Capital Requirements and Claims Recovery: A New Perspective on Solvency Regulation

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Abstract

Protection of creditors is a key objective of financial regulation. Where the protection needs are high, i.e., in banking and insurance, regulatory solvency requirements are an instrument to prevent that creditors incur losses on their claims. The current regulatory requirements based on Value at Risk and Average Value at Risk limit the probability of default of financial institutions, but they fail to control the size of recovery on creditors' claims in the case of default. We resolve this failure by developing a novel risk measure, Recovery Value at Risk. Our conceptual approach is flexible and allows the construction of general recovery risk measures for various risk management purposes. We provide detailed case studies and applications. We show that recovery risk measures can be used for performance-based management of business divisions of firms and discuss how to calibrate recovery risk measures to historical regulatory standards. Finally, we analyze how recovery risk measures react to the joint distributions of assets and liabilities on firms' balance sheets and compare the corresponding capital requirements with the current regulatory benchmarks based on Value at Risk and Average Value at Risk.

Keywords: Risk Measures, Capital Requirements, Solvency Regulation, Recovery on Liabilities.

1 Introduction

Banks and insurance companies are subject to a variety of regulatory constraints. A key objective of financial regulation is the appropriate protection of creditors, e.g., depositors, policyholders, and other counterparties. Corporate governance, reporting, and transparency are cornerstones of regulatory schemes, but equally important is capital regulation. Financial companies are required to respect solvency capital requirements that define a minimum for their current net asset value. Firms that fail to meet these requirements are subject to supervisory interventions.

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The computation of solvency capital requirements is often based on some pre-specified notion of acceptable default risk. Banks and insurance companies must hold enough capital to meet their obligations in a sufficient number of future economic scenarios. Regulators typically focus on quantities such as the change of net asset value over a specific time horizon — for example, one year — and require that a suitable risk measure applied to such quantities is below the current level of available capital. The risk measure implicitly defines a notion of acceptable default risk. Different risk measures are applied in practice.

The standard example are solvency capital requirements defined in terms of Value at Risk. In this case, a company is adequately capitalized if its default probability is lower than a given threshold. The upcoming regulatory framework for the internationally active insurance groups uses a Value at Risk at the level 0.5%. In Europe, insurance companies and groups are subject to the same requirement under the Solvency II regime. Value at Risk has been strongly criticized due to its tail blindness and its lack of convexity – not encouraging diversification.

An alternative to Value at Risk is the coherent risk measure Average Value at Risk, also called Conditional or Tail Value at Risk or Expected Shortfall. The market risk standards in Basel III, the international regulatory framework for banks, and the Swiss Solvency Test, the Swiss regulatory framework for insurance companies, are both based on Average Value at Risk with levels 2.5% and 1%, respectively. In this case, a company or portfolio is deemed adequately capitalized, if it generates profits on average conditional on its tail distribution below the chosen level. Average Value at Risk is sensitive to the tail, and, being convex, it does not penalize diversification. It is also a tractable ingredient to optimization problems in the context of asset-liability-management and provides an instrument for decentralized risk management, e.g., limit systems within firms.

Despite all its merits, Average Value at Risk fails — just as Value at Risk — at one central task: It cannot control recovery in the case of default, i.e., the probabilities that creditors recover prespecified fractions of their claims. This goal is, of course, important from a regulatory point of view. Recovering, say, 80% instead of 0% in the case of default makes a big difference to creditors such as depositors or policyholders. This failure is apparent when we consider Value at Risk. By design, the corresponding solvency tests only limit the probability of insolvency and are incapable of imposing any stricter bound on the loss given default.

But the same failure is shared by Average Value at Risk. In spite of being sensitive to tail losses, Average Value at Risk still leaves too many degrees of freedom to control recovery. This is because the loss given default is captured by way of an average loss, which is too gross to exert a fine control on the recovery probability. An additional key deficiency is that all monetary risk measures in current solvency regulation focus on a residual quantity, i.e., the difference between assets and liabilities, that is owned by shareholders. This quantity is insufficient to adequately capture what will happen in the case of default.

The goal of this paper is to address the question:

How should regulators design solvency tests in order to control the recovery on creditors' claims in the case of default?

Our contributions are the following:

I. We demonstrate that classical monetary risk measures such as Value at Risk and Average Value at Risk are unable to control recovery on creditors' claims in the case of default. In fact, we argue that, to capture this important aspect of tail risk, one has to abandon solvency tests based on the net asset value only and consider more articulated solvency tests based on both the net asset value and the firm's liabilities. This is discussed in Section 2.

II. We develop a novel risk measure, Recovery Value at Risk, to successfully address the goal of controlling recovery risk. We demonstrate that Recovery Value at Risk can serve as the basis of solvency tests and discuss its operational interpretation as a capital requirement rule. This new risk measure can be applied to both external and internal risk management and helps to quantify how far standard regulatory risk measures are from controlling recovery risk, thereby improving our understanding of (the limitations of) these standard risk measures. This is discussed in Section 3.

III. Our conceptual approach is flexible and leads to the construction of general recovery risk measures that include Recovery Average Value at Risk. This allows to integrate the ability to control the recovery on creditors' claims with other desirable properties such as convexity or subadditivity. Convexity facilitates applications to optimization problems such as portfolio choice under risk constraints. Subadditivity provides incentives for the diversification of positions and enables limit systems within firms for decentralized risk management. This is discussed in Section 4.1 and in Section 4.2.

IV. We demonstrate how recovery risk measures can be applied to performance-based management of business divisions of firms. We define and investigate the appropriate notion of RoRaC-compatibility. This is discussed in Section 5.2.

V. We discuss a possible strategy to calibrate recovery risk measures consistently with existing regulatory standards, following a common methodology chosen by regulators in the context of classical risk measures. We refer to Section 5.3.

VI. In order to better understand the behavior of recovery risk measures we illustrate how they react to changes of the joint distribution of assets and liabilities on the firm's balance sheet. We focus on two characteristics – marginal distributions and stochastic dependence – and compare risk measurements to the classical solvency benchmarks, i.e., Value at Risk and Average Value at Risk. This is discussed in Section 5.4.

The paper is structured as follows. Section 2 reviews solvency regulation based on Value at Risk and Average Value at Risk with a focus on recovery risk. In Section 3 we introduce the new risk measure Recovery Value at Risk and discuss its main properties. In the parallel Section 4.1 we introduce the convex risk measure Recovery Average Value at Risk. Section 5 focuses on a number of related applications including risk allocation in the context of decentralized risk and performance management of firms. We also discuss calibration issues that arise when solvency regimes are modified. Section 5.4 features detailed case studies providing insights on how risk measures react to the shape of distributions and stochastic dependence. General recovery risk measures are discussed in Section 4.2 in the appendix. All proofs and further technical supplements are collected in Section A of the appendix.

Literature

Solvency capital requirements impose constraints on the operations of businesses such as banks and insurance companies. Their purpose is to protect creditors from excessive downside risk. Capital requirements are an integral part of broader regulatory frameworks that allow companies to freely operate within pre-specified legal boundaries. Historically, regulatory deliberations like Basel I and Solvency I formulated simple rules. However, these could be exploited by regulatory arbitrage, see, e.g., Basel Committee on Banking Supervision (1988), The European Parliament and the Council of the European Union (2002*a*), The European Parliament and the Council of the European Union (2002*b*), and Jones (2000). Regulatory frameworks have been modified multiple times during the past decades, but – as we will demonstrate in this paper – serious problems remain.

A key issue is how to define the required solvency capital in an appropriate manner. Basel II, Solvency II, and the upcoming international Insurance Capital Standard compute solvency capital on the basis of Value at Risk, while Basel III and the Swiss Solvency Test use Average

Value at Risk. The risk measure Value at Risk has been criticized in the context of solvency regulation since the 1990s, in particular due to its tail blindness and lack of convexity. Alternatives are provided by the axiomatic theory of risk measures — initiated in a seminal paper by Artzner, Delbaen, Eber & Heath (1999) — that systematically analyzes properties of risk measures, implications, and examples. The notion of coherent risk measures is introduced in Artzner et al. (1999) and is generalized to the class of convex risk measures in Frittelli & Gianin (2002) and Föllmer & Schied (2002). Key developments are discussed in the monograph Föllmer & Schied (2016) and the surveys Föllmer, Schied & Weber (2009) and Föllmer & Weber (2015). The coherence of Average Value at Risk is first established in Acerbi & Tasche (2002). We refer to Wang & Zitikis (2020) for a recent axiomatic characterization of Average Value at Risk.

Monetary risk measures are based on the notion of acceptability. While preferences rank distributions, random variables, or processes, acceptance sets divide the universe of such objects into acceptable and unacceptable ones. Monetary risk measures can be interpreted as numerical representations of acceptance sets and parallel in this respect utility functionals that represent preferences. Within the theory of choice, risk measures provide a model of guard rails for the actions of financial firms. At the same time, they possess an operational interpretation as capital requirement rules, measuring the distance from acceptability in terms of cash or, more generally, eligible assets. We refer to Föllmer & Schied (2016) for a broad discussion on these aspects and to Filipović & Svindland (2008), Artzner, Delbaen & Koch-Medina (2009), Farkas, Koch-Medina & Munari (2014), Feinstein, Rudloff & Weber (2017), and Biagini, Fouque, Frittelli & Meyer-Brandis (2019) for specific applications to capital adequacy, hedging, risk sharing, and systemic risk. Our paper follows the same approach, i.e., taking the notion of acceptability as the starting point when formalizing recovery-based solvency tests. A different approach is pursued by the literature on acceptability indices that mainly focus on performance measurement, see Aumann & Serrano (2008), Cherny & Madan (2009), Foster & Hart (2009), Brown, Giorgi & Sim (2012), Drapeau & Kupper (2013), Rosazza Gianin & Sgarra (2013), Bielecki, Cialenco & Zhang (2014).[1] A related concept is also the notion of Loss Value at Risk introduced by Bignozzi, Burzoni & Munari (2020). Monetary risk measures have also natural applications to risk allocation problems; see, e.g., Tasche (2000), Kalkbrener (2005), Tasche (2008), Dhaene, Tsanakas, Valdez & Vanduffel (2012), Bauer & Zanjani (2013), Bauer & Zanjani (2016), Embrechts, Liu & Wang (2018), Weber (2018), Hamm, Knispel & Weber (2020), and Guo, Bauer & Zanjani (2020).

To the best of our knowledge, this paper is the first to introduce and study solvency capital requirements that are designed to control the recovery on creditors' claims. The literature on recovery rates has historically focused on explaining the determinants of recovery rates in specific settings, e.g., for corporate and government bonds or bank loans. We refer to Duffie & Singleton (1999), Altman, Brady, Resti & Sironi (2005), and Guo, Jarrow & Zeng (2009) for a presentation of different models for recovery rates and to Khieu, Mullineaux & Yi (2012), Jankowitsch, Nagler & Subrahmanyam (2014), and Ivashina, Iverson & Smith (2016) for some recent empirical investigations.

2 Solvency Regulation and Claims Recovery

The protection of creditors is a key goal of capital regulation. To achieve this goal, financial institutions are required to hold a certain amount of capital as a buffer against future losses. The regulatory capital is chosen such that it ensures an acceptable level of safety against the risk of default. The standard rules used in practice to compute solvency capital requirements

[1] Parametric families of Value at Risk were previously studied in this literature. But acceptability indices are applied to fixed univariate positions (modelling net asset values). In our case, parametric families of Value at Risk or different risk measures are applied to bivariate positions (modelling net asset values jointly with liabilities). As a consequence, the formal construction of acceptability and their financial interpretation differs substantially from our approach.

are based on risk measures such as Value at Risk or Average Value at Risk. We demonstrate that these rules are insufficient to provide a satisfactory control on the recovery on creditors' claims and suggest an alternative approach that achieves this goal.

From now on, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we denote by $\mathcal{X}$ some vector space of random variables. We assume that $\mathcal{X}$ contains all bounded random variables with discrete distribution. In particular, $\mathcal{X}$ includes all constant random variables. The subset of positive random variables is denoted by $\mathcal{X}_+$. As usual, we do not distinguish two random variables that coincide on a set of full probability.

2.1 Risk-Sensitive Solvency Regimes

Most existing regulatory frameworks share a "balance sheet approach" to determine capital requirements. The random evolution of assets and liabilities of a financial institution is captured at time horizons specified by regulators, typically one year.[2] The following table displays a stylized balance sheet of a company at a generic time t:

Assets	**Liabilities**
A_t	L_t
	$E_t = A_t - L_t$

The quantity E_t represents the net asset value of the firm and can be either positive or negative depending on whether the asset value A_t is larger than the liability value L_t or not. In the typical setting of a one-year horizon we have two reference dates which are denoted by $t = 0$ (beginning of the year) and $t = 1$ (end of the year). The quantities at time $t = 0$ are known whereas the quantities at time $t = 1$ are random variables. In a risk-sensitive solvency framework, a company is deemed adequately capitalized if its *available capital* E_0 is larger than a suitable *solvency capital requirement* that reflects the inherent risk in the evolution of the balance sheet. This is typically captured by applying a suitable risk measure ρ to the net asset value variation $\Delta E_1 := E_1 - E_0$.[3] The corresponding *solvency test* is formally defined by:[4]

$$\rho(\Delta E_1) \;\leq\; E_0. \tag{1}$$

If ρ is a monetary risk measure such as Value at Risk or Average Value at Risk, condition (1) can be equivalently expressed in terms of the future net asset value E_1 only as

$$\rho(E_1) \;\leq\; 0. \tag{2}$$

The standard risk measures *Value at Risk* (V@R) and *Average Value at Risk* (AV@R) at some level $\alpha \in [0, 1)$ are defined for a random variable X by[5]

$$\mathrm{V@R}_\alpha(X) := \inf\{x \in \mathbb{R}\,;\ \mathbb{P}(X + x < 0) \leq \alpha\},$$

[2]A balance sheet approach requires an internal model of the stochastic evolution of the balance sheet of the financial firm or insurance company that is subject to capital regulation. Many firms do not have sufficient capacities and expertise to implement and analyze such models. For this reason, in practice, simplifications are admissible which may substantially deviate from the original objectives of the regulator. Examples are the standard approach in the Insurance Capital Standard or the standard formula in Solvency II.

[3]In practice, solvency capital requirements may only refer to "unexpected" losses. In this case, E_0 is replaced by the expected value of (the suitably discounted) E_1. In this respect, the European regulatory framework for insurance companies Solvency II is contradictory in itself. We refer to Hamm et al. (2020) for a detailed discussion.

[4]For simplicity, we assume in this paper that interest rates over the one-year horizon are approximately zero. For adjustments on the definition of the solvency tests if interest rates are not zero see Christiansen & Niemeyer (2014).

[5]Throughout the paper we apply the following sign convention: Positive values of X represent a profit or a positive balance, negative values of X represent a loss or a negative balance.

$$\mathrm{AV@R}_\alpha(X) := \begin{cases} \frac{1}{\alpha}\int_0^\alpha \mathrm{V@R}_\beta(X)d\beta, & \text{if } \alpha \in (0,1), \\ \mathrm{esssup}(-X), & \text{if } \alpha = 0. \end{cases}$$

Note that V@R at level $\alpha > 0$ coincides, up to a sign, with the upper α-quantile of the probability distribution of X. Equivalently, it coincides with the lower $(1-\alpha)$-quantile of the distribution of $-X$. Note also that V@R and AV@R at level $\alpha = 0$ correspond to the so-called *worst-case risk measure* which equals the essential supremum of $-X$, i.e., up to a sign, the smallest realization of X.

In insurance regulation, V@R at level $\alpha = 0.5\%$ is used in the Insurance Capital Standard and in Solvency II while AV@R at level $\alpha = 1\%$ is adopted in the Swiss Solvency Test. In banking regulation, AV@R with level $\alpha = 2.5\%$ has recently become the reference risk measure in Basel III, where it replaces V@R at level $\alpha = 1\%$. In a V@R setting, the solvency test (2) can equivalently be reformulated as

$$\mathrm{V@R}_\alpha(E_1) \le 0 \iff \mathbb{P}(E_1 < 0) \le \alpha \iff \mathbb{P}(E_1 \ge 0) \ge 1-\alpha. \tag{3}$$

This shows that a company is adequately capitalized under V@R if it is able to maintain its default probability below a certain level. Similarly, in an AV@R setting, we can equivalently rewrite the solvency test (2) for $\alpha > 0$ as[6]

$$\mathrm{AV@R}_\alpha(E_1) \le 0 \iff \int_0^\alpha \mathrm{V@R}_\beta(E_1)d\beta \le 0 \iff \mathbb{E}(E_1 | E_1 \le -\mathrm{V@R}_\alpha(E_1)) \ge 0. \tag{4}$$

Hence, a company is adequately capitalized under AV@R if on the lower tail beyond the α-quantile it is solvent on average. In this case, we automatically have $\mathbb{P}(E_1 < 0) \le \alpha$. In other words, if we fix the same probability level α, capital adequacy under AV@R is more conservative than capital adequacy under V@R.

It is often stressed that — in contrast to V@R — AV@R is a tail-sensitive risk measure and, hence, captures tail risk in a more comprehensive way. In fact, V@R is completely blind to the tail of the reference loss distribution beyond a certain quantile level. While this is correct, one should bear in mind that AV@R captures tail risk in a *specific* way, namely via expected losses in the tail, thereby leaving many degrees of freedom to the behavior of the tail distribution.

2.2 Claims Recovery Under V@R and AV@R

The point of departure of our contribution is to highlight that risk measures such as V@R and AV@R fail to provide a direct control on a fundamental aspect of tail risk, namely the recovery on creditors' claims. The basic problem is that both risk measures are functions of the net asset value E_1 only. The net asset value summarizes the financial resources of the equity holders without any reference to leverage, i.e., without imposing any direct constraints on the liabilities L_1. However, controlling the recovery on claims requires to deal explicitly with L_1.

This failure is documented by the next proposition. To motivate it, observe that, for given $\alpha \in (0,1)$, the solvency test based on V@R as described in (3) guarantees that the probability of solvency is at least $1-\alpha$. The same is true for the solvency test based on AV@R at the same level because AV@R dominates V@R. The question we ask is if and how the probability $\mathbb{P}(A_1 \ge \lambda L_1)$ of recovering at least a fraction $\lambda \in (0,1)$ of claims can be made higher than the probability of solvency. For V@R and AV@R the answer is negative: The lower bound $1-\alpha$ is sharp for any target fraction of claims payments. In other words, both V@R and AV@R impose the same weak lower bound on recovery probabilities, and this bound cannot be improved upon, even if the target recovery fraction is arbitrarily small.

[6] The second equivalence holds provided the cumulative distribution function of E_1 is, e.g., continuous.

Proposition 1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be nonatomic. For all $\alpha \in (0,1)$ and $\lambda \in (0,1)$ we have*

$$\begin{aligned} 1-\alpha &= \inf\{\mathbb{P}(A \geq \lambda L)\,;\ A, L \in \mathcal{X}_+,\ \mathrm{AV@R}_\alpha(A-L) \leq 0\} \\ &= \inf\{\mathbb{P}(A \geq \lambda L)\,;\ A, L \in \mathcal{X}_+,\ \mathrm{V@R}_\alpha(A-L) \leq 0\}. \end{aligned}$$

Proof. See Section A.1. □

The preceding proposition shows that, from the perspective of controlling the probability of claims recovery (beyond the probability of solvency), there is little difference between V@R and AV@R. This lack of control is not desirable for a financial regulator, as companies that seek to boost the payoff to shareholders are not prevented from taking on excessive risk thereby significantly reducing recovery payments to creditors in the case of their own default. This is illustrated by the following stylized but insightful example.

Example 2. *We consider a scenario space Ω consisting of two states, g (the good state) and b (the bad state). The probability of the bad state is $\mathbb{P}(b) = \frac{\alpha}{2}$ with α close to zero, say $\alpha = 0.5\%$ or $\alpha = 1\%$. An insurance firm sells a policy that results in the following liability schedule:*

$$L_1(\omega) = \begin{cases} 0 & \text{if } \omega = g, \\ 100 & \text{if } \omega = b. \end{cases}$$

The firm can manage its assets by engaging in a stylized financial contract, for instance an internal reinsurance contract, transferring dollars from the good state to the bad state with zero initial cost. More specifically, we assume that the firm can choose one of the following asset profiles at time 1:

$$A_1^k(\omega) = \begin{cases} 100-k & \text{if } \omega = g, \\ k & \text{if } \omega = b, \end{cases} \qquad \text{with} \quad k \in [0, 100].$$

Hedging its liabilities completely would require the firm to choose $k = 100$. However, since the contract transforms dollars in the high probability scenario into dollars in the low probability scenario, this is not attractive from the point of view of the firm. Indeed, for any $k \in [0, 100]$, the firm's net asset value is given by

$$E_1^k(\omega) = \begin{cases} 100-k & \text{if } \omega = g, \\ k-100 & \text{if } \omega = b. \end{cases}$$

Due to limited liability, the corresponding shareholder value is

$$\max\{E_1^k(\omega), 0\} = \begin{cases} 100-k & \text{if } \omega = g, \\ 0 & \text{if } \omega = b. \end{cases}$$

Hence, the choice $k = 0$ is optimal from the perspective of shareholders. We show that this choice is possible under capital requirements based on V@R *and* AV@R*. In fact, the firm is adequately capitalized under* V@R *and* AV@R *at level α regardless of the size of k. Indeed,*

$$\mathrm{V@R}_\alpha(E_1^k) = k - 100 \leq 0, \quad \mathrm{AV@R}_\alpha(E_1^k) = \frac{1}{\alpha}\left(\frac{\alpha}{2}(100-k) + \frac{\alpha}{2}(k-100)\right) = 0.$$

At the same time, this choice is detrimental for the policyholders because it leads to no recovery on the expected claims payment. Indeed, in the default state b, the policyholders' recovery on their claims is equal to $\frac{k}{100}$ and may take any value between 0 and 1, depending on the level of k. For the optimal choice from the perspective of shareholders, namely $k = 0$, the recovery fraction in state b is minimal, in fact zero.

The example shows that pursuing the interests of shareholders might trigger investment decisions with adverse effects on creditors. A solvency framework based on V@R and AV@R fails to disincentivize firms from taking investment decisions that increase shareholders' value at the price of jeopardizing their ability to cover liabilities. In the next section we show how to mitigate this deficiency of current regulatory frameworks.

3 Recovery Value at Risk

In this section we introduce a solvency test that controls the loss given default by imposing suitable bounds on the recovery on creditors' claims. The test is based on a new risk measure called *Recovery Value at Risk*. The main difference with respect to standard risk measures like V@R and AV@R is that Recovery Value at Risk is not a function of the net asset value E_1 only but also of the liabilities L_1. As shown in the previous section, this extension is necessary if we want to explicitly control the recovery on claims. Throughout the section we continue to use the balance sheet notation introduced in Section 2.

3.1 Introducing RecV@R

Creditors receive at least a recovery fraction $\lambda \in [0,1]$ on their claims payments if[7]

$$A_1 \geq \lambda L_1 \iff E_1 + (1-\lambda)L_1 \geq 0. \tag{5}$$

In this event, assets may not be sufficient to meet all obligations, but they cover at least a fraction λ of liabilities. We control recovery by imposing lower bounds on the recovery probabilities

$$\mathbb{P}(A_1 \geq \lambda L_1)$$

for all recovery fractions $\lambda \in [0,1]$.[8] For this purpose, we introduce the following risk measure.

Definition 3. *Let $\gamma : [0,1] \to [0,1)$ be an increasing function. The* Recovery Value at Risk

$$\mathrm{RecV@R}_\gamma : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \cup \{\infty\}$$

with level function γ *is defined by*

$$\mathrm{RecV@R}_\gamma(X,Y) := \sup_{\lambda \in [0,1]} \mathrm{V@R}_{\gamma(\lambda)}(X + (1-\lambda)Y). \tag{6}$$

If the random variables X and Y in Definition 3 are interpreted, respectively, as the net asset value E_1 and liabilities L_1 in a company's balance sheet,[9] the risk measure RecV@R can be used to formulate a solvency test of the form (1), namely

$$\mathrm{RecV@R}_\gamma(\Delta E_1, L_1) \leq E_0. \tag{7}$$

To better interpret the solvency test based on RecV@R, observe that

$$\begin{aligned}
\mathrm{RecV@R}_\gamma(\Delta E_1, L_1) \leq E_0 \quad &\iff \quad \mathrm{RecV@R}_\gamma(E_1, L_1) \leq 0 \\
&\iff \quad \forall\, \lambda \in [0,1] : \quad \mathrm{V@R}_{\gamma(\lambda)}(A_1 - \lambda L_1) \leq 0 \\
&\iff \quad \forall\, \lambda \in [0,1] : \quad \mathbb{P}(A_1 < \lambda L_1) \leq \gamma(\lambda) \\
&\iff \quad \forall\, \lambda \in [0,1] : \quad \mathbb{P}(A_1 \geq \lambda L_1) \geq 1 - \gamma(\lambda).
\end{aligned}$$

[7] For simplicity, we neglect bankruptcy costs (administrative expenses, legal fees, etc.) which can substantially impair the size of recovery. Regulators may improve the efficiency of bankruptcy procedures and thereby decrease their costs, e.g., by requiring *last wills of financial institutions*.

[8] We can rewrite the event of recovering a fraction of λ in different ways, i.e., $\{A_1 \geq \lambda L_1\} = \{A_1 - \lambda L_1 \geq 0\} = \{A_1/L_1 \geq \lambda\}$, where in the last formulation we assume that $L_1 > 0$. In particular, our approach can also be interpreted in terms of target probabilities for future leverage ratios. Focusing on the modified net asset value $A_1 - \lambda L_1$ instead of the ratio A_1/L_1 is more aligned with current regulation and has the mathematical advantage to avoid divisions by zero.

[9] To allow for different applications, we mathematically define recovery risk measures over generic pairs (X,Y) without any restriction on the sign of X and Y and without any specific assumptions about their relationship and interpretation. In the relevant applications, we have $X = E_1$ or $X = \Delta E_1$ and $Y = L_1$.

This shows that (7) is equivalent to requiring that the recovery probabilities satisfy

$$\mathbb{P}(A_1 < \lambda L_1) \leq \gamma(\lambda) \iff \mathbb{P}(A_1 \geq \lambda L_1) \geq 1 - \gamma(\lambda) \tag{8}$$

for every $\lambda \in [0,1]$. This guarantees the desired control on the loss given default. More precisely, the risk measure RecV@R controls the probability with which any given fraction of liabilities is recovered: The level function γ specifies, for each recovery level $\lambda \in [0,1]$, an upper bound $\gamma(\lambda)$ on the probability that the realized recovery level turns out to be lower than λ, or equivalently a lower bound $1-\gamma(\lambda)$ on the probability that the realized recovery level is higher than λ. In particular, the solvency test (7) can be seen as a refinement of the standard solvency test (3) based on V@R where the probability bound α is replaced by a bound that depends on the target recovery fraction through the function γ. The assumption that γ is increasing captures the basic requirement that smaller recovery fractions on liabilities should be guaranteed at higher probability levels. It should be noted that, as soon as $\gamma(0+) := \lim_{\lambda\to 0}\gamma(\lambda) > 0$, a control on recovery probabilities does not imply that recovery fractions are controlled on all events. More precisely, RecV@R does not impose a restriction on the ratio between assets and liabilities on some rare event that occurs with a probability of $\gamma(0+)$. While this is true, we argue that this aspect can be safely accounted for. Indeed, the recovery function γ is a normative choice of the regulator, and so is the choice of $\gamma(0+)$. In this respect, there are two natural options. On the one hand, the regulator may impose a normative cap on this probability, making it so small that it is negligible in practical situations. On the other hand, by setting $\gamma(0+) = 0$, or, if γ is continuous at 0, by setting $\gamma(0) = 0$, the occurrence of an event with no restriction on the ratio between assets and liabilities can simply be excluded (almost surely).[10]

The recovery-adjusted solvency test (7) can easily be combined with a standard solvency test based on V@R at level α. Indeed, setting $\gamma(1) = \alpha$, it follows that

$$\mathrm{RecV@R}_\gamma(\Delta E_1, L_1) \geq \mathrm{V@R}_\alpha(\Delta E_1), \tag{9}$$

showing that recovery-based capital requirements are more stringent than the standard ones. The standard V@R test can be reproduced by setting $\gamma(\lambda) = \alpha$ for all recovery fractions $\lambda \in [0,1]$, in which case the inequality in (9) becomes an equality. It is worth highlighting that the level $\gamma(1)$ may also be strictly larger than a regulatory level α. In this case, the inequality in (9) may be reversed. The recovery-based risk measure RecV@R can be viewed as a flexible generalization of V@R that reacts to the entire loss tail as specified by the recovery function γ.

Remark 4. *The solvency test* (7) *also controls the conditional recovery probabilities given default. Indeed, assuming that* $\mathbb{P}(E_1 < 0) > 0$, *for all fractions* $\lambda \in [0,1]$ *we have*

$$\mathbb{P}(A_1 \geq \lambda L_1 \,|\, E_1 < 0) = \frac{\mathbb{P}(\lambda L_1 \leq A_1 < L_1)}{\mathbb{P}(E_1 < 0)} = 1 - \frac{\mathbb{P}(A_1 < \lambda L_1)}{\mathbb{P}(E_1 < 0)}.$$

This implies the following equivalent formulation of the recovery-adjusted solvency test:

$$\mathrm{RecV@R}_\gamma(E_1, L_1) \leq 0 \iff \forall\,\lambda \in [0,1]: \quad \mathbb{P}(A_1 \geq \lambda L_1 \,|\, A_1 < L_1) \geq 1 - \frac{\gamma(\lambda)}{\mathbb{P}(E_1 < 0)}.$$

In particular, if the company's unconditional default probability $\mathbb{P}(E_1 < 0)$ *attains* $\gamma(1)$, *then the lower bound on conditional recovery probabilities depends only on* γ.

[10] We thank a referee for pointing out that one always has

$$\mathrm{RecV@R}_\gamma(\Delta E_1, L_1) \leq \mathrm{V@R}_{\gamma(0)}(\Delta E_1).$$

This implies that the undesirable behavior of V@R discussed in Example 2 can be in principle observed also under RecV@R. To this effect, however, the parameter α in that example has to be taken to coincide with $\gamma(0)$. Once again, the regulatory choice of $\gamma(0)$ is critical. If $\gamma(0) = 0$, the issue simply does not arise. If $\gamma(0) > 0$, the issue is possible in theory but will hardly materialize in practice as soon as $\gamma(0)$ is very close to 0.

Remark 5. *Contrary to* V@R *and* AV@R*, the risk measure* RecV@R *depends on the joint distribution of the tuple* (E_1, L_1)*. In particular, the marginal distributions of* E_1 *and* L_1 *are not a sufficient statistic for* RecV@R *but knowledge of the dependence structure, as captured, e.g., by the copula of the pair, is additionally required.*[11] *The evaluation of* RecV@R *is technically not more complicated than the computation of standard solvency capital requirements, since it only requires the computation of a supremum of distribution-based risk measures, namely* V@R*'s. In practical situations, knowledge of the precise joint distribution between assets and liabilities is challenging. We refer to Section 5 for a detailed numerical illustration.*

3.2 Choosing the Recovery Function

The normative choice of the recovery function γ is a critical step in our model and should reflect the risk preferences of the (external or internal) regulators. In this section we describe a class of parametric recovery functions[12] that provides an ideal compromise between flexibility and tractability and can be successfully tailored to different applications as demonstrated in Section 5.

We consider step-wise recovery functions[13] of the form

$$\gamma(\lambda) = \begin{cases} \alpha_1 & \text{if } 0 = r_0 \le \lambda < r_1, \\ \alpha_2 & \text{if } r_1 \le \lambda < r_2, \\ \vdots & \\ \alpha_n & \text{if } r_{n-1} \le \lambda < r_n, \\ \alpha_{n+1} & \text{if } r_n \le \lambda \le r_{n+1} = 1, \end{cases} \tag{10}$$

with $0 \le \alpha_1 < \cdots < \alpha_{n+1} < 1$ and $0 < r_1 < \cdots < r_n < 1$. The parameters r_i correspond to critical target recovery fractions while the parameters α_i define bounds on the corresponding recovery probabilities for every $i = 1, \ldots, n+1$. This type of recovery functions requires regulators to specify a finite number of parameters only and might be used to approximate more complicated recovery functions taking infinitely many values.

As shown in the next proposition, the RecV@R induced by a piecewise linear recovery function can be expressed as a maximum of finitely many V@R's.

Proposition 6. *Let* γ *be defined as in* (10)*. Then, for all* $X \in \mathcal{X}$ *and* $Y \in \mathcal{X}_+$

$$\mathrm{RecV@R}_\gamma(X, Y) = \max_{i=1,\ldots,n+1} \mathrm{V@R}_{\alpha_i}(X + (1 - r_i)Y).$$

Proof. See Section A.2. □

The preceding proposition shows that, under a recovery function of the form (10), the recovery-based solvency test (7) takes the particularly simple form:

$$\mathbb{P}(A_1 \ge r_i L_1) \ge 1 - \alpha_i, \quad i = 1, \ldots, n+1. \tag{11}$$

[11]The problem is akin to risk estimation in the presence of aggregate positions where a model for the joint distribution is also needed. The structure of recovery risk measures opens up a variety of interesting technical questions related to dependence modelling that, however, go beyond the scope of the current work. The rich and growing literature on the topic is a good starting point to address such questions, see, e.g., Embrechts, Puccetti & Rüschendorf (2013), Bernard, Jiang & Wang (2014), Bernard, Rüschendorf, Vanduffel & Wang (2017), Cai, Liu & Wang (2018).

[12]We describe a methodology to calibrate γ to an existing regulatory framework in Section 5.3. This mirrors a common strategy chosen by regulators when adapting a new solvency setting to replace a pre-existing one. Another possibility to choose γ is to elicit it from the risk appetite of risk managers or customers, e.g., by way of a questionnaire targeting recovery distributions. This would raise a number of interesting questions for future research that are, however, beyond the scope of the paper.

[13]This choice is not particularly restrictive, since an increasing function can be well approximated from below by step functions.

In this case, a company is adequately capitalized under RecV@R if, for every $i = 1, \dots, n+1$, assets are sufficient to cover a fraction r_i of liabilities with a probability of at least $1 - \alpha_i$. The largest recovery probability α_{n+1} with target recovery fraction $r_{n+1} = 1$ caps the default probability and could correspond to the level of V@R in a classical solvency test. This shows that a solvency test of the form (11) can easily be harmonized with the solvency tests currently used in solvency regulation.

3.3 Basic Properties of RecV@R

We ask which basic properties of V@R are inherited by its recovery counterpart RecV@R. For a comprehensive survey on scalar monetary risk measures we refer to Föllmer & Schied (2016). A monetary risk measure is a function $\rho : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ that satisfies the following two properties:

- *Cash invariance*: $\rho(X + m) = \rho(X) - m$ for all $X \in \mathcal{X}$ and $m \in \mathbb{R}$;
- *Monotonicity*: $\rho(X_1) \leq \rho(X_2)$ for all $X_1, X_2 \in \mathcal{X}$ with $X_1 \geq X_2$ $\mathbb{P}$-almost surely.

The cash invariance property formalizes that adding cash to a capital position reduces risk by exactly the same amount and implies that risk is measured on a monetary scale. In particular, cash invariance allows to rewrite the risk measure as a capital requirement rule:

$$\rho(X) = \inf\{m \in \mathbb{R};\ \rho(X + m) \leq 0\},$$

i.e., the quantity $\rho(X)$ can be interpreted as the minimal amount of cash that needs to be injected into the position X in order to pass the solvency test in (2). If the position already fulfills this solvency condition, then $-\rho(X)$ corresponds to the maximal amount of capital that can be extracted from the balance sheet without compromising capital adequacy. Cash invariance guarantees that risk measures possess an operational interpretation in the context of solvency tests. Monotonicity reflects that larger capital positions correspond to lower risk and to lower capital requirements. In addition to its defining properties, a monetary risk measure may possess the following properties:

- *Convexity*: $\rho(aX_1 + (1-a)X_2) \leq a\rho(X_1) + (1-a)\rho(X_2)$ for all $X_1, X_2 \in \mathcal{X}$ and $a \in [0, 1]$;
- *Subadditivity*: $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$ for all $X_1, X_2 \in \mathcal{X}$;
- *Positive homogeneity*: $\rho(aX) = a\rho(X)$ for all $X \in \mathcal{X}$ and $a \in (0, \infty)$.
- *Normalization*: $\rho(0) = 0$.

The first two properties characterize the behavior of the risk measure with respect to aggregation and require that diversification is not penalized. The third property specifies that risk measurements scale with the size of positions.

The next proposition records elementary properties of RecV@R. In particular, RecV@R is a standard monetary risk measure if the second argument is fixed. The proof follows from the general result in Proposition 19.

Proposition 7. *The risk measure* $\mathrm{RecV@R}_\gamma$ *has the following properties:*

(a) Cash invariance in the first component: *For all* $X, Y \in \mathcal{X}$ *and* $m \in \mathbb{R}$

$$\mathrm{RecV@R}_\gamma(X + m, Y) = \mathrm{RecV@R}_\gamma(X, Y) - m.$$

(b) Monotonicity: *For all* $X_1, X_2, Y_1, Y_2 \in \mathcal{X}$ *with* $X_1 \geq X_2$ *and* $Y_1 \geq Y_2$ $\mathbb{P}$*-almost surely*[14]

$$\mathrm{RecV@R}_\gamma(X_1, Y_1) \leq \mathrm{RecV@R}_\gamma(X_2, Y_2).$$

(c) Positive homogeneity: *For all* $X, Y \in \mathcal{X}$ *and* $a \in [0, \infty)$

$$\mathrm{RecV@R}_\gamma(aX, aY) = a\mathrm{RecV@R}_\gamma(X, Y).$$

(d) Star-shapedness[15] in the first component: *For all* $X \in \mathcal{X}$*,* $Y \in \mathcal{X}_+$*, and* $a \in [1, \infty)$

$$\mathrm{RecV@R}_\gamma(aX, Y) \geq a\mathrm{RecV@R}_\gamma(X, Y).$$

(e) Normalization: *For every* $Y \in \mathcal{X}_+$ *we have* $\mathrm{RecV@R}_\gamma(0, Y) = 0$.

(f) Finiteness: *For all* $X \in \mathcal{X}$ *and* $Y \in \mathcal{X}_+$ *we have* $\mathrm{RecV@R}_\gamma(X, Y) < \infty$ *if* $\gamma(0) > 0$ *or if* X *is bounded from below.*

The previous proposition shows that RecV@R is a standard monetary risk measure in its first component and can conveniently be expressed as a capital requirement:

$$\begin{aligned}\mathrm{RecV@R}_\gamma(E_1, L_1) &= \inf\{m \in \mathbb{R}\,;\ \mathrm{RecV@R}_\gamma(E_1 + m, L_1) \leq 0\} \\ &= \inf\{m \in \mathbb{R}\,;\ \mathbb{P}(A_1 + m < \lambda L_1) \leq \gamma(\lambda),\ \forall \lambda \in [0, 1]\} \\ &= \inf\{m \in \mathbb{R}\,;\ \mathbb{P}(A_1 + m \geq \lambda L_1) \geq 1 - \gamma(\lambda),\ \forall \lambda \in [0, 1]\}.\end{aligned}$$

This leads to the following operational interpretation of RecV@R: If $\mathrm{RecV@R}_\gamma(E_1, L_1) > 0$, the company fails to pass the recovery-based solvency test (7) and $\mathrm{RecV@R}_\gamma(E_1, L_1)$ is the minimal amount of cash that needs to be added to its assets in order to become adequately capitalized. If $\mathrm{RecV@R}_\gamma(E_1, L_1) < 0$, the company is adequately capitalized according to the recovery-based solvency test (7) and $-\mathrm{RecV@R}_\gamma(E_1, L_1)$ is the maximal amount of cash that may be extracted from the asset side without compromising capital adequacy.

Remark 8. *From an operational perspective the interpretation of monetary risk measures as capital requirement rules relies on the cash invariance property.* RecV@R *is cash invariant in the first but not in the second argument. If one intends to modify the liabilities, e.g. by transferring them to another institution, instead of the assets on the balance sheet, an alternative definition of* RecV@R *is appropriate, namely ("L" stands for "liabilities")*

$$\mathrm{LRecV@R}_\gamma(A_1, L_1) := \sup_{\lambda \in (0,1]} \frac{1}{\lambda} \cdot \mathrm{V@R}_{\gamma(\lambda)}(A_1 - \lambda L_1). \tag{12}$$

In this case, the correct way to express the solvency test (7) *is*

$$\mathrm{LRecV@R}_\gamma(A_1, L_1) \leq 0, \tag{13}$$

[14]Increasing leverage does not necessarily increase risk. However, this will be the case in two situations: If assets are held constant, increasing the value of the liabilities will increase risk and, similarly, if liabilities are fixed, decreasing the value of assets will increase risk. Monotonicity of RecV@R does, in this respect, not differ from the standard monotonicity properties of classical monetary risk measures. Less obvious is the behavior of leverage and risk, if asset and liability positions are modified at the same time. In these situations, their joint behavior needs to be considered in detail. Leverage is a rather imprecise quantity to characterize risk, while recovery risk measures have a solid foundation in terms of the notion of acceptability. At the same time, risk measures possess a simple operational interpretation that directly links them to solvency tests.

[15]We refer to the recent preprint Castagnoli, Cattelan, Maccheroni, Tebaldi & Wang (2021) for a study of star-shaped risk measures.

which is still equivalent to condition (8)*. Note that* LRecV@R *is cash invariant (in the appropriate sense) with respect to its second argument, i.e., for all* $A_1, L_1 \in \mathcal{X}_+$ *and* $m \in \mathbb{R}$

$$\mathrm{LRecV@R}_\gamma(A_1, L_1 + m) = \mathrm{LRecV@R}_\gamma(A_1, L_1) + m.$$

This leads to the following operational interpretation: If $\mathrm{LRecV@R}_\gamma(A_1, L_1) > 0$*, the company fails the solvency test* (13) *and* $\mathrm{LRecV@R}_\gamma(A_1, L_1)$ *is the minimal nominal amount of liabilities that needs to be removed from the balance sheet in order to pass the test, e.g., by transferring these liabilities to suitable equity holders outside the firm. If* $\mathrm{LRecV@R}_\gamma(A_1, L_1) < 0$*, the company is adequately capitalized. The company may at most create an additional amount* $-\mathrm{LRecV@R}_\gamma(A_1, L_1)$ *of liabilities, e.g., via additional debt, and immediately distribute the same amount of cash to its shareholders.*

Observe that assets A_1 *and liabilities* L_1 *are used in the definition of* LRecV@R *instead of the net asset value* E_1 *and liabilities* L_1 *in order to obtain a simple cash-invariant recovery risk measure with a transparent operational interpretation.*[16]

4 Recovery Risk Measures

The risk measure RecV@R allows to control the loss given default by prescribing suitable bounds on the probability that part of the creditors' claims can be recovered. If one replaces V@R with other monetary risk measures, e.g., convex risk measures, one obtains recovery risk measures of a different type. In particular, by choosing appropriate monetary risk measures as the basic ingredients, it is possible to construct convex recovery risk measures. In this section we describe the general structure of recovery risk measures and their main properties. We continue to use the balance sheet notation introduced in Section 2.

4.1 Recovery Average Value at Risk

We start by focusing on the natural recovery-based version of AV@R, which is arguably the most important convex risk measure used in practice.

Definition 9. *Let* $\gamma : [0,1] \to [0,1)$ *be an increasing function. The* Recovery Average Value at Risk

$$\mathrm{RecAV@R}_\gamma : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \cup \{\infty\}$$

with level function γ *is defined by*

$$\mathrm{RecAV@R}_\gamma(X, Y) := \sup_{\lambda \in [0,1]} \mathrm{AV@R}_{\gamma(\lambda)}(X + (1-\lambda)Y). \tag{14}$$

If the random variables X and Y in Definition 9 are interpreted, respectively, as the net asset value E_1 and liabilities L_1 in a company's balance sheet, the recovery risk measure RecAV@R can be used to formulate the solvency test (17):

$$\mathrm{RecAV@R}_\gamma(\Delta E_1, L_1) \le E_0 \quad \Longleftrightarrow \quad \forall\, \lambda \in [0,1] \,:\, \mathrm{AV@R}_{\gamma(\lambda)}(A_1 - \lambda L_1) \le 0. \tag{15}$$

A company will thus be adequately capitalized according to RecAV@R with level function γ if for all recovery fractions $\lambda \in [0,1]$ the modified net asset value $A_1 - \lambda L_1$ is positive on average on the lower tail beyond the $\gamma(\lambda)$-quantile. Since AV@R dominates V@R at the same level, domination is inherited by their recovery-based versions, i.e., for all $X, Y \in L^1$

$$\mathrm{RecAV@R}_\gamma(X, Y) \ge \mathrm{RecV@R}_\gamma(X, Y).$$

[16] Combining RecV@R and LRecV@R leads to the question of how to combine asset and liability management for capital adequacy purposes, which, however, goes beyond the scope of this paper. The literature on set-valued risk measures may help to address this question.

The solvency test (15) is stricter than (7) and the recovery probabilities are still controlled as described in equation (8).

The next proposition states some basic properties of RecAV@R, which follow from a general result in Proposition 19 in the appendix.[17]

Proposition 10. *The risk measure* $\mathrm{RecAV@R}_\gamma$ *is cash invariant in its first component, monotone, convex, subadditive, positively homogeneous, star shaped in its first component, and normalized. Moreover,* $\mathrm{RecAV@R}_\gamma(X,Y)<\infty$ *for all integrable* $X\in\mathcal{X}$ *and* $Y\in\mathcal{X}_+$ *if* $\gamma(0)>0$ *or if* X *is bounded from below.*

The subadditivity of RecAV@R makes it suitable to serve as a basis for limit systems that enable decentralized risk management within firms. We consider a bank or an insurance company that consists of N subentities. For each date $t=0,1$ their assets, liabilities, and net asset value are denoted by A_t^i, L_t^i, and E_t^i, $i=1,\dots,N$. The consolidated figures are denoted by

$$A_t=\sum_{i=1}^N A_t^i,\quad L_t=\sum_{i=1}^N L_t^i,\quad E_t=\sum_{i=1}^N E_t^i.$$

The firm may enforce entity-based risk constraints of the form

$$\mathrm{RecAV@R}_\gamma(E_1^i,L_1^i)\le c^i,\quad i=1,\dots,N,$$

where $c^1,\dots,c^N\in\mathbb{R}$ are given risk limits. If the limits are chosen to satisfy $\sum_{i=1}^N c^i\le 0$, then

$$\mathrm{RecAV@R}_\gamma(E_1,L_1)\le\sum_{i=1}^N\mathrm{RecAV@R}_\gamma(E_1^i,L_1^i)\le\sum_{i=1}^N c^i\le 0$$

by subadditivity. This shows that imposing risk constraints at the level of subentities allows to fulfill the “global” solvency test (15). A closely related issue is performance measurement and adaptive management of the balance sheets of firms, as often seen in practice. This is discussed for general recovery risk measures in Section 5.2.

If the recovery function γ is piecewise constant, then RecAV@R is the maximum of finitely many AV@R’s. This parallels the representation of RecV@R recorded in Proposition 6.

Proposition 11. *Let* γ *be defined as in* (10)*. Then, for all* $X\in\mathcal{X}$ *and* $Y\in\mathcal{X}_+$

$$\mathrm{RecAV@R}_\gamma(X,Y)=\max_{i=1,\dots,n+1}\mathrm{AV@R}_{\alpha_i}(X+(1-r_i)Y).$$

Proof. See Section A.3. □

Remark 12. *A modified version of* RecAV@R *that is cash invariant with respect to its second component can be easily constructed as in Remark 8.*

4.2 General Recovery Risk Measures

To motivate the general definition of a recovery risk measure, we observe that RecV@R may be expressed in terms of a decreasing family of monetary risk measures indexed by recovery fractions $\lambda\in[0,1]$. Indeed, for a given level function γ, the collection of monetary risk measures $\rho_\lambda:\mathcal{X}\to\mathbb{R}$ given by

$$\rho_\lambda(X)=\mathrm{V@R}_{\gamma(\lambda)}(X),\quad\lambda\in[0,1],$$

[17] We refer to Section A.4 in the appendix for a dual representation of RecAV@R. Duality results play an important role in applications such as optimization problems involving risk measures.

defines the associated RecV@R by setting

$$\mathrm{RecV@R}_\gamma(X,Y) = \sup_{\lambda\in[0,1]} \rho_\lambda(X+(1-\lambda)Y).$$

The same holds for AV@R and its recovery counterpart. By construction, smaller recovery fractions are guaranteed with higher probability, which is captured by γ being increasing. As a consequence, the family of maps ρ_λ, $\lambda \in [0,1]$, is decreasing in the sense that $\rho_{\lambda_1} \geq \rho_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$. A smaller recovery fraction corresponds to a more conservative risk measure. This motivates the general definition of a recovery risk measure.

Definition 13. *For every $\lambda \in [0,1]$ consider a map $\rho_\lambda : \mathcal{X} \to \mathbb{R}\cup\{\infty\}$ and assume that $\rho_{\lambda_1} \geq \rho_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$. The* recovery risk measure

$$\mathrm{Rec}\rho : \mathcal{X}\times\mathcal{X} \to \mathbb{R}\cup\{\infty\}$$

is defined by

$$\mathrm{Rec}\rho(X,Y) := \sup_{\lambda\in[0,1]} \rho_\lambda(X+(1-\lambda)Y). \tag{16}$$

In line with our discussion on RecV@R, if the random variables X and Y in Definition 13 are respectively interpreted as the net asset value E_1 and liabilities L_1 in a company's balance sheet, the recovery risk measure Recρ can be employed to formulate a solvency test of the form (1). Indeed, similarly to what we have shown in Section 3, we have

$$\mathrm{Rec}\rho(\Delta E_1, L_1) \leq E_0 \quad\Longleftrightarrow\quad \forall\,\lambda\in[0,1]\,:\,\rho_\lambda(A_1-\lambda L_1)\leq 0. \tag{17}$$

The specific interpretation of this recovery-based solvency test will, of course, depend on the choice of the monetary risk measures used to build Recρ. In Section A.5 in the appendix we show how a number of standard properties of monetary risk measures are inherited by their recovery counterparts and discuss dual representations of convex recovery risk measures.

5 Applications

We complement the foundations on recovery risk measures with detailed case studies and applications. In Section 5.1 we demonstrate that recovery-based solvency requirements may help align the decisions of the management of firms with the interest of creditors in protecting their claims in the case of default. In Section 5.2 we focus on performance-based management of business divisions of firms for recovery risk measures. In Section 5.3 we address the problem of calibrating the recovery function to pre-specified benchmarks, an issue that is relevant in the context of regulatory regime changes. Finally, in Section 5.4 we study the impact of the distributions of the underlying balance sheet figures on capital adjustments.

5.1 Protecting the Interests of Creditors

We demonstrated in Example 2 that capital requirements based on V@R and AV@R may fail to provide an adequate protection to creditors. We return to this example and show that RecV@R and RecAV@R can successfully be employed to enforce guarantees on claims recovery.[18]

Example 14. *We consider the situation of Example 2, but with a different risk constraint in terms of* RecV@R. *While solvency constraints in terms of* V@R *or* AV@R *led to recovery* 0*, the recovery risk measure* RecV@R *is able to guarantee a pre-specified recovery level.*

[18] For detailed calculations we refer to Section A.6 in the appendix.

We fix a recovery function in the class described in Section 3.2 with $n = 1$. For a probability level $\beta \in (0,\alpha)$ and a recovery level $r \in (0,1)$, we set

$$\gamma(\lambda) = \begin{cases} \beta & \text{if } \lambda \in [0,r), \\ \alpha & \text{if } \lambda \in [r,1]. \end{cases}$$

For every choice of $k \in [0,100]$ we obtain from Proposition 6 that

$$\mathrm{RecV@R}_\gamma(E_1^k, L_1) = \max\{\mathrm{V@R}_\alpha(E_1^k), \mathrm{V@R}_\beta(E_1^k + (1-r)L_1)\}.$$

A direct computation shows that

$$\mathrm{RecV@R}_\gamma(E_1^k, L_1) = \begin{cases} 100r - k & \text{if } \beta < \frac{\alpha}{2},\ k \le 50(r+1), \\ k - 100 & \text{otherwise.} \end{cases}$$

According to (7) *the company is adequately capitalized if*

$$\mathrm{RecV@R}_\gamma(E_1^k, L_1) \le 0 \iff \begin{cases} k \ge 100r & \text{if } \beta < \frac{\alpha}{2}, \\ k \ge 0 & \text{if } \beta \ge \frac{\alpha}{2}. \end{cases}$$

A maximal shareholder value under the recovery-based solvency constraint is attained with $k = 100r$ when $\beta < \frac{\alpha}{2}$ and with $k = 0$ otherwise. The first case corresponds to successfully controlling recovery. Hence, the regulator may choose a suitable recovery function such that RecV@R *is more stringent than* V@R *and the recovery fraction in the default state is equal to r. This is in contrast to Example 2 with solvency constraints in terms of* V@R *or* AV@R *that led to recovery* 0 *when the management maximizes shareholder value.*

Example 15. *We consider the same situation as in Example 14, but replace* RecV@R *by* RecAV@R *with the same recovery function. We will demonstrate that the recovery risk measure* RecAV@R *is also able to guarantee a pre-specified recovery level.*

To be more specific, it follows from Proposition 11 for every choice of $k \in [0,100]$ that

$$\mathrm{RecAV@R}_\gamma(E_1^k, L_1) = \max\{\mathrm{AV@R}_\alpha(E_1^k), \mathrm{AV@R}_\beta(E_1^k + (1-r)L_1)\}.$$

A direct computation shows that

$$\mathrm{RecAV@R}_\gamma(E_1^k, L_1) = \begin{cases} 100r - k & \text{if } \beta < \frac{\alpha}{2},\ k \le 100r, \\ r - 101 + \frac{\alpha}{2\beta}(101 + 99r) + (1 - \frac{\alpha}{\beta})k & \text{if } \beta \ge \frac{\alpha}{2},\ k \le \frac{(99\alpha+2\beta)r - 101(2\beta-\alpha)}{2(\alpha-\beta)}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the company is adequately capitalized under (15) *if*

$$\mathrm{RecAV@R}_\gamma(E_1^k, L_1) \le 0 \iff \begin{cases} k \ge 100r & \text{if } \beta < \frac{\alpha}{2}, \\ k \ge \max\left\{\frac{(99\alpha+2\beta)r - 101(2\beta-\alpha)}{2(\alpha-\beta)}, 0\right\} & \text{if } \beta \ge \frac{\alpha}{2}. \end{cases}$$

If $\beta < \frac{\alpha}{2}$ and the management selects the individually optimal admissible level of k, the recovery fraction in the default state is equal to r as observed in Example 14. In this case, there is no difference between RecAV@R *and* RecV@R.

Interestingly enough, contrary to RecV@R, *claims recovery can be controlled under* RecAV@R *even in the situation where $\beta \ge \frac{\alpha}{2}$. In this case, under the assumption that shareholder value is maximized, the fraction of claims recovered in the default state equals*

$$\max\left\{\frac{(99\alpha + 2\beta)r - 101(2\beta - \alpha)}{200(\alpha - \beta)}, 0\right\}.$$

This expression is strictly positive as soon as r is strictly larger than the bound $\frac{101(2\beta-\alpha)}{99\alpha+2\beta} \in [0,1)$. (For example, taking $\beta = \frac{\alpha}{2}$ always ensures a recovery equal to r). Solvency capital requirements based on RecAV@R *are more effective in controlling claims recovery in comparison to those based on* RecV@R *in Example 14.*

5.2 Performance Measurement

An important issue that is closely related to solvency capital requirements is performance measurement. We show that this task can be implemented on the basis of recovery risk measures. A popular metric in practice is the *return on risk-adjusted capital* (RoRaC). Consider a recovery risk measure $\mathrm{Rec}\rho$ that is subadditive and positively homogeneous as discussed in Section A.5. The associated RoRaC is defined by

$$\mathrm{RoRaC}(\Delta E_1, L_1) \ := \ \frac{\mathbb{E}(\Delta E_1)}{\mathrm{Rec}\rho(\Delta E_1, L_1)}.$$

This quantity measures the *expected return per unit of economic capital* expressed in terms of the risk measure $\mathrm{Rec}\rho$. We suppose that $\mathrm{Rec}\rho(\Delta E_1, L_1) > 0$.

The goal of the firm is to improve its RoRaC. We assume that the company is composed of different subentities labelled $i = 1, \dots, n$. The central management may impose risk limits and adjust the size of different business units. A key question is which allocation of economic capital to business units and corresponding performance measurements provide appropriate information to improve the overall performance of the firm. We denote the net asset values and liabilities of the subentities at time $t = 0, 1$ by E_t^i and L_t^i for $i = 1, \dots, N$, respectively. Note that for $t = 0, 1$

$$L_t = \sum_{i=1}^{N} L_t^i, \quad E_t = \sum_{i=1}^{N} E_t^i.$$

The subadditivity of $\mathrm{Rec}\rho$ implies

$$\mathrm{Rec}\rho(\Delta E_1, L_1) \ \leq \ \sum_{i=1}^{N} \mathrm{Rec}\rho(\Delta E_1^i, L_1^i).$$

We seek an allocation of economic capital $\kappa^i := \mathrm{Rec}\rho^{\Delta E_1, L_1}(\Delta E_1^i, L_1^i)$, $i = 1, \dots, N$, satisfying:

- *Full allocation*: $\sum_{i=1}^{N} \kappa^i = \mathrm{Rec}\rho(\Delta E_1, L_1)$;
- *Diversification*: $\kappa^i \leq \mathrm{Rec}\rho(\Delta E_1^i, L_1^i)$ for all $i = 1, \dots, N$;
- *RoRaC-compatibility*: If for some $i = 1, \dots, N$ we have

  $$\mathrm{RoRaC}^i := \frac{\mathbb{E}(\Delta E_1^i)}{\kappa^i} \ > \ \mathrm{RoRaC}(\Delta E_1, L_1) \quad (\text{resp. } <),$$

  then there exists $\varepsilon > 0$ such that for every $h \in (0, \varepsilon)$

  $$\mathrm{RoRaC}(\Delta E_1 + h\Delta E_1^i, L_1 + hL_1^i) \ > \ \mathrm{RoRaC}(\Delta E_1, L_1) \quad (\text{resp. } <).$$

The full allocation property requires that the entire solvency capital is allocated to the individual subentities. The diversification property specifies that no more capital is allocated to the individual subentities than their stand-alone solvency capital, taking beneficial diversification effects into account, which is feasible due to the subadditivity of $\mathrm{Rec}\rho$. Finally, RoRaC-compatibility guarantees that performance measurement based on the chosen capital allocation provides the correct information to the management of the firm to improve the overall performance of the firm. To be more precise, if the performance of subentity i — as captured by RoRaC^i — is better than the overall RoRaC, the performance of the entire firm can be improved by growing subentity i. An allocation fulfilling the above three properties is called a *suitable allocation*.

The existence of suitable allocations has been extensively studied in the literature, see, e.g., Tasche (2000), Tasche (2004), Kalkbrener (2005), Tasche (2008), Dhaene et al. (2012), Bauer &

Zanjani (2013), and the general review by Guo et al. (2020). It follows from the general results in these papers that the only suitable allocation in the above sense is the *Euler allocation*

$$\kappa^i \;=\; \frac{d}{dh}\,\mathrm{Rec}\rho(\Delta E_1 + h\Delta E_1^i, L_1 + hL_1^i)_{|h=0}$$

for all $i = 1, \ldots, N$. In the specific case of $\mathrm{Rec}\rho = \mathrm{RecAV@R}$ with a simple piecewise constant level function, the Euler allocation can explicitly be computed.

Proposition 16. *Let* γ *be defined as in* (10) *and let* $j \in \{1, \ldots, N\}$ *satisfy*

$$\mathrm{AV@R}_{\alpha_j}(\Delta E_1 + (1 - r_j)L_1) > \max_{i=1,\ldots,N,\; i\neq j} \mathrm{AV@R}_{\alpha_i}(\Delta E_1 + (1 - r_i)L_1).$$

Under suitable assumptions on the joint distribution of $\Delta E_1^1 + (1-r_j)L_1^1, \ldots, \Delta E_1^N + (1-r_j)L_1^N$ *(see Section A.8), the Euler allocation based on* $\mathrm{RecAV@R}_\gamma$ *is given for every* $i = 1, \ldots, N$ *by*

$$\kappa^i \;=\; -E\big(\Delta E_1^i + (1 - r_j)L_1^i \,|\, E_1 + (1 - r_j)L_1 \leq -\mathrm{V@R}_{\alpha_j}(E_1 + (1 - r_j)L_1)\big).$$

Proof. See Section A.8. □

In summary, performance measurement inside firms can be based on recovery risk measures. Notions such as return on risk-adjusted capital (RoRaC) and RoRaC-compatible allocations may be extended to solvency regimes that control the size of recovery on creditors' claims in the case of default.

5.3 Calibrating the Recovery Function

When regulatory solvency standards in practice are modified and improved, the old regulatory framework is often used as a numerical benchmark for the new one. New and old requirements will, of course, differ for many distributions of assets and liabilities at the considered time horizon, and many companies might experience corresponding changes in solvency requirements. For this reason, a common approach in practice is to calibrate the new standards in such a way that they produce the same solvency requirement for a prototypical benchmark company. The rational behind this strategy is to ensure some form of continuity in the sense that benchmark firms are not too much affected over short time horizons. At the same time, regulatory standards are ideally modified in such a way that their new design is more efficient in achieving key regulatory goals in the long run. Choosing a benchmark balance sheet for calibration naturally remains a political decision.

In this section, we explain in the context of an example how recovery risk measures could be calibrated to existing regulatory standards. We begin by recalling the transition from Basel II to Basel III. Basel II was based on V@R at level $\alpha = 1\%$ while the new Basel III has adopted AV@R at level $\beta = 2.5\%$. The choice of $\beta = 2.5\%$ was justified a) by assuming in a benchmark model that changes in net asset value ΔE_1 are normally distributed and b) by requiring[19]

$$\mathrm{AV@R}_\beta(\Delta E_1) \approx \mathrm{V@R}_\alpha(\Delta E_1).$$

The new regulatory level equates capital requirements of normally distributed positions for old and new standards. In the same spirit, we describe how to calibrate the recovery level function of RecV@R. A challenge is that we deal with a function γ instead of a single parameter β as well as with a pair of random variables, $E_1 - E_0$ and L_1, instead of just one random variable, $E_1 - E_0$. The aim is to equate, for given $\alpha \in (0, 1)$ and for ΔE_1 being normally distributed,

$$\mathrm{RecV@R}_\gamma(\Delta E_1, L_1) = \mathrm{V@R}_\alpha(\Delta E_1). \tag{18}$$

[19] We refer to Li & Wang (2019) for a general study on calibration of V@R and AV@R.

The choice of a benchmark is a political decision of the regulator. In our recovery-based setting, we consider a particularly simple choice. As discussed before, we assume that $\Delta E := E_1 - E_0$ is normally distributed with mean $\mu_{\Delta E} \in \mathbb{R}$ and standard deviation $\sigma_{\Delta E} > 0$. In addition, we suppose that $L := L_1$ is normally distributed with mean $\mu_L \in \mathbb{R}$ and standard deviation $\sigma_L > 0$ and is independent of ΔE. The latter assumption is not meant to capture realistic balance sheets, but is simply chosen for illustration as it leads to explicit calculations. By independence, for every $\lambda \in [0,1]$ the random variable $\Delta E + (1-\lambda)L$ is also normal. We denote by Φ the distribution function of a standard normal random variable.[20]

We seek a function γ such that (18) holds under the assumptions above. A sufficient requirement is that

$$\mathrm{V@R}_{\gamma(\lambda)}(\Delta E + (1-\lambda)L) = \mathrm{V@R}_{\alpha}(\Delta E), \quad \text{for every } \lambda \in [0,1].$$

A direct calculations yields

$$\gamma(\lambda) = \Phi\left(\frac{\sigma_{\Delta E}\Phi^{-1}(\alpha) - (1-\lambda)\mu_L}{\sqrt{\sigma_{\Delta E}^2 + (1-\lambda)^2\sigma_L^2}}\right), \quad \lambda \in [0,1],$$

with $\gamma(1) = \alpha$. This function might be inconsistent with the requirements in Definition 3, namely with γ being increasing. A potential remedy to this problem could be to modify the choice of γ as follows. Since our solution for γ is differentiable with respect to λ, by taking derivatives, its increasing part can easily be characterized by

$$\gamma'(\lambda) \geq 0 \iff (1-\lambda)\Phi^{-1}(\alpha)\sigma_L^2 + \mu_L\sigma_{\Delta E} \geq 0, \quad \lambda \in (0,1).$$

Here, we have used that $\Phi^{-1}(\alpha) < 0$ because α is assumed to be close to zero. Hence, γ is increasing on the interval $[\lambda^*, 1]$ where

$$\lambda^* := \max\left\{1 + \frac{\mu_L\sigma_{\Delta E}}{\sigma_L^2\Phi^{-1}(\alpha)}, 0\right\} < 1.$$

If $\lambda^* = 0$, the function γ is increasing on the whole interval $[0,1]$. Otherwise, we compute

$$\gamma(\lambda^*) = \Phi\left(-\sqrt{\Phi^{-1}(\alpha)^2 + \frac{\mu_L^2}{\sigma_L^2}}\right).$$

and redefine γ as follows:

$$\gamma(\lambda) := \begin{cases} \Phi\left(-\sqrt{\Phi^{-1}(\alpha)^2 + \frac{\mu_L^2}{\sigma_L^2}}\right) & \text{if } \lambda \in [0, \lambda^*), \\ \Phi\left(\frac{\sigma_{\Delta E}\Phi^{-1}(\alpha) - (1-\lambda)\mu_L}{\sqrt{\sigma_{\Delta E}^2 + (1-\lambda)^2\sigma_L^2}}\right) & \text{if } \lambda \in [\lambda^*, 1]. \end{cases}$$

Finally, if a piecewise-constant recovery function is sought, see Section 3.2, a suitable approximation of γ may be chosen.

5.4 Numerical Case Studies

Standard solvency capital requirements based on V@R and AV@R cannot control the probability of recovering certain pre-specified fractions of claims. Additional capital is required which needs to be computed on the basis of recovery risk measures such as RecV@R and RecAV@R. In this

[20] Note that a positive random variable like L cannot have a normal distribution. In practice, this can be taken into account by imposing the condition $\Phi(-\frac{\mu_L}{\sigma_L}) = \mathbb{P}(L < 0) \leq \varepsilon$ for a sufficiently small $\varepsilon > 0$.

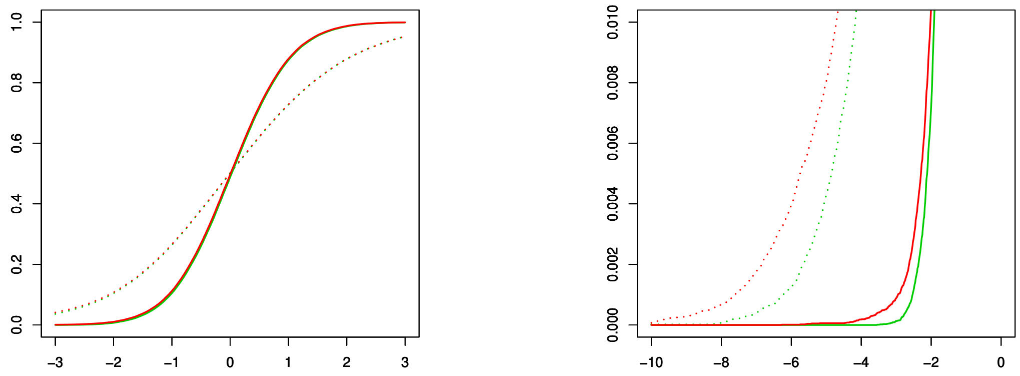

Figure 1: Probability distribution function of ΔE_1 (left) and a detail of its tail (right) for $\rho = 0.1$ (dotted) and $\rho = 0.9$ (plain) and for $\tau = 1$ (green) and $\tau = 5$ (red).

section, we study the impact of a variation in the distribution of the underlying balance sheet figures on the size of necessary capital adjustments. Section 5.4.1 numerically illustrates this for standard parametric distributions. This allows to understand the influence of correlation between assets and liabilities and the tail size of liabilities. Section 5.4.2 presents a stylized example demonstrating that under standard solvency regimes sophisticated asset-liability-management may hide substantial tail risk. These situations correspond to high capital adjustments, if the required capital is instead computed by recovery risk measures.

Throughout the section, we consider a financial institution with assets A_t, liabilities L_t, and net asset value $E_t = A_t - L_t$ at dates $t = 0, 1$. The changes of the net asset value over the considered time window or, equivalently, the corresponding cash flows are $\Delta E_1 = E_1 - E_0$.

5.4.1 Parametric Distributions

In this section, we consider parametric distributions that model the evolution of the company's assets and liabilities and show how the gap between standard capital requirements and those based on recovery risk measures is influenced by the dependence between assets and liabilities and by their marginal distributions, in particular the liability tail size. We refer to Section A.9 for further details.

Distribution of assets and liabilities. We assume that A_1 possesses a lognormal distribution with log-mean $\mu \in \mathbb{R}$ and log-standard deviation $\sigma > 0$. This specification for the asset distribution is standard in the finance literature and compatible, e.g., with the Black-Scholes setting. We fix $\mu = 2$ and $\sigma = 0.2$. Liabilities L_1 follow a mixture gamma distribution. More precisely, up to the 95% quantile L_1 possesses a gamma distribution with shape parameter $\tau_0 > 0$ and rate parameter $\delta_0 > 0$; beyond the 95% quantile L_1 is determined by a gamma distribution with shape parameter $\tau > 0$ and rate parameter $\delta > 0$. This specification is encountered in many applications, including insurance, and allows a flexible control on the tail distribution (heavier tails correspond to higher levels of τ). Setting $\delta_0 = \delta = 1$ and $\tau_0 = 1$, we focus on the range $\tau \in [1, 5]$.

Assets and liabilities are linked by a Gaussian copula. This choice allows to capture dependence by a single parameter, the correlation coefficient $\rho \in [-1, 1]$. Under positive dependence ($\rho > 0$), shocks increasing the value of liabilities are more frequently accompanied by increased asset values. In this case, the asset position may be considered a reasonable hedge of the liability position. We focus on the range $\rho \in [0, 1]$.

Simulated distribution of assets and liabilities. Our computations are implemented using the software R. We resort to standard Monte Carlo simulation based on quantile inversion

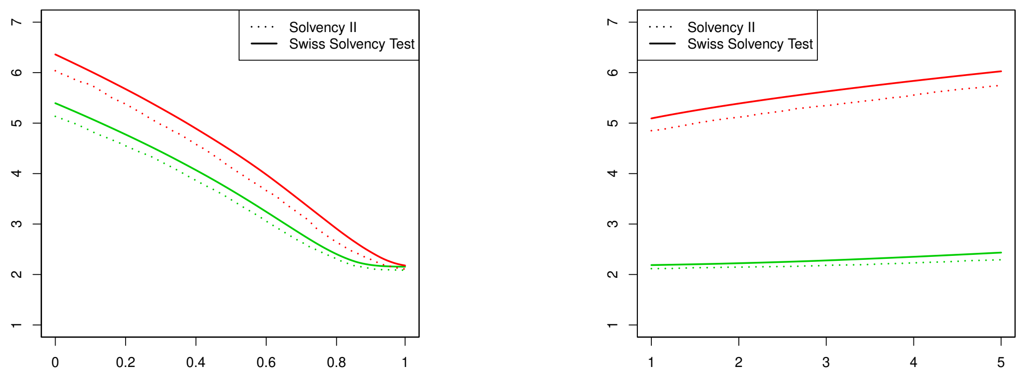

Figure 2: The solvency capital requirement $\rho_{reg}(\Delta E_1)$ as a function of ρ (left) for $\tau = 1$ (green) and $\tau = 5$ (red) and as a function of τ (right) for $\rho = 0.1$ (red) and $\rho = 0.9$ (green).

(for the marginal distributions) and Cholesky decomposition (for the joint distribution); see, e.g., Glasserman (2013). The simulated cash flow distribution is displayed in Figure 1. The choice of E_0 is made to ensure a realistic probability of observing negative cash flows over the considered period of time, i.e., $\mathbb{P}(E_1 < E_0)$; we target a value of about 50%. This constraint is met in our case if, e.g., $E_0 = 6.5$. As expected, increasing the correlation level between assets and liabilities leads to a more concentrated cash flow distribution. Increasing the size of the liability tail leads to a heavier cash flow tail. Probabilities of negative cash flows are decreasing functions of the correlation level and increasing functions of the liability tail size. This is illustrated in Figure 5 in Section A.11. The first observation is due to the fact that the more positive the dependence, the more effective are the assets as a hedge against liabilities and the lower the probability of negative cash flows.

Regulatory capital requirements. We focus on the two most prominent solvency regimes in insurance, Solvency II and the Swiss Solvency Test, with regulatory capital requirements

$$\rho_{reg}(\Delta E_0) = \begin{cases} \mathrm{V@R}_{0.5\%}(\Delta E_0) & \text{under Solvency II,} \\ \mathrm{AV@R}_{1\%}(\Delta E_0) & \text{under the Swiss Solvency Test.} \end{cases}$$

Figure 2 displays solvency capital requirements as functions of the correlation level between assets and liabilities and of the liability tail size. In line with our previous discussion, the level of regulatory capital is a decreasing function of correlation and an increasing function of tail size. The risk measure $\rho_{reg}(E_1)$ is always negative under our specifications, indicating that we are focusing on companies that are technically solvent with respect to the regulatory solvency tests under consideration. In addition, the solvency ratio $\frac{E_0}{\rho_{reg}(\Delta E_0)}$ lies in the interval $[1, 3]$, which is the relevant range in practice. This is illustrated in Figure 6 and Figure 7 in Section A.11.

Recovery-based capital requirements. We consider recovery risk measures with a simple parametric recovery function belonging to the class described in Section 3.2 with $n = 1$. We fix a regulatory level $\alpha \in (0, 1)$ and consider a piecewise constant recovery function

$$\gamma(\lambda) = \begin{cases} \beta & \text{if } \lambda \in [0, r) \\ \alpha & \text{if } \lambda \in [r, 1] \end{cases} \tag{19}$$

for suitable $\beta \in (0, \alpha)$ and $r \in (0, 1)$. In line with the Solvency II standards we take $\alpha = 0.5\%$. The solvency capital requirement induced by the corresponding RecV@R is given by

$$\mathrm{RecV@R}_{\gamma}(\Delta E_1, L_1) = \max\{\mathrm{V@R}_{\alpha}(\Delta E_1), \mathrm{V@R}_{\beta}(\Delta E_1 + (1 - r)L_1)\},$$

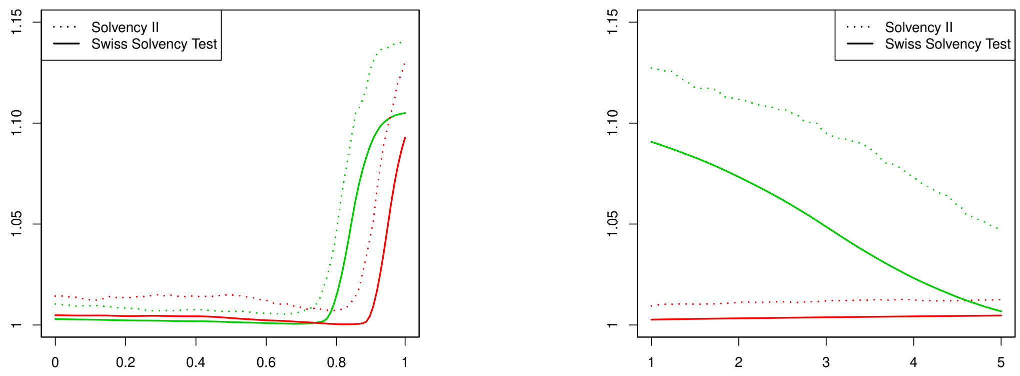

Figure 3: The aggregate recovery adjustment AggRecAdj as a function of ρ (left) for $\tau = 1$ (green) and $\tau = 5$ (red) and as a function of τ (right) for $\rho = 0.1$ (red) and $\rho = 0.9$ (green).

the solvency test (7) is equivalent to

$$\mathbb{P}(A_1 < L_1) \leq \alpha \quad \text{and} \quad \mathbb{P}(A_1 < rL_1) \leq \beta.$$

Besides controlling the default probability $\mathbb{P}(A_1 < L_1)$ at the pre-specified regulatory level α, the recovery risk measure additionally bounds the probability $\mathbb{P}(A_1 < rL_1)$ of covering less than a fraction r of liabilities by a more stringent level β.

Recovery adjustments. In order to capture the extent to which the regulatory solvency capital requirements fail to control the recovery on liabilities we define the *recovery adjustment*

$$\text{RecAdj}_\gamma(\Delta E_1, L_1) := \max\left\{\frac{\text{RecV@R}_\gamma(\Delta E_1, L_1)}{\rho_{reg}(\Delta E_1)}, 1\right\}. \tag{20}$$

This quantity is the maximum of 1 and the multiplicative factor by which regulatory requirements would have to be adjusted to guarantee the considered recovery levels. Recovery adjustments may also be conveniently expressed as a function of the regulatory level β and the recovery rate r as

$$\text{RecAdj}(\beta, r) := \max\left\{\frac{\max\{\text{V@R}_\alpha(\Delta E_1), \text{V@R}_\beta(\Delta E_1 + (1-r)L_1)\}}{\rho_{reg}(\Delta E_1)}, 1\right\}.$$

We consider the *aggregate recovery adjustment*

$$\text{AggRecAdj} := \int_{\beta_{min}}^{\beta_{max}} \int_{r_{min}}^{r_{max}} \text{RecAdj}(\beta, r) d\beta dr$$

with $(\beta_{min}, \beta_{max}) = (0.1\%, 0.25\%)$ and $(r_{min}, r_{max}) = (80\%, 90\%)$. Apart from a normalization constant, this quantity corresponds to the average recovery adjustment over the chosen range of the recovery parameters β and r. Figure 3 displays the aggregate recovery adjustment as a function of the correlation between assets and liabilities and of the size of the liability tail. Figures 8 and 9 in Section A.11 display recovery adjustments for specific choices of β and r.

Observations. The qualitative behavior of recovery adjustments can be described as follows:

(a) Recovery adjustments are typically larger than 1, indicating that regulatory solvency requirements are too low to fulfill the target recovery-based solvency condition.

(b) The size of the recovery adjustment depends on the recovery-based regulatory level β and the recovery rate r as expected: It is larger if β is lower (a tighter constraint on the recovery probability) and if r is higher (a larger portion of liabilities to recover). This relation holds across liability tail sizes and correlation levels but is more pronounced in the presence of lighter liability tails and higher correlations.

(c) For sufficiently large correlation levels, recovery adjustments are increasing functions of the correlation level between assets and liabilities, suggesting that the failure of regulatory capital requirements to control the recovery on liabilities is more pronounced in the presence of large correlation levels. This relation holds across all liability tail sizes.

(d) For sufficiently light tails, recovery adjustments are decreasing functions of the liability tail size, suggesting that the failure of regulatory capital requirements to control the recovery on liabilities is stronger in the presence of lighter liability tails. This relation holds across different correlation levels.

(e) These observations hold for both regulatory frameworks under investigation. In comparison, recovery adjustments in the Swiss Solvency Test are lower than those induced in Solvency II under our distributional specifications.

Our observations demonstrate the importance of recovery risk measures from a risk management perspective. First, we observe that, under standard distributional assumptions, there may exist a considerable gap between the standard risk measures used in practice and our reference recovery risk measure. Second, this gap tends to be wider in the presence of lighter liability tails and higher levels of correlation between assets and liabilities. In situations when assets appear to better hedge liability claims, standard risk measures show lower ability to control recovery rates.

5.4.2 Sophisticated Asset-Liability Management

In this section we consider a firm with a stylized balance sheet. Assets are deterministic, but the firm is capable of controlling the shape of the liability distribution in a sophisticated way. Our case study provides another perspective on the failure of standard solvency regulation to control recovery and highlights that this deficiency might be associated with large recovery adjustments. For detailed calculations we refer to Section A.10.

Distribution of assets and liabilities. We assume that assets evolve in a deterministic way with A_1 being equal to a constant $k > 0$. The future value of liabilities L_1 follows a probability density function with two peaks as displayed in Figure 4. The probability of falling in the light tail peak is equal to 99.5% and that of falling in the heavy tail peak is equal to 0.5%.[21]

Regulatory capital requirements. In line with Solvency II and the Swiss Solvency Test, we focus on VaR at level 0.5% and AVaR at level 1%. The chosen regulatory risk measure is denoted by ρ_{reg}. The corresponding solvency capital requirements admit analytic solutions:

$$\rho_{reg}(\Delta E_1) = \begin{cases} \text{V@R}_{0.5\%}(\Delta E_1) = a - k + E_0, \\ \text{AV@R}_{1\%}(\Delta E_1) = \left(\frac{1}{2} - \frac{1}{3}\sqrt{\frac{\alpha}{2(1-\alpha)}}\right)a + \frac{b+c}{4} - k + E_0. \end{cases}$$

The capital requirements based on V@R are blind to all liability payments beyond the first peak while capital requirements based on AV@R react to the entire distribution of liabilities.

[21] The explicit expression of the density function is provided in Section A.10.

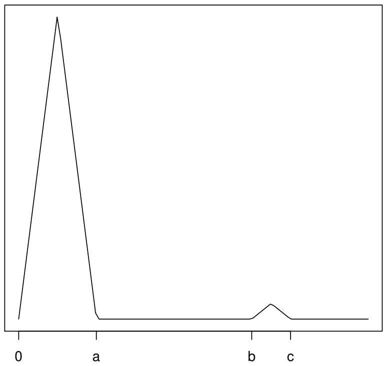

Figure 4: Qualitative plot of the probability density function of L_1. The area below the left peak equals 99.5% while the area below the right peak equals 0.5%.

Recovery-based capital requirements. Fixing a regulatory level $\alpha \in (0,1)$, consider a piecewise constant recovery function belonging to the class described in Section 3.2 with $n = 1$:

$$\gamma(\lambda) = \begin{cases} \beta & \text{if } \lambda \in [0, r) \\ \alpha & \text{if } \lambda \in [r, 1] \end{cases} \tag{21}$$

with $\beta \in (0, \alpha)$, $r \in (0,1)$, and $\alpha = 0.5\%$. The choice of α is motivated by the standards implemented in Solvency II. The solvency capital requirement corresponding to RecV@R is

$$\text{RecV@R}_\gamma(\Delta E_1, L_1) = \max\{\text{V@R}_\alpha(\Delta E_1), \text{V@R}_\beta(\Delta E_1 + (1-r)L_1)\} = \max\left\{a, r\frac{b+c}{2}\right\} - k + E_0$$

and depends on the entire distribution of liabilities. This paralles AV@R, but tail risk is captured in a more sophisticated way: The recovery level r determines the relative importance of the peaks of the liability distribution.

Recovery adjustments. As in Section 5.4.1 we consider recovery adjustments as introduced in (20). We will answer the question how large the recovery adjustments may become, if a firm's asset-liability-management is constrained by the following conditions:

(1)	Solvent profile under ρ_{reg}	$\rho_{reg}(E_1) \leq 0$
(2)	Capital requirement under ρ_{reg}	$\rho_{reg}(\Delta E_1) > 0$
(3)	Solvent profile under RecV@R$_\gamma$	$\text{RecV@R}_\gamma(E_1, L_1) \leq 0$
(4)	Capital requirement under RecV@R$_\gamma$	$\text{RecV@R}_\gamma(\Delta E_1, L_1) > 0$
(5)	V@R$_\alpha$ insufficient to control claims recovery	$\text{RecV@R}_\gamma(E_1, L_1) > \text{V@R}_\alpha(E_1)$
(6)	Range of admissible regulatory solvency ratios	$s_{min} \leq \frac{E_0}{\rho_{reg}(\Delta E_1)} \leq s_{max}$

More precisely, we focus on the optimization problem

$$\max \ \text{RecAdj}_\gamma(\Delta E_1, L_1) \quad \text{over } A_1 \text{ and } L_1 \text{ as specified above} \tag{22}$$

under the constraints (1) to (6). The solvency ratios in (6) are in practice typically in the range between 1.2 and 3.[22] It turns out that in many cases the highest admissible recovery adjustment coincides with $s_{\max}$, i.e., capital requirements under RecV@R can be as large as $s_{\max}$ times the capital requirements under V@R or AV@R. This implies that the difference between the current capital requirements and their recovery-based versions in our asset-liability setting may be substantial. The next proposition states this in detail.

[22]In general, we assume that $1 < s_{min} < s_{max}$.

Proposition 17. *The optimal value of* (22) *is bounded from above by* s_{max}*. If* $\rho_{reg} = \mathrm{V@R}_{0.5\%}$*, this upper bound is attained for every choice of* γ*. If* $\rho_{reg} = \mathrm{AV@R}_{1\%}$*, this upper bound is attained for special choices of* γ*, e.g., when* $\beta \geq \frac{\alpha}{2}$ *and* $\frac{1}{4}\sqrt{\frac{2\alpha}{\alpha-\beta}} < r \leq \frac{1}{4}\frac{\sqrt{2\alpha}}{\sqrt{2\alpha}-\sqrt{\alpha-\beta}}\frac{1}{\frac{1}{2}+\frac{1}{3}\sqrt{\frac{\alpha}{2(1-\alpha)}}}$.[23]

Proof. See Section A.7. □

The preceding result suggests that companies subject to capital requirements based on V@R and AV@R may be far from guaranteeing acceptable recovery rates on their creditors' claims. In the V@R case, this is a consequence of tail blindness, which, in the absence of external controls, allows companies to accumulate tail risk without any regulatory cost. Also in the AV@R case this problem does not disappear because increased tail risk may often be compensated by a suitable shift in the asset distribution or in the body of the liability distribution. In our example, a more dispersed distribution beyond the 99.5% quantile may leave the AV@R unchanged provided the distribution within the same quantile level shrinks.[24]

6 Conclusion

Risk measures used in solvency regulation specify guard rails for financial firms such as banks or insurance companies. Within their legal boundaries firms can otherwise freely choose their actions, e.g., in order to maximize shareholder value. As a consequence, an axiomatic theory of risk measures for solvency regulation should carefully formulate and capture the goals of regulation, and determine and investigate suitable instruments to meet them. The issue of recovery on creditors' claims has not yet been considered in sufficient detail, and the existing literature lacks solvency requirements that provide adequate protection of customers and counterparties in the case of default. In this paper, we propose the novel concept of recovery risk measures to resolve this issue. We analyze the properties of these risk measures and describe how to apply them in the context of solvency regulation, performance measurement, and portfolio optimization. Our findings suggest that recovery risk measures add value to the current risk management toolkit. They are tractable tools for both internal risk management and solvency regulation and can be employed to provide a more comprehensive picture on tail risk with a focus on safeguarding the interests of creditors and policyholders.

Various extensions of the suggested framework are possible. Our framework focuses on a static setting as common in solvency regulation where relevant time horizons are fixed. Dynamic or conditional solvency risk measures would be an interesting extension; see Bielecki, Cialenco & Pitera (2017) for a survey on previous research on this topic. Recovery risk measures are applied to single firms in this paper. From this perspective, another natural extension is the regulation of financial systems. As outlined in the literature, systemic risk measures should capture the local and global interaction of economic agents and operationalize the emerging risk at the level of the entire system; see, e.g., Chen, Iyengar & Moallemi (2013), Kromer, Overbeck & Zilch (2016), Feinstein et al. (2017), Biagini et al. (2019). A related issue is the study of optimal investment under risk constraints in the spirit of Markowitz (1952), where the variance is replaced by a recovery risk measure. Our approach to model firms' balance sheets offers a natural and flexible starting point to address optimal asset-liability problems of this type.

[23]For instance, if $\beta = \frac{\alpha}{2}$, then we can take $50\% < r \leq 95\%$.

[24]This phenomenon refers to the lack of *surplus invariance* and was studied in Koch-Medina & Munari (2016).

A Appendix

A.1 Proof of Proposition 1

Proof. Fix $\lambda \in (0,1)$. By nonatomicity, for every $p \in (0,\alpha)$ we find an event $F_p \in \mathcal{F}$ such that $\mathbb{P}(F_p) = p$. Set $A_p = \frac{p}{\alpha-p} 1_{F_p^c}$ and $L_p = 1_{F_p}$ and note that $A_p - L_p = -1_{F_p} + \frac{p}{\alpha-p} 1_{F_p^c}$. Note also that both A_p and L_p belong to $\mathcal{X}$. A simple computation shows that

$$\text{V@R}_\alpha(A_p - L_p) \leq \text{AV@R}_\alpha(A_p - L_p) = \frac{1}{\alpha}\left(p - (\alpha - p)\frac{p}{\alpha - p}\right) = 0.$$

Moreover, we have $\mathbb{P}(A_p \geq \lambda L_p) = \mathbb{P}(F_p^c) = 1 - p$. As a result,

$$\begin{aligned} 1 - \alpha &\leq \inf\{\mathbb{P}(A \geq L)\,;\ A, L \in \mathcal{X}_+,\ \text{V@R}_\alpha(A - L) \leq 0\} \\ &\leq \inf\{\mathbb{P}(A \geq \lambda L)\,;\ A, L \in \mathcal{X}_+,\ \text{V@R}_\alpha(A - L) \leq 0\} \\ &\leq \inf\{\mathbb{P}(A \geq \lambda L)\,;\ A, L \in \mathcal{X}_+,\ \text{AV@R}_\alpha(A - L) \leq 0\} \\ &\leq \inf\{\mathbb{P}(A_p \geq \lambda L_p)\,;\ 0 < p < \alpha\} \\ &= \inf\{1 - p\,;\ 0 < p < \alpha\} = 1 - \alpha. \end{aligned}$$

This yields the desired statements. □

A.2 Proof of Proposition 6

Proof. Fix $i = 1, \dots, n$ and observe that γ is constant and equal to α_i on the interval $[r_{i-1}, r_i)$. Hence, we get

$$\text{V@R}_{\gamma(\lambda)}(X + (1-\lambda)Y) = \text{V@R}_{\alpha_i}(X + (1-\lambda)Y) \leq \text{V@R}_{\alpha_i}(X + (1-r_i)Y)$$

for every $\lambda \in [r_{i-1}, r_i)$ by positivity of Y and monotonicity of V@R. As a result,

$$\sup_{\lambda \in [r_{i-1}, r_i)} \text{V@R}_{\gamma(\lambda)}(X + (1-\lambda)Y) = \text{V@R}_{\alpha_i}(X + (1-r_i)Y).$$

Similarly, observe that γ is constant and equal to α_{n+1} on the interval $[r_n, 1]$. Hence, we get

$$\text{V@R}_{\gamma(\lambda)}(X + (1-\lambda)Y) = \text{V@R}_{\alpha_{n+1}}(X + (1-\lambda)Y) \leq \text{V@R}_{\alpha_{n+1}}(X)$$

for every $\lambda \in [r_n, 1]$ by positivity of Y and monotonicity of V@R. As a result,

$$\sup_{\lambda \in [r_n, 1]} \text{V@R}_{\gamma(\lambda)}(X + (1-\lambda)Y) = \text{V@R}_{\alpha_{n+1}}(X).$$

The desired assertion is a direct consequence of the above identities. □

A.3 Proof of Proposition 11

Proof. Fix $i = 1, \dots, n$ and observe that γ is equal to α_i on the interval $[r_{i-1}, r_i)$. Hence,

$$\text{AV@R}_{\gamma(\lambda)}(X + (1-\lambda)Y) = \text{AV@R}_{\alpha_i}(X + (1-\lambda)Y) \leq \text{AV@R}_{\alpha_i}(X + (1-r_i)Y)$$

for every $\lambda \in [r_{i-1}, r_i)$ by positivity of Y and monotonicity of AV@R. As a result,

$$\sup_{\lambda \in [r_{i-1}, r_i)} \text{AV@R}_{\gamma(\lambda)}(X + (1-\lambda)Y) = \text{AV@R}_{\alpha_i}(X + (1-r_i)Y).$$

Similarly, observe that γ is constant and equal to α_{n+1} on the interval $[r_n, 1]$. Hence, we get

$$\text{AV@R}_{\gamma(\lambda)}(X + (1-\lambda)Y) = \text{AV@R}_{\alpha_{n+1}}(X + (1-\lambda)Y) \leq \text{AV@R}_{\alpha_{n+1}}(X)$$

for every $\lambda \in [r_n, 1]$ by positivity of Y and monotonicity of AV@R. As a result,

$$\sup_{\lambda \in [r_n, 1]} \text{AV@R}_{\gamma(\lambda)}(X + (1-\lambda)Y) = \text{AV@R}_{\alpha_{n+1}}(X).$$

The desired assertion is a direct consequence of the above identities. □

A.4 Dual Representation for Recovery Average Value at Risk

We establish a dual representation of RecAV@R in terms of probability measures as in the classical framework of monetary risk measures. Here, we denote by $\mathcal{M}_1^\infty(\mathbb{P})$ the set of all probability measures $\mathbb{Q}$ over $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}$ and such that the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ belongs to L^∞.

Proposition 18. *For all $X, Y \in L^1$ the following representation holds:*

$$\mathrm{RecAV@R}_\gamma(X,Y) = \sup_{\mathbb{Q}\in\mathcal{M}_1^\infty(\mathbb{P})} \big(\mathbb{E}_\mathbb{Q}(-X) - (1-\lambda(\mathbb{Q}))\mathbb{E}_\mathbb{Q}(Y)\big)$$

where for each $\mathbb{Q} \in \mathcal{M}_1^\infty(\mathbb{P})$ we set

$$\lambda(\mathbb{Q}) = \sup\left\{\lambda \in [0,1]\,;\ \gamma(\lambda) \le \left\|\frac{d\mathbb{Q}}{d\mathbb{P}}\right\|_\infty^{-1}\right\}.$$

Proof. It is well-known, see e.g. Theorem 4.52 in Föllmer & Schied (2016) for the classical statement in L^∞, that for all $\lambda \in [0,1]$ and $X \in L^1$

$$\mathrm{AV@R}_{\gamma(\lambda)}(X) = \sup_{\mathbb{Q}\in\mathcal{M}_1^\infty(\mathbb{P})} \big(\mathbb{E}_\mathbb{Q}(-X) - \alpha_\lambda(\mathbb{Q})\big)$$

where

$$\alpha_\lambda(\mathbb{Q}) = \begin{cases} 0 & \text{if } \frac{d\mathbb{Q}}{d\mathbb{P}} \le \frac{1}{\gamma(\lambda)}, \\ \infty & \text{otherwise.} \end{cases}$$

Exchanging the supremum in the definition of RecAV@R with that in the dual representation of AV@R yields

$$\mathrm{RecAV@R}_\gamma(X,Y) = \sup_{\mathbb{Q}\in\mathcal{M}_1^\infty(\mathbb{P})} \big(\mathbb{E}_\mathbb{Q}(-X) - \alpha_Y(\mathbb{Q})\big)$$

where

$$\alpha_Y(\mathbb{Q}) = \mathbb{E}_\mathbb{Q}(Y) + \inf_{\lambda\in[0,1]} \big(\alpha_\lambda(\mathbb{Q}) - \lambda\mathbb{E}_\mathbb{Q}(Y)\big).$$

Now, for every $\mathbb{Q} \in \mathcal{M}_1^\infty(\mathbb{P})$ define

$$\Lambda(\mathbb{Q}) = \left\{\lambda \in [0,1]\,;\ \left\|\frac{d\mathbb{Q}}{d\mathbb{P}}\right\|_\infty \le \frac{1}{\gamma(\lambda)}\right\}$$

and set $\lambda(\mathbb{Q}) = \sup \Lambda(\mathbb{Q})$. Since Y is positive, it is easy to see that

$$\inf_{\lambda\in[0,1]} \big(\alpha_\lambda(\mathbb{Q}) - \lambda\mathbb{E}_\mathbb{Q}(Y)\big) = \inf_{\lambda\in\Lambda(\mathbb{Q})} \big(-\lambda\mathbb{E}_\mathbb{Q}(Y)\big) = -\lambda(\mathbb{Q})\mathbb{E}_\mathbb{Q}(Y)$$

for every $\mathbb{Q} \in \mathcal{M}_1^\infty(\mathbb{P})$. This implies that

$$\mathrm{RecAV@R}_\gamma(X,Y) = \sup_{\mathbb{Q}\in\mathcal{M}_1^\infty(\mathbb{P})} \big(\mathbb{E}_\mathbb{Q}(-X) - (1-\lambda(\mathbb{Q}))\mathbb{E}_\mathbb{Q}(Y)\big). \qquad \square$$

As discussed in (5), the term $E_1 + (1-\lambda)L_1$ represents the available resources of the firm at time 1 beyond a recovery level λ. For a fixed probability measure $\mathbb{Q} \in \mathcal{M}_1^\infty(\mathbb{P})$, we can thus interpret $\mathbb{E}_\mathbb{Q}(-E_1) - (1-\lambda)\mathbb{E}_\mathbb{Q}(L_1)$ as the expected shortfall below the recovery level λ with respect to the measure $\mathbb{Q}$. Proposition 18 thus represents $\mathrm{RecAV@R}_\gamma$ as a supremum over expected shortfalls below different recovery levels over the collection of absolutely continuous probability measures with bounded Radon-Nikodym density: The recovery levels depend on $\mathbb{Q}$ and are given by the generalized inverse of γ evaluated at the inverse of the supremum norm of the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to the reference measure $\mathbb{P}$. The robust representation in Proposition 18 can thus be interpreted as a worst-case approach in the face of Knightian uncertainty: The recovery risk measure $\mathrm{RecAV@R}_\gamma$ is the worst-case expected shortfall beyond different recovery levels over a class of probability measures. The size of the recovery level encodes to what extent different probability measures contribute to the worst-case.

A.5 Properties of Recovery Risk Measures

The next result collects some basic properties of recovery risk measures. In particular, we analyze how a recovery risk measure inherits the key properties of its underlying building blocks.

Proposition 19. *A recovery risk measure* $\mathrm{Rec}\rho : \mathcal{X}\times\mathcal{X} \to \mathbb{R}\cup\{\infty\}$ *has the following properties:*

(a) Cash invariance in the first component: *If* ρ_λ *is cash invariant for every* $\lambda \in [0,1]$*, then for all* $X, Y \in \mathcal{X}$ *and* $m \in \mathbb{R}$

$$\mathrm{Rec}\rho(X+m, Y) = \mathrm{Rec}\rho(X, Y) - m.$$

(b) Monotonicity: *If* ρ_λ *is monotone for every* $\lambda \in [0,1]$*, then for all* $X_1, X_2, Y_1, Y_2 \in \mathcal{X}$ *such that* $X_1 \geq X_2$ *and* $Y_1 \geq Y_2$ $\mathbb{P}$*-almost surely*

$$\mathrm{Rec}\rho(X_1, Y_1) \leq \mathrm{Rec}\rho(X_2, Y_2).$$

(c) Convexity: *If* ρ_λ *is convex for every* $\lambda \in [0,1]$*, then for all* $X_1, X_2, Y_1, Y_2 \in \mathcal{X}$ *and* $a \in [0,1]$

$$\mathrm{Rec}\rho(aX_1 + (1-a)X_2, aY_1 + (1-a)Y_2) \leq a\,\mathrm{Rec}\rho(X_1, Y_1) + (1-a)\,\mathrm{Rec}\rho(X_2, Y_2).$$

(d) Subadditivity: *If* ρ_λ *is subadditive for every* $\lambda \in [0,1]$*, then for all* $X_1, X_2, Y_1, Y_2 \in \mathcal{X}$

$$\mathrm{Rec}\rho(X_1 + X_2, Y_1 + Y_2) \leq \mathrm{Rec}\rho(X_1, Y_1) + \mathrm{Rec}\rho(X_2, Y_2).$$

(e) Positive homogeneity: *If* ρ_λ *is positively homogeneous for every* $\lambda \in [0,1]$*, then for all* $X, Y \in \mathcal{X}$ *and* $a \in [0,\infty)$

$$\mathrm{Rec}\rho(aX, aY) = a\,\mathrm{Rec}\rho(X, Y).$$

(f) Star-shapedness in the first component: *If* ρ_λ *is monotone and positively homogeneous for every* $\lambda \in [0,1]$*, then for all* $X \in \mathcal{X}$*,* $Y \in \mathcal{X}_+$*, and* $a \in [1,\infty)$

$$\mathrm{Rec}\rho(aX, Y) \geq a\,\mathrm{Rec}\rho(X, Y).$$

(g) Normalization: *If* ρ_λ *is monotone and* $\rho_\lambda(0) = 0$ *for every* $\lambda \in [0,1]$*, then* $\mathrm{Rec}\rho(0, Y) = 0$ *for every* $Y \in \mathcal{X}_+$*.*

(h) Finiteness: *If* ρ_λ *is monotone for every* $\lambda \in [0,1]$*, then for every* $X \in \mathcal{X}$ *with* $\rho_0(X) < \infty$ *and for every* $Y \in \mathcal{X}_+$ *we have* $\mathrm{Rec}\rho(X, Y) < \infty$*.*

Proof. (a) For every $X \in \mathcal{X}$ and $m \in \mathbb{R}$ the cash invariance of ρ_λ readily implies

$$\mathrm{Rec}\rho(X+m, Y) = \sup_{\lambda\in[0,1]} \rho_\lambda(X+m+(1-\lambda)Y) = \sup_{\lambda\in[0,1]} \rho_\lambda(X+(1-\lambda)Y) - m = \mathrm{Rec}\rho(X, Y) - m.$$

(b) Since $X_1 + (1-\lambda)Y_1 \geq X_2 + (1-\lambda)Y_2$ for every $\lambda \in [0,1]$, the monotonicity of ρ_λ yields

$$\mathrm{Rec}\rho(X_1, Y_1) = \sup_{\lambda\in[0,1]} \rho_\lambda(X_1 + (1-\lambda)Y_1) \leq \sup_{\lambda\in[0,1]} \rho_\lambda(X_2 + (1-\lambda)Y_2) = \mathrm{Rec}\rho(X_2, Y_2).$$

(c) It follows from the convexity of ρ_λ that

$$\begin{aligned}
\mathrm{Rec}\rho(aX_1 + (1-a)X_2, aY_1 + (1-a)Y_2) &= \sup_{\lambda\in[0,1]} \rho_\lambda(a(X_1 + (1-\lambda)Y_1) + (1-a)(X_2 + (1-\lambda)Y_2)) \\
&\leq \sup_{\lambda\in[0,1]} \big(a\rho_\lambda(X_1 + (1-\lambda)Y_1) + (1-a)\rho_\lambda(X_2 + (1-\lambda)Y_2)\big) \\
&\leq a \sup_{\lambda\in[0,1]} \rho_\lambda(X_1 + (1-\lambda)Y_1) + (1-a) \sup_{\lambda\in[0,1]} \rho_\lambda(X_2 + (1-\lambda)Y_2) \\
&= a\,\mathrm{Rec}\rho(X_1, Y_1) + (1-a)\,\mathrm{Rec}\rho(X_2, Y_2).
\end{aligned}$$

(d) Similarly, we can use the subadditivity of ρ_λ to get

$$\begin{aligned}\mathrm{Rec}\rho(X_1+X_2,Y_1+Y_2) &= \sup_{\lambda\in[0,1]} \rho_\lambda(X_1+(1-\lambda)Y_1+X_2+(1-\lambda)Y_2)\\ &\leq \sup_{\lambda\in[0,1]} \{\rho_\lambda(X_1+(1-\lambda)Y_1)+\rho_\lambda(X_2+(1-\lambda)Y_2))\}\\ &\leq \sup_{\lambda\in[0,1]} \rho_\lambda(X_1+(1-\lambda)Y_1)+\sup_{\lambda\in[0,1]} \rho_\lambda(X_2+(1-\lambda)Y_2))\\ &= \mathrm{Rec}\rho(X_1,Y_1)+\mathrm{Rec}\rho(X_2,Y_2).\end{aligned}$$

(e) Using the positive homogeneity of ρ_λ, we easily see that

$$\mathrm{Rec}\rho(aX,aY) = \sup_{\lambda\in[0,1]} \rho_\lambda(aX+(1-\lambda)aY) = \sup_{\lambda\in[0,1]} a\rho_\lambda(X+(1-\lambda)Y) = a\,\mathrm{Rec}\rho(X,Y).$$

(f) Observe that $a(1-\lambda)Y \geq (1-\lambda)Y$ for every $\lambda\in[0,1]$. This is because Y is positive. Then, it follows from the monotonicity and positive homogeneity of ρ_λ that

$$\mathrm{Rec}\rho(aX,Y) = \sup_{\lambda\in[0,1]} \rho_\lambda(aX+(1-\lambda)Y) \geq \sup_{\lambda\in[0,1]} \rho_\lambda(a(X+(1-\lambda)Y)) = a\,\mathrm{Rec}\rho(X,Y).$$

(g) It follows from monotonicity that $\rho_\lambda((1-\lambda)Y) \leq \rho_\lambda(0)$ for every $\lambda\in[0,1]$. This is because Y is positive. Then, normalization yields

$$0 = \rho_1(0) \leq \mathrm{Rec}\rho(0,Y) = \sup_{\lambda\in[0,1]} \rho_\lambda((1-\lambda)Y) \leq \sup_{\lambda\in[0,1]} \rho_\lambda(0) = 0.$$

(h) Take $X\in\mathcal{X}$ such that $\rho_0(X)<\infty$ and observe that $X+(1-\lambda)Y \geq X$ for every $\lambda\in[0,1]$. This is because Y is positive. Then, monotonicity implies

$$\mathrm{Rec}\rho(X,Y) = \sup_{\lambda\in[0,1]} \rho_\lambda(X+(1-\lambda)Y) \leq \sup_{\lambda\in[0,1]} \rho_\lambda(X) = \rho_0(X) < \infty,$$

where we used that $\rho_0(X) \geq \rho_\lambda(X)$ for every $\lambda\in[0,1]$ by assumption. □

If the risk measures ρ_λ's are convex, the recovery risk measure Recρ admits a dual representation similar to the classical representations discussed in Föllmer & Schied (2016). Suppose $\mathcal{X} = L^p$ for some $p\in[1,\infty]$. For every $q\in[1,\infty]$ we denote by $\mathcal{M}_1^q(\mathbb{P})$ the set of probability measures $\mathbb{Q}$ over $(\Omega,\mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}$ and satisfy $\frac{d\mathbb{Q}}{d\mathbb{P}}\in L^q$. Recall that a map $\rho:\mathcal{X}\to\mathbb{R}\cup\{\infty\}$ satisfies the *Fatou property* if for every sequence $(X_n)\subset\mathcal{X}$ and every $X\in\mathcal{X}$ we have

$$X_n\to X \ \mathbb{P}\text{-almost surely},\ \sup_{n\in\mathbb{N}}|X_n|\in\mathcal{X} \quad\Longrightarrow\quad \rho(X)\leq\liminf_{n\to\infty}\rho(X_n).$$

The Fatou property corresponds to a weak form of continuity, namely lower semicontinuity, with respect to dominated almost-sure convergence. A well-known result by Jouini, Schachermayer & Touzi (2006) shows that every distribution-based monetary risk measure defined on L^∞ has the Fatou property. We refer to Gao, Leung, Munari & Xanthos (2018) for a general result beyond bounded random variables. It follows at once from Theorem 4.33 in Föllmer & Schied (2016) (if $p=\infty$) or from Corollary 7 in Frittelli & Gianin (2002) (if $p<\infty$) that, if ρ_λ is a convex monetary risk measure with the Fatou property for every $\lambda\in[0,1]$, then for all $X,Y\in\mathcal{X}$

$$\mathrm{Rec}\rho(X,Y) = \sup_{\mathbb{Q}\in\mathcal{M}_1^q(\mathbb{P})} \big(\mathbb{E}_\mathbb{Q}(-X)-\alpha_Y(\mathbb{Q})\big)$$

where $q=\frac{p}{1-p}$ and for every $\mathbb{Q}\in\mathcal{M}_1^q(\mathbb{P})$ we set

$$\alpha_Y(\mathbb{Q}) = \mathbb{E}_\mathbb{Q}(Y) + \inf_{\lambda\in[0,1]}\left(\sup_{X\in\mathcal{A}_\lambda}\mathbb{E}_\mathbb{Q}(-X)-\lambda\mathbb{E}_\mathbb{Q}(Y)\right)$$

where $\mathcal{A}_\lambda=\{X\in L^p\,;\ \rho_\lambda(X)\leq 0\}$. The proof is similar to that of Proposition 18 and is thus omitted.

A.6 Complementary Material for Section 5.1

Fix $k \in [1, 100]$. We know from Example 2 that $\mathrm{V@R}_\alpha(E_1^k) = k - 100$. Moreover,

$$\mathrm{V@R}_\beta(E_1^k + (1-r)L_1) = \begin{cases} 100r - k & \text{if } \beta < \frac{\alpha}{2},\ k \le \frac{101+99r}{2}, \\ k + r - 101 & \text{otherwise.} \end{cases}$$

It follows from Proposition 6 that

$$\mathrm{RecV@R}_\gamma(E_1^k, L_1) = \max\{\mathrm{V@R}_\alpha(E_1^k), \mathrm{V@R}_\beta(E_1^k + (1-r)L_1).$$

As a result, a direct calculation yields

$$\mathrm{RecV@R}_\gamma(E_1^k, L_1) = \begin{cases} 100r - k & \text{if } \beta < \frac{\alpha}{2},\ k \le 50(r+1), \\ k - 100 & \text{otherwise.} \end{cases}$$

This shows that

$$\mathrm{RecV@R}_\gamma(E_1^k, L_1) \le 0 \iff \begin{cases} k \ge 100r & \text{if } \beta < \frac{\alpha}{2}, \\ k \ge 0 & \text{if } \beta \ge \frac{\alpha}{2}. \end{cases}$$

We turn to RecAV@R. We know from Example 2 that $\mathrm{AV@R}_\alpha(E_1^k) = 0$. Moreover,

$$\mathrm{AV@R}_\beta(E_1^k + (1-r)L_1) = \begin{cases} 100r - k & \text{if } \beta < \frac{\alpha}{2},\ k \le \frac{101+99r}{2}, \\ r - 101 + \frac{\alpha}{2\beta}(101 + 99r) + (1 - \frac{\alpha}{\beta})k & \text{if } \beta \ge \frac{\alpha}{2},\ k \le \frac{101+99r}{2}, \\ k + r - 101 & \text{otherwise.} \end{cases}$$

It follows from Proposition 11 that

$$\mathrm{RecAV@R}_\gamma(E_1^k, L_1) = \max\{\mathrm{AV@R}_\alpha(E_1^k), \mathrm{AV@R}_\beta(E_1^k + (1-r)L_1)\}.$$

As a result, we obtain

$$\mathrm{RecAV@R}_\gamma(E_1^k, L_1) = \begin{cases} 100r - k & \text{if } \beta < \frac{\alpha}{2},\ k \le 100r, \\ r - 101 + \frac{\alpha}{2\beta}(101 + 99r) + (1 - \frac{\alpha}{\beta})k & \text{if } \beta \ge \frac{\alpha}{2},\ k \le \frac{(99\alpha+2\beta)r - 101(2\beta-\alpha)}{2(\alpha-\beta)}, \\ 0 & \text{otherwise.} \end{cases}$$

We infer that

$$\mathrm{RecAV@R}_\gamma(E_1^k, L_1) \le 0 \iff \begin{cases} k \ge 100r & \text{if } \beta < \frac{\alpha}{2}, \\ k \ge \max\left\{\frac{(99\alpha+2\beta)r - 101(2\beta-\alpha)}{2(\alpha-\beta)}, 0\right\} & \text{if } \beta \ge \frac{\alpha}{2}. \end{cases}$$

A.7 Proof of Proposition 17

Proof. Throughout the proof we use the notation from Section A.10. In the V@R case, the optimization problem can be reformulated in explicit terms as

$$\begin{aligned} \max \quad & \frac{rq_\beta(b,c) - k + E_0}{a - k + E_0} \\ \text{s.t.} \quad & (1)\ \ k \ge a, \\ & (2)\ \ k < a + E_0, \\ & (3)\ \ k \ge rq_\beta(b,c), \\ & (4)\ \ k < rq_\beta(b,c) + E_0, \\ & (5)\ \ rq_\beta(b,c) > a, \\ & (6)\ \ s_{min} \le \frac{E_0}{a - k + E_0} \le s_{max}, \\ \text{over} \quad & k > 0 \ \text{ and } \ 0 < a < b < c. \end{aligned}$$

It is clear that, due to (5), both conditions (1) and (4) can be dropped as they are implied by conditions (3) and (2), respectively. Moreover, (6) clearly implies (2). Since the objective function is increasing in the term $q_\beta(b,c)$, the maximum is achieved by taking $rq_\beta(b,c) = k$ in (3). In this case, condition (5) is implied by (6). By conveniently rewriting condition (6), we thus obtain the equivalent problem

$$\begin{aligned} &\max \frac{E_0}{a-k+E_0} \\ &\qquad \text{s.t.} \quad a + \frac{s_{min}-1}{s_{min}} E_0 \leq k \leq a + \frac{s_{max}-1}{s_{max}} E_0, \\ &\qquad \text{over} \quad k > 0 \text{ and } a > 0. \end{aligned}$$

Note that the new objective function is increasing in k. As a result, the maximum is achieved by taking $k = a + \frac{s_{max}-1}{s_{max}} E_0$, which yields a recovery adjustment equal to

$$\frac{E_0}{a - \left(a + \frac{s_{max}-1}{s_{max}} E_0\right) + E_0} = s_{max}.$$

(The parameter a can be selected to ensure a realistic loss probability. Indeed, we have

$$\mathbb{P}(E_1 < E_0) = P\left(L_1 > a - \frac{1}{s_{max}} E_0\right).$$

It is then clear that we can always choose a so as to ensure a loss probability around 50%. To this effect, it suffices to have $a - \frac{1}{s_{max}} E_0 \approx \frac{a}{2}$). To deal with the AV@R case, it is first convenient to define the quantity

$$\xi = \frac{1}{2} - \frac{1}{3}\sqrt{\frac{\alpha}{2(1-\alpha)}} = 0.47\ldots.$$

The corresponding optimization problem can be rewritten in explicit terms as

$$\begin{aligned} &\max \frac{rq_\beta(b,c) - k + E_0}{\xi a + \frac{b+c}{4} - k + E_0} \\ &\qquad \text{s.t.} \quad (1)\ \ k \geq \xi a + \frac{b+c}{4}, \\ &\qquad \qquad \ \ (2)\ \ k < \xi a + \frac{b+c}{4} + E_0, \\ &\qquad \qquad \ \ (3)\ \ k \geq rq_\beta(b,c), \\ &\qquad \qquad \ \ (4)\ \ k < rq_\beta(b,c) + E_0, \\ &\qquad \qquad \ \ (5)\ \ rq_\beta(b,c) > a, \\ &\qquad \qquad \ \ (6)\ \ s_{min} \leq \frac{E_0}{\xi a + \frac{b+c}{4} - k + E_0} \leq s_{max}, \\ &\qquad \text{over} \quad k > 0 \text{ and } 0 < a < b < c. \end{aligned}$$

Observe that we cannot proceed as in the V@R case because of a more complex dependence on the parameters b and c. As a first step, note that condition (6) implies both conditions (1) and (2). The objective function is decreasing in a. As a result, the maximum is achieved by taking

$a = \frac{1}{\xi}\left(k - \frac{b+c}{4} - \frac{s_{max}-1}{s_{max}}E_0\right)$ in (6). We thus obtain the equivalent problem

$$\max \ \frac{rq_\beta(b,c) - k + E_0}{\frac{E_0}{s_{max}}}$$

$$\begin{aligned}
\text{s.t.} \quad & (1') \ \ k \geq rq_\beta(b,c), \\
& (2') \ \ k < rq_\beta(b,c) + E_0, \\
& (3') \ \ rq_\beta(b,c) > \frac{1}{\xi}\left(k - \frac{b+c}{4} - \frac{s_{max}-1}{s_{max}}E_0\right), \\
& (4') \ \ 0 < \frac{1}{\xi}\left(k - \frac{b+c}{4} - \frac{s_{max}-1}{s_{max}}E_0\right) < b, \\
\text{over} \quad & k > 0 \ \text{ and } \ 0 < b < c.
\end{aligned}$$

The new objective function is decreasing in k. This implies that the maximum is achieved by taking $k = rq_\beta(b,c)$ in (1'), which entails a recovery adjustment equal to

$$\frac{rq_\beta(b,c) - rq_\beta(b,c) + E_0}{\frac{E_0}{s_{max}}} = s_{max}.$$

In this case, condition (2') is automatically satisfied. We need to show when there exist $0 < b < c$ satisfying all the remaining conditions, namely (3') and (4'), under $k = rq_\beta(b,c)$. We focus on the case $\beta \geq \frac{\alpha}{2}$. In this case, setting $\lambda = \sqrt{\frac{\alpha-\beta}{2\alpha}} \in (0, \frac{1}{2}]$, we can write $q_\beta(b,c) = (1-\lambda)b + \lambda c$. Condition (3') can equivalently be written as

$$\left((1-\xi)r(1-\lambda) - \frac{1}{4}\right)b + \left((1-\xi)r\lambda - \frac{1}{4}\right)c < \frac{s_{max}-1}{s_{max}}E_0.$$

This holds for all $0 < b < c$ provided that the two expressions multiplying b and c are both negative, i.e.

$$r \leq \frac{1}{4}\frac{1}{(1-\lambda)(1-\xi)}. \tag{23}$$

Similarly, condition (4') is equivalent to

$$\frac{s_{max}-1}{s_{max}}E_0 - \left(r(1-\lambda) - \frac{1}{4}\right)b < \left(r\lambda - \frac{1}{4}\right)c < \frac{s_{max}-1}{s_{max}}E_0.$$

For every $b > 0$, this holds for a suitable $c > 0$ provided that the expression in parenthesis multiplying b and the expression multiplying c are both strictly positive, i.e.

$$r > \frac{1}{4\lambda}. \tag{24}$$

To ensure that c can be taken to satisfy $c > b$, it suffices to impose the bound $b < \frac{1}{r\lambda - \frac{1}{4}}\frac{s_{max}-1}{s_{max}}E_0$. This shows that we can indeed find $0 < b < c$ satisfying (3') and (4') under $k = rq_\beta(b,c)$ provided that (23) and (24) hold. □

A.8 Proof of Proposition 16

Proof. We rely on Lemma 5.6 in Tasche (2000). To this effect, the random variables

$$X^1 = -(\Delta E_1^1 + (1-r_j)L_1^1), \ldots, X^N = -(\Delta E_1^N + (1-r_j)L_1^N)$$

have to satisfy the so-called (S)-Assumption in that paper. This stipulates some requirements on the joint distribution of the above random variables, namely:

- $X^1,\dots,X^N$ are integrable and continuously distributed;
- the conditional distribution of X^1 given $X^2,\dots,X^N$ has a density ϕ;
- $x^1 \mapsto \phi(x^1,x^2,\dots,x^N)$ is continuous for all $x^2,\dots,x^N \in \mathbb{R}$;
- the maps $\Phi_1,\dots,\Phi_N : \mathbb{R}\times\mathbb{R}\setminus\{0\}\times\mathbb{R}^{N-1} \to \mathbb{R}$ defined by

$$\Phi_1(x^1,u_1,\dots,u_N) = E_P\left(\phi\Big(u_1^{-1}\Big(x^1-\sum_{i=2}^N u_iX^i\Big),X^2,\dots,X^N\Big)\right),$$

$$\Psi_l(x^1,u_1,\dots,u_N) = E_P\left(X^l\phi\Big(u_1^{-1}\Big(x^1-\sum_{i=2}^N u_iX^i\Big),X^2,\dots,X^N\Big)\right),\quad l=2,\dots,N,$$

  are finite valued and continuous;
- for every $u=(u_1,\dots,u_N)\in\mathbb{R}\setminus\{0\}\times\mathbb{R}^{N-1}$ we have

$$E_P\left(\phi\Big(u_1^{-1}\Big(q_{\alpha_j}(u)-\sum_{i=2}^N u_iX^i\Big),X^2,\dots,X^N\Big)\right)>0,$$

  where $q_{\alpha_j}(u)=\inf\{x\in\mathbb{R}\,;\ \mathbb{P}(\sum_{i=1}^N u_iX^i\le x)\ge 1-\alpha_j\}$.

Now, fix $i=1,\dots,N$ and for every $k=1,\dots,n+1$ consider the convex (hence, continuous) function $f^{i,k}:\mathbb{R}\to\mathbb{R}$ defined by setting

$$f^{i,k}(h)=\mathrm{AV@R}_{\alpha_k}(\Delta E_1+(1-r_k)L_1+h(\Delta E_1^i+(1-r_k)L_1^i)).$$

By assumption we have that

$$f^{i,j}(0)>\max_{k=1,\dots,n+1,\ k\neq j} f^{i,k}(0).$$

It follows from continuity that there exists $\varepsilon>0$ such that

$$f^{i,j}(h)>\max_{k=1,\dots,n+1,\ k\neq j} f^{i,k}(h).$$

for every $h\in(-\varepsilon,\varepsilon)$. As a result,

$$\mathrm{RecAV@R}_\gamma(\Delta E_1+h\Delta E_1^i,L_1+hL_1^i)=\max_{k=1,\dots,n+1} f^{i,k}(h)=f^{i,j}(h)$$

for every $h\in(-\varepsilon,\varepsilon)$. This immediately yields

$$\frac{d}{dh}\mathrm{RecAV@R}_\gamma(\Delta E_1+h\Delta E_1^i,L_1+hL_1^i)_{|h=0}=\frac{df^{i,j}}{dh}(0).$$

Since the (S)-Assumption holds, we infer from Theorem 4.4 in Tasche (2000) that

$$\frac{df^{i,j}}{dh}(0)=-E\big(\Delta E_1^i+(1-r_j)L_1^i\,|\,\Delta E_1+(1-r_j)L_1\le -\mathrm{V@R}_{\alpha_j}(\Delta E_1+(1-r_j)L_1)\big).$$

This delivers the desired statement. □

A.9 Complementary Material for Section 5.4.1

The probability distribution function of A_1 is specified by

$$\mathbb{P}(A_1 \le x) = \begin{cases} \Phi\left(\frac{\log(x)-\mu}{\sigma}\right) & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases}$$

where Φ is the distribution function of a standard normal random variable. To define the probability distribution function of L_1, recall that the gamma distribution with rate $a > 0$ and shape $b > 0$ is given by

$$G_{a,b}(x) := \begin{cases} \frac{a^b}{\Gamma(b)} \int_0^x y^{b-1} e^{-ay} dy & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases}$$

where Γ is the gamma function. For every $p \in (0,1)$ set

$$q_p(G_{a,b}) := \inf\{x \in \mathbb{R}\,;\ G_{a,b}(x) \ge p\}.$$

The probability distribution function of L_1 is then specified by

$$\mathbb{P}(L_1 \le x) = \begin{cases} G_{\delta_0,\tau_0}(x) & \text{if } x < q_{97.5\%}(G_{\delta_0,\tau_0}), \\ G_{\delta,\tau}(x + q_{97.5\%}(G_{\delta,\tau}) - q_{97.5\%}(G_{\delta_0,\tau_0})) & \text{if } x \ge q_{97.5\%}(G_{\delta_0,\tau_0}). \end{cases}$$

Recall that, by Sklar's Theorem, see, e.g., Theorem 2.3.3 in Nelsen (2007), the joint distribution of A_1 and L_1 can be expressed through a suitable copula function $C : [0,1] \times [0,1] \to [0,1]$ as

$$\mathbb{P}(A_1 \le x, L_1 \le y) = C(\mathbb{P}(A_1 \le x), \mathbb{P}(L_1 \le y)).$$

The assets and liabilities are linked through the Gaussian copula

$$C(p,q) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\Phi^{-1}(p)} \int_{-\infty}^{\Phi^{-1}(q)} e^{-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}} dudv$$

where $\rho \in (-1,1)$ is the correlation coefficient.

A.10 Complementary Material for Section 5.4.2

Distribution of assets and liabilities. The probability density function of L_1 is explicitly given by

$$f(x) = \begin{cases} \frac{4(1-\alpha)}{a^2} x & \text{if } 0 \le x \le \frac{a}{2}, \\ -\frac{4(1-\alpha)}{a^2} x + \frac{4(1-\alpha)}{a} & \text{if } \frac{a}{2} < x \le a, \\ \frac{4\alpha}{(c-b)^2} x - \frac{4\alpha b}{(c-b)^2} & \text{if } b \le x \le \frac{b+c}{2}, \\ -\frac{4\alpha}{(c-b)^2} x + \frac{4\alpha c}{(c-b)^2} & \text{if } \frac{b+c}{2} < x \le c, \\ 0 & \text{otherwise}, \end{cases}$$

for $\alpha = 0.5\%$ and for fixed parameters $0 < a < b < c$.

Regulatory benchmarks. Note that $\mathbb{P}(L_1 \le a) = 1 - \alpha$ and $\mathbb{P}(L_1 \le x) < 1 - \alpha$ for every $x < a$. This yields

$$\begin{aligned} \mathrm{V@R}_{0.5\%}(E_1) &= \mathrm{V@R}_{\alpha}(-L_1) - k = \inf\{x \in \mathbb{R}\,;\ \mathbb{P}(-L_1 + x < 0) \le \alpha\} - k \\ &= \inf\{x \in \mathbb{R}\,;\ \mathbb{P}(L_1 \le x) \ge 1 - \alpha\} - k = a - k. \end{aligned}$$

The computation of $\mathrm{AV@R}_{1\%}(E_1)$ requires a bit more effort. First of all, define

$$q = \mathrm{V@R}_{2\alpha}(-L_1) = \inf\{x \in \mathbb{R}\,;\ \mathbb{P}(L_1 \le x) \ge 1 - 2\alpha.$$

Note that q must satisfy

$$\frac{1}{2}(a-q)\left(-\frac{4(1-\alpha)}{a^2}q+\frac{4(1-\alpha)}{a}\right)=\alpha.$$

This gives $q=a\big(1\pm\sqrt{\frac{\alpha}{2(1-\alpha)}}\big)$. Since $q<a$ must hold, we obtain

$$q=a\left(1-\sqrt{\frac{\alpha}{2(1-\alpha)}}\right).$$

As a next step, observe that

$$\text{AV@R}_{2\alpha}(-L_1)=\mathbb{E}(L_1|L_1\geq \text{V@R}_{2\alpha}(-L_1))=\mathbb{E}(L_1|L_1\geq q)=\frac{\mathbb{E}(L_1 1_{\{L_1\geq q\}})}{\mathbb{P}(L\geq q)}.$$

We clearly have $\mathbb{P}(L_1\geq q)=2\alpha$. Moreover, noting that $q>\frac{a}{2}$, we get

$$\begin{aligned}
\mathbb{E}(L_1 1_{\{L_1\geq q\}}) &= \int_q^\infty xf(x)dx=\int_q^a xf(x)dx+\int_b^{\frac{b+c}{2}}xf(x)dx+\int_{\frac{b+c}{2}}^c xf(x)dx\\
&=\frac{4(1-\alpha)}{a^2}\int_q^a(ax-x^2)dx+\frac{4\alpha}{(b-c)^2}\left(\int_b^{\frac{b+c}{2}}(x^2-bx)dx+\int_{\frac{b+c}{2}}^c(cx-x^2)dx\right)\\
&=\frac{4(1-\alpha)}{a^2}\left(\frac{1}{6}a^3+\frac{1}{3}q^3-\frac{1}{2}aq^2\right)+\frac{4\alpha}{(b-c)^2}\left(\frac{1}{6}(b^3+c^3)-\frac{1}{24}(b+c)^3\right)\\
&=\frac{4(1-\alpha)}{a^2}\frac{\alpha}{2(1-\alpha)}\left(\frac{1}{2}-\frac{1}{3}\sqrt{\frac{\alpha}{2(1-\alpha)}}\right)a^3+\frac{4\alpha}{(b-c)^2}\frac{1}{8}(b+c)(b-c)^2\\
&=2\alpha\left(\frac{1}{2}-\frac{1}{3}\sqrt{\frac{\alpha}{2(1-\alpha)}}\right)a+\alpha\frac{b+c}{2}.
\end{aligned}$$

As a result, it follows that

$$\text{AV@R}_{1\%}(E_1)=\text{AV@R}_{2\alpha}(-L_1)-k=\left(\frac{1}{2}-\frac{1}{3}\sqrt{\frac{\alpha}{2(1-\alpha)}}\right)a+\frac{b+c}{4}-k.$$

Recovery-based capital requirements. We turn to the computation of $\text{RecV@R}_\gamma(E_1,L_1)$. To this effect, define

$$q_\beta=\text{V@R}_\beta(-L_1)=\inf\{x\in\mathbb{R};\ \mathbb{P}(L_1\leq x)\geq 1-\beta.$$

If $\beta<0.25\%$, then q_β must satisfy

$$\frac{1}{2}(c-q_\beta)\left(-\frac{4\alpha}{(b-c)^2}q_\beta+\frac{4\alpha c}{(b-c)^2}\right)=\beta.$$

This gives $q_\beta=c\pm\sqrt{\frac{\beta}{2\alpha}}(c-b)$. Since $q_\beta<c$ must hold, we obtain

$$q_\beta=\sqrt{\frac{\beta}{2\alpha}}b+\left(1-\sqrt{\frac{\beta}{2\alpha}}\right)c.$$

Similarly, if $\beta\geq 0.25\%$, then q_β must satisfy

$$\frac{1}{2}(q_\beta-b)\left(\frac{4\alpha}{(b-c)^2}q_\beta-\frac{4\alpha b}{(b-c)^2}\right)=\alpha-\beta.$$

This gives $q_\beta = b \pm \sqrt{\frac{\alpha-\beta}{2\alpha}}(c-b)$. Since $q_\beta > b$ must hold, we obtain

$$q_\beta = \left(1 - \sqrt{\frac{\alpha-\beta}{2\alpha}}\right) b + \sqrt{\frac{\alpha-\beta}{2\alpha}} c.$$

For convenience, we set

$$q_\beta(b,c) = \begin{cases} \sqrt{\frac{\beta}{2\alpha}} b + \left(1 - \sqrt{\frac{\beta}{2\alpha}}\right) c & \text{if } \beta < 0.25\%, \\ \left(1 - \sqrt{\frac{\alpha-\beta}{2\alpha}}\right) b + \sqrt{\frac{\alpha-\beta}{2\alpha}} c & \text{if } \beta \geq 0.25\%. \end{cases}$$

It follows that

$$\mathrm{V@R}_\beta(A_1 - rL_1) = r\,\mathrm{V@R}_\beta(-L_1) - k = rq_\beta(b,c) - k.$$

As a result, we conclude that

$$\mathrm{RecV@R}_\gamma(E_1, L_1) = \max\{\mathrm{V@R}_\alpha(E_1), \mathrm{V@R}_\beta(A_1 - rL_1)\} = \max\{a, rq_\beta(b,c)\} - k.$$

A.11 Additional Plots for Section 5.4.1

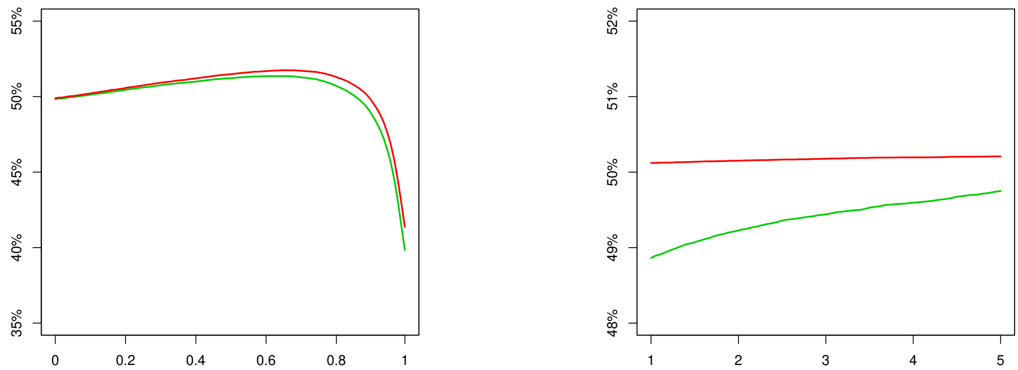

Figure 5: The loss probability $\mathbb{P}(\Delta E_1 < 0)$ as a function of ρ (left) for $\tau = 1$ (green) and $\tau = 5$ (red) and as a function of τ (right) for $\rho = 0.1$ (red) and $\rho = 0.9$ (green).

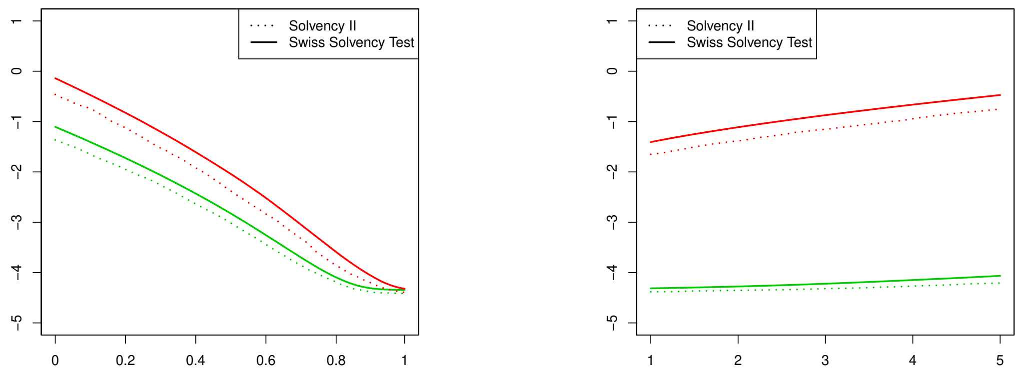

Figure 6: The regulatory risk measure $\rho_{reg}(E_1)$ as a function of ρ (left) for $\tau = 1$ (green) and $\tau = 5$ (red) and as a function of τ (right) for $\rho = 0.1$ (red) and $\rho = 0.9$ (green).

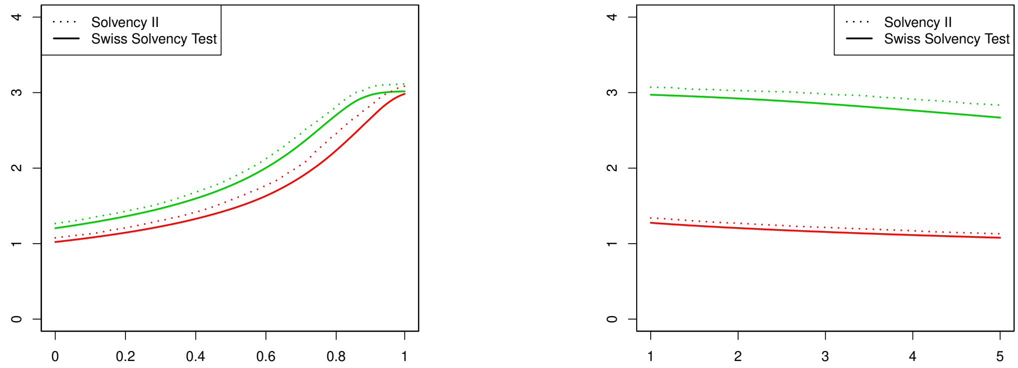

Figure 7: The solvency ratio $\frac{E_0}{\rho_{reg}(\Delta E_1)}$ as a function of ρ (left) for $\tau = 1$ (green) and $\tau = 5$ (red) and as a function of τ (right) for $\rho = 0.1$ (red) and $\rho = 0.9$ (green).

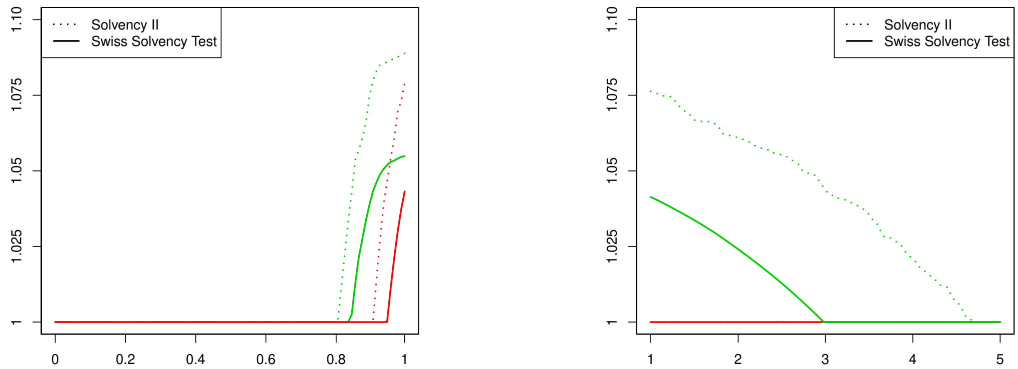

Figure 8: The recovery adjustment $\text{RecAdj}(\beta, r)$ for $\beta = \beta_{max} = 0.25\%$ and $r = r_{min} = 50\%$ as a function of ρ (left) for $\tau = 1$ (green) and $\tau = 5$ (red) and as a function of τ (right) for $\rho = 0.1$ (red) and $\rho = 0.9$ (green).

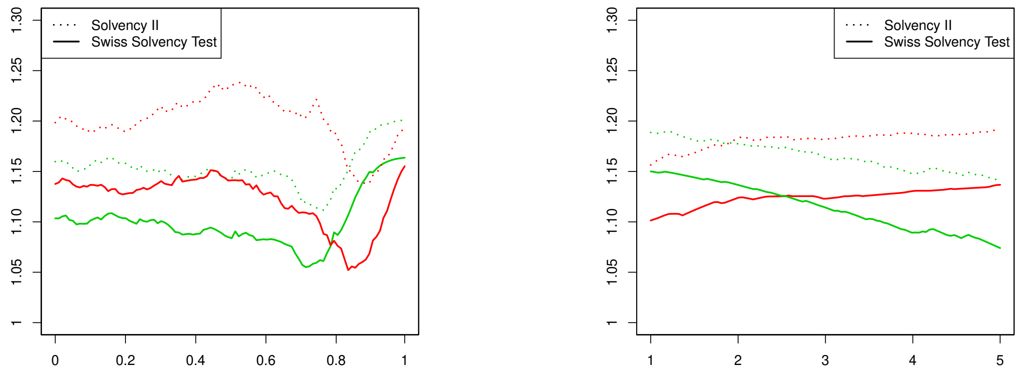

Figure 9: The recovery adjustment $\text{RecAdj}(\beta, r)$ for $\beta = \beta_{min} = 0.1\%$ and $r = r_{max} = 90\%$ as a function of ρ (left) for $\tau = 1$ (green) and $\tau = 5$ (red) and as a function of τ (right) for $\rho = 0.1$ (red) and $\rho = 0.9$ (green).

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