Constructive Convex Optimisation

Josef Berger and Gregor Svindland

May 5, 2021

Abstract

We present first steps towards a constructive theory of convex optimisation. Our results indicate that mathematics in convex environments has some innate constructive nature.

1 Introduction

This contribution is a survey of our research on a constructive approach to convex optimisation. The results we present are taken from [3, 4, 5, 7, 8, 9]. We also refer to [6] for an earlier detailed survey of [3, 4, 5] in which we also present essential parts of the underlying theory in BISH. In this contribution, however, we assume that the reader is familiar with basic terminology and results from constructive analysis such as presented in [13]. We will only briefly introduce some notation, conventions, and notions related to convexity in Section 2 . Many of the results we discuss will not be proved here, we only refer to the respective papers. Nevertheless, where proofs are not too tedious, we will present them, in particular to illustrate applications of our main results. Section 3 considers results on existence of infima and minima for convex functions whereas Section 4 provides the corresponding background in the framework of Brouwer's Fan Theorem. Section 5 discusses some recent results on Lemmas of Alternatives.

2 Some Definitions and Notation

Throughout this article $\|\cdot\|$ will denote the Euclidean norm on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ we denote by x_i, $i = 1, \ldots, n$, the ith coordinate of x, that is $x = (x_1, \ldots, x_n)$. Moreover, we write

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

where $x, y \in \mathbb{R}^n$ for the Euclidean scalar product. If $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is a real matrix and $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, then $A \cdot x$ is the vector in $\mathbb{R}^m$ with ith coordinate

$$(A \cdot x)_i = \sum_{j=1}^{n} a_{ij} x_j$$

whereas $y \cdot A$ is the vector in $\mathbb{R}^n$ with jth coordinate

$$(y \cdot A)_j = \sum_{i=1}^{m} a_{ij} y_i.$$

Whenever $C \subseteq \mathbb{R}^n$ is located

$$d(x, C) := \inf\{\|x - y\| \mid y \in C\}$$

denotes the distance from $x \in \mathbb{R}^n$ to C. In this contribution located sets are always inhabited. Also totally bounded sets, and thus compact sets, are always assumed to be inhabited. A set $C \subseteq \mathbb{R}^n$ is *convex* if it is inhabited and if

$$\forall x, y \in C \, \forall \lambda \in [0, 1] \, (\lambda x + (1 - \lambda) y \in C).$$

This in fact implies that C is closed under finite convex combinations. Let

$$\mathcal{X}_m := \left\{ \lambda \in \mathbb{R}^m \mid \lambda_i \geq 0 (i = 1, \ldots, m), \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

For m points $x^1, \ldots, x^m \in \mathbb{R}^n$ we define the *convex hull*

$$\mathrm{co}(x^1, \ldots, x^m) := \left\{ \sum_{i=1}^{m} \lambda_i x^i \mid \lambda \in \mathcal{X}_m \right\},$$

the *convex cone*

$$\mathrm{cone}(x^1, \ldots, x^m) := \left\{ \sum_{i=1}^{m} \lambda_i x^i \mid \lambda \in \mathbb{R}^m, \lambda_i \geq 0 (i = 1, \ldots, m) \right\},$$

and the *linear space*

$$\mathrm{span}(x^1, \ldots, x^m) := \left\{ \sum_{i=1}^{m} \lambda_i x^i \mid \lambda \in \mathbb{R}^m \right\}$$

generated by $x^1, \dots, x^m$. Let $C \subseteq \mathbb{R}^n$ be convex. A function $f : C \to \mathbb{R}$ is called *convex* if

$$\forall x, y \in C \, \forall \lambda \in [0,1] \; f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

and *quasi-convex* if

$$\forall x, y \in C \, \forall \lambda \in [0,1] \; f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

Clearly, convex functions are quasi-convex.

3 Convexity and Existence of Infima and Minima

In this section we give an overview of results on infima and minima of quasi-convex functions. We recall that a uniformly continuous function $f : C \to \mathbb{R}$ on a compact set C always admits an infimum, see [13, Corollary 2.2.7]. However, if $f : C \to \mathbb{R}^+ := (0, \infty)$, the statement that $\inf f > 0$ is equivalent to Brouwer's Fan Theorem, see [1, 14]. The same holds for the statement that if f admits at most one minimum it has a minimum point, see [1, 2]. Nevertheless, it turns out that if we add convexity to the picture, suddenly those statements are constructively verifiable. The underlying reason is that in fact Brouwer's Fan Theorem is constructively verifiable for so-called co-convex bars, these are bars in $\{0,1\}^*$, the set of all finite binary sequences, possessing a convexity property which we will discuss in Section 3.

Theorem 1. *(see [4, Theorem 1] and [3, Proposition 1]) If $C \subseteq \mathbb{R}^n$ is compact and convex and*

$$f : C \to \mathbb{R}^+$$

is quasi-convex and uniformly continuous, then $\inf f > 0$.

As a first consequence we obtain the following version of Theorem 1 for convex hulls which in contrast to classical mathematics cannot be verified to be closed and thus compact in general. Only special cases like $\mathcal{X}_m$ are indeed compact.

Corollary 1. *(see [3, Corollary 1]) Let $x^1, \dots, x^m \in \mathbb{R}^n$ and suppose that $f : \mathrm{co}(x^1, \dots, x^m) \to \mathbb{R}^+$ is quasi-convex and uniformly continuous. Then* $\inf f > 0$.

Proof. Consider the function

$$\kappa : \mathcal{X}_m \to \mathrm{co}(x^1, \dots, x^m), \;\; \lambda \mapsto \sum_{i=1}^m \lambda_i x^i.$$

The composition $f \circ \kappa$ satisfies the requirements of Theorem 1. □

Theorem 1 also implies the following separating hyperplane result for disjoint convex sets which does not require locatedness of the algebraic difference $Y - C$ such as in [13, Theorem 5.2.9].

Theorem 2. *(see [4, Theorem 2]) Let C and Y be subsets of $\mathbb{R}^n$ and suppose that*

1. *C is convex and compact*
2. *Y is convex, complete, and located*
3. *$\|c - y\| > 0$ for all $c \in C$ and $y \in Y$.*

Then there exist $p \in \mathbb{R}^n$ and reals α, β such that

$$p \cdot c < \alpha < \beta < p \cdot y$$

for all $c \in C$ and $y \in Y$. In particular, the sets C and Y are strictly separated by the hyperplane

$$H = \{x \in \mathbb{R}^n \mid p \cdot x = \gamma\},$$

with $\gamma = \frac{1}{2}(\alpha + \beta)$.

Next we consider existence of minima. A function $f : C \to \mathbb{R}$, where $C \subseteq \mathbb{R}^n$ is inhabited, is said to have *at most one minimum point* if $\inf f$ exists and

$$\forall x, y \in C \; (\|x - y\| > 0 \;\Rightarrow f(x) > \inf f \;\vee\; f(y) > \inf f).$$

Theorem 3. *[7, Theorem 1] Let C be a convex and compact subset of $\mathbb{R}^n$. Then every quasi-convex, uniformly continuous function $f : C \to \mathbb{R}$ with at most one minimum point has a minimum point, that is*

$$\exists x \in C \; f(x) = \inf f.$$

As consequence we obtain supporting hyperplanes for compact, strictly convex sets, see Proposition 1 below. To this end, note that an inhabited subset C of $\mathbb{R}^n$ is *strictly convex* if

$$\lambda x + (1 - \lambda) y \in C^\circ$$

for all $x, y \in C$ with $\|x - y\| > 0$ and all $\lambda \in (0, 1)$. Here the set C°, the *interior* of C, is defined as usual:

$$x \in C^\circ \;\Leftrightarrow\; \exists \varepsilon > 0 \, \forall y \in \mathbb{R}^n \, (\|y - x\| < \varepsilon \;\Rightarrow\; y \in C)$$

Lemma 1. *Fix a subset C of $\mathbb{R}^n$.*

i) If C is convex and open, then it is strictly convex.

ii) If C is strictly convex and closed, then it is convex.

Proof. We only prove ii). Let $x, y \in C$ and $\lambda \in (0,1)$. Define an increasing sequence $(a_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $a_n = 0$ implies $\|x-y\| < \frac{1}{n}$ whereas $a_n = 1$ implies $\|x-y\| > 0$. Next define a sequence

$$x_n = \begin{cases} x & \text{if } a_n = 0 \\ \lambda x + (1-\lambda)y & \text{if } a_n = 1 \end{cases}, \; n \in \mathbb{N}.$$

Note that $(x_n)_{n\in\mathbb{N}} \subseteq C$ is a Cauchy sequence. As C is closed its limit $\lambda x + (1-\lambda)y$ lies in C. For general $\lambda \in [0,1]$ choose a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq (0,1)$ such that $\lambda_n \to \lambda$ and note that by closedness of C we have

$$\lambda x + (1-\lambda)y = \lim_{n\to\infty} \lambda_n x + (1-\lambda_n)y \in C.$$

□

Proposition 1. *(see [7, Proposition 1]) Let $C \subseteq \mathbb{R}^n$ be a compact and strictly convex set. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a linear function such that $g(v) > 0$ for some $v \in \mathbb{R}^n$. Then the restriction of g to C has a minimum point w.*

Proof. Let f denote the restriction of g to C. Note that linear functions are quasi-convex. We will prove that f has at most one minimum point. To this end, consider $x, y \in C$ with $\|x-y\| > 0$. Set $z = (x+y)/2$. Since C is strictly convex, there exists $\delta > 0$ such that $z - \delta v \in C$. We obtain

$$f(z-\delta v) < f(z) \le \max\{f(x), f(y)\}.$$

Thus f has at most one minimum point. By Theorem 3, f has a minimum point. □

In the situation of Proposition 1, the set

$$\mathcal{H} := \{x \in \mathbb{R}^n \mid g(x) = g(w)\}$$

is called a *supporting hyperplane* of C. Indeed, C lies on one side of $\mathcal{H}$, since $\forall x \in C \; g(x) \ge g(w)$, and $\mathcal{H}$ touches C in the point w.
Another consequence of Theorem 3 is that a strictly quasi-convex function defined on a convex compact set possesses a (unique) minimum point. To

this end note that a function $f : C \to \mathbb{R}$ defined on a convex set $C \subseteq \mathbb{R}^n$ is called *strictly quasi-convex* if

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}$$

for all $\lambda \in (0,1)$ and $x, y \in C$ such that $\|x - y\| > 0$. Since strictly quasi-convex functions have at most one minimum point, the following result follows from Theorem 3.

Proposition 2. *(see [7, Proposition 2]) Let $C \subseteq \mathbb{R}^n$ be convex and compact. Then every strictly quasi-convex, uniformly continuous function $f : C \to \mathbb{R}$ has a minimum point.*

Another application of Theorems 1 and 3 in game theory will be given at the end of Section 5.

4 Convexity and Brouwer's Fan Theorem

In this section we give a résumé on the deeper reason why statements equivalent to Brouwer's Fan Theorem become constructively verifiable once we add some convexity assumption. We will see in Theorem 4 that in fact the Fan Theorem is constructively verifiable for so-called co-convex bars. Before we can state this result we need to introduce some further notions and notation related to the Fan Theorem.
We write $\{0,1\}^*$ for the set of all finite binary sequences u, v, w. Let $\emptyset$ be the empty sequence and let $\{0,1\}^{\mathbb{N}}$ be the set of all infinite binary sequences α, β, γ. For every u let $|u|$ be the *length* of u, that is $|\emptyset| = 0$ and for $u = (u_0, \ldots, u_{n-1})$ we have $|u| = n$. For $u = (u_0, \ldots, u_{n-1})$ and $v = (v_0, \ldots, v_{m-1})$ the *concatenation* $u * v$ of u and v is defined by

$$u * v = (u_0, \ldots, u_{n-1}, v_0, \ldots, v_{m-1}).$$

A subset B of $\{0,1\}^*$ is *closed under extension* if $u * v \in B$ for all $u \in B$ and for all v. The *restriction* $\overline{\alpha}n$ of α to n bits is given by $\overline{\alpha}0 = \emptyset$ and

$$\overline{\alpha}n = (\alpha_0, \ldots, \alpha_{n-1}) \quad \text{whenever } n \geq 1.$$

For u with $n \leq |u|$, the restriction $\overline{u}n$ is defined analogously. A sequence α *hits* B if there exists n such that $\overline{\alpha}n \in B$. B is a *bar* if every α hits B. B is a *uniform bar* if there exists N such that for every α there exists $n \leq N$ such that $\overline{\alpha}n \in B$. Often one requires B to be *detachable*, that is for every u the statement $u \in B$ is decidable. *Brouwer's Fan Theorem* for detachable bars is

FAN Every detachable bar is a uniform bar.

FAN is neither provable nor falsifiable in BISH, see [12, Section 3 of Chapter 5]. In their seminal paper [14] Julian and Richman established a correspondence between FAN and functions on $[0,1]$ as follows.

Proposition 3. *For every detachable subset B of $\{0,1\}^*$ there exists a uniformly continuous function $f:[0,1]\to[0,\infty)$ such that*

1) B is a bar $\Leftrightarrow$ f is positive-valued

2) B is a uniform bar $\Leftrightarrow$ $\inf f > 0$.

Conversely, for every uniformly continuous function $f:[0,1]\to[0,\infty)$ there exists a detachable subset B of $\{0,1\}^*$ such that 1) and 2) hold.

Consequently, FAN is equivalent to the statement that every uniformly continuous, positive-valued function defined on the unit interval has positive infimum. In fact, in the latter statement the unit interval may be replaced by compact subsets of $\mathbb{R}^n$, see [1]. Now, in view of Theorem 1, the question arises whether there is a constructively valid convex version of Brouwer's Fan Theorem. To this end, we define

$$u < v \;:\Leftrightarrow\; |u| = |v| \wedge \exists k < |u| \,(\overline{u}k = \overline{v}k \wedge u_k = 0 \wedge v_k = 1)$$

and

$$u \leq v \;:\Leftrightarrow\; u = v \vee u < v.$$

A subset B of $\{0,1\}^*$ is *co-convex* if for every α which hits B there exists n such that either

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B \quad \text{or} \quad \{v \mid \overline{\alpha}n \leq v\} \subseteq B\,.$$

Note that for detachable B co-convexity follows from the convexity of the complement of B, where $C \subseteq \{0,1\}^*$ is *convex* if for all u, v, w we have

$$u \leq v \leq w \wedge u, w \in C \Rightarrow v \in C.$$

The following is the already advertised fan theorem for co-convex bars:

Theorem 4. *(see [5, Theorem 2.1]) Every co-convex bar is a uniform bar.*

Proof. Fix a co-convex bar B. We can and will assume that B is closed under extension, see [5] for the details. Define

$$C = \{u \mid \exists n\, \forall w \in \{0,1\}^n \, (u * w \in B)\} .$$

Note that C consists of the set of nodes beyond which B is uniform, $B \subseteq C$ and that C is closed under extension as well. Moreover, B is a uniform bar if and only if there exists n such that $\{0,1\}^n \subseteq C$.
First, we show that

$$\forall u\, \exists i \in \{0,1\} \, (u * i \in C) . \tag{1}$$

Fix u. For

$$\beta = u * 1 * 0 * 0 * 0 * \ldots$$

there exist an l such that either

$$\left\{v \mid v \leq \overline{\beta}l\right\} \subseteq B,$$

or

$$\left\{v \mid \overline{\beta}l \leq v\right\} \subseteq B.$$

Since B is closed under extension, we can assume that $l > |u| + 1$. Let $m = l - |u| - 1$. If $\left\{v \mid v \leq \overline{\beta}l\right\} \subseteq B$, we can conclude that

$$u * 0 * w \in B$$

for every w of length m, which implies that $u * 0 \in C$. If $\left\{v \mid \overline{\beta}l \leq v\right\} \subseteq B$, we obtain

$$u * 1 * w \in B$$

for every w of length m, which implies that $u * 1 \in C$. This concludes the proof of (1).

By countable choice, there exists a function $F : \{0,1\}^* \to \{0,1\}$ such that

$$\forall u \; (u * F(u) \in C) .$$

Define α by

$$\alpha_n = 1 - F(\overline{\alpha}n).$$

Next, we show by induction on n that

$$\forall n \geq 1\, \forall u \in \{0,1\}^n \, (u \neq \overline{\alpha}n \Rightarrow u \in C) . \tag{2}$$

The case $n = 1$ is easily verified. Now fix some $n \geq 1$ such that (2) holds. Moreover, fix $w \in \{0,1\}^{n+1}$ such that $w \neq \overline{\alpha}(n+1)$.

case 1. $\overline{w}n \neq \overline{\alpha}n$. Then $\overline{w}n \in C$ and therefore $w \in C$.

case 2. $w = \overline{\alpha}n * (1 - \alpha_n) = \overline{\alpha}n * F(\overline{\alpha}n)$. This implies $w \in C$. So we have established (2).

There exists n such that $\overline{\alpha}n \in B$. Applying (2) to this n, we can conclude that every u of length n is an element of C, thus B is a uniform bar. $\square$

Remark 1. *Note that we do not need to require that the co-convex bar in Theorem 4 is detachable.*

In order to include convexity in the list of Proposition 3, we introduce the notion of weakly convex functions. Let S be an inhabited subset of $\mathbb{R}$. A function $f : S \to \mathbb{R}$ is *weakly convex* if for all $t \in S$ with $f(t) > 0$ there exists $\varepsilon > 0$ such that either

$$\forall s \in S \ (s \leq t \Rightarrow f(s) \geq \varepsilon) \quad \text{or} \quad \forall s \in S \ (t \leq s \Rightarrow f(s) \geq \varepsilon) \, .$$

Note that weak convexity is a generalisation of convexity in that uniformly continuous (quasi-) convex functions $f : [0,1] \to \mathbb{R}$ are weakly convex, see [5, Lemma 3.3]. For convex functions we can even drop uniform continuity:

Proposition 4. *(see [8, Proposition 3]) Every convex function $f : [0,1] \to \mathbb{R}$ is weakly convex.*

The following generalisation of Proposition 3 links Theorem 1 with Theorem 4.

Theorem 5. *(see [5, Theorem 3.4]) For every detachable subset B of $\{0,1\}^*$ which is closed under extension there exists a uniformly continuous function $f : [0,1] \to \mathbb{R}$ such that*

1) B is a bar $\Leftrightarrow$ f is positive-valued

2) B is a uniform bar $\Leftrightarrow$ $\inf f > 0$

3) B is co-convex $\Leftrightarrow$ f is weakly convex.

Conversely, for every uniformly continuous function $f : [0,1] \to \mathbb{R}$ there exists a detachable subset B of $\{0,1\}^*$ which is closed under extension such that 1), 2), and 3) hold.

Hence, by Theorems 4 and 5 uniformly continuous and weakly convex functions $f : [0,1] \to \mathbb{R}^+$ have positive infimum. This is a generalisation of Theorem 1 in the one-dimensional case. Indeed, an inspection of the proof of Theorem 1 given in [4] shows that one can replace the required quasi-convexity by the even weaker notion of weak convexity in that proof to directly obtain Theorem 1 for weakly convex functions, which then in conjunction with Theorem 5 again implies Theorem 4. Without studying the fan theorem for bars in $\{0,1\}^*$ we would, however, never have spotted weak convexity as the essential property behind positive infima, see also the discussion at the end of [5].

5 Lemmas of the Alternative and Consequences

Lemmas of the alternative such as Farkas' lemma play an important role in convex optimisation. Constructively valid versions of those results are therefore of great interest. In this section we will present two types of such constructive versions: The first type of results replace the alternatives by equivalences and make some stronger assumptions on the appearing objects. These are useful in applications such as solvability criteria for systems of linear equations, see Propositions 7, 8 and Corollary 2. The second type of constructively valid versions concludes the classical formulation as alternatives from the detachability of a suited set from $\{1,\ldots,k\}$ for some $k \in \mathbb{N}$. We will call these results *conditionally constructive*. The rule of intuitionistic propositional logic

$$((\varphi \vee \neg\varphi) \Rightarrow \neg\psi) \Rightarrow \neg\psi$$

implies that conditionally constructive formulas ν may be used to prove negated statements:

$$(\nu \Rightarrow \neg\psi) \Rightarrow \neg\psi, \tag{3}$$

see [9]. This observation is very useful because lemmas of the alternative often come into play when we wish to derive falsum. Indeed, based on this we will prove a constructive version of optimality criteria in linear programming, see Proposition 9, and we will also provide a simple proof of a constructive version of the von Neumann minimax theorem which first appeared in [11]. As regards the lemmas of the alternative, as in [9], our main reference point will be Farkas' lemma. Throughout this section we will need the following notation: For $x, y \in \mathbb{R}^n$ we write

$$x \leq y \;:\Leftrightarrow\; \forall i \in \{1,\ldots,n\}\,(x_i \leq y_i)\,, \quad y \geq x \;:\Leftrightarrow\; x \leq y,$$

$$x < y \;:\Leftrightarrow\; \forall i \in \{1,\ldots,n\}\,(x_i < y_i)\,, \quad y > x \;:\Leftrightarrow\; x < y,$$

and

$$x \lneq y \;:\Leftrightarrow\; x \leq y \wedge \exists i \in \{1,\ldots,n\}\,(x_i < y_i)\,, \quad x \gneq y \;:\Leftrightarrow\; y \lneq x.$$

Farkas' lemma in its classical formulation states the following: For any real matrix $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ we have

FAR(A,b) Exactly one of the following statements is true.

i) $\exists \xi \in \mathbb{R}^m \, (\xi \cdot A \geq 0 \, \wedge \, \xi \cdot b < 0)$

ii) $\exists q = (q_1, \dots q_n) \in \mathbb{R}^n \, (q_i \geq 0 \, (i = 1, \dots, n) \, \wedge \, A \cdot q = b)$

Farkas' lemma is not constructively verifiable, indeed:

Proposition 5. *(see [9, Proposition 2]) Equivalent are:*

i) $\mathrm{FAR} : \forall A \in \mathbb{R}^{m\times n} \; \forall b \in \mathbb{R}^m \; \mathrm{FAR}(A,b)$

ii) LPO

The following two propositions are constructive versions of Farkas' lemma of the first kind discussed above. We write cone(A) for the convex cone generated by the columns of A, that is $\mathrm{cone}(A) = \mathrm{cone}(a^1, \dots, a^n)$ where $a^i \in \mathbb{R}^m$ denotes the ith column of A. Similarly we will write span(A) for the linear space generated by the columns of A.

Proposition 6. *(see [9, Proposition 3]) Fix a matrix $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. If cone(A) is located, the following are equivalent:*

i) $\exists \xi \in \mathbb{R}^m \, (\xi \cdot A \geq 0 \, \wedge \, \xi \cdot b < 0)$

ii) $d(b, \mathrm{cone}(A)) > 0$

Proof. $i) \Rightarrow ii)$: As $\mathbb{R}^m \ni x \mapsto \xi \cdot x$ is continuous and $\xi \cdot b < 0$, there exists $\delta > 0$ such that

$$\forall x \in \mathbb{R}^m \, (\|b - x\| < \delta \, \Rightarrow \, \xi \cdot x < 0) \, .$$

Fix $x \in \mathrm{cone}(A)$. If $\|b - x\| < \delta$, we can conclude that $\xi \cdot x < 0$, a contradiction. Thus,

$$\forall x \in \mathrm{cone}(A) \, (\|b - x\| \geq \delta) \, .$$

This implies $ii)$.

$ii) \Rightarrow i)$: Set $d := d(b, \mathrm{cone}(A))$. By [3, Lemma 6], there exists $\xi \in \mathbb{R}^n$ such that

$$\forall x \in \mathrm{cone}(A) \left(\xi \cdot (x - b) \geq d^2 \right) .$$

Thus,

$$\forall x \in \mathrm{cone}(A) \left(\xi \cdot x \geq d^2 + \xi \cdot b \right) .$$

Since $0 \in \mathrm{cone}(A)$, we conclude that $\xi \cdot b < 0$. Finally, $\mathrm{cone}(A)$ being a cone implies

$$\forall x \in \mathrm{cone}(A)\,(\xi \cdot x \geq 0)\,.$$

□

Proposition 7. *(see [9, Proposition 4]) Fix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. If* $\mathrm{cone}(A)$ *is located and closed, then the following are equivalent:*

i) $\forall \xi \in \mathbb{R}^m\,(\xi \cdot A \geq 0 \;\Rightarrow\; \xi \cdot b \geq 0)$

ii) $\exists q \in \mathbb{R}^n\;(q_i \geq 0 (i = 1, \ldots, n) \;\wedge\; A \cdot q = b)$

Proof. Since $\mathrm{cone}(A)$ is located and closed, ii) is equivalent to $d(b, \mathrm{cone}(A)) = 0$, that is $\neg(d(b, \mathrm{cone}(A)) > 0)$. Thus the assertion follows from Proposition 6. □

For instance, Proposition 7 implies the following constructive version of the so-called Fredholm alternative:

Proposition 8. *(see [9, Proposition 8]) Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Suppose that* $\mathrm{span}(A)$ *is located and closed. Equivalent are:*

i) $\forall \xi \in \mathbb{R}^m\,(\xi \cdot A = 0 \Rightarrow \xi \cdot b = 0)$*,*

ii) $\exists x \in \mathbb{R}^n\,(A \cdot x = b)$*.*

Proof. Consider the matrix $B := (A \; -A)$, then $\mathrm{cone}(B) = \mathrm{span}(A)$ is closed and located. Hence, by Proposition 7 the following are equivalent

1) $\forall \xi \in \mathbb{R}^m\,(\xi \cdot B \geq 0 \Rightarrow \xi \cdot b \geq 0)$

2) $\exists q \in X_{2n}\,(B \cdot q) = b$.

Now i) is equivalent to 1) and ii) is equivalent to 2). □

As a consequence we obtain a constructive version of the Fredholm alternative for solvability of systems of linear equations.

Corollary 2. *(see [9, Corollary 2]) Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Suppose* $\mathrm{span}(A)$ *is located and closed. If the homogeneous equation $\xi \cdot A = 0$ has a unique solution, then there exists a solution to the system of linear equations $A \cdot x = b$.*

Proof. The unique solution to $\xi \cdot A = 0$ is of course $\xi = 0$, so i) of Proposition 8 is satisfied which implies ii). □

For the second type of constructive versions of lemmas of the alternative we have to introduce the following notion. A formula φ is *conditionally constructive* if there exists a $k \in \mathbb{N}$ and a subset M of $\{1, \ldots, k\}$ such that the detachability of M from $\{1, \ldots, k\}$ implies φ.

Theorem 6. *(see [9, Propositions 5, 7, 11, and 13]) Fix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are conditionally constructive:*

i) $\mathrm{FAR(A, b)}$

ii) The Fredholm alternative for A and b, that is exactly one of the following statements is true:

a) $\exists \xi \in \mathbb{R}^m \, (\xi \cdot A = 0 \wedge |\xi \cdot b| > 0)$

b) $\exists x \in \mathbb{R}^n \, (A \cdot x = b)$

iii) Stiemke's Lemma for A and b, that is exactly one of the following alternatives is true:

a) $\exists \xi \in \mathbb{R}^m \, (\xi \cdot A \gneq 0)$

b) $\exists p \in \mathcal{X}_n \, (p_i > 0 (i = 1, \ldots, n) \ \wedge \ (A \cdot p = 0))$

iv) Exactly one of the following statements is true:

a) $\exists p \in \mathcal{X}_m \, (p \cdot A \geq 0)$

b) $\exists q \in \mathcal{X}_n \, (A \cdot q < 0)$

Theorem 6 allows to derive a number of constructive versions of prominent classical results from convex programming, see [9]. We review a few of those in the following. The first result is on optimlity criteria for linear programming. To this end, consider the following linear optimisation problems: Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. The primal problem is

$$(P) \quad \text{minimise } c \cdot x \quad \text{subject to } x \in \mathcal{P} := \{y \in X_n \mid A \cdot y = b\},$$

whereas the dual problem is

$$(D) \quad \text{maximise } b \cdot u \quad \text{subject to } u \in \mathcal{D} := \{v \in \mathbb{R}^m \mid v \cdot A \leq c\}.$$

Proposition 9. *(see [9, Proposition 10]) Consider the $(m+1) \times n$-matrix*

$$A' = \begin{pmatrix} A \\ c \end{pmatrix}$$

and suppose that $\mathrm{cone}(A')$ is closed and located. If there is a solution u to (D), then there exists a solution x to (P) and $c \cdot x = b \cdot u$.

The well-known von Neumann minimax theorem states that for any matrix $A \in \mathbb{R}^{m\times n}$

$$\max_{p\in\mathcal{X}_m} \min_{q\in\mathcal{X}_n} p\cdot A\cdot q = \min_{q\in\mathcal{X}_n} \max_{p\in\mathcal{X}_m} p\cdot A\cdot q.$$

A thorough discussion of this result in BISH is given in [11]. In that article also the following constructive version of von Neumann's minimax theorem was introduced, see [11, Theorem 2.3]. Here, to illustrate the applicability of conditionally convex statements, we provide a short proof of this result based on Theorem 6 and (3).

Proposition 10. *(see [9, Proposition 14]) Let $A \in \mathbb{R}^{m\times n}$. Then*

$$\sup_{p\in\mathcal{X}_m} \inf_{q\in\mathcal{X}_n} p\cdot A\cdot q = \inf_{q\in\mathcal{X}_n} \sup_{p\in\mathcal{X}_m} p\cdot A\cdot q.$$

Proof. Note that $\inf_{q\in\mathcal{X}_n} p\cdot A\cdot q = \min\{(p\cdot A)_i \mid i = 1,\dots,n\}$, $p \in \mathcal{X}_m$, and $\sup_{p\in\mathcal{X}_m} p\cdot A\cdot q = \max\{(A\cdot q)_j \mid j = 1,\dots,m\}$, $q \in \mathcal{X}_n$, and that the functions

$$\mathcal{X}_m \ni p \mapsto \inf_{q\in\mathcal{X}_n} p\cdot A\cdot q \quad \text{and} \quad \mathcal{X}_n \ni q \mapsto \sup_{p\in\mathcal{X}_m} p\cdot A\cdot q$$

are uniformly continuous, whence

$$\sup_{p\in\mathcal{X}_m} \inf_{q\in\mathcal{X}_n} p\cdot A\cdot q \quad \text{and} \quad \inf_{q\in\mathcal{X}_n} \sup_{p\in\mathcal{X}_m} p\cdot A\cdot q$$

exist, see [13, Corollary 2.2.7]. Clearly,

$$\sup_{p\in\mathcal{X}_m} \inf_{q\in\mathcal{X}_n} p\cdot A\cdot q \le \inf_{q\in\mathcal{X}_n} \sup_{p\in\mathcal{X}_m} p\cdot A\cdot q,$$

so it remains to show that

$$\neg\left(\sup_{p\in\mathcal{X}_m} \inf_{q\in\mathcal{X}_n} p\cdot A\cdot q < \inf_{q\in\mathcal{X}_n} \sup_{p\in\mathcal{X}_m} p\cdot A\cdot q\right).$$

Suppose

$$\sup_{p\in\mathcal{X}_m} \inf_{q\in\mathcal{X}_n} p\cdot A\cdot q < \inf_{q\in\mathcal{X}_n} \sup_{p\in\mathcal{X}_m} p\cdot A\cdot q.$$

Without loss of generality, by suitable translation, we may assume that there exists $\iota > 0$ such that

$$\sup_{p\in\mathcal{X}_m} \inf_{q\in\mathcal{X}_n} p\cdot A\cdot q \le -\iota \quad \text{and} \quad \iota \le \inf_{q\in\mathcal{X}_n} \sup_{p\in\mathcal{X}_m} p\cdot A\cdot q. \tag{4}$$

As we aim at proving falsum, by (3) and Theorem 6 (see [9] for the details) it suffices to consider the cases

$$\exists p \in \mathcal{X}_m \, (p \cdot A \geq 0) \quad \text{or} \quad \exists q \in \mathcal{X}_n \, (A \cdot q < 0) \, .$$

In the first case

$$\sup_{p \in \mathcal{X}_m} \inf_{q \in \mathcal{X}_n} p \cdot A \cdot q \geq 0 > -\iota,$$

a contradiction, and in the second case

$$\inf_{q \in \mathcal{X}_n} \sup_{p \in \mathcal{X}_m} p \cdot A \cdot q \leq 0 < \iota,$$

also a contradiction. □

Now based on Proposition 10 and as a further consequence of Theorem 3 we obtain the following existence result for solutions to two-person zero-sum games; see for instance [15] for a classical discussion of such games.

Proposition 11. *(see [9, Proposition 15]) Let $A \in \mathbb{R}^{m \times n}$. Suppose that*

$$f_A : \, \mathcal{X}_n \ni q \mapsto \sup_{p \in \mathcal{X}_m} p \cdot A \cdot q$$

admits at most one minimum point, and that

$$g_A : \, \mathcal{X}_m \ni p \mapsto \inf_{q \in \mathcal{X}_n} p \cdot A \cdot q$$

admits at most one maximum point, that is $-g_A$ admits at most one minimum point. Then there exists $(\hat{p}, \hat{q}) \in \mathcal{X}_m \times \mathcal{X}_n$ such that

$$\hat{p} \cdot A \cdot \hat{q} = \sup_{p \in \mathcal{X}_m} \inf_{q \in \mathcal{X}_n} p \cdot A \cdot q = \inf_{q \in \mathcal{X}_n} \sup_{p \in \mathcal{X}_m} p \cdot A \cdot q.$$

Proof. Note that $\mathcal{X}_n$ and $\mathcal{X}_m$ are compact and that f_A is convex whereas g_A is concave, that is $-g_A$ is convex. Hence, according to Theorem 3 there exists a minimiser $\hat{q} \in \mathcal{X}_n$ of f_A and a minimiser $\hat{p} \in \mathcal{X}_m$ of $-g_A$, i.e. $\hat{p}$ is a maximiser of g_A. We have

$$\sup_{p \in \mathcal{X}_m} \inf_{q \in \mathcal{X}_n} p \cdot A \cdot q = \inf_{q \in \mathcal{X}_n} \hat{p} \cdot A \cdot q \leq \hat{p} \cdot A \cdot \hat{q} \leq \sup_{p \in \mathcal{X}_m} p \cdot A \cdot \hat{q} = \inf_{q \in \mathcal{X}_n} \sup_{p \in \mathcal{X}_m} p \cdot A \cdot q.$$

Now apply Proposition 10. □

Saddle points $(\hat{p}, \hat{q})$ as in Proposition 11 are called solutions to the two-person zero-sum game given by A. The following Corollary 3 generalises [11, Theorem 3.2] and verifies the conjecture as regards existence of solutions to two-person zero-sum games made at the end of [11]. We will apply Theorem 1 in its proof.

Corollary 3. *(see [9, Corollary 3]) Let $A \in \mathbb{R}^{m\times n}$, and suppose that the associated two-person zero-sum game has at most one solution in the sense of [11], that is, denoting*

$$\alpha := \sup_{p\in\mathcal{X}_m} \inf_{q\in\mathcal{X}_n} p \cdot A \cdot q = \inf_{q\in\mathcal{X}_n} \sup_{p\in\mathcal{X}_m} p \cdot A \cdot q,$$

we have for any pairs $(p, q), (p', q') \in \mathcal{X}_m \times \mathcal{X}_n$ with $\|p - p'\| + \|q - q'\| > 0$ that either $|p \cdot A \cdot q - \alpha| > 0$ or $|p' \cdot A \cdot q' - \alpha| > 0$. Then the game has a unique solution, that is there exists a unique $(\hat{p}, \hat{q}) \in \mathcal{X}_m \times \mathcal{X}_n$ such that

$$\hat{p} \cdot A \cdot \hat{q} = \alpha.$$

Proof. For uniqueness, assume that $(p, q), (p', q') \in \mathcal{X}_m \times \mathcal{X}_n$ are two solutions to the game. Then, as the game has at most one solution, $\|p - p'\| + \|q - q'\| > 0$ is absurd, which implies $(p, q) = (p', q')$.
As regards existence of solutions, we show that the function f_A defined in Proposition 11 admits at most one minimum. Note that $\inf_{q\in\mathcal{X}_n} f_A(q) = \alpha$ and

$$\forall \delta > 0\, \forall q \in \mathcal{X}_n\, \exists p \in \mathcal{X}_m\, (|p \cdot A \cdot q - f_A(q)| < \delta)\,. \tag{5}$$

Fix $q, q' \in \mathcal{X}_n$ and suppose that $\|q - q'\| > 0$. The function

$$\begin{array}{rrcl} h & : \mathcal{X}_m \times \mathcal{X}_m & \to & \mathbb{R} \\ & (p, p') & \mapsto & |p \cdot A \cdot q - \alpha| + |p' \cdot A \cdot q' - \alpha| \end{array}$$

is uniformly continuous, convex, and positive-valued. The latter follows from the assumption that the game has at most one solution. Thus, according to Theorem 1 there exists $\varepsilon > 0$ such that

$$\inf_{(p,p')\in\mathcal{X}_m\times\mathcal{X}_m} h(p, p') > \varepsilon. \tag{6}$$

We have that either $f_A(q) < \alpha+\varepsilon/4$ or $f_A(q) > \alpha$ and either $f_A(q') < \alpha+\varepsilon/4$ or $f_A(q') > \alpha$. Assume that

$$f_A(q) < \alpha + \frac{\varepsilon}{4} \quad \text{and} \quad f_A(q') < \alpha + \frac{\varepsilon}{4}.$$

Then there are $p, p' \in \mathcal{X}_m$ such that

$$|p \cdot A \cdot q - \alpha| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|p' \cdot A \cdot q' - \alpha\right| < \frac{\varepsilon}{2}.$$

This is a contradiction to (6). Thus, either

$$f_A(q) > \alpha \quad \text{or} \quad f_A(q') > \alpha.$$

Similarly, one verifies that g_A defined in Proposition 11 admits at most one maximum. Hence, the assertion follows from Proposition 11. □

References

[1] Josef Berger and Hajime Ishihara, Brouwer's fan theorem and unique existence in constructive analysis. *Mathematical Logic Quarterly*, 51(4), 360–364, 2005.

[2] Josef Berger, Douglas Bridges, Peter Schuster, The fan theorem and unique existence of maxima. *The Journal of Symbolic Logic*, 71(2), 713–720, 2006.

[3] Josef Berger and Gregor Svindland, A separating hyperplane theorem, the fundamental theorem of asset pricing, and Markov's principle. *Annals of Pure and Applied Logic*, 167, 1161–1170, 2016

[4] Josef Berger and Gregor Svindland, Convexity and constructive infima. *Archive for Mathematical Logic*, 55, 873–881, 2016.

[5] Josef Berger and Gregor Svindland, Brouwer's Fan Theorem and convexity. *Journal of Symbolic Logic*, 83(4), 1363–1375, 2018.

[6] Josef Berger and Gregor Svindland, Constructive convex programming. *Proof and Computation: Digitalization in Mathematics, Computer Science, and Philosophy*, World Scientific Publishing Co. Pte. Ltd., 2018.

[7] Josef Berger and Gregor Svindland, Convexity and unique minimum points. *Archive for Mathematical Logic*, 58(1-2), 27–34, 2019.

[8] Josef Berger and Gregor Svindland, Constructive proofs of negated statements. *Mathesis Universalis, Computability and Proof, Springer*, 2019.

[9] Josef Berger and Gregor Svindland, On Farkas' Lemma and related propositions in BISH. *preprint available at* *https://arxiv.org/pdf/2101.03424.pdf*, 2021.

[10] Errett Bishop and Douglas Bridges, *Constructive Analysis.* Springer-Verlag, 1985.

[11] Douglas Bridges, First steps in constructive game theory, *Mathematical Logic Quarterly*, 50, 501–506, 2004.

[12] Douglas Bridges and Fred Richman, *Varieties of Constructive Mathematics.* London Math. Soc. Lecture Notes 97, Cambridge Univ. Press, 1987.

[13] Douglas S. Bridges and Luminiţa Simona Vîţă, *Techniques of Constructive Analysis.* Universitext, Springer-Verlag New York, 2006.

[14] William H. Julian and Fred Richman, A uniformly continuous function on $[0, 1]$ that is everywhere different from its infimum. *Pacific Journal of Mathematics*, 111(2), 333–340, 1984.

[15] David Gale, *The Theory of Linear Economic Models*, University of Chicago Press, 1989.