies $Q(A_Q^H(X) \triangle A_Q^H(Y)) = 0$ whenever $Q(X = Y) = 1$. Then $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$.

Proof. Assume $\mathcal{C} \neq \emptyset$ and let $X \in L_c^0$ such that $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$. Fix $Q \in \mathcal{Q}$. Then there exists $Y \in \mathcal{C}$ such that $j_Q(X) = j_Q(Y)$, that is $Q(X = Y) = 1$. Hence, dependent on the quantifier, there either exists an $H \in \mathcal{H}$ such that, or it holds for all $H \in \mathcal{H}$ that

$$Q(A_Q^H(X)) = Q(A_Q^H(Y)) = 1.$$

As $Q \in \mathcal{Q}$ was arbitrary, $X \in \mathcal{C}$. $\square$

Example 5.2. Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$.

1. (local boundedness condition) Set $\mathcal{H} := \mathbb{N}$ and $A_Q^n(X) := \{\omega \in \Omega \mid f(\omega) \leq n\}$ for some $f \in X$. Then

$$\mathcal{C} := \{X \in L^0_{c+} \mid \forall Q \in \mathcal{Q}\, \exists n \in \mathbb{N} \colon Q(X \leq n) = 1\}$$

is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$ and even convex and solid. However, $\mathcal{C}$ is not sequentially order closed, as we can easily see that $\mathcal{C}$ is not $\mathcal{Q}$-closed.

2. (uniform local boundedness condition) Let $Y_Q \in L^0_{c+}$ for each $Q \in \mathcal{Q}$. Set $\mathcal{H} := \{0\}$ and $A_Q^0(X) := \{\omega \in \Omega \mid f(\omega) \leq g(\omega)\}$ for some $f \in X$ and $g \in Y_Q$. Then

$$\mathcal{C} := \{X \in L^0_{c+} \mid \forall Q \in \mathcal{Q} \colon Q(X \leq Y_Q) = 1\}$$

is $\mathcal{P}$-sensitive, convex, and solid. Clearly, $\mathcal{C}$ is also $\mathcal{Q}$-closed and hence sequentially order closed.

3. (uniform martingale condition) Let $\mathcal{H} := \{(Y, \mathcal{G})\}$ for some sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ and some $Y \in L^0_{c+}$ which admits a $\mathcal{G}$-measurable representative $g \in Y$. The set

$$\mathcal{C} := \{X \in L^0_{c+} \mid \forall Q \in \mathcal{Q}\, \forall f \in E_Q[X \mid \mathcal{G}] \colon f = g\ Q\text{-a.s.}\}$$

is $\mathcal{P}$-sensitive. Here $E_Q[X \mid \mathcal{G}] \in L^0_c(\Omega, \mathcal{G}, Q)$ denotes the equivalence class of conditional expectations under Q of (any representative of) X given $\mathcal{G}$. We could for instance set

$$A_Q^{(Y,\mathcal{G})}(X) := \{\omega \in \Omega \mid f(\omega) = g(\omega)\}$$

for some arbitrary choice $f \in E_Q[X \mid \mathcal{G}]$. Then

$$\mathcal{C} = \{X \in L^0_{c+} \mid \forall Q \in \mathcal{Q} \colon Q(A_Q^{(Y,\mathcal{G})}(X)) = 1\}$$

4. (uniform supermartingale condition) Again let $\mathcal{H} := \{(Y, \mathcal{G})\}$ for some sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ and some $Y \in L^0_{c+}$ which admits a $\mathcal{G}$-measurable representative $g \in Y$. The set

$$\mathcal{C} := \{X \in L^0_{c+} \mid \forall Q \in \mathcal{Q}\, \forall f \in E_Q[X \mid \mathcal{G}] \colon f \leq g\ Q\text{-a.s.}\}$$

is $\mathcal{P}$-sensitive ($A_Q^{(Y,\mathcal{G})}(X) := \{\omega \in \Omega \mid f(\omega) \leq g(\omega)\}$ for some arbitrary choice $f \in E_Q[X \mid \mathcal{G}]$). Moreover, $\mathcal{C}$ is solid, convex, $\mathcal{Q}$-closed. Hence, by Theorems 4.9 and 4.12 $\mathcal{C}$ is sequentially order closed and even order closed if $\mathcal{Q} \subset sca_c$.

5. Let $Y \in L^0_{c+}$. Then the set

$$\mathcal{C} := \{X \in L^0_{c+} \mid X \preccurlyeq Y\} = \{X \in L^0_{c+} \mid \forall P \in \mathcal{P} \colon P(X \leq Y) = 1\}$$

is convex, solid, and sequentially order closed. $\mathcal{C}$ is also $\mathcal{P}$-sensitive according Proposition 5.1. Indeed, set $\mathcal{H} := \{Y\}$ and $A_P^Y(X) := \{\omega \in \Omega \mid f(\omega) \leq g(\omega)\}$, $P \in \mathcal{P} = \mathcal{Q}$, $X \in L^0_c$, where $f \in X$ and $g \in Y$.

5.2 $\mathcal{P}$-Sensitivity and Aggregation

In the following we relate $\mathcal{P}$-sensitivity to the concept of aggregation (cf. [11, 16]).

Definition 5.3. Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$.

1. A family $(X^Q)_{Q\in\mathcal{Q}} \subset L^0_c$ is $\mathcal{Q}$-coherent if there is $X^{\mathcal{Q}} \in L^0_c$ such that

$$\forall Q \in \mathcal{Q} \quad Q(X^Q = X^{\mathcal{Q}}) = 1.$$

   The equivalence class $X^{\mathcal{Q}}$ is called a $\mathcal{Q}$-aggregator of $(X^Q)_{Q\in\mathcal{Q}}$.

2. A set $\mathcal{C} \subset L^0_c$ is called $\mathcal{Q}$-stable if for any $\mathcal{Q}$-coherent family $(X^Q)_{Q\in\mathcal{Q}} \subset \mathcal{C}$ the set $\mathcal{C}$ contains all $\mathcal{Q}$-aggregators of $(X^Q)_{Q\in\mathcal{Q}}$.

Proposition 5.4. *Let $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$. Then a non-empty set $\mathcal{C} \subset L^0_c$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$ if and only if $\mathcal{C}$ is $\mathcal{Q}$-stable.*

Proof. Let $\mathcal{C}$ be $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$. Suppose that $(X^Q)_{Q\in\mathcal{Q}} \subset \mathcal{C}$ is $\mathcal{Q}$-coherent and let $X^{\mathcal{Q}} \in L^0_c$ be a $\mathcal{Q}$-aggregator. It then holds that $j_Q(X^{\mathcal{Q}}) = j_Q(X^Q) \in \mathcal{C}_Q$ for all $Q \in \mathcal{Q}$. Hence, as $\mathcal{Q}$ is a reduction set for $\mathcal{C}$, $X^{\mathcal{Q}} \in \mathcal{C}$ and $\mathcal{C}$ is $\mathcal{Q}$-stable.
Now suppose that $\mathcal{C}$ is $\mathcal{Q}$-stable. Let $X \in \bigcap_{Q\in\mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{C})$. Then there exist $(X^Q)_{Q\in\mathcal{Q}} \subset \mathcal{C}$ such that $j_Q(X^Q) = j_Q(X) \;\Leftrightarrow\; Q(X = X^Q) = 1$ for all $Q \in \mathcal{Q}$. Thus, X is a $\mathcal{Q}$-aggregator for $(X^Q)_{Q\in\mathcal{Q}} \subset \mathcal{C}$ and therefore $X \in \mathcal{C}$. Hence, $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$. □

Example 5.5 (Superhedging)**.** Suppose that the (multivariate) process S in continuous or discrete time describes the discounted price evolution of some financial assets. Let $\mathcal{H}$ be a set of investment strategies and denote the portfolio wealth at terminal time $T > 0$ of some $H \in \mathcal{H}$ as $(H \cdot S)_T$ which is a random variable. The latter will typically coincide with a stochastic integral at time T, and $(H \cdot S)_0 = 0$. The set of superhedgeable claims at cost less than 1 is given by

$$\mathcal{C} := \{X \in L^0_{c+} \mid \exists H \in \mathcal{H} \colon X \preccurlyeq 1 + (H \cdot S)_T\}.$$

A bipolar representation of $\mathcal{C}$ is closely related to so-called martingale measures, i.e. probability measures under which the discounted price process S is a martingale, see Section 7.4. Hence, we are interested in criteria which ensure that $\mathcal{C}$ is $\mathcal{P}$-sensitive. Indeed, according to Proposition 5.4 $\mathcal{C}$ is $\mathcal{P}$-sensitive if and only if $\mathcal{C}$ is $\mathcal{Q}$-stable for some $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$. This however requires some aggregation property of the portfolio wealths $(H \cdot S)_T$. For instance, suppose that $\mathcal{P}$ is of class (S) and that L^0_c is Dedekind complete. Let $\mathcal{Q}$ be a disjoint supported alternative to $\mathcal{P}$, see Lemma 2.4. Then any family $(X^Q)_{Q\in\mathcal{Q}} \subset \mathcal{C}$ is $\mathcal{Q}$-coherent, see Lemma 5.6 below. Let X be a $\mathcal{Q}$-aggregator of $(X^Q)_{Q\in\mathcal{Q}}$ and let $H^Q \in \mathcal{H}$ be such that $X^Q \preccurlyeq 1 + (H^Q \cdot S)_T$. Consider any $\mathcal{Q}$-aggregator Y of the terminal wealths $((H^Q \cdot S)_T)_{Q\in\mathcal{Q}}$ which exists by Lemma 5.6. Then

$$X \preccurlyeq 1 + Y.$$

A sufficient condition for $\mathcal{P}$-sensitivity is thus that for any such $\mathcal{Q}$-aggregator of terminal wealths Y there is $H \in \mathcal{H}$ such that $Y = (H \cdot S)_T$.

Lemma 5.6. *Suppose that $\mathcal{P}$ is of class (S) and L^0_c is Dedekind complete. Let $\mathcal{Q}$ denote a disjoint supported alternative to $\mathcal{P}$ (Lemma 2.4). Then any choice $(X^Q)_{Q\in\mathcal{Q}} \subset L^0_{c+}$ is $\mathcal{Q}$-coherent. Moreover, any $\mathcal{Q}$-aggregator X of $(X^Q)_{Q\in\mathcal{Q}}$ satisfies $X\mathbf{1}_{S(Q)} = X^Q\mathbf{1}_{S(Q)}$ for all $Q \in \mathcal{Q}$.*

Proof. The last assertion follows from $X\mathbf{1}_{S(Q)} = X^Q\mathbf{1}_{S(Q)}$ if and only if $Q(X = X^Q) = 1$. For the first assertion let $(X^Q)_{Q\in\mathcal{Q}} \subset L^0_c$. For $n \in \mathbb{N}$ let $X^n \in L^0_{c+}$ denote the least upper bound of the bounded family $(X^Q \wedge n)\mathbf{1}_{S(Q)}$, $Q \in \mathcal{Q}$. Then it follows that $Q(X^n = X^Q \wedge n) = 1$ for all $Q \in \mathcal{Q}$ and thus

$$X^n\mathbf{1}_{S(Q)} = (X^Q \wedge n)\mathbf{1}_{S(Q)} \preccurlyeq X^Q\mathbf{1}_{S(Q)}.$$

Therefore $X^n \preccurlyeq X^{n+1}$ for all $n \in \mathbb{N}$ and the $\mathcal{P}$-quasi sure limit[2] $X := \lim_{n\to\infty} X^n \in L^0_c$ exists and

$$X\mathbf{1}_{S(Q)} = X^Q\mathbf{1}_{S(Q)}.$$

Hence, X is a $\mathcal{Q}$-aggregator of $(X^Q)_{Q\in\mathcal{Q}}$. □

5.3 $\mathcal{P}$-sensitivity as a Consequence of Weak Closedness

Recall the following classical bipolar theorem for locally convex topologies.

Theorem 5.7 (see, e.g., [1, Theorem 5.103])**.** *Let $\langle \mathcal{X}, \mathcal{Y}\rangle$ be a dual pair, see [1, Definition 5.90], and let $\emptyset \neq \mathcal{C} \subset \mathcal{X}$. Define $\mathcal{C}^\circ := \{Y \in \mathcal{Y} \mid \forall X \in \mathcal{C}\colon \langle X, Y\rangle \leq 1\}$ and $\mathcal{C}^{\circ\circ} := \{X \in \mathcal{X} \mid \forall Y \in \mathcal{C}^\circ\colon \langle X, Y\rangle \leq 1\}$. $\mathcal{C} = \mathcal{C}^{\circ\circ}$ if and only if $\mathcal{C}$ is convex, $\sigma(\mathcal{X}, \mathcal{Y})$-closed, and $0 \in \mathcal{C}$.*

The following result shows that $\sigma(\mathcal{X}, \mathcal{Y})$-closedness with respect to some dual pair $\langle \mathcal{X}, \mathcal{Y}\rangle$ where $\mathcal{X} \subset L^0_c$ and $\mathcal{Y} \subset \mathrm{ca}_c$ already implies $\mathcal{P}$-sensitivity.

Theorem 5.8. *Let $\mathcal{X} \subset L^0_c$ and $\mathcal{Y} \subset \mathrm{ca}_c$ be subspaces such that $\langle \mathcal{X}, \mathcal{Y}\rangle$ is a dual pair. Suppose that $\mathcal{C} \subset \mathrm{X}$ is non-empty, convex and $\sigma(\mathcal{X}, \mathcal{Y})$-closed. Then $\mathcal{C}$ is $\mathcal{P}$-sensitive, and we may find a reduction set $\mathcal{Q} \subset \mathcal{Y}$ of $\mathcal{C}$ (in particular $\mathcal{Q} = \mathfrak{P}_c(\Omega) \cap \mathcal{Y}$ does the job).*

Proof. The convex indicator function $f\colon \mathcal{X} \to [0, \infty]$ defined as

$$f(X) := \delta(X \mid \mathcal{C}) = \begin{cases} 0, & X \in \mathcal{C}, \\ \infty, & X \notin \mathcal{C}, \end{cases}$$

is convex and $\sigma(\mathcal{X}, \mathcal{Y})$-lower semi-continuous and thus, by the Fenchel-Moreau theorem,

$$f(X) = f^{**}(X) = \sup_{\mu\in\mathcal{Y}} \int X d\mu - f^*(\mu)$$

where $f^* : \mathcal{Y} \to (-\infty, \infty]$ is given by

$$f^*(\mu) = \sup_{X\in\mathcal{X}} \int X d\mu - f(X).$$

We may thus represent $\mathcal{C}$ as

$$\mathcal{C} = \{X \in \mathcal{X} \mid f(X) = 0\} \; = \bigcap_{\mu\in\mathrm{dom} f^*\setminus\{0\}} \{X \in \mathcal{X} \mid \int X d\mu - f^*(\mu) \leq 0\}, \tag{9}$$

[2] $(X_n) \subset L^0_c$ is said to converge to $X \in L^0_c$ $\mathcal{P}$-quasi surely if $P(X_n \to X) = 1$ for all $P \in \mathcal{P}$.

where $\text{dom} f^* := \{\mu \in \mathcal{Y} \mid f^*(\mu) < \infty\}$ and the last step follows from the fact that for $\mu = 0$

$$f^*(\mu) = -\inf_{Y \in \mathcal{X}} f(Y) = 0 = \int X d\mu$$

for all $X \in \mathcal{X}$. Let $\mathcal{Q} := \{\frac{|\mu|}{|\mu|(\Omega)} \mid \mu \in \text{dom} f^* \setminus \{0\}\}$. We claim that $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$. Indeed let $j_Q(X) \in j_Q(\mathcal{C})$ for all $Q \in \mathcal{Q}$, and $\mu \in \text{dom} f^* \setminus \{0\}$. For $Q := \frac{|\mu|}{|\mu|(\Omega)} \in \mathcal{Q}$ pick $Y \in \mathcal{C}$ such that $j_Q(X) = j_Q(Y)$. As $\mu \ll Q$ it follows that

$$\int X d\mu = \int j_Q(X) d\mu = \int j_Q(Y) d\mu = \int Y d\mu \leq f^*(\mu).$$

Since $\mu \in \text{dom} f^* \setminus \{0\}$ was arbitrary and by (9) we infer that $X \in \mathcal{C}$. $\square$

The next simple example shows that even in a dominated framework the $\mathcal{P}$-sensitive sets in L^0_c do not all coincide with weakly closed sets in some locally convex subspace $\mathcal{X}$ of L^0_c.

Example 5.9. Let $\mathcal{P} = \{P\}$ for a non-atomic probability measure $P \in \mathfrak{P}(\Omega)$. In this case, it is well-known that there is no subspace $\mathcal{Y} \subset ca_P \simeq L^1_P$ such that $\langle L^0_P, \mathcal{Y}\rangle$ is a dual pair. Indeed, for any $\mu \in ca_P \setminus \{0\}$ there is $X \in L^0_{P+}$ such that $\int X d\mu$ is not well-defined or infinite. However, $\mathcal{C} := L^0_{P+}$ is convex, solid, and trivially P-sensitive with reduction set $\{P\}$. Also $\mathcal{C}$ admits a bipolar representation with polar set $\mathcal{C}^\circ = \{\mu \in ca_{P+} \mid \forall X \in \mathcal{C}\colon \int X d\mu \leq 1\} = \{0\}$ and $\mathcal{C}^{\circ\circ} = \{X \in L^0_{c+} \mid 0 \leq 1\} = L^0_{c+} = \mathcal{C}$, see Section 6.

5.4 $\mathcal{P}$-Sensitivity as a Consequence of Class (S) and Order Closedness

As mentioned previously a widely used closedness requirement in robust frameworks is order closedness, see [10, 13] . Supposing that $\mathcal{P}$ is of class (S), we will in the following show that order closedness already implies $\mathcal{P}$-sensitivity.

Lemma 5.10. *Suppose that $\mathcal{P}$ is of class (S) and let $\mathcal{Y} \subset sca_c$ be any linear space separating the points of L^∞_c.*[3] *Moreover, let $\mathcal{C} \subset L^0_{c+}$ be convex, solid, and order closed. Then $\mathcal{C} \cap L^\infty_c$ is $\sigma(L^\infty_c, \mathcal{Y})$-closed.*

Proof. $\tau := |\sigma|(L^\infty_c, \mathcal{Y})$ is a locally convex-solid Hausdorff topology with the Lebesgue property[4] since sca_c may be identified with the order continuous dual of L^∞_c, see, e.g., [13]. Suppose $\mathcal{C} \neq \emptyset$. Consider the set $\mathcal{D} := \mathcal{C} \cap L^\infty_c$. $\mathcal{D}$ is non-empty (because for each $X \in \mathcal{C}$ and $k \in \mathbb{N}$, $X \wedge k \in \mathcal{D}$ by solidity), convex, solid, and order closed. Using [2, Lemma 4.2 and Lemma 4.20], we infer that $\mathcal{D}$ is $|\sigma|(L^\infty_c, \mathcal{Y})$-closed. As $|\sigma|(L^\infty_c, \mathcal{Y})$ and $\sigma(L^\infty_c, \mathcal{Y})$ share the same closed convex sets (see [1, Theorem 8.49 and Corollary 5.83]), $\mathcal{D}$ is $\sigma(L^\infty_c, \mathcal{Y})$-closed. $\square$

Corollary 5.11. *Suppose that $\mathcal{P}$ is of class (S) and let $\mathcal{Y} \subset sca_c$ be any linear space separating the points of L^∞_c. Suppose that $\mathcal{C} \subset L^0_{c+}$ is convex, solid, and order closed. Then $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} \subset \mathfrak{P}_c(\Omega) \cap \mathcal{Y}$.*

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[3]that means for any $X, Y \in L^\infty_c$ such that $X \neq Y$ there is $\mu \in \mathcal{Y}$ such that $\int X d\mu \neq \int Y d\mu$.

[4]For the definition of absolute weak topologies $|\sigma|(\mathcal{X}, \mathcal{Y})$, locally convex-solid topologies, and the Lebesgue property, see [2].

Note that in particular sca_c always separates the points of L^∞_c when $\mathcal{P}$ is of class (S), see [13, Proposition B.5].

Proof. The previous lemma shows that $\mathcal{C} \cap L^\infty_c$ is $\sigma(L^\infty_c, \mathcal{Y})$-closed. According to Theorem 5.8 $\mathcal{C} \cap L^\infty_c$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} \subset \mathcal{Y}$. Suppose that

$$X \in \bigcap_{Q \in \mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{C}).$$

Let $n \in \mathbb{N}$. For all $Q \in \mathcal{Q}$ there is $Y \in \mathcal{C}$ such that $j_Q(Y) = j_Q(X)$. As $\mathcal{C}$ is solid, we have that $Y \wedge n \in \mathcal{C}$ which implies $j_Q(X \wedge n) = j_Q(Y \wedge n) \in j_Q(\mathcal{C})$. As $Q \in \mathcal{Q}$ was arbitrary, and as $\mathcal{Q}$ is a reduction set for $\mathcal{C} \cap L^\infty_c$, we have that $X \wedge n \in \mathcal{C}$ for all $n \in \mathbb{N}$. By order closedness of $\mathcal{C}$ we conclude that indeed $X \in \mathcal{C}$. □

The next example, which can originally be found in [14], gives us an example of a convex, solid, and sequentially order closed set which is not $\mathcal{P}$-sensitive. Moreover, $\mathcal{P}$ will be of class (S) and L^0_c will be Dedekind complete. However, the example is based on assuming the continuum hypothesis, i.e., there is no set $\mathfrak{X}$ whose cardinality satisfies $|\mathbb{N}| = \aleph_0 < |\mathfrak{X}| < 2^{\aleph_0} = |\mathbb{R}|$.

Example 5.12 ([14, Example 3.7])**.** Consider $(\Omega, \mathcal{F}) = ([0,1], \mathbb{P}([0,1]))$, where $\mathbb{P}([0,1])$ denotes the power set of $[0,1]$. Let $\mathcal{P} := \{\delta_\omega \mid \omega \in [0,1]\}$ be the set of all Dirac measures. Apparently, every probability measure in $\mathcal{P}$ is supported, and L^0_c is easily seen to be Dedekind complete. Assume the continuum hypothesis. Banach and Kuratowski have shown that for any set Λ with the same cardinality as $\mathbb{R}$ there is no measure μ on $(\Lambda, \mathbb{P}(\Lambda))$ such that $\mu(\Lambda) = 1$ and $\mu(\{\omega\}) = 0$ for all $\omega \in \Lambda$, see for instance [8, Theorem C.1]. It follows that any probability measure μ on $(\Omega, \mathcal{F})$ must be a countable sum of weighted Dirac-measures, i.e., $\mu = \sum_{i=1}^\infty a_i \delta_{\omega_i}$, where $\sum_{i=1}^\infty a_i = 1$, $a_i \geq 0$, and $\omega_i \in \Omega$ for all $i \in \mathbb{N}$. In this case any probability measure has a countable support, and in particular $ca = ca_c = sca = sca_c$. Now consider the set

$$\mathcal{D} := \{\mathbf{1}_A \mid \emptyset \neq A \subset [0,1] \text{ is countable}\}.$$

and let $\mathcal{C}$ be the solid hull of $\mathcal{D}$. $\mathcal{C}$ can then be written as

$$\mathcal{C} = \{X \in L^0_{c+} \mid \exists Y \in \mathcal{D} \colon 0 \leq X \leq Y\}.$$

$\mathcal{C}$ is clearly convex and solid. Note that every $X \in \mathcal{C}$ is countably supported. Now let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $X_n \xrightarrow[c]{o} X \in L^0_{c+}$. For each $X_n \in \mathcal{C}$ there exists a countable set $A_n \subset [0,1]$ such that $0 \leq X_n \leq \mathbf{1}_{A_n}$. Set $A := \bigcup_{n\in\mathbb{N}} A_n$. A is still countable and it holds that $0 \leq X_n \leq \mathbf{1}_A$ for all $n \in \mathbb{N}$. Hence, $0 \leq X \leq \mathbf{1}_A$ and therefore $X \in \mathcal{C}$. Thus, $\mathcal{C}$ is sequentially order closed.
Let $Q \in \mathfrak{P}_c(\Omega) = \mathfrak{P}(\Omega)$. Q has a countable support $S(Q)$ and therefore $\mathbf{1}_{S(Q)} \in \mathcal{C}$ by definition. Then $j_Q(\mathbf{1}_\Omega) = j_Q(\mathbf{1}_{S(Q)}) \in \mathcal{C}_Q$. As $Q \in \mathfrak{P}_c(\Omega)$ was arbitrary, we have

$$\mathbf{1}_\Omega \in \bigcap_{Q \in \mathfrak{P}_c(\Omega)} j_Q^{-1} \circ j_Q(\mathcal{C}).$$

However, $\mathbf{1}_\Omega \notin \mathcal{C}$. Hence, $\mathcal{C}$ is not $\mathcal{P}$-sensitive. By Corollary 5.11, $\mathcal{C}$ cannot be order closed either. This fact can also be easily directly verified. Indeed, set $I := \{A \subset [0,1] \text{ finite}\}$. For $\alpha, \beta \in I$ we let $\alpha \leq \beta$ if $\alpha \subset \beta$ and set $X_\alpha = \mathbf{1}_\alpha$. Then $(X_\alpha)_{\alpha \in I}$ converges in order to $\mathbf{1}_\Omega \notin \mathcal{C}$. Hence, $\mathcal{C}$ is not order closed.

Example 5.12 also implies that there is no proof of the statement that convexity, solidness, and sequential order closedness imply $\mathcal{P}$-sensitivity:

Corollary 5.13. *Let $\mathcal{C} \subset L^0_{c+}$ be convex, solid, and sequentially order closed. Without further assumptions, there exists no proof that the assumed properties of $\mathcal{C}$ imply $\mathcal{P}$-sensitivity.*

Proof. This follows from Example 5.12 and the fact that the continuum hypothesis is consistent with the standard mathematical axioms (ZFC). □

6 Bipolar Theorems on L^0_{c+}

We will now apply Theorem 3.7 to extend Theorems 3.1 and 3.3 to L^0_{c+}.

Theorem 6.1 (Extension of [12, Corollary 2.7])**.** *Suppose that $\mathcal{C} \subset L^0_{c+}$ is non-empty. Let*

$$\mathcal{C}^\circ := \{(Q,Z) \in \mathfrak{P}_c(\Omega) \times L^\infty_{c+} \mid \forall X \in \mathcal{C}\colon E_Q[ZX] \leq 1\}$$

and

$$\mathcal{C}^{\circ\circ} := \{X \in L^0_{c+} \mid \forall (Q,Z) \in \mathcal{C}^\circ\colon E_Q[ZX] \leq 1\}.$$

Then $\mathcal{C}^{\circ\circ}$ is the smallest $\mathcal{P}$-sensitive, convex, solid, and sequentially order closed subset of L^0_{c+} containing $\mathcal{C}$. In particular, $\mathcal{C} = \mathcal{C}^{\circ\circ}$ if and only if $\mathcal{C}$ is $\mathcal{P}$-sensitive, convex, solid, and sequentially order closed.

Proof. Clearly, $\mathcal{C} \subset \mathcal{C}^{\circ\circ}$, and $\mathcal{P}$-sensitivity, convexity, solidness, and sequentially order closedness of $\mathcal{C}^{\circ\circ}$ have already been proved in Corollary 3.6 and Theorem 4.9.

Now suppose that $\mathcal{C}$ is $\mathcal{P}$-sensitive, convex, solid, and sequentially order closed. Consider any $Q \in \mathfrak{P}_c(\Omega)$. $j_Q(\mathcal{C})$ is clearly convex in L^0_{Q+} and also solid by Lemma 4.7. Moreover, by Theorem 4.9 $j_Q(\mathcal{C})$ is Q-closed. Hence, according to Theorem 3.3, the requirement of Theorem 3.7 is satisfied. This proves $\mathcal{C} = \mathcal{C}^{\circ\circ}$ once we verify that the polar set given in (7) may be identified with $\mathcal{C}^\circ$ as defined in the theorem. To this end, consider the composition $h = E_Q[Z\cdot] \circ j_Q$ where $Q \in \mathfrak{P}_c(\Omega)$ and $Z \in L^\infty_Q$ is an element of the polar $\mathcal{C}^\circ_Q$ of $\mathcal{C}_Q := j_Q(\mathcal{C})$ under Q given in Theorem 3.3, that is h is an element of the polar given in (7). Then for any $\tilde{Z} \in j_Q^{-1}(Z) \cap L^\infty_{c+}$ we have $h(X) = E_Q[Zj_Q(X)] = E_Q[\tilde{Z}X]$, $X \in L^0_{c+}$. In particular $E_Q[\tilde{Z}X] = E_Q[Zj_Q(X)] \leq 1$ for all $X \in \mathcal{C}$ because $Z \in \mathcal{C}^\circ_Q$. Hence, $h = E_Q[\tilde{Z}\cdot]$ and $(Q,\tilde{Z}) \in \mathcal{C}^\circ$. Conversely, let $(Q,Z) \in \mathcal{C}^\circ$, then one verifies that $j_Q(Z) \in \mathcal{C}^\circ_Q$. Therefore, $(L^0_{c+} \ni X \mapsto E_Q[j_Q(Z)j_Q(X)] = E_Q[ZX])$ is an element of the polar given in (7).

Minimality of $\mathcal{C}^{\circ\circ}$ follows by standard arguments. □

In fact, replacing $\mathfrak{P}_c(\Omega)$ by an arbitrary reduction set $\mathcal{Q}$ of $\mathcal{C}$ in the proof of Theorem 6.1 shows that we may even conclude the following representation:

Corollary 6.2. *Suppose that $\mathcal{C} \subset L^0_{c+}$ is non-empty and $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$. Let*

$$\mathcal{C}^{\circ\circ}_{\mathcal{Q}} := \{X \in L^0_{c+} \mid \forall (Q,Z) \in \mathcal{C}^\circ_{\mathcal{Q}}\colon E_Q[ZX] \leq 1\}$$

where

$$\mathcal{C}^\circ_{\mathcal{Q}} := \{(Q,Z) \in \mathcal{Q} \times L^\infty_{c+} \mid \forall X \in \mathcal{C}\colon E_Q[ZX] \leq 1\}.$$

Then $\mathcal{C}^{\circ\circ} = \mathcal{C}^{\circ\circ}_{\mathcal{Q}}$ *where* $\mathcal{C}^{\circ\circ}$ *is given in Theorem 6.1. Moreover, if* $\mathcal{Q} \subset sca_c$ *is disjoint, then*

$$\mathcal{C}^{\circ\circ} = \mathcal{C}^{**}_{\mathcal{Q}} := \{X \in L^0_{c+} \mid \forall Z \in \mathcal{C}^*_{\mathcal{Q}} \colon \sup_{Q \in \mathcal{Q}} E_Q[ZX] \leq 1\}$$

where

$$\mathcal{C}^*_{\mathcal{Q}} := \{Z \in L^\infty_{c+} \mid \forall X \in \mathcal{C} : \sup_{Q \in \mathcal{Q}} E_Q[ZX] \leq 1\}.$$

Proof. Replacing $\mathfrak{P}_c(\Omega)$ by an arbitrary reduction set $\mathcal{Q}$ of $\mathcal{C}$ in the proof of Theorem 6.1 shows that $\mathcal{C}^{\circ\circ}_{\mathcal{Q}}$ is the smallest $\mathcal{P}$-sensitive, convex, solid, and sequentially order closed subset of L^0_{c+} containing $\mathcal{C}$, so it must coincide with $\mathcal{C}^{\circ\circ}$.

Finally, as

$$\mathcal{C}^{**}_{\mathcal{Q}} = \{X \in L^0_{c+} \mid \forall Z \in \mathcal{C}^*_{\mathcal{Q}} \forall Q \in \mathcal{Q} \colon E_Q[ZX] \leq 1\},$$

Corollary 3.6 and Theorem 4.9 show that $\mathcal{C}^{**}_{\mathcal{Q}}$ is a $\mathcal{P}$-sensitive, convex, solid, and sequentially order closed subset of L^0_{c+} containing $\mathcal{C}$. It remains to show that $\mathcal{C}^{**}_{\mathcal{Q}} \subset \mathcal{C}^{\circ\circ}$. To this end, let $X \in \mathcal{C}^{**}_{\mathcal{Q}}$ and $(Q, Z) \in \mathcal{C}^{\circ}_{\mathcal{Q}}$, then $Z\mathbf{1}_{S(Q)} \in \mathcal{C}^*_{\mathcal{Q}}$. Indeed, by disjointness of the supports and as $\mathcal{C} \subset L^0_{c+}$, we obtain

$$\sup_{\tilde{Q} \in \mathcal{Q}} E_{\tilde{Q}}[Z\mathbf{1}_{S(Q)}Y] = E_Q[Z\mathbf{1}_{S(Q)}Y] = E_Q[ZY] \leq 1$$

for all $Y \in \mathcal{C}$. Hence, $Z\mathbf{1}_{S(Q)} \in \mathcal{C}^*_{\mathcal{Q}}$ and thus

$$E_Q[ZX] = \sup_{\tilde{Q} \in \mathcal{Q}} E_{\tilde{Q}}[Z\mathbf{1}_{S(Q)}X] \leq 1.$$

As $(Q, Z) \in \mathcal{C}^{\circ}_{\mathcal{Q}}$ was arbitrary, this implies $X \in \mathcal{C}^{\circ\circ}$. □

Analogously to the proof of Theorem 6.1 we could obtain a lifting of Theorem 3.1, which involves, however, unbounded elements in the polar, or we simply conclude it from Theorem 6.1:

Theorem 6.3 (Extension of [5, Theorem 1.3])**.** *Suppose that* $\mathcal{C} \subset L^0_{c+}$ *is non-empty. Let*

$$\mathcal{C}^{\diamond} := \{(Q, Z) \in \mathfrak{P}_c(\Omega) \times L^0_{c+} \mid \forall X \in \mathcal{C} \colon E_Q[ZX] \leq 1\}$$

and

$$\mathcal{C}^{\diamond\diamond} := \{X \in L^0_{c+} \mid \forall (Q, Z) \in \mathcal{C}^{\diamond} \colon E_Q[ZX] \leq 1\}.$$

Then $\mathcal{C}^{\diamond\diamond}$ *is the smallest* $\mathcal{P}$*-sensitive, convex, solid, and sequentially order closed subset of* L^0_{c+} *containing* $\mathcal{C}$*. In particular,* $\mathcal{C}^{\diamond\diamond} = \mathcal{C}^{\circ\circ}$ *where* $\mathcal{C}^{\circ\circ}$ *is given in Theorem 6.1, and* $\mathcal{C} = \mathcal{C}^{\diamond\diamond}$ *if and only if* $\mathcal{C}$ *is* $\mathcal{P}$*-sensitive, convex, solid, and sequentially order closed.*

Proof. This follows from $\mathcal{C} \subset \mathcal{C}^{\diamond\diamond} \subset \mathcal{C}^{\circ\circ}$ (since $\mathcal{C}^{\circ} \subset \mathcal{C}^{\diamond}$), Corollary 3.6, and Theorems 4.9 and 6.1. □

Of course, also in the case of Theorem 6.3 we may prove a result corresponding to Corollary 6.2, which we, however, leave to the reader. The advantage of the bipolar representation in Theorem 6.1 compared to Theorem 6.3 is that it implies a representation over finite measures:

Corollary 6.4. *Suppose that $\mathcal{C} \subset L^0_{c+}$ is non-empty. Let*

$$\mathcal{C}^{\circ\circ}_{ca} := \{X \in L^0_{c+} \mid \forall \mu \in \mathcal{C}^{\circ}_{ca} \colon \int X d\mu \leq 1\}$$

where

$$\mathcal{C}^{\circ}_{ca} := \{\mu \in ca_{c+} \mid \forall X \in \mathcal{C} \colon \int X d\mu \leq 1\}.$$

Then $\mathcal{C}^{\circ\circ}_{ca} = \mathcal{C}^{\circ\circ}$ where $\mathcal{C}^{\circ\circ}$ is given in Theorem 6.1. Furthermore, if $\mathcal{C}$ is $\mathcal{P}$-sensitive and there is a reduction set $\mathcal{Q} \subset sca_c$, then

$$\mathcal{C}^{\circ\circ} = \mathcal{C}^{\circ\circ}_{sca} := \{X \in L^0_{c+} \mid \forall \mu \in \mathcal{C}^{\circ}_{sca} \colon \int X d\mu \leq 1\},$$

where

$$\mathcal{C}^{\circ}_{sca} := \{\mu \in sca_{c+} \mid \forall X \in \mathcal{C} \colon \int X d\mu \leq 1\}.$$

Both $\mathcal{C}^{\circ}_{ca}$ and $\mathcal{C}^{\circ}_{sca}$ are convex, solid, and $\sigma(ca_c, L^\infty_c)$-closed or $\sigma(sca_c, L^\infty_c)$-closed, respectively. Here solid means that $\mu \in \mathcal{C}^{\circ}_{ca}$ (resp. $\mu \in \mathcal{C}^{\circ}_{sca}$) and $\nu \in ca_{c+}$ (resp. $\nu \in sca_{c+}$) such that $\nu(A) \leq \mu(A)$ for all $A \in \mathcal{F}$ imply $\nu \in \mathcal{C}^{\circ}_{ca}$ (resp. $\nu \in \mathcal{C}^{\circ}_{sca}$)

Proof. Note that any $(Q, Z) \in \mathcal{C}^{\circ}$ from Theorem 6.1 can be identified with a measure $\mu \in ca_c$ given by $\mu(A) = E_Q[Z\mathbf{1}_A]$, $A \in \mathcal{F}$. Hence, we may view $\mathcal{C}^{\circ}$ as a subset of $\mathcal{C}^{\circ}_{ca}$ and therefore

$$\mathcal{C} \subset \mathcal{C}^{\circ\circ}_{ca} \subset \mathcal{C}^{\circ\circ}.$$

$\mathcal{C}^{\circ\circ}_{ca}$ is clearly convex and solid, and also sequentially order closed by the monotone convergence theorem. $\mathcal{P}$-sensitivity of $\mathcal{C}^{\circ\circ}_{ca}$ was shown in Proposition 3.4. Hence, $\mathcal{C}^{\circ\circ}_{ca} = \mathcal{C}^{\circ\circ}$ follows from Theorem 6.1.

The assertion for the case that $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} \subset sca_c$ follows similarly from Corollary 6.2.

Convexity of $\mathcal{C}^{\circ}_{ca}$ and $\mathcal{C}^{\circ}_{sca}$ is easily verified. Regarding solidness, note that if $\nu, \mu \in ca_{c+}$ are such that $\nu(A) \leq \mu(A)$ for all $A \in \mathcal{F}$, then $\int X d\nu \leq \int X d\mu$ for all $X \in L^0_{c+}$. We proceed to prove $\sigma(ca_c, L^\infty_c)$-closedness of $\mathcal{C}^{\circ}_{ca}$: Consider a net $(\mu_\alpha)_{\alpha \in I} \subset \mathcal{C}^{\circ}_{ca}$ such that $\mu_\alpha \to \mu$ with respect to $\sigma(ca_c, L^\infty_c)$. Then for all $X \in \mathcal{C}$ and all $n \in \mathbb{N}$ we have $\int (X \wedge n) d\mu_\alpha \leq \int X d\mu_\alpha \leq 1$ by monotonicity of the integral. Moreover,

$$\int (X \wedge n) d\mu = \lim_\alpha \int (X \wedge n) d\mu_\alpha \leq 1$$

since $(X \wedge n) \in L^\infty_c$. As necessarily $\mu \in ca_{c+}$, the monotone convergence theorem now implies $\int X d\mu \leq 1$. Hence, $\mu \in \mathcal{C}^{\circ}_{ca}$. The same argument shows $\sigma(sca_c, L^\infty_c)$-closedness of $\mathcal{C}^{\circ}_{sca}$. □

Finally, we give the following standard result on $\mathcal{C}^{\circ}_{ca}$ which will be needed in Section 7.6.

Lemma 6.5. *Let $\mathcal{M} \subset ca_{c+}$ be non-empty and define*

$$\mathcal{C} := \{X \in L^0_{c+} \mid \forall \mu \in \mathcal{M} \colon \int X d\mu \leq 1\}.$$

Then $\mathcal{C}^{\circ}_{ca}$ is the smallest solid convex $\sigma(ca_c, L^\infty_c)$-closed subset of ca_{c+} containing $\mathcal{M}$.

The same assertion holds if ca is replaced by sca.

Proof. Clearly, $\mathcal{M} \subset \mathcal{C}_{ca}^{\circ}$. Suppose there is another solid convex $\sigma(ca_c, L_c^\infty)$-closed subset $\mathcal{D}$ of ca_{c+} such that $\mathcal{M} \subset \mathcal{D} \subsetneq \mathcal{C}_{ca}^{\circ}$. Let $\mu \in \mathcal{C}_{ca}^{\circ} \setminus \mathcal{D}$. Then by an appropriate version of the Hahn-Banach separation theorem there is $X \in L_c^\infty$ such that

$$\sup_{\nu \in \mathcal{D}} \int X d\nu =: \beta < \int X d\mu.$$

Note that

$$\beta = \sup_{\nu \in \mathcal{D}} \int X^+ d\nu$$

where $X^+ = \max\{X, 0\}$. Indeed, let $A := \{X \geq 0\}$. By solidness of $\mathcal{D}$, for all $\nu \in \mathcal{D}$ we also have $\nu_A \in \mathcal{D}$ where ν_A is given by $\nu_A(\cdot) = \nu(\cdot \cap A)$ ($\nu_A = 0$ in case $\nu(A) = 0$). Clearly,

$$\int X^+ d\nu = \int X d\nu_A \geq \int X d\nu.$$

Since $\int X d\mu \leq \int X^+ d\mu$, we may from now on assume that $X \in L_{c+}^0$. If $\beta = 0$, then $tX \in \mathcal{C}$ for all $t > 0$. However, there is $t > 0$ such that $\int tX d\mu > 1$, so $tX \notin \mathcal{C}_{ca}^{\circ\circ}$. But this contradicts $\mathcal{C} = \mathcal{C}_{ca}^{\circ\circ}$ (Theorem 6.1 and Corollary 6.4). Similarly, if $\beta > 0$, then $\frac{X}{\beta} \in \mathcal{C}$, but $\frac{X}{\beta} \notin \mathcal{C}_{ca}^{\circ\circ}$ which again contradicts $\mathcal{C} = \mathcal{C}_{ca}^{\circ\circ}$. Hence, μ cannot exist. □

7 Applications

7.1 A Bipolar Theorem given in [10]

Our results imply the following bipolar theorem given in [10]:

Corollary 7.1 ([10, Theorem 14])**.** *Assume that $ca_c^* = L_c^\infty$, i.e. the norm dual space of ca_c can be identified with L_c^∞. Let $\mathcal{C} \subset L_{c+}^0$ be non-empty, convex, order closed, and solid in L_{c+}^0. Set*

$$ca_c^\infty := \operatorname{span}\{\mu_{P,Z} \mid P \in \mathcal{P},\ Z \in L_c^\infty\},$$

the linear space spanned by signed measures of type $\mu_{P,Z}(A) := E_P[Z\mathbf{1}_A]$, $A \in \mathcal{F}$. Then we have

$$\mathcal{C} = \mathcal{C}^{**} := \{X \in L_{c+}^0 \mid \forall \mu \in \mathcal{C}^* \colon \int X d\mu \leq 1\},$$

where

$$\mathcal{C}^* := \{\mu \in ca_{c+}^\infty \mid \forall X \in \mathcal{C} \colon \int X d\mu \leq 1\}.$$

Proof. The condition $ca_c^* = L_c^\infty$ implies that $\mathcal{P}$ is of class (S) ([13, Lemma 5.15]) and that $sca_c = ca_c$ (see [3, Theorem 4.60]). Therefore, in particular, $ca_c^\infty \subset sca_c$. As ca_c^∞ is separating the points of L_c^∞, Corollary 5.11 implies that $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} \subset ca_{c+}^\infty$. The polar $\mathcal{C}_{\mathcal{Q}}^{\circ}$ given in Corollary 6.2 may be viewed as a subset of $\mathcal{C}^*$. Using Theorem 6.1 and Corollary 6.2 it follows that

$$\mathcal{C} \subset \mathcal{C}^{**} \subset \mathcal{C}_{ca}^{\circ\circ} = \mathcal{C}.$$

□

7.2 Another Bipolar Theorem provided in [13]

Our results also imply the following robust bipolar theorem which can be found in [13]:

Theorem 7.2 ([13, Theorem 4.2]). *Suppose that $\mathcal{P}$ is of class (S). Then for all convex and solid sets $\emptyset \neq \mathcal{C} \subset L^0_{c+}$, order closedness of $\mathcal{C}$ is equivalent to $\mathcal{C} = \mathcal{C}^{\circ\circ}_{sca}$ where $\mathcal{C}^{\circ\circ}_{sca}$ is given in Corollary 6.4.*

Proof. According to Corollary 5.11 the set $\mathcal{C}$ is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q} \subset \mathfrak{P}_c(\Omega) \cap sca_c$. Apply Corollary 6.4. □

7.3 Yet another Bipolar Theorem given in [4]

Consider the case $\mathcal{P} = (\delta_\omega)_{\omega\in\Omega}$, so that $\preceq$ coincides with the pointwise order and $L^0_c = \mathcal{L}^0$ and $ca_c = ca$. In [4] the following pointwise bipolar theorem is proved:

Theorem 7.3 ([4, Theorem 1]). *Let $\mathcal{C}$ be a non-empty solid regular subset of $\mathcal{L}^0_+$. Then $\mathcal{C} = \mathcal{C}^{\circ\circ}_{ca}$ (where $\mathcal{C}^{\circ\circ}_{ca}$ is given in Corollary 6.4) if and only if $\mathcal{C}$ is convex and closed under* $\liminf$.

In [4], $\mathcal{C}$ is called regular if

$$\forall \mu \in ca_+ \colon \sup_{h\in\mathcal{C}\cap U_b} \int h d\mu = \sup_{h\in\mathcal{C}\cap C_b} \int h d\mu \tag{10}$$

where C_b and U_b denote the spaces of bounded functions $f \in \mathcal{L}^0$ which are in addition continuous or upper semi-continuous, respectively. Involving continuity properties of course requires that Ω carries a topology, and in fact [4] assume that Ω be a σ-compact metric space, and $\mathcal{F}$ is the corresponding Borel σ-algebra. $\mathcal{C}$ is said to be closed under $\liminf$ whenever $\liminf_{n\to\infty} h_n \in \mathcal{C}$ for any sequence $(h_n)_{n\in\mathbb{N}} \subset \mathcal{C}$. One verifies that for solid sets being closed under $\liminf$ is equivalent to sequential order closedness. In view of Theorem 6.1 and Corollary 6.4 we observe that the rather technical assumption of regularity (10) simply implies $\mathcal{P}$-sensitivity of $\mathcal{C}$. The opposite is generally not true, as the following example shows.

Example 7.4. Suppose that $\Omega = [0, 1]$. Let

$$\mathcal{C} := \{X \in \mathcal{L}^0_+ \mid X \preccurlyeq \mathbf{1}_{[\frac{1}{2},1]}\}.$$

Note that $\mathcal{C}$ is $\mathcal{P}$-sensitive (see Example 5.2 (5)), convex, solid, and sequentially order closed. However, $\mathcal{C}$ is not regular, because $x \mapsto \mathbf{1}_{[\frac{1}{2},1]}$ is upper semi-continuous and for $\mu = \delta_{\frac{1}{2}}$ we have

$$\sup_{X\in\mathcal{C}\cap U_b} \int X d\mu = 1 > 0 = \sup_{X\in\mathcal{C}\cap C_b} \int X d\mu.$$

7.4 Superhedging and Martingale Measures

Recall Example 5.5 and the set of superhedgeable claims at cost less than 1

$$\mathcal{C} = \{X \in L^0_{c+} \mid \exists H \in \mathcal{H} \colon X \preccurlyeq 1 + (H \cdot S)_T\}.$$

Clearly, $\mathcal{C}$ is non-empty, convex, and solid. Suppose that $\mathcal{C}$ is also $\mathcal{P}$-sensitive, see Example 5.5, and sequentially order closed. Then according to Corollary 6.4 $\mathcal{C} = \mathcal{C}^{\circ\circ}_{ca}$. Under some conditions on S and $\mathcal{H}$ the set $\mathcal{C}_{ca}^\circ \cap \mathfrak{P}_c(\Omega)$ is known to coincide with the set of martingale measures for S: A probability measure Q is called a martingale measure for S if the process S is a martingale under Q with respect to some suitable filtration $(\mathcal{F}_t)_{t\geq 0}$ to which S is adapted. The existence of martingale measures is closely related to the arbitrage-freeness of the financial market model S via the Fundamental Theorem of Asset Pricing, see for instance [7, 9] for the details and more information on mathematical finance modeling. To illustrate the basic ideas, in the following suppose for simplicity that S is one-dimensional and bounded and that $(H \cdot S)$ are stochastic integrals. Note that any $Q \in \mathcal{C}_{ca}^\circ \cap \mathfrak{P}_c(\Omega)$ satisfies

$$E_Q[(H \cdot S)_T] \leq 0 \quad \text{for all } H \in \mathcal{H} \text{ such that } -1 \preccurlyeq (H \cdot S)_T. \tag{11}$$

Suppose that $\mathcal{H}$ is rich enough in the sense that all processes $H_a^{A,t}(s,\omega) := a1_A(\omega)1_{(t,T]}(s)$ where $A \in \mathcal{F}_t$, $a > 0$, and $t \in [0,T]$ are elements of $\mathcal{H}$. Note that $(H_a^{A,t} \cdot S)_T = a1_A(S_T - S_t)$. By boundedness of S we find $a > 0$ such that

$$-1 \preccurlyeq (H_a^{A,t} \cdot S)_T = a1_A(S_T - S_t) \preccurlyeq 1.$$

From (11) it follows that $E_Q[1_A(S_T - S_t)] = 0$ and hence the martingale property of S under Q. Conversely, for any martingale measure $Q \in \mathfrak{P}_c(\Omega)$ for S, the stochastic integrals $(H \cdot S)$ are local martingales under Q and the lower bound $-1 \preccurlyeq (H \cdot S)_T$ implies that $(H \cdot S)$ is in fact a supermartingale. Hence,

$$E_Q[(H \cdot S)_T] \leq (H \cdot S)_0 = 0.$$

Thus for any $X \in \mathcal{C}$ it follows that

$$E_Q[X] \leq 1 + E_Q[(H \cdot S)_T] \leq 1,$$

so $Q \in \mathcal{C}_{ca}^\circ \cap \mathfrak{P}_c(\Omega)$.

7.5 Acceptability Criteria for Random Costs/Losses

Identify L_{c+}^0 with random costs/losses. Consider a non-empty set $\mathcal{C} \subset L_{c+}^0$ of acceptable random costs. Assuming that $\mathcal{C}$ is solid means that if some costs are acceptable then less costs are too. Convexity means that cost diversification is not penalised, and sequential order closedness implies that for an order convergent increasing sequence of acceptable losses the limit remains acceptable. Finally, $\mathcal{P}$-sensitivity can be seen as the requirement that acceptability of costs is determined by acceptability under each probability measure $Q \in \mathcal{Q}$ where $\mathcal{Q} \subset \mathfrak{P}_c(\Omega)$ is a test set/reduction set of $\mathcal{C}$. Equivalently $\mathcal{P}$-sensitivity means that aggregated acceptable losses remain acceptable, see Proposition 5.4. Under those conditions Corollary 6.2 provides a dual characterisation of acceptability

$$X \in \mathcal{C} \quad \Leftrightarrow \quad \sup_{(Q,Z) \in \mathcal{C}_{\mathcal{Q}}^\circ} E_Q[ZX] \leq 1$$

where the Z are test functions associated to some test probability $Q \in \mathcal{Q}$.

7.6 A Mass Transport Type Duality

This application is inspired by [4] and a straightforward generalisation of [4, Section 4]. Consider two measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$. Let $\Omega := \Omega_1 \times \Omega_2$ and $\mathcal{F} := \mathcal{F}_1 \otimes \mathcal{F}_2$ denote the product space. Consider probability measures P_1 on $(\Omega_1, \mathcal{F}_1)$ and P_2 on $(\Omega_2, \mathcal{F}_2)$ and the set of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{F})$ consisting of all $P \in \mathfrak{P}(\Omega)$ with marginals $P(\cdot \times \Omega_2) = P_1$ and $P(\Omega_1 \times \cdot) = P_2$. Any $f \in \mathcal{L}^0_+(\Omega)$, which serves as a goal function, gives rise to the optimal mass transport (or Monge-Kantorovich) problem

$$\int f dP \to \max \quad \text{subject to} \quad P \in \mathcal{P}.$$

In fact, as we have been practising so far, we may identify f with the equivalence class $X = [f]_c$ generated by f in $L^0_c(\Omega)$ and write

$$\int X dP \to \max \quad \text{subject to} \quad P \in \mathcal{P} \tag{12}$$

where $c(A) = \sup_{P \in \mathcal{P}} P(A)$, $A \in \mathcal{F}$, is the upper probability corresponding to $\mathcal{P}$ on the product space $(\Omega, \mathcal{F})$, and $L^0_c(\Omega)$ is the space of equivalence classes of $\mathcal{P}$-q.s. equal random variables on $(\Omega, \mathcal{F})$.
A robustification of this problem is obtained by replacing the marginals P_1 and P_2 with sets of marginals $\mathcal{P}_1 \subset \mathfrak{P}(\Omega_1)$ and $\mathcal{P}_2 \subset \mathfrak{P}(\Omega_2)$. We thus obtain the upper probabilities

$$c_1(A) = \sup_{P \in \mathcal{P}_1} P(A),\ A \in \mathcal{F}_1, \quad \text{and} \quad c_2(A) = \sup_{P \in \mathcal{P}_2} P(A),\ A \in \mathcal{F}_2,$$

and the corresponding spaces $L^0_{c_1}(\Omega_1)$ and $L^0_{c_2}(\Omega_2)$ of $\mathcal{P}_i$-q.s. equivalence classes of random variables $\Omega_i \to \mathbb{R}$, $i = 1, 2$, respectively. For $X_1 \in L^0_{c_1}(\Omega_1)$ and $X_2 \in L^0_{c_2}(\Omega_2)$ we write $X_1 \oplus X_2 \in L^0_c(\Omega)$ for the $\mathcal{P}$-q.s. equivalence class given by $f_1 \oplus f_2(\omega) := f_1(\omega_1) + f_2(\omega_2)$, $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ where $f_1 \in X_1$ and $f_2 \in X_2$. Note that the latter is well-defined.
Unfortunately, before we can state our duality result, we have to relax the mass transport problem as follows: Let $\mathcal{M}_i \subset ca_{c_i+}(\Omega_i)$ be a set such that $\mathcal{M}_i = \mathcal{C}^{\circ}_{i,ca}$ for some non-empty, convex, solid, $\mathcal{P}_i$-sensitive, and sequentially order closed sets $\mathcal{C}_i \subset L^0_{c_i}(\Omega_i)$, i=1,2. Then we consider the problem

$$\int X d\mu \to \max \quad \text{subject to} \quad \mu \in \mathcal{M} \tag{13}$$

where $\mathcal{M} \subset ca_{c+}(\Omega)$ is the set of finite measures μ on $(\Omega, \mathcal{F})$ such that the marginals satisfy $\mu(\cdot \times \Omega_2) \in \mathcal{M}_1$ and $\mu(\Omega_1 \times \cdot) \in \mathcal{M}_2$. The dual problem to (13) is given by

$$\sup_{\mu_1 \in \mathcal{M}_1} \int X_1 d\mu_1 + \sup_{\mu_2 \in \mathcal{M}_2} \int X_2 d\mu_2 \to \min \quad \text{subject to} \quad (X_1, X_2) \in \Psi_X \tag{14}$$

where

$$\Psi_X := \{(X_1, X_2) \in L^0_{c_1+}(\Omega_1) \times L^0_{c_2+}(\Omega_2) \mid X \preccurlyeq X_1 \oplus X_2\}.$$

Suppose that the problem (13) is non-trivial in the sense that $\sup_{\mu \in \mathcal{M}} \int X d\mu > 0$. Further suppose that (13) is well-posed in the sense that $\sup_{\mu \in \mathcal{M}} \int X d\mu < \infty$. Then, after a suitable normalisation, we may assume that $\sup_{\mu \in \mathcal{M}} \int X d\mu = 1$. Hence, X is an element of the following set

$$\mathcal{D} := \{Y \in L^0_{c+} \mid \sup_{\mu \in \mathcal{M}} \int Y d\mu \leq 1\}.$$

Consider the set

$$\mathcal{C} := \{Y \in L^0_{c+} \mid \exists (Y_1, Y_2) \in \Psi_Y : \sup_{\mu_1 \in \mathcal{M}_1} \int Y_1 d\mu_1 + \sup_{\mu_2 \in \mathcal{M}_2} \int Y_2 d\mu_2 \leq 1\}.$$

If we are able to show that $\mathcal{C} = \mathcal{D}$, then there is $(X_1, X_2) \in \Psi_X$ such that

$$1 \geq \sup_{\mu_1 \in \mathcal{M}_1} \int X_1 d\mu_1 + \sup_{\mu_2 \in \mathcal{M}_2} \int X_2 d\mu_2 \geq \sup_{\mu \in \mathcal{M}} \int X d\mu = 1.$$

In other words, the dual problem (14) admits a solution (X_1, X_2) and there is no duality gap, i.e.

$$\min_{(X_1, X_2) \in \Psi_X} \sup_{\mu_1 \in \mathcal{M}_1} \int X_1 d\mu_1 + \sup_{\mu_2 \in \mathcal{M}_2} \int X_2 d\mu_2 = \sup_{\mu \in \mathcal{M}} \int X d\mu.$$

Theorem 7.5. *$\mathcal{C} = \mathcal{D}$ if and only if $\mathcal{C}$ is $\mathcal{P}$-sensitive and sequentially order closed.*

Before we prove Theorem 7.5 consider the following auxiliary lemma.

Lemma 7.6. *Let $\mu \in ca_{c+}(\Omega)$ and denote by $\mu_1(\cdot) = \mu(\cdot \times \Omega_2) \in ca_{c_1+}(\Omega_1)$ and $\mu_2(\cdot) = \mu(\Omega_1 \times \cdot) \in ca_{c_2+}(\Omega_2)$ the corresponding marginal distributions. Then*

$$\sup_{X \in \mathcal{C}} \int X d\mu = \max_{i \in \{1,2\}} \sup_{X_i \in \mathcal{C}_i} \int X_i d\mu_i.$$

Consequently,

$$\mathcal{C}^\circ_{ca} = \{\mu \in ca_{c+}(\Omega) \mid \mu_i \in \mathcal{M}_i, i \in \{1,2\}\} = \mathcal{M}.$$

Proof. Consider $X \in \mathcal{C}$ and let $(X_1, X_2) \in \Psi_X$ such that

$$\sup_{\nu_1 \in \mathcal{M}_1} \int X_1 d\nu_1 + \sup_{\nu_2 \in \mathcal{M}_2} \int X_2 d\nu_2 \leq 1.$$

Suppose that $\sup_{\nu_i \in \mathcal{M}_i} \int X_i d\nu_i > 0$, $i = 1, 2$, then

$$\begin{aligned}
\int X d\mu &\leq \int X_1 \oplus X_2 d\mu = \int X_1 d\mu_1 + \int X_2 d\mu_2 \\
&= \sup_{\nu_1 \in \mathcal{M}_1} \int X_1 d\nu_1 \int \frac{X_1}{\sup\limits_{\nu_1 \in \mathcal{M}_1} \int X_1 d\nu_1} d\mu_1 + \sup_{\nu_2 \in \mathcal{M}_2} \int X_2 d\nu_2 \int \frac{X_2}{\sup\limits_{\nu_2 \in \mathcal{M}_2} \int X_2 d\nu_2} d\mu_2 \\
&\leq \sup_{\nu_1 \in \mathcal{M}_1} \int X_1 d\nu_1 \sup_{Y_1 \in \mathcal{C}_1} \int Y_1 d\mu_1 + \sup_{\nu_2 \in \mathcal{M}_2} \int X_2 d\nu_2 \sup_{Y_2 \in \mathcal{C}_2} \int Y_2 d\mu_2 \\
&\leq \max_{i \in \{1,2\}} \sup_{Y_i \in \mathcal{C}_i} \int Y_i d\mu_i
\end{aligned}$$

where we used that

$$\frac{X_i}{\sup\limits_{\nu_i \in \mathcal{M}_i} \int X_i d\nu_i} \in \mathcal{C}^{\circ\circ}_{i,ca} = \mathcal{C}_i \;, i = 1, 2, \qquad \text{(Theorem 6.1 and Corollary 6.4)}$$

for the second inequality. If $\sup_{\nu_i\in\mathcal{M}_i}\int X_i d\nu_i = 0$, then $X_i \in \mathcal{C}_i$ and additionally, for all $t > 0$, $X_i/t \in \mathcal{C}_i$. Without loss of generality assume now that $\sup_{\nu_1\in\mathcal{M}_1}\int X_1 d\nu_1 = 0$. Then $\sup_{\nu_2\in\mathcal{M}_2}\int X_2 d\nu_2 \leq 1$ and therefore $X_2 \in \mathcal{C}_2$. Thus, for all $t > 0$

$$\begin{aligned}\int X d\mu \leq \int X_1 \oplus X_2 d\mu = \int X_1 d\mu_1 + \int X_2 d\mu_2 = t\int \frac{1}{t}X_1 d\mu_1 + \int X_2 d\mu_2 \\ \leq t \sup_{Y_1\in\mathcal{C}_1}\int Y_1 d\mu_1 + \sup_{Y_2\in\mathcal{C}_2}\int Y_2 d\mu_2 \\ \leq (1+t)\max_{i\in\{1,2\}}\sup_{Y_i\in\mathcal{C}_i}\int Y_i d\mu_i\end{aligned}$$

Letting $t \to 0$ shows that indeed $\int X d\mu \leq \max_{i\in\{1,2\}}\sup_{Y_i\in\mathcal{C}_i}\int Y_i d\mu_i$. Hence,

$$\sup_{X\in\mathcal{C}}\int X d\mu \leq \max_{i\in\{1,2\}}\sup_{X_i\in\mathcal{C}_i}\int X_i d\mu_i.$$

In order to show the reverse inequality, for $X_1 \in \mathcal{C}_1$ let $X := X_1 \oplus 0 \in \mathcal{C}$ and for $X_2 \in \mathcal{C}_2$ let $\tilde{X} = 0 \oplus X_2 \in \mathcal{C}$. Then

$$\int X_1 d\mu_1 = \int X d\mu \leq \sup_{Y\in\mathcal{C}}\int Y d\mu \quad \text{and} \quad \int X_2 d\mu_2 = \int \tilde{X} d\mu \leq \sup_{Y\in\mathcal{C}}\int Y d\mu.$$

It follows that

$$\max_{i\in\{1,2\}}\sup_{X_i\in\mathcal{C}_i}\int X_i d\mu_i \leq \sup_{X\in\mathcal{C}}\int X d\mu.$$

Finally,

$$\begin{aligned}\mathcal{C}_{ca}^{\circ} &= \{\mu \in ca_{c+}(\Omega) \mid \forall X \in \mathcal{C}\colon \int X d\mu \leq 1\} \\ &= \{\mu \in ca_{c+}(\Omega) \mid \sup_{X\in\mathcal{C}}\int X d\mu \leq 1\} \\ &= \{\mu \in ca_{c+}(\Omega) \mid \max_{i\in\{1,2\}}\sup_{X_i\in\mathcal{C}_i}\int X_i d\mu_i \leq 1\} \\ &= \{\mu \in ca_{c+}(\Omega) \mid \mu_i \in \mathcal{C}_{i,ca}^{\circ}, i \in \{1,2\}\} = \mathcal{M}.\end{aligned}$$

□

Corollary 7.7. $\mathcal{C}_{ca}^{\circ} = \mathcal{M} = \mathcal{D}_{ca}^{\circ}$.

Proof. This follows from Lemma 6.5, the definition of $\mathcal{D}$, and the the fact that $\mathcal{C}_{ca}^{\circ}$ is solid, convex, and $\sigma(ca_c, L_c^{\infty})$-closed by Corollary 6.4. □

Proof of Theorem 7.5. As $\mathcal{D}$ is $\mathcal{P}$-sensitive and sequentially order closed, see Corollary 3.6 and Theorem 4.9, necessity follows.

Now suppose that $\mathcal{C}$ is $\mathcal{P}$-sensitive and sequentially order closed. It is clear that $\mathcal{C}$ is also non-empty, convex, and solid. $\mathcal{D}$ is non-empty, convex, solid, $\mathcal{P}$-sensitive, and sequentially order closed by definition (see also Proposition 3.4). Hence, by Theorem 6.1 and Corollaries 6.4 and 7.7 we have

$$\mathcal{C} = \mathcal{C}_{ca}^{\circ\circ} = \{X \in L^0_{c+}(\Omega) \mid \forall \mu \in \mathcal{C}_{ca}^{\circ} \colon \int X d\mu \leq 1\} = \mathcal{D}_{ca}^{\circ\circ} = D.$$

□

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