ROBUST ORLICZ SPACES: OBSERVATIONS AND CAVEATS

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ABSTRACT. We investigate two different constructions of robust Orlicz spaces as a generalisation of robust L^p-spaces. Our first construction is top-down and considers the maximal domain of a worst-case Luxemburg norm. From an applied persepective, this approach can be justified by a uniform-boundedness-type result. In typical situations, the worst-case Orlicz space agrees with the intersection of the corresponding individual Orlicz spaces. Our second construction produces the closure of a space of test random variables with respect to the worst-case Luxemburg norm. We show that separability of such spaces or their subspaces has very strong implications in terms of dominatedness of the set of priors and thus for applications in the field of robust finance. For instance, norm closures of bounded continuous functions as considered in the G-framework lead to spaces which are lattice-isomorphic to sublattices of a classical L^1-space lacking, however, Dedekind σ-completeness. We further show that the topological spanning power of options is always limited under nondominated uncertainty.

Keywords: Orlicz space, model uncertainty, nonlinear expectation, Banach lattice, Dedekind completeness

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1. INTRODUCTION

Since the beginning of this century, the simultaneous consideration of families of prior distributions instead of a single probability measure has become of fundamental importance for the risk assessment of financial positions. In this context, one often speaks of *model uncertainty* or *ambiguity*, where the uncertainty is modeled by a set $\mathfrak{P}$ of probability measures. Especially after the subprime mortgage crisis, the desire for mathematical models based on *nondominated* families of priors arose: no single reference probability measure can be chosen which determines whether an event is deemed certain or negligible. To date, the most prominent example for a model consisting of a nondominated set of probability distributions is a Brownian motion with uncertain volatility, the so-called G-Brownian motion, cf. Peng [34, 35].[1] The latter is intimately related to the theory of second order backward stochastic differential equations, cf. Cheridito et al. [11], and an extensive strand of literature has formed around this model. Another seminal contribution to nondominated sets of priors is Bouchard & Nutz [9].

On another note, there has been renewed interest in the role of *Orlicz spaces* in mathematical finance in the past few years. They have, for instance, appeared as canonical model spaces for risk measures, premium principles, and utility maximisation problems; see Bellini & Rosazza Gianin [4], Biagini & Černý [5], Cheridito & Li [10], Gao et al. [16, 19], and many others.

The aim of the present manuscript is to investigate robust Orlicz spaces in a setting of potentially nondominated sets $\mathfrak{P}$ of probability priors instead of a single reference probability measure $\mathbb{P}$. *A priori*, such a construction faces a choice among two paths, though, which often lead to the same result if a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is underlying. As an illustrative example, consider the case of a classical $L^p(\mathbb{P})$-space, for $p \in [1, \infty)$. It can be obtained either by a *top-down* approach, considering the maximal set of all equivalence classes of real-valued measurable functions on which the norm

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[1] More precisely, the resulting measures are mutually singular.

$\|\cdot\|_{L^p(\mathbb{P})} := \mathbb{E}_{\mathbb{P}}[|\cdot|^p]^{1/p}$ is finite. Or, equivalently, by proceeding in a *bottom-up* manner, closing a smaller test space of, say, bounded random variables w.r.t. the norm $\|\cdot\|_{L^p(\mathbb{P})}$. If the underlying state space is topological, one has even more degrees of freedom and may select suitable spaces of *continuous functions* as a test space. Both approaches lead to a *Banach lattice*, which naturally carries the $\mathbb{P}$-almost-sure order, and this turns out to be $L^p(\mathbb{P})$ in both cases. Morally speaking, the reason for this equivalence is that $\|\cdot\|_{L^p(\mathbb{P})}$ is not very robust and rather insensitive to the tail behaviour of a given random variable. It may therefore come as no surprise that the two paths tend to diverge substantially for *robust Orlicz spaces*, and this note may be understood as a comparison of the two approaches.

As top-down approach, we suggest to consider a fixed measurable space $(\Omega, \mathcal{F})$, a nonempty set of probability measures $\mathfrak{P}$ on $(\Omega, \mathcal{F})$, and a family $\Phi = (\phi_{\mathbb{P}})_{\mathbb{P}\in\mathfrak{P}}$ of Orlicz functions. As usual, we consider the quotient space $L^0(\mathfrak{P})$ of all real-valued measurable functions on $(\Omega, \mathcal{F})$ up to $\mathfrak{P}$-quasi-sure ($\mathfrak{P}$-q.s.) equality. On $L^0(\mathfrak{P})$, we define the robust Luxemburg norm

$$\|X\|_{L^\Phi(\mathfrak{P})} := \sup_{\mathbb{P}\in\mathfrak{P}} \|X\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})} \in [0,\infty], \quad \text{for } X \in L^0(\mathfrak{P}), \tag{1.1}$$

where $\|\cdot\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}$ is the Luxemburg seminorm for $\phi_{\mathbb{P}}$ under the probability measure $\mathbb{P} \in \mathfrak{P}$. In line with Pitcher [38] and Roy & Chakraborty [40, 41], the robust Orlicz space $L^\Phi(\mathfrak{P})$ is then defined to be the space of all $X \in L^0(\mathfrak{P})$ with $\|X\|_{L^\Phi(\mathfrak{P})} < \infty$. Notice that these spaces arise naturally in the context of *variational preferences* as axiomatised by Maccheroni et al. [26]. These encompass prominent classes of preferences, such as multiple prior preferences introduced by Gilboa & Schmeidler [20] and multiplier preferences introduced by Hansen & Sargent [22] – see also [26, Section 4.2.1]. More precisely, one considers a cloud of agents operating on, say, bounded random variables X, endowed with variational preferences represented by expressions of the form

$$\inf_{\mathbb{Q}\in\mathfrak{Q}} \mathbb{E}_{\mathbb{Q}}[u(X)] + c(\mathbb{Q}),$$

where u is a utility function, $\mathfrak{Q}$ is a set of probability priors, and c is a prior-dependent cost function. Then, robust Orlicz spaces arise as a canonical maximal model space to which all individual preferences can be extended continuously. For the details, we refer to Section 4.4. Special cases of robust Orlicz spaces have also been studied in the G-Framework by, e.g., Nutz & Soner [31] and Soner et al. [44], and for general measurable spaces by Gao & Munari [17], Kupper & Svindland [24], Liebrich & Svindland [25], Maggis et al. [27], Owari [33], and Svindland [45]. Note that they are Banach lattices when endowed with the $\mathfrak{P}$-q.s. order which additionally have the desirable order completeness property of *Dedekind σ-completeness*, the existence of *suprema* for bounded countable families of contingent claims. We point out that σ-order properties are of fundamental interest in many financial applications, for example, in the context of *weak free lunch with vanishing risk*. We refer to Obłój & Wiesel [32] for an overview on no-arbitrage conditions. Moreover, the set of all regular pricing rules on $L^\Phi(\mathfrak{P})$ reflects the full uncertainty given by $\mathfrak{P}$; cf. Section 4.3.

An alternative top-down approach to robust Orlicz spaces present in the literature, cf. Nutz [30] and Soner et al. [43], is given by the space

$$\mathfrak{L}^\Phi(\mathfrak{P}) := \big\{X \in L^0(\mathfrak{P}) \,\big|\, \forall\, \mathbb{P} \in \mathfrak{P}\, \exists\, \alpha > 0 : \ \mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(\alpha|X|)] < \infty\big\},$$

the "intersection" of the individual Orlicz spaces. In the situation of a cloud of variational preference agents above, this space collects minimal agreement among all agents under consideration, that is, they all can attach a well-defined utility to each of the objects in $\mathfrak{L}^\Phi(\mathfrak{P})$. One may therefore be tempted to prefer $\mathfrak{L}^\Phi(\mathfrak{P})$ over $L^\Phi(\mathfrak{P})$ and drop the modelling choice of the worst-case approach represented by the supremum over all priors in $\mathfrak{P}$ in (1.1). In Proposition 2.11 and Theorem 2.12, we show that, in many situations,

$$\mathfrak{L}^\Phi(\mathfrak{P}) = L^\Phi(\mathfrak{P}),$$

a uniform-boundedness type result, which proves the equivalence of both constructions in terms of the extension of the resulting spaces in $L^0(\mathfrak{P})$.

In order to maintain a certain degree of analytic tractability while still allowing for uncertainty in terms of nondominated priors, an huge strand of literature has preferred the bottom-up construction of robust Orlicz spaces (at least for special cases of Orlicz functions). First, Owari [33] shows that, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the largest ideal of $L^1(\mathbb{P})$ on which a convex finite function on $L^\infty(\mathbb{P})$ has a finite extension with the Lebesgue property is the closure of $L^\infty(\mathbb{P})$ w.r.t. a multiplicatively weighted L^1-norm (see also Lemma 3.6 below). Second, closures $\mathcal{C}^p$ of the space C_b of bounded continuous functions under robust L^p-norms $\|\cdot\|_{L^p(\mathfrak{P})}$, for a nonempty set of priors $\mathfrak{P}$ – which we shall cover in Section 4.1 – have become a frequent choice for commodity spaces or spaces of contingent claims in the context of a G-Brownian motion, see, for instance, Beissner & Denis [2], Beissner & Riedel [3], Bion-Nadal & Kervarec [6], Denis et al. [13], or Denis & Kervarec [14]. One reason for this choice is certainly that, roughly speaking, all "nice" analytic properties of C_b carry over to the $\|\cdot\|_{L^p(\mathfrak{P})}$-closure. As a consequence, in the past decades, the analytic properties of these spaces have been studied extensively, see, e.g., Beissner & Denis [2] or Denis et al. [13], and a complete stochastic calculus has been developed based on these spaces, cf. Peng [36]. However, very little is known about their properties as Banach lattices. Third, the bottom-up approach appears naturally in the study of option spanning under uncertainty in the spirit of Ross [39], cf. Section 4.2.

The guiding question in their studies is the following: Which properties of the larger robust Orlicz space $L^\Phi(\mathfrak{P})$ do these smaller (Banach) sublattices share, and which properties do not carry over? The second set of results of the present manuscript addresses this question from an order-theoretic perspective and with a particular view towards a possibly nondominated underlying set of priors. We sometimes reduce our attention to Banach lattices which arise from closures w.r.t. a robust Luxemburg norm introduced in equation (1.1) below of certain classes of *bounded* functions, e.g. bounded continuous functions. We would like to draw attention to two types of results.

Our first main result in this direction, Theorem 2.7, states that every separable subspace of a robust Orlicz space is order-isomorphic to a subspace of a classical L^1-space without changing the measurable space. As a consequence, its elements are still dominated by a single probability measure, and the $\mathfrak{P}$-q.s. order collapses to an almost sure order – even for nondominated sets $\mathfrak{P}$ of priors.

In particular, Theorem 2.7 applies to the situation considered in the G-framework. We show in Proposition 4.1 that, under very mild and typically satisfied conditions, robust closures of C_b are *separable*. Therefore, the special robust closure $\mathcal{C}^p$ not only inherits all nice analytic properties from C_b, but also its dominatedness, a result that, in this special case, has already been obtained by Bion-Nadal & Kervarec [6]. Theorem 2.7 and Proposition 4.1 therefore provide decisive generalisations of results by Bion-Nadal & Kervarec [6], showing that the very general condition of separability is what causes a collapse of the $\mathfrak{P}$-q.s. order.

In a similar spirit, our second main result, Theorem 3.7, concerns Dedekind σ-completeness of sublattices of robust Orlicz spaces. Theorem 3.7 states that, in typical situations, there exists at most one separable Dedekind σ-complete Banach sublattice that generates the σ-algebra, and if it exists the family of priors $\mathfrak{P}$ is already dominated with uniformly integrable densities. We thereby qualify that what *prevents* nondominated models from being dominated is the lack of all order completeness properties for separable Banach sublattices that generate the σ-algebra. In the case of the robust closure $\mathcal{C}^p$ of C_b, one concludes that this space is too similar to the original space C_b in terms of its order completeness properties.

In conclusion, our results highlight both the advantages and the cost of taking a top-down or a bottom-up approach to robust Orlicz spaces, respectively. Whereas the former may lack good dual behaviour, it has reasonable order completeness properties and reflects the full nondominated nature of the underlying uncertainty structure. The latter may be handy analytically, but either ignores the nondominated uncertainty structure *a posteriori*, or tends to lead to a complete breakdown of almost all lattice properties.

Structure of the Paper: In Section 2, we start with the announced top-down construction of robust Orlicz spaces and discuss its basic properties. We derive equivalent conditions for a robust Orlicz space to coincide with a robust multiplicatively penalised L^1-space, cf. Theorem 2.6, and show that every separable subspace of $L^\Phi(\mathfrak{P})$ is order-isomorphic to a sublattice of $L^1(\mathbb{P}^*)$ for a suitable probability measure $\mathbb{P}^*$ (Theorem 2.7). We further provide sufficient conditions for the equality $L^\Phi(\mathfrak{P}) = \mathfrak{L}^\Phi(\mathfrak{P})$ to be valid, see Proposition 2.11 and Theorem 2.12. In Section 3, we consider sublattices of $L^\Phi(\mathfrak{P})$. Theorem 3.7, Proposition 3.9, and Proposition 3.10 provide a set of equivalent conditions for the Dedekind σ-completeness of sublattices of $L^\Phi(\mathfrak{P})$. In particular, we prove that separability together with Dedekind σ-completeness for *any* generating sublattice already implies the dominatedness of the set of priors $\mathfrak{P}$. We further give, in special yet relevant cases, an explicit description of the dual space of sublattices of $L^\Phi(\mathfrak{P})$, see Proposition 3.5. In Section 4, we discuss applications and implications of the results obtained in Section 2 and Section 3 to the field of robust finance. In particular, we discuss the relation to aggregating variational preferences, regular pricing rules, and implications for option spanning, arbitrage theory, and the G-Framework. The proofs of Section 2 can be found in the Appendix A, the proofs of Section 3 are collected in the Appendix B, and the proofs of Section 4 are collected in the Appendix C.

Notation: For a set $S \neq \emptyset$ and a function $f\colon S \to (-\infty, \infty]$, the *effective domain* of f will be denoted by $\mathrm{dom}(f) := \{s \in S \mid f(s) < \infty\}$. For a normed vector space $\mathcal{H}$ we denote by $\mathcal{H}^*$ its dual space and by $\|\cdot\|_{\mathcal{H}^*}$ the operator norm.
Throughout, we consider a measurable space $(\Omega, \mathcal{F})$ and a nonempty set $\mathfrak{P}$ of probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$. The latter give rise to an equivalence relation on the real vector space $\mathcal{L}^0(\Omega, \mathcal{F})$ of all real-valued random variables on $(\Omega, \mathcal{F})$:

$$f \sim g \quad :\Longleftrightarrow \quad \forall\, \mathbb{P} \in \mathfrak{P}:\ \mathbb{P}(f = g) = 1.$$

The quotient space $L^0(\mathfrak{P}) := \mathcal{L}^0(\Omega, \mathcal{F})/\sim$ is the space of all real-valued random variables on $(\Omega, \mathcal{F})$ up to $\mathfrak{P}$-quasi-sure ($\mathfrak{P}$-q.s.) equality. The elements $f\colon \Omega \to \mathbb{R}$ in the equivalence class $X \in L^0(\mathfrak{P})$ are called *representatives*, and are denoted by $f \in X$. Conversely, for $f \in \mathcal{L}^0(\Omega, \mathcal{F})$, $[f]$ denotes the equivalence class in $L^0(\mathfrak{P})$ generated by f. For X and Y in $L^0(\mathfrak{P})$,

$$X \preceq Y \quad :\Longleftrightarrow \quad \forall\, f \in X\, \forall\, g \in Y\, \forall\, \mathbb{P} \in \mathfrak{P}:\ \mathbb{P}(f \leq g) = 1,$$

defines a vector space order on $L^0(\mathfrak{P})$, the $\mathfrak{P}$-*q.s. order* on $L^0(\mathfrak{P})$, and $(L^0(\mathfrak{P}), \preceq)$ is a vector lattice. In fact, for $X, Y \in L^0(\mathfrak{P})$ and representatives $f \in X, g \in Y$, the formulae

$$X \wedge Y = [f \wedge g] \quad \text{and} \quad X \vee Y = [f \vee g]$$

hold for the minimum and the maximum, respectively. We denote the vector sublattice of all bounded real-valued random variables up to $\mathfrak{P}$-q.s. equality by $L^\infty(\mathfrak{P})$. The latter is a Banach lattice, when endowed with the norm

$$\|X\|_{L^\infty(\mathfrak{P})} := \inf\big\{m > 0 \,\big|\, X \preceq m\mathbf{1}_\Omega\big\}, \quad X \in L^\infty(\mathfrak{P}).$$

As usual, $\mathbf{ca}$ denotes the space of all signed measures with finite total variation. We denote by $\mathbf{ca}_+$ or $\mathbf{ca}_+^1$ the subset of all finite measures or probability measures, respectively. For $\mu \in \mathbf{ca}$, let $|\mu|$ denote the total variation measure of μ. Given two nonempty sets $\mathfrak{Q}, \mathfrak{R} \subset \mathbf{ca}$, we write $\mathfrak{Q} \ll \mathfrak{R}$ if $\sup_{\mu \in \mathfrak{Q}} |\mu|(N) = 0$ for all events $N \in \mathcal{F}$ with $\sup_{\nu \in \mathfrak{R}} |\nu|(N) = 0$. We write, $\mathfrak{Q} \approx \mathfrak{R}$ if $\mathfrak{Q} \ll \mathfrak{R}$ and $\mathfrak{R} \ll \mathfrak{Q}$. For singletons

$\mathfrak{Q} = \{\mu\}$, we use the notation $\mu \ll \mathfrak{R}$, $\mathfrak{R} \ll \mu$, and $\mathfrak{R} \approx \mu$. Finally, $\mathbf{ca}(\mathfrak{P}) := \{\mu \in \mathbf{ca} \,|\, \mu \ll \mathfrak{P}\}$ denotes the space of all countably additive signed measures, which are absolutely continuous with respect to $\mathfrak{P}$. The subsets $\mathbf{ca}_+(\mathfrak{P})$ and $\mathbf{ca}_+^1(\mathfrak{P})$ are defined in an analogous way. For all $\mu \in \mathbf{ca}$, $X \in L^0(\mathfrak{P})_+$, and $f, g \in X$, $\int f\,d\mu$ and $\int g\,d\mu$ are well-defined and satisfy

$$\int f\,d\mu = \int g\,d\mu.$$

We shall therefore henceforth write

$$\mu X := \int X\,d\mu := \int f\,d\mu, \quad \text{for } f \in X,$$

if $X \in L^0(\mathfrak{P})_+$ or it has $|\mu|$-integrable representatives.

2. Robust Orlicz spaces: definition and first properties

In this section, we introduce the main objects of this manuscript, robust versions of Orlicz spaces, and investigate their basic properties. For the theory of classical Orlicz spaces, we refer to [15, Chapter 2]. An *Orlicz function* is a function $\phi\colon [0,\infty) \to [0,\infty]$ with the following three properties:

(i) ϕ is lower semicontinuous, nondecreasing, and convex.
(ii) $\phi(0) = 0$.
(iii) there are $x_0, x_1 > 0$ with $\phi(x_0) \in [0,\infty)$ and $\phi(x_1) \in (0,\infty]$.[2]

Throughout this section, we consider a general measurable space $(\Omega, \mathcal{F})$, a nonempty set of probability measures $\mathfrak{P}$, a family $\Phi = (\phi_{\mathbb{P}})_{\mathbb{P}\in\mathfrak{P}}$ of Orlicz functions, and define

$$\phi_{\mathrm{Max}}(x) := \sup_{\mathbb{P}\in\mathfrak{P}} \phi_{\mathbb{P}}(x), \quad \text{for all } x \in [0,\infty). \tag{2.1}$$

Notice that, by definition, ϕ_{Max} is a lower semicontinuous, nondecreasing, and convex function $[0,\infty) \to [0,\infty]$ with $\phi_{\mathrm{Max}}(0) = 0$. However, in general, ϕ_{Max} is not an Orlicz function, since $\phi_{\mathrm{Max}}(x_0) \in [0,\infty)$ for some $x_0 \in (0,\infty)$ cannot be guaranteed.

2.1. Robust Orlicz spaces and penalised versions of robust L^p-spaces.

Definition 2.1. For $X \in L^0(\mathfrak{P})$, the *(Φ-)Luxemburg norm* is defined via

$$\|X\|_{L^\Phi(\mathfrak{P})} := \inf\left\{\lambda > 0 \,\Big|\, \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_{\mathbb{P}}\left[\phi_{\mathbb{P}}(\lambda^{-1}|X|)\right] \le 1\right\} = \sup_{\mathbb{P}\in\mathfrak{P}} \|X\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})} \in [0,\infty].^3$$

We define by $L^\Phi(\mathfrak{P}) := \mathrm{dom}(\|\cdot\|_{L^\Phi(\mathfrak{P})})$ the *(Φ-)robust Orlicz space.*

Example 2.2. Let $(\Omega, \mathcal{F})$ be a measurable space, $\mathfrak{P}$ a nonempty set of probability priors, and $\phi\colon [0,\infty) \to [0,\infty]$ be an Orlicz function.

(1) For an arbitrary function $\gamma\colon \mathfrak{P} \to [0,\infty)$, consider

$$\phi_{\mathbb{P}}(x) := \frac{\phi(x)}{1+\gamma(\mathbb{P})}, \quad \text{for } x \ge 0.$$

This leads to an *additively penalised robust Orlicz space* with Luxemburg norm

$$\|X\|_{L^\Phi(\mathfrak{P})} = \inf\left\{\lambda > 0 \,\Big|\, \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_{\mathbb{P}}\left[\phi(\lambda^{-1}|X|)\right] - \gamma(\mathbb{P}) \le 1\right\}, \quad \text{for } X \in L^0(\mathfrak{P}).$$

For $\phi := \infty \cdot \mathbf{1}_{(1,\infty)}$, the Luxemburg norm is, independently of γ, given by

$$\|X\|_{L^\Phi(\mathfrak{P})} = \sup_{\mathbb{P}\in\mathfrak{P}} \|X\|_{L^\infty(\mathbb{P})} = \|X\|_{L^\infty(\mathfrak{P})}, \quad \text{for } X \in L^0(\mathfrak{P}).$$

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[2] This definition precludes triviality of ϕ, i.e. the cases $\phi \equiv 0$ and $\phi = \infty \cdot 1_{(0,\infty)}$.

[3] Defining $\|\cdot\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}$ in the usual way, we obtain a seminorm on $L^\Phi(\mathfrak{P})$, not a norm as on the classical Orlicz space $L^{\phi_{\mathbb{P}}}(\mathbb{P})$.

Introducing the, up to a sign, convex monetary risk measure

$$\rho(X) := \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_{\mathbb{P}}[X] - \gamma(\mathbb{P}) \in [0,\infty], \quad \text{for } X \in L^0(\mathfrak{P})_+,$$

the robust Luxemburg norm can be expressed as

$$\|X\|_{L^\Phi(\mathfrak{P})} = \inf\left\{\lambda > 0 \;\middle|\; \rho\left(\phi(\lambda^{-1}|X|)\right) \le 1\right\}, \quad \text{for } X \in L^0(\mathfrak{P}).$$

(2) For $\theta\colon \mathfrak{P} \to (0,\infty)$ with $\sup_{\mathbb{P}\in\mathfrak{P}} \theta(\mathbb{P}) < \infty$, we consider

$$\phi_{\mathbb{P}}(x) := \phi\big(\theta(\mathbb{P})x\big), \quad \text{for } \mathbb{P} \in \mathfrak{P} \text{ and } x \ge 0.$$

This leads to a *multiplicatively penalised robust Orlicz space* with Luxemburg norm

$$\|X\|_{L^\Phi(\mathfrak{P})} = \sup_{\mathbb{P}\in\mathfrak{P}} \theta(\mathbb{P})\|X\|_{L^\phi(\mathbb{P})}, \quad \text{for } X \in L^0(\mathfrak{P}).$$

For $p \in [1,\infty)$ and $\phi(x) = x^p$, $x \ge 0$, we obtain the weighted robust L^p-norm

$$\|X\|_{L^\phi(\mathfrak{P})} = \sup_{\mathbb{P}\in\mathfrak{P}} \theta(\mathbb{P})\|X\|_{L^p(\mathbb{P})}, \quad \text{for } X \in L^0(\mathfrak{P}),$$

and, for $\phi(x) = \infty \cdot 1_{(1,\infty)}$, the Luxemburg norm is given by

$$\|X\|_{L^\Phi(\mathfrak{P})} = \sup_{\mathbb{P}\in\mathfrak{P}} \theta(\mathbb{P})\|X\|_{L^\infty(\mathbb{P})}, \quad \text{for } X \in L^0(\mathfrak{P}).$$

The resulting spaces will be referred to as *weighted robust L^p-spaces*, for $1 \le p \le \infty$.

Before we prove that, as in the classical case, robust Orlicz spaces are Banach lattices, we introduce $\mathbf{ca}\big(L^\Phi(\mathfrak{P})\big)$ as the set of all signed measures $\mu \in \mathbf{ca}(\mathfrak{P})$ for which each $X \in L^\Phi(\mathfrak{P})$ is $|\mu|$-integrable and the map

$$L^\Phi(\mathfrak{P}) \to \mathbb{R}, \quad X \mapsto |\mu|X \tag{2.2}$$

is continuous. Moreover, we set $\mathbf{ca}_+\big(L^\Phi(\mathfrak{P})\big) := \mathbf{ca}\big(L^\Phi(\mathfrak{P})\big) \cap \mathbf{ca}_+$ and $\mathbf{ca}_+^1\big(L^\Phi(\mathfrak{P})\big) := \mathbf{ca}\big(L^\Phi(\mathfrak{P})\big) \cap \mathbf{ca}_+^1$.

Proposition 2.3. *The space $\big(L^\Phi(\mathfrak{P}), \preceq, \|\cdot\|_{L^\Phi(\mathfrak{P})}\big)$ is a Dedekind σ-complete Banach lattice and $L^\Phi(\mathfrak{P}) \subset L^0(\mathfrak{P})$ is an ideal. Moreover, for all $\mathbb{P} \in \mathfrak{P}$, $a_{\mathbb{P}} > 0$, and $b_{\mathbb{P}} \le 0$ with $a_{\mathbb{P}}x + b_{\mathbb{P}} \le \phi_{\mathbb{P}}(x)$ for all $x \ge 0$,*

$$\|\mathbb{P}\|_{L^\Phi(\mathfrak{P})^*} \le \frac{1 - b_{\mathbb{P}}}{a_{\mathbb{P}}}. \tag{2.3}$$

In particular, $\mathfrak{P} \subset \mathbf{ca}_+^1\big(L^\Phi(\mathfrak{P})\big)$.

Remark 2.4. The lattice norm property of $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ which is proved in the preceding proposition admits two conclusions: (i) For each $\mu \in \mathbf{ca}(L^\Phi(\mathfrak{P}))$, the functional $L^\Phi(\mathfrak{P}) \ni X \mapsto \mu X$ is continuous. This is due to the fact that the Radon-Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}$ takes values in $[-1,1]$ $|\mu|$-almost everywhere. (ii) $\mathbf{ca}(L^\Phi(\mathfrak{P}))$ is a vector sublattice of $\mathbf{ca}(\mathfrak{P})$.

Example 2.5. Suppose $\mathcal{H} \subset L^0(\mathfrak{P})$ is an ideal which is a Banach lattice when endowed with a norm $\|\cdot\|_{\mathcal{H}}$. Furthermore assume the norm is completely determined by σ-finite measures, i.e., there is a set $\mathfrak{D} \ll \mathfrak{P}$ of σ-finite measures such that, for all $X \in \mathcal{H}$,

$$\|X\|_{\mathcal{H}} = \sup_{\mu\in\mathfrak{D}} \mu|X|.$$

Then $\mathcal{H}$ is a robust Orlicz space after a potential modification of $\mathfrak{P}$.

A robust Orlicz space may be reduced to a weighted robust L^1-space if and only if it contains all bounded random variables.

Theorem 2.6. *The following statements are equivalent:*

(1) $L^\infty(\mathfrak{P}) \subset L^\Phi(\mathfrak{P})$,
(2) ϕ_{Max} *is an Orlicz function, i.e., there exists some* $x_0 \in (0,\infty)$ *with* $\phi_{\mathrm{Max}}(x_0) \in [0,\infty)$,
(3) *There exists a nonempty set* $\mathfrak{Q} \subset \mathbf{ca}^1_+\big(L^\Phi(\mathfrak{P})\big)$ *of probability measures with* $\mathfrak{P} \subset \mathfrak{Q}$ *and a weight function* $\theta\colon \mathfrak{Q} \to (0,\infty)$ *with* $\sup_{\mathbb{Q}\in\mathfrak{Q}} \theta(\mathbb{Q}) < \infty$ *such that* $\|\cdot\|_{L^\Phi(\mathfrak{P})} = \sup_{\mathbb{Q}\in\mathfrak{Q}} \theta(\mathbb{Q})\|\cdot\|_{L^1(\mathbb{Q})}$.

In this case, $L^\Phi(\mathfrak{P})$ *is a weighted robust* L^1*-space, and there is a constant* $\kappa > 0$ *such that*

$$\|X\|_{L^\Phi(\mathfrak{P})} \le \kappa \|X\|_{L^\infty(\mathfrak{P})}, \quad \textit{for } X \in L^\infty(\mathfrak{P}).$$

For $\mathbb{P} \in \mathbf{ca}^1_+(L^\Phi(\mathfrak{P}))$, we define the *canonical projection* $J_\mathbb{P}\colon L^\Phi(\mathfrak{P}) \to L^1(\mathbb{P})$ via

$$J_\mathbb{P}(X) := \big\{ g \in \mathcal{L}^0(\Omega,\mathcal{F}) \,\big|\, \exists\, f \in X\colon \mathbb{P}(f \neq g) = 0 \big\}, \quad \text{for } X \in L^\Phi(\mathfrak{P}).$$

Since $\mathbb{P} \in \mathbf{ca}^1_+\big(L^\Phi(\mathfrak{P})\big)$, $J_\mathbb{P}$ is well-defined, linear, continuous, and a lattice homomorphism, i.e., it is order-preserving in that

$$J_\mathbb{P}(X \wedge Y) = J_\mathbb{P}(X) \wedge J_\mathbb{P}(Y), \quad \text{for } X, Y \in L^\Phi(\mathfrak{P}).$$

However, in general, it fails to be a lattice isomorphism onto its image, i.e., it is not injective. Still, the following surprising result holds.

Theorem 2.7. *Suppose* $\mathcal{H}$ *is a separable subspace of* $L^\Phi(\mathfrak{P})$*. Then, there is a probability measure* $\mathbb{P}^* \in \mathbf{ca}^1_+\big(L^\Phi(\mathfrak{P})\big)$ *such that* $\mathcal{H}$ *is isomorphic to a subspace of* $L^1(\mathbb{P}^*)$ *via the canonical projection* $J_{\mathbb{P}^*}$*. In particular, the following assertions hold:*

(1) $\mathbb{P}^*$ *defines a strictly positive linear functional on* $\mathcal{H}$*.*
(2) *The* $\mathfrak{P}$*-q.s. order and the* $\mathbb{P}^*$*-a.s. order coincide on* $\mathcal{H}$*.*
(3) *If* $\mathfrak{P}$ *is countably convex,* $\mathbb{P}^*$ *can be chosen as an element of* $\mathfrak{P}$*.*

We could rephrase the previous theorem as the fact that, on separable subspaces of $L^\Phi(\mathfrak{P})$, the $\mathfrak{P}$-q.s. order collapses to a $\mathbb{P}^*$-a.s. order for some $\mathbb{P}^* \in \mathbf{ca}^1_+(L^\Phi(\mathfrak{P}))$.

Corollary 2.8. *Assume that one of the three equivalent conditions of Theorem 2.6 is satisfied. Then, for every separable subspace* $\mathcal{H}$ *of* $L^\Phi(\mathfrak{P})$*, there exists a countable set* $\mathfrak{Q}_\mathcal{H} \subset \mathfrak{Q}$ *such that*

$$\|X\|_{L^\Phi(\mathfrak{P})} = \sup_{\mathbb{Q}\in\mathfrak{Q}_\mathcal{H}} \theta(\mathbb{Q})\|X\|_{L^1(\mathbb{Q})}, \quad \textit{for all } X \in \mathcal{H}.$$

Example 2.9.

(1) We consider the setup of Example 2.2. Let $\theta\colon \mathfrak{P} \to (0,\infty)$ with $c := \sup_{\mathbb{P}\in\mathfrak{P}} \theta(\mathbb{P}) < \infty$, $\gamma\colon \mathfrak{P} \to [0,\infty)$, and ϕ be a joint Orlicz function. Let

$$\phi_\mathbb{P}(x) := \frac{\phi\big(\theta(\mathbb{P})x\big)}{1+\gamma(\mathbb{P})}, \quad \text{for } x \ge 0,$$

corresponding to the case of a *doubly penalised robust Orlicz space*. Then, for $x_0 \in (0,\infty)$ with $cx_0 \in \mathrm{dom}(\phi)$,

$$\phi_{\mathrm{Max}}(x_0) = \sup_{\mathbb{P}\in\mathfrak{P}} \phi_\mathbb{P}(x_0) = \sup_{\mathbb{P}\in\mathfrak{P}} \frac{\phi\big(\theta(\mathbb{P})x_0\big)}{1+\gamma(\mathbb{P})} \le \phi(cx_0) < \infty.$$

By Proposition 2.6, we obtain that $L^\Phi(\mathfrak{P})$ is a weighted robust L^1-space.

(2) For each fixed probability measure $\mathbb{P}^*$ on $(\Omega,\mathcal{F})$, Proposition 2.6 shows that the classical space $L^\infty(\mathbb{P}^*)$ is a robust L^1-space, although this result could, of course, also be obtained in a more direct manner. Let $\mathfrak{P}$ be the set of all probability measures $\mathbb{P}$ on $(\Omega,\mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}^*$. Consider $\phi_\mathbb{P}(x) = x$ for all $x \ge 0$ and $\mathbb{P} \in \mathfrak{P}$, leading a robust L^1-space over $\mathfrak{P}$. Then,

$$\|X\|_{L^\Phi(\mathfrak{P})} = \|X\|_{L^\infty(\mathbb{P}^*)}, \quad \text{for } X \in L^0(\mathfrak{P}) = L^0(\mathbb{P}^*).$$

(3) Let $\mathbb{P}^*$ be a probability measure on $(\Omega, \mathcal{F})$, and consider a convex monetary risk measure $\rho\colon L^\infty(\mathbb{P}^*) \to \mathbb{R}$, which enjoys the Fatou property, and satisfies $\rho(0) = 0$. The dual representation, up to a sign,

$$\rho(X) = \sup_{Z \in \mathrm{dom}(\rho^*) \cap L^1(\mathbb{P}^*)} \mathbb{E}[ZX] - \rho^*(Z), \quad \text{for } X \in L^\infty(\mathbb{P}^*),$$

is a well-known consequence, where ρ^* is the convex conjugate of ρ. In the situation of Example 2.2 (1), set

$$\mathfrak{P} := \big\{ Z\mathrm{d}\mathbb{P}^* \,\big|\, Z \in \mathrm{dom}(\rho^*) \cap L^1(\mathbb{P}^*) \big\},$$
$$\gamma\big(Z\mathrm{d}\mathbb{P}^*\big) := \rho^*(Z), \quad \text{for } Z \in \mathrm{dom}(\rho^*) \cap L^1(\mathbb{P}^*),$$
$$\phi_{\mathbb{P}}(x) := x, \quad \text{for } x \geq 0 \text{ and } \mathbb{P} \in \mathfrak{P}.$$

Then, $L^\Phi(\mathfrak{P})$ contains $L^\infty(\mathfrak{P})$ as a sublattice. In general, we have $\mathfrak{P} \ll \mathbb{P}^*$, but $\mathfrak{P} \approx \mathbb{P}^*$ may fail without further conditions on ρ. We can always define the "projection"

$$\widehat{\rho}(Y) := \rho\big(J^{-1}(Y)\big), \quad \text{for } Y \in L^\infty(\mathfrak{P}),$$

though, where $J\colon L^\infty(\mathbb{P}^*) \to L^\infty(\mathfrak{P})$ is the natural projection. In that case, $L^\Phi(\mathfrak{P})$ serves as the maximal sensible domain of definition of $\widehat{\rho}$. Various aspects of such spaces have been studied in [24, 25, 33, 37, 45].

2.2. **An alternative path to robust Orlicz spaces.** In this section, we focus on a way to translate the concept of Orlicz spaces to a robust setting *without* using a robust Luxemburg norm and the worst-case approach represented by the supremum over all models $\mathbb{P} \in \mathfrak{P}$. One may indeed wonder if this modelling assumption is actually necessary to produce the largest commodity space on which the analytic behaviour of utility can be captured well with respect to any model considered in the uncertainty profile. An alternative would be provided by the space

$$\mathfrak{L}^\Phi(\mathfrak{P}) := \big\{ X \in L^0(\mathfrak{P}) \,\big|\, \forall\, \mathbb{P} \in \mathfrak{P} \,\exists\, \alpha_{\mathbb{P}} > 0 : \ \mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(\alpha_{\mathbb{P}}|X|)] < \infty \big\}. \tag{2.4}$$

A special case of this space has, e.g., been studied in [30] and [43]. One can show that $\mathfrak{L}^\Phi(\mathfrak{P})$ is a vector sublattice of $L^0(\mathfrak{P})$. Moreover, independent of Φ, $L^\Phi(\mathfrak{P}) \subset \mathfrak{L}^\Phi(\mathfrak{P})$ holds *a priori*, and the inclusion can be strict as the following example demonstrates.

Example 2.10. Fix two constants $0 < c < 1 < C$ and consider the case where $\Omega = \mathbb{R}$ is endowed with the Borel σ-algebra $\mathcal{F}$, and $\mathfrak{P}$ is given by the set of all probability measures $\mathbb{P}$, which are equivalent to $\mathbb{P}^* := \mathcal{N}(0,1)$ with bounded density $c \leq \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^*} \leq C$. Moreover, fix a partition $(\mathfrak{P}_n)_{n\in\mathbb{N}}$ of $\mathfrak{P}$ into nonempty sets. We set

$$\phi_{\mathbb{P}}(x) := x^n, \quad \text{for } x \geq 0,\ n \in \mathbb{N}, \text{ and } \mathbb{P} \in \mathfrak{P}_n.$$

Then,

$$\mathfrak{L}^\Phi(\mathfrak{P}) = \big\{ X \in L^0(\mathbb{P}^*) \,\big|\, \forall\, n \in \mathbb{N} : \ \mathbb{E}_{\mathbb{P}^*}[|X|^n] < \infty \big\},$$

and thus $U \in \mathfrak{L}^\Phi(\mathfrak{P})$ if $U\colon \Omega \to \mathbb{R}$ is the identity, i.e., $U \sim \mathcal{N}(0,1)$ under $\mathbb{P}^*$. However, Stirling's formula implies that, for all $\alpha > 0$,

$$\sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(\alpha|U|)] \geq c \sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}^*}[\alpha^n |U|^n] = \infty,$$

and $U \notin L^\Phi(\mathfrak{P})$ follows. It is easy to see that $\mathfrak{L}^\Phi(\mathfrak{P})$ is a Fréchet space, but not a Banach space.

The next proposition shows that $L^\Phi(\mathfrak{P})$ can always be seen as a space of type (2.4) if $L^\infty(\mathfrak{P}) \subset L^\Phi(\mathfrak{P})$, and more can be said if $\mathfrak{P}$ is countably convex.

Proposition 2.11. *The following are equivalent:*

(1) $L^\infty(\mathfrak{P}) \subset L^\Phi(\mathfrak{P})$.

(2) *There is a set of probability measures* $\mathfrak{R} \subset \mathbf{ca}^1_+(L^\Phi(\mathfrak{P}))$ *and a family* $\Psi = (\psi_\mathbb{Q})_{\mathbb{Q}\in\mathfrak{R}}$ *of Orlicz functions such that* $\mathfrak{R} \approx \mathfrak{P}$ *and*

$$L^\Phi(\mathfrak{P}) = \mathfrak{L}^\Psi(\mathfrak{R}).$$

In particular, if $\mathfrak{P}$ *is countably convex and there exist constants* $(c_\mathbb{P})_{\mathbb{P}\in\mathfrak{P}} \subset (0,\infty)$ *such that*

$$\phi_{\mathrm{Max}}(x) \le \phi_\mathbb{P}(c_\mathbb{P} x), \quad \textit{for all } x \ge 0 \textit{ and } \mathbb{P} \in \mathfrak{P}, \tag{2.5}$$

then (1) *and* (2) *hold and one can choose* $\mathfrak{R} = \mathfrak{P}$ *as well as* $\Psi = \Phi$ *or* $\Psi = (\phi_{\mathrm{Max}})_{\mathbb{P}\in\mathfrak{P}}$.

The next theorem, which generalises [25, Proposition 4.2(ii)] and [41, Theorem 4.4], shows that, for doubly penalised Orlicz spaces, cf. Example 2.9(1), the assumption of countable convexity of the set $\mathfrak{P}$ can be further relaxed.

Theorem 2.12. *Suppose that* $\mathfrak{P}$ *is convex. Assume that* Φ *is doubly penalised with joint Orlicz function* ϕ*, multiplicative penalisation* θ*, and convex additive penalty function* $\gamma\colon \mathfrak{P} \to [0,\infty)$ *with countably convex lower level sets. Then,*

$$\mathfrak{L}^\Phi(\mathfrak{P}) = L^\Phi(\mathfrak{P}).$$

Example 2.13. An example for an additive penalty function as demanded in Theorem 2.12 is given by the set $\mathfrak{P}$ of all probability measures in $\mathrm{dom}(\rho^*)$ for a convex monetary risk measure $\rho : \mathcal{L}^\infty(\Omega,\mathcal{F}) \to \mathbb{R}$ with $\mathrm{dom}(\rho^*) \cap \mathbf{ca}^1_+ \neq \emptyset$. However, this set is typically not countably convex, as the choice $\Omega = \mathbb{N}$, $\mathcal{F} = 2^\mathbb{N}$, and

$$\rho(f) = \sup_{n\in\mathbb{N}} f(n) - 2^{2n}, \quad \text{for } f \in \mathcal{L}^\infty(\Omega,\mathcal{F}),$$

demonstrates. Indeed, the Dirac measure δ_n lies in $\mathrm{dom}(\rho^*) \cap \mathbf{ca}^1_+$, $n \in \mathbb{N}$, but $\rho^*(\sum_{n=1}^\infty 2^{-n}\delta_n) = \infty$.

Remark 2.14. Assume that, in the situation of Theorem 2.12, the multiplicative penalty is $\theta \equiv 1$. Then, there are two equally consistent ways to translate convergence in $L^\phi(\mathbb{P})$ to a robust setting given by the set $\mathfrak{P}$ of priors. One could either declare a net $(X_\alpha)_{\alpha\in I}$ to be convergent if it (i) converges with respect to each seminorm $\|\cdot\|_{L^\phi(\mathbb{P})}$, for $\mathbb{P} \in \mathfrak{P}$, at equal or comparable speed to the same limit, or (ii) converges to the same limit with respect to each seminorm $\|\cdot\|_{L^\phi(\mathbb{P})}$, for $\mathbb{P} \in \mathfrak{P}$. Convergence (i) is reflected by the norm $\|\cdot\|_{L^\Phi(\mathfrak{P})}$, and the equality of speeds may be relaxed by the additive penalty, whereas (ii) would be the natural choice of a topology on $\mathfrak{L}^\Phi(\mathfrak{P})$. Even though $\mathfrak{L}^\Phi(\mathfrak{P}) = L^\Phi(\mathfrak{P})$ holds, convergence (ii) might not be normable or even sequential. However, having both options at hand provides a degree of freedom to reflect different economic phenomena on an applied level.

3. Generating sublattices of Φ-robust Orlicz spaces

By construction, Φ-robust Orlicz spaces are ideals in $L^0(\mathfrak{P})$ with respect to the $\mathfrak{P}$-q.s. order, and therefore particularly well-behaved with respect to order properties. Each Φ-robust Orlicz space is Dedekind σ-complete. Moreover, using arguments as in [17, Lemma 8], (super) Dedekind completeness of $L^0(\mathfrak{P})$ implies (super) Dedekind completeness of $L^\Phi(\mathfrak{P})$, and the converse implications hold in the situation of Theorem 2.6.[4] In conclusion, Φ-robust Orlicz spaces not only have the desirable Banach space property, but also behave reasonably well as vector lattices.

In contrast to the top-down construction of Φ-robust Orlicz spaces, one could also build a robust space bottom-up, a path taken in, e.g., [2, 6, 13]. Starting with a space of *test random variables*, one could consider closing the test space in the larger ambient space $L^\Phi(\mathfrak{P})$ with respect to the risk-uncertainty structure as given by $\|\cdot\|_{L^\Phi(\mathfrak{P})}$. Such a procedure leads to smaller spaces in general. The existing literature typically discusses (special cases of) these spaces as *Banach spaces* without further going into detail on their order-theoretic properties. This section therefore fills this gap, and explores their

[4] For the definition of these notions, we refer to [1].

properties as Banach lattices. We shall observe that they tend to be not very tractable as vector lattices. If they behave well with respect to order properties, this usually has strong consequences.

Assumption 3.1. Throughout this section, we assume that there exists some $x_0 \in (0,\infty)$ with $\phi_{\mathrm{Max}}(x_0) \in [0,\infty)$, or, equivalently, that $L^\Phi(\mathfrak{P})$ contains the equivalence class of the constant function $\mathbf{1}_\Omega$.

In the following, we consider a sublattice $\mathcal{H}$ of $L^\Phi(\mathfrak{P})$ containing the equivalence class of the constant function $\mathbf{1}_\Omega$. We assume that $\mathcal{H}$ *generates* $\mathcal{F}$ in that the σ-algebra $\sigma(\mathcal{L})$ generated by the lattice $\mathcal{L} := \{f \in \mathcal{L}^0(\Omega,\mathcal{F}) \mid [f] \in \mathcal{H}\}$ equals $\mathcal{F}$. Note that the latter assumption does not restrict generality and merely simplifies the exposition of our results. They transfer to smaller σ-algebras otherwise. By $\mathcal{C}$ we denote the $\|\cdot\|_{L^\Phi(\mathfrak{P})}$-closure of $\mathcal{H}$ in $L^\Phi(\mathfrak{P})$, i.e.

$$\mathcal{C} = \mathrm{cl}(\mathcal{H}).$$

We define the subspaces $\mathbf{ca}(\mathcal{H})$ and $\mathbf{ca}(\mathcal{C})$ of $\mathbf{ca}(\mathfrak{P})$ in complete analogy with $\mathbf{ca}(L^\Phi(\mathfrak{P}))$ (see equation (2.2)). Using Remark 2.4, one can show that, for each $\mu \in \mathbf{ca}(\mathcal{H})$, the functional $\mathcal{H} \ni X \mapsto \mu X$ is continuous.

Definition 3.2. Let $(\mathcal{X},\preceq)$ be a vector lattice. A possibly nonlinear functional $\ell\colon \mathcal{X} \to \mathbb{R}$ is *σ-order continuous* if the following two properties hold: (i) for all $x,y \in \mathcal{X}$, the set $\{\ell(z) \mid x \preceq z \preceq y\} \subset \mathbb{R}$ is bounded, (ii) whenever for a sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{X}$ there is another sequence $(y_n)_{n\in\mathbb{N}} \subset \mathcal{X}_+$ such that $|x_n| \preceq y_n$ for all $n \in \mathbb{N}$, $y_n \downarrow$, i.e., $y_{n+1} \preceq y_n$ for all $n \in \mathbb{N}$, and $\inf_{n\in\mathbb{N}} y_n = 0$ in $\mathcal{X}$, then $\lim_{n\to\infty} |\ell(x_n)| = 0$.

In a first step, we characterise *σ-order continuous functionals* on $\mathcal{C}$ and $\mathcal{H}$.

Lemma 3.3. *For each σ-order continuous functional $\ell\colon \mathcal{H} \to \mathbb{R}$ there is a unique signed measure $\mu \in \mathbf{ca}(\mathfrak{P})$ such that, for all $X \in \mathcal{H}$, all representatives of X are $|\mu|$-integrable and*

$$\ell(X) = \mu X.$$

In particular, ℓ satisfies $\ell(X) \geq 0$ for all $X \in \mathcal{H}_+$ if and only if the associated measure μ lies in $\mathbf{ca}_+(\mathfrak{P})$.

This motivates to denote the space of all σ-order continuous linear functionals on $\mathcal{H}$ by $\mathbf{ca}^\sigma(\mathcal{H})$ and the cone of positive σ-order continuous linear functionals by $\mathbf{ca}^\sigma_+(\mathcal{H}) := \mathbf{ca}^\sigma(\mathcal{H}) \cap \mathbf{ca}_+$.

Lemma 3.4. *Assume that $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ is σ-order continuous on $\mathcal{H}$. Then,*

$$\mathcal{H}^* = \mathbf{ca}(\mathcal{H}) = \mathbf{ca}^\sigma(\mathcal{H}) \cap \mathcal{H}^*.$$

Proposition 3.5. *The space $\big(\mathcal{C},\preceq,\|\cdot\|_{L^\Phi(\mathfrak{P})}\big)$ is a Banach lattice and $\mathbf{ca}(\mathcal{C}) = \mathbf{ca}(\mathcal{H})$. If $\mathcal{H} \subset L^\infty(\mathfrak{P})$ and $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ is σ-order continuous on $\mathcal{H}$, then*

$$\mathcal{C}^* = \mathbf{ca}(\mathcal{C}) = \mathbf{ca}(\mathcal{H}) = \mathbf{ca}^\sigma(\mathcal{H}) \cap \mathcal{C}^*.$$

Throughout the remainder of this section, the closure $\mathrm{cl}\big(L^\infty(\mathfrak{P})\big)$ of $L^\infty(\mathfrak{P})$ plays a fundamental role. The following lemma is a slight generalisation of [13, Proposition 18] and provides an explicit description of the closure of $L^\infty(\mathfrak{P})$ in our setup.

Lemma 3.6. *Let $X \in L^\Phi(\mathfrak{P})$. Then, $X \in \mathrm{cl}\big(L^\infty(\mathfrak{P})\big)$ if and only if, for all $\alpha > 0$,*

$$\lim_{n\to\infty} \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_\mathbb{P}\big[\phi_\mathbb{P}(\alpha|X|)\mathbf{1}_{\{|X|>n\}}\big] = 0.$$

That is,

$$\mathrm{cl}\big(L^\infty(\mathfrak{P})\big) = \Big\{X \in L^\Phi(\mathfrak{P}) \;\Big|\; \lim_{n\to\infty} \big\|X\mathbf{1}_{\{|X|>n\}}\big\|_{L^\Phi(\mathfrak{P})} = 0\Big\}.$$

For the remaining results of this section, we emphasise that, if we view $\mathcal{H}$ or $\mathcal{C}$ as spaces of *measurable functions*, two properties should not be far-fetched: (i) Dedekind σ-completeness, (ii) many positive functionals which are integrals with respect to a measure are σ-order continuous.[5]
The following theorem shows that, if $\mathfrak{P}$ is nondominated, the Banach lattice $\mathcal{C}$ cannot be separable *and* simultaneously have the mild order completeness property of Dedekind σ-completeness.

Theorem 3.7. *Suppose that the Banach lattice $\big(\mathcal{C}, \preceq, \|\cdot\|_{L^\Phi(\mathfrak{P})}\big)$ is separable, and let $\mathbb{P}^*$ be a probability measure as in Theorem 2.7. Then, the following are equivalent:*

(1) *$\mathcal{C}$ is Dedekind σ-complete.*
(2) *$\mathcal{C}$ is super Dedekind complete.*
(3) *$\mathcal{C} = \mathrm{cl}\big(L^\infty(\mathfrak{P})\big)$.*
(4) *$\mathcal{C}$ is an ideal in $L^\Phi(\mathfrak{P})$.*
(5) *$\mathcal{C}^* = \mathbf{ca}(\mathcal{C}) = \mathbf{ca}(L^\Phi(\mathfrak{P})) \approx \mathbb{P}^*$ and the unit ball therein is weakly compact in $L^1(\mathbb{P}^*)$.*

Moreover, they imply both of the following assertions:

(6) *$\mathfrak{P} \approx \mathbb{P}^*$.*
(7) *If, additionally,*
$$\inf_{\mathbb{P}\in\mathfrak{P}} \phi_{\mathbb{P}}(x_0) \in (0,\infty] \quad \text{for some } x_0 \in (0,\infty), \tag{3.1}$$
the set $\big\{\frac{d\mathbb{P}}{d\mathbb{P}^*} \,\big|\, \mathbb{P}\in\mathfrak{P}\big\}$ of densities of priors in $\mathfrak{P}$ is uniformly $\mathbb{P}^*$-integrable.

We thus see that, in typical situations encountered in the literature, all order completeness properties agree, and their validity usually implies dominatedness of the underlying set of priors in a particularly strong form. Although separability is a desirable property from an analytic point of view, we have hereby shown that it has very strong implications for uncertainty robust spaces. One may wonder what happens if one drops this assumption. We start with the following version of the Monotone Class Theorem.

Lemma 3.8. *Assume that $\mathcal{H}$ is Dedekind σ-complete and $\mathfrak{P} \approx \mathbf{ca}_+^\sigma(\mathcal{H})$. Then, $L^\infty(\mathfrak{P}) \subset \mathcal{H}$.*

The next proposition now shows that the only (generating) sublattice of $L^\infty(\mathfrak{P})$ satisfying the requirements (i) and (ii) above is $L^\infty(\mathfrak{P})$ itself.

Proposition 3.9. *The following statements are equivalent:*

(1) *$\mathcal{H}$ is Dedekind σ-complete and $\mathfrak{P} \approx \mathbf{ca}_+^\sigma(\mathcal{H})$,*
(2) *$\mathcal{H}$ is Dedekind σ-complete and $\mathbf{ca}(\mathcal{H}) = \mathbf{ca}^\sigma(\mathcal{H}) \cap \mathcal{H}^*$,*
(3) *$\mathcal{H}$ is an ideal in $L^\Phi(\mathfrak{P})$.*

If $\mathcal{H} \subset L^\infty(\mathfrak{P})$, (1)–(3) are furthermore equivalent to

(4) *$\mathcal{H} = L^\infty(\mathfrak{P})$.*

Considering $\mathcal{C}$ instead of $\mathcal{H}$ does not change the picture, since the closure of any (generating) sublattice of $L^\infty(\mathfrak{P})$ satisfying (i) and (ii) leads to the same Banach lattice, the closure of $L^\infty(\mathfrak{P})$.

Proposition 3.10. *The following statements are equivalent:*

(1) *$\mathcal{C}$ is Dedekind σ-complete and $\mathfrak{P} \approx \mathbf{ca}_+^\sigma(\mathcal{C})$,*
(2) *$\mathcal{C}$ is Dedekind σ-complete and $\mathbf{ca}(\mathcal{C}) = \mathbf{ca}^\sigma(\mathcal{C})$,*
(3) *$\mathcal{C}$ is an ideal in $L^\Phi(\mathfrak{P})$.*

If $\mathcal{H} \subset L^\infty(\mathfrak{P})$, (1)–(3) are furthermore equivalent to

(4) *$\mathcal{C} = \mathrm{cl}\big(L^\infty(\mathfrak{P})\big)$.*

[5] Tellingly, the early literature on vector lattices refers to σ-order continuous linear functionals as "integrals".

4. Applications

This section is devoted to the discussion of the financial and economic implications of our theoretical results.

4.1. Closures of continuous functions.

Prominent sublattices of $L^\Phi(\mathfrak{P})$ appearing in the literature – at least for special cases of Φ – are $\|\cdot\|_{L^\Phi(\mathfrak{P})}$-closures of sets of continuous functions on a separable metrisable space Ω, such as in [6], where a general lattice of bounded continuous functions generating the Borel-σ-algebra and containing $\mathbf{1}_\Omega$ is considered. Other examples include bounded Lipschitz functions, or bounded cylindrical Lipschitz functions, respectively, cf. [13, 23]. The usual minimal assumption on $\mathfrak{P}$ is tightness, sometimes one imposes that $\mathfrak{P}$ is convex and weakly compact, cf. [2]. In that case, $\mathfrak{P}$ has the stronger property of being countably convex.

Throughout this subsection, we assume that Ω is a separable and metrisable topological space, endowed with the Borel σ-algebra $\mathcal{F}$. Let C_b be the space of bounded continuous functions on Ω, and let $\mathcal{H}$ be a lattice of bounded continuous functions containing $\mathbf{1}_\Omega$. We shall again impose Assumption 3.1, which yields that

$$\iota\colon C_b \to L^\Phi(\mathfrak{P}), \quad f \mapsto [f].$$

is a well-defined, continuous, and injective lattice homomorphism. We shall abuse notation and also refer to $\iota(C_b)$ as C_b, to the equivalence classes by capital letters though. As before, let

$$\mathcal{C} := \mathrm{cl}\big(\mathcal{H}\big),$$

endowed with $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ and the $\mathfrak{P}$-q.s. order.

Our first main observation is that the results in [2, 6] are based on separability of the primal space, which holds under a comparatively mild tightness condition. The following result is a decisive generalisation of [6, Proposition 2.6].

Proposition 4.1. *Suppose that, for every $\varepsilon > 0$, there exists a compact set $K \subset \Omega$ with*

$$\|\mathbf{1}_{\Omega\setminus K}\|_{L^\Phi(\mathfrak{P})} < \varepsilon. \tag{4.1}$$

Then, $\mathcal{C}$ is separable.

Lemma 4.2. *Condition* (4.1) *is met in any of the following situations:*

(1) *Ω is compact.*
(2) $\mathrm{dom}(\phi_{\mathrm{Max}}) = [0, \infty)$ *and, for all $t > 0$, the set $\mathfrak{P}_t := \{\mathbb{P} \in \mathfrak{P} \mid \phi_{\mathbb{P}}(t) > 1\}$ is tight.*
(3) $\mathrm{dom}(\phi_{\mathrm{Max}}) = [0, \infty)$ *and $\mathfrak{P}$ is tight.*

If Φ satisfies (3.1)*, the validity of* (4.1) *implies that $\mathfrak{P}$ is tight.*

We emphasise that (3) is the typical minimal assumption in the literature. It is, in particular, satisfied in the G-framework, see [36, Theorem 2.5].

Example 4.3.

(1) Let $p \in [1, \infty)$, and consider the case, where $\phi_{\mathbb{P}}(x) = x^p$ for all $x \geq 0$ and $\mathbb{P} \in \mathfrak{P}$. Then, Lemma 4.2 implies that (4.1) holds if and only if $\mathfrak{P}$ is tight.
(2) Consider the case of a doubly penalised robust Orlicz space as in Example 2.9(1) with bounded multiplicative penalty $\theta\colon \mathfrak{P} \to (0, \infty)$ and additive penalty $\gamma\colon \mathfrak{P} \to [0, \infty)$. Then, $\mathrm{dom}(\phi_{\mathrm{Max}}) = [0, \infty)$ if and only if the joint Orlicz function ϕ satisfies $\mathrm{dom}(\phi) = [0, \infty)$. Moreover, condition (2) in Lemma 4.2 is met if the lower level sets of γ are tight. Notice that, in this case, the validity of (3.1) implies the boundedness of γ, and thus, naturally, the tightness of $\mathfrak{P}$.

Remark 4.4.

(1) Strictly speaking, Bion-Nadal & Kervarec [6] work with the *Lebesgue prolongation* of a capacity $\mathfrak{c}$ defined on a generating lattice of continuous functions. In most of their results, they assume that $\mathfrak{c}$ is a *Prokhorov capacity* on a separable metrisable space. As $\mathbf{1}_{\Omega\setminus K}$ is l.s.c. for every compact $K \subset \Omega$, one thus obtains, for each $\varepsilon > 0$, the existence of a compact $K \subset \Omega$ such that

$$\mathfrak{c}(\mathbf{1}_{\Omega\setminus K}) \leq \varepsilon.$$

This counterpart of (4.1) admits to perform our proof of Proposition 4.1 in their framework, and the result transfers.

(2) We comment here on the role of Proposition 3.10 and Theorem 3.7 in the present setting. As is remarked after [12, Corollary 5.6], C_b over a Polish space does not admit any nontrivial σ-order continuous linear functional when endowed with the pointwise order. One could therefore interpret Proposition 3.10 as a *dichotomy*: either the closure $\mathcal{C}$ of C_b in $L^\Phi(\mathfrak{P})$ behaves very much like the space of continuous functions, or it is an ideal of $L^\Phi(\mathfrak{P})$, which could be obtained more directly as the closure of $L^\infty(\mathfrak{P})$ and to which in most typical cases Theorem 3.7 applies.
As an illustrative example, consider $\Omega = [0,1]$ endowed with its σ-algebra $\mathcal{F}$ of Borel sets and set $\mathfrak{P}$ to be the set of all atomless probability measures. Consider the robust weighted L^1-space, where $\theta \equiv 1$. One shows that each $X \in \mathcal{C}$ has a unique continuous representative f and satisfies $\|X\|_{L^\phi(\mathfrak{P})} = \|f\|_\infty$. In this setting, the inclusions

$$\{0\} = \mathbf{ca}^\sigma(\mathcal{C}) \subsetneq \mathbf{ca}(\mathcal{C}) \subsetneq \mathcal{C}^* \cap \mathbf{ca}$$

hold. For the first equality, note that $\mathcal{C}$ is lattice-isometric to C_b, and the existence of a nontrivial σ-order continuous linear functional would contradict the result cited above. For the second strict inclusion, consider the linear bounded functional $\ell(X) := f(1)$, $X \in \mathcal{C}$, where $f \in X$ is a continuous representative. Although it corresponds to the Dirac measure concentrated at 1, it cannot be identified with a measure absolutely continuous with respect to $\mathfrak{P}$.

We conclude with a Riesz representation result for the dual of $\mathcal{C}$, which follows directly from the more general observations in Section 3 and extends [2, Proposition 4] to our setting.

Corollary 4.5. *Assume that* $\mathcal{H} = C_b$, $\mathfrak{P}$ *is weakly compact, and that* $\mathrm{dom}(\phi_{\mathrm{Max}}) = [0,\infty)$. *Then,*

$$\mathcal{C}^* = \mathbf{ca}\big(\mathcal{C}\big).$$

4.2. **Option spanning under uncertainty.** A rich strand of literature deals with the power of options to complete a market, at least in an approximate sense. These studies date back to [21, 39] for finite and arbitrary numbers of future states of the economy, respectively. They have since been extended to a multitude of model spaces, such as the space of continuous functions over a compact Hausdorff space, L^p-spaces, or even more general ideals of L^0 over a probability space. We refer to [18] and the references therein. In the present example we study option spanning under potentially nondominated uncertainty.

Fix a *limited liability claim* $X \in L^\Phi(\mathfrak{P})$, i.e., $X \succeq 0$ holds. Its *option space*

$$\mathcal{H}_X := \mathrm{span}\left(\{\mathbf{1}_\Omega\} \cup \{(X - k\mathbf{1}_\Omega)^+ \mid k \in \mathbb{R}\}\right)$$

is the collection of all portfolios of call and put options written on X. In line with the simplifying assumption in Section 3, we will assume w.l.o.g. that

$$\mathcal{F} = \sigma(\{f \mid f \in X\}),$$

a condition studied in detail in the existing literature on option spanning. We also introduce the norm closure

$$\mathcal{C}_X := \mathrm{cl}(\mathcal{H}_X),$$

the space of all contingent claims, which can be approximated by linear combinations of call and put options.

Proposition 4.6. *$\big(\mathcal{H}_X, \preceq, \|\cdot\|_{L^\Phi(\mathfrak{P})}\big)$ is a separable normed sublattice of $L^\Phi(\mathfrak{P})$, and $\big(\mathcal{C}_X, \preceq, \|\cdot\|_{L^\Phi(\mathfrak{P})}\big)$ is a separable Banach sublattice of $L^\Phi(\mathfrak{P})$. Thus, Theorem 2.7 and Theorem 3.7 apply.*

We may thus draw two interesting financial conclusions from our results. First, by Theorem 2.7, *nondominated* uncertainty collapses both over the option space $\mathcal{H}_X$ and its closure $\mathcal{C}_X$. In fact, the same reference probability measure $\mathbb{P}^*$ can be chosen for both spaces (Corollary 2.8). $\mathbb{P}^*$ can be interpreted as intrinsic to and $J_{\mathbb{P}^*}(\mathcal{H}_X) \subset L^1(\mathbb{P}^*)$ as a copy of the original option space $\mathcal{H}_X$. This motivates the following corollary.

Corollary 4.7. *Let $\mathbb{P}^* \in \mathbf{ca}^1_+(L^\Phi(\mathfrak{P}))$ be a dominating probability measure for $\mathcal{H}_X$ as constructed in Theorem 2.7. Then, for each $Y \in L^\Phi(\mathfrak{P})$, there is a sequence $(Y_n)_{n\in\mathbb{N}} \subset \mathcal{H}_X$ such that $Y_n \to Y$ $\mathbb{P}^*$-a.s. as $n \to \infty$.*

The second conclusion concerns the *topological spanning power of X* and follows directly from Theorem 3.7.

Corollary 4.8. *Suppose X has topological spanning power in that*

$$\mathcal{C}_X \text{ is an ideal of } L^\Phi(\mathfrak{P}). \tag{4.2}$$

Let $\mathbb{P}^*$ be a probability measure as in Theorem 2.7. Then, the following assertions hold:

1. *$\mathcal{C}_X$ is super Dedekind complete.*
2. *$\mathbf{ca}(\mathcal{C}_X) \approx \mathfrak{P} \approx \mathbb{P}^*$ and the unit ball of $\mathbf{ca}(\mathcal{C}_X)$ is weakly compact in $L^1(\mathbb{P}^*)$.*
3. *$\mathcal{C}_X$ is lattice-isomorphic to an ideal of $L^1(\mathbb{P}^*)$.*
4. *$X \in \mathrm{cl}(L^\infty(\mathfrak{P})) = \mathcal{C}_X$.*

In particular, the topological spanning power of limited liability claims is always weaker than (4.2) *unless $\mathfrak{P}$ is dominated.*

Moreover, under the mild growth condition (3.1) on Φ, which does not depend on the concrete choice of the limited liability claim X whatsoever, (4.2) implies that all densities of priors in $\mathfrak{P}$ w.r.t. $\mathbb{P}^*$ are uniformly $\mathbb{P}^*$-integrable. This can be seen as a converse to the spanning power results on classical L^p-spaces, $1 \le p < \infty$. In conclusion, the topological spanning power of options under nondominated uncertainty is *always* weaker than (4.2), whereas X always has full spanning power with respect to the reference measure $\mathbb{P}^*$ by Corollary 4.7.

4.3. **Regular pricing rules and the Fatou property.** *Positive linear functionals* on a space of contingent claims are commonly interpreted as linear pricing rules. Applying this to a generating lattice $\mathcal{H} \subset L^\Phi(\mathfrak{P})$ as studied in Section 3 and a positive linear functional $\ell\colon \mathcal{H} \to \mathbb{R}$, σ-order continuity of ℓ now has the following economic interpretation. Whenever a sequence of contingent claims $(X_n)_{n\in\mathbb{N}}$ satisfies $X_n \downarrow$ and $\inf_{n\in\mathbb{N}} X_n = 0$, that is, the payoffs X_n become arbitrarily invaluable in the objective $\mathfrak{P}$-q.s. order, their prices $\ell(X_n)$ under ℓ vanish:

$$\lim_{n\to\infty} \ell(X_n) = 0.$$

Pricing with such functionals does not exaggerate the value of (objectively) increasingly invaluable contingent claims.

We have seen in Lemma 3.3 that such functionals correspond to measures. The condition

$$\mathfrak{P} \approx \mathbf{ca}^\sigma_+(\mathcal{H})$$

encountered in Lemma 3.8 and Propositions 3.9 and 3.10 means that the set of all regular pricing rules (in the sense described above) holds the same information about certainty and impossibility of events

as the set $\mathfrak{P}$ of "physical priors". In the case $\mathcal{H} = L^\Phi(\mathfrak{P})$, Proposition 3.10 shows $\mathbf{ca}^\sigma(L^\Phi(\mathfrak{P})) = \mathbf{ca}(L^\Phi(\mathfrak{P}))$. *A fortiori*, $\mathbf{ca}^\sigma_+(L^\Phi(\mathfrak{P})) \approx \mathfrak{P}$ holds.
For smaller generating lattices, our results thus describe a dichotomy in the case of full information: either the lattice does not generally admit aggregation even of countable order bounded families of contingent claims, or it is an ideal. In the case where bounded contingent claims are dense in $\mathcal{H}$, the latter further specialises to $\mathcal{H} = L^\infty(\mathfrak{P})$, or, if $\mathcal{H}$ is closed in $L^\Phi(\mathfrak{P})$, $\mathcal{H} = \mathrm{cl}(L^\infty(\mathfrak{P}))$.
We would like to make another comment on the conjunction of $\mathfrak{P} \approx \mathbf{ca}^\sigma_+(\mathcal{H})$ and Dedekind σ-completeness of $\mathcal{H}$ concerning the *Fatou property*. The latter is one of the most prominent phenomena studied in theoretical mathematical finance and relates order closedness properties of convex sets to dual representations of these in terms of measures. In a dominated framework, say, $L^\infty(\mathbb{P})$ for a single reference measure $\mathbb{P}$, a subset $\mathcal{B} \subset L^\infty(\mathbb{P})$ is called *Fatou closed* if, for each sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ which converges $\mathbb{P}$-a.s. to some $X \in L^\infty(\mathfrak{P})$ and whose moduli are dominated by some $Y \in L^\infty(\mathfrak{P})$, the limit satisfies $X \in \mathcal{B}$. Due to the super Dedekind completeness of $L^\infty(\mathfrak{P})$, Fatou closedness is equivalent to the so-called order closedness of $\mathcal{B}$. It is well known that $\mathcal{B} \subset L^\infty(\mathbb{P})$ is Fatou closed if and only if it has a representation of shape

$$\mathcal{B} = \{X \in L^\infty \mid \forall\, \mu \in \mathfrak{D} :\ \mu X \le h(\mu)\}$$

for a suitable set $\mathfrak{D} \subset \mathbf{ca}(\mathbb{P})$ and a function $h : \mathfrak{D} \to \mathbb{R}$.
This observation does not directly transfer to nondominated frameworks. In our setting, consider the family of seminorms

$$\rho_\mu : \mathcal{H} \to [0,\infty), \quad X \mapsto |\mu|(|X|), \quad \text{for } \mu \in \mathbf{ca}^\sigma_+(\mathcal{H}),$$

and let τ be the locally convex topology on $\mathcal{H}$ generated by them. In the literature, τ is often referred to as the *absolute weak topology* $|\sigma|(\mathcal{H}, \mathbf{ca}^\sigma_+(\mathcal{H}))$.

Observation 4.9. If $\mathfrak{P} \approx \mathbf{ca}^\sigma_+(\mathcal{H})$, τ is a locally convex Hausdorff topology and – by Kaplan's Theorem [1, Theorem 3.50] and [1, Theorem 1.57] – the dual of $(\mathcal{H}, \tau)^*$ is $\mathbf{ca}^\sigma(\mathcal{H})$. In particular, τ admits the application of separating hyperplane theorems.

Let $\mathcal{B} \subset \mathcal{H}$ be a nonempty convex set.

Observation 4.10. If $\mathcal{H}$ is Dedekind σ-complete, the following are equivalent:
(1) $\mathcal{B}$ is sequentially order closed.
(2) For all sequences $(X_n)_{n\in\mathbb{N}}$ whose moduli $(|X_n|)_{n\in\mathbb{N}}$ admit some upper bound $Y \in \mathcal{H}$ and which converge $\mathfrak{P}$-q.s. to some $X \in \mathcal{H}$, this limit X lies in $\mathcal{B}$.

Note that (2) is the direct counterpart of Fatou closedness as formulated above.
Hence, in the outlined situation, there is a locally convex Hausdorff topology τ on $\mathcal{H}$ such that each τ-closed convex set $\mathcal{B} \subset \mathcal{H}$ is sequentially order closed. Although the power of the Fatou property lies in the *converse* implication, the assumptions in the preceding two observations appear to be analytic minimal requirements for a fruitful study. Accepting this, Proposition 3.9 states that studying the (sequential) Fatou property only makes sense on *ideals* of robust Orlicz spaces.
However, in robust frameworks, there is a caveat concerning the Fatou property: sequences are usually not sufficient to unfold its full analytic power, and intuitive reasoning learned in dominated frameworks usually fails. We refer to [27] for a detailed discussion on this issue.

4.4. **Utility theory for multiple agents.** As anticipated in the introduction, robust Orlicz spaces are canonical model spaces for aggregating *variational preferences* of a cloud of agents. Variational preferences encompass other prominent classes of preferences, such as multiple prior preferences introduced by [20] and multiplier preferences introduced by [22] – see also [26, Section 4.2.1]. One of the most appealing qualities of variational preference relations is the handy separation of risk attitudes (measured by the prior-wise expected utility approach) and ambiguity or uncertainty attitudes (as expressed by the choice of the underlying set of priors and their additive penalisation). Aggregating expert opinions or the preferences of a cloud of variational preference agents, however, requires to consider more than one utility function.

Consider a nonempty set $\mathcal{I}$ of agents. Each agent $i \in \mathcal{I}$ has preferences over, say, $\mathcal{L}^\infty(\Omega, \mathcal{F})$, the space of bounded real-valued random variables on $(\Omega, \mathcal{F})$, given by

$$f \preceq_i g \quad :\Longleftrightarrow \quad \inf_{\mathbb{P} \in \mathfrak{P}_i} \mathbb{E}_\mathbb{P}[u_i(f)] + c_i(\mathbb{P}) \leq \inf_{\mathbb{P} \in \mathfrak{P}_i} \mathbb{E}_\mathbb{P}[u_i(g)] + c_i(\mathbb{P}).$$

We shall assume that $\mathfrak{P}_i$ is a nonempty set of probability measures on $(\Omega, \mathcal{F})$ equivalent to a reference probability measure $\mathbb{P}_i^*$, $c_i \colon \mathfrak{P}_i \to [0, \infty)$ satisfies $\inf_{\mathbb{P} \in \mathfrak{P}_i} c_i(\mathbb{P}) = 0$, and that the utility function $u_i \colon \mathbb{R} \to \mathbb{R}$ is concave, upper semicontinuous, nondecreasing, and satisfies $u_i(0) = 0$. For the details of this axiomatisation, we refer to [26]. For the sake of brevity, we set

$$\mathfrak{U}_i \colon \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}, \quad f \mapsto \inf_{\mathbb{P} \in \mathfrak{P}_i} \mathbb{E}_\mathbb{P}[u_i(f)] + c_i(\mathbb{P}).$$

By an affine transformation, we can w.l.o.g. assume that

$$\mathfrak{U}_i(-\mathbf{1}_\Omega) = \inf_{\mathbb{P} \in \mathfrak{P}_i} u_i(-1) + c_i(\mathbb{P}) = -1, \quad \text{for all } i \in \mathcal{I}.$$

Aggregating the preferences of all agents in $\mathcal{I}$ leads to the *unanimous preference relation*

$$f \trianglelefteq g \quad :\Longleftrightarrow \quad \forall\, i \in \mathcal{I} : \; f \preceq_i g.$$

For its closer study, we first observe that, setting $\mathfrak{P} := \bigcup_{i \in \mathcal{I}} \mathfrak{P}_i$, $f = g$ $\mathfrak{P}$-q.s. implies $f \trianglelefteq g$ and $g \trianglelefteq f$, i.e., all agents are indifferent between f and g. Hence, we may consider the preference relations $\preceq_i$ on the space $L^\infty(\mathfrak{P})$ instead without losing any information. The definition of $\mathfrak{U}_i$ on $L^\infty(\mathfrak{P})$ is immediate. Next, for $\mathbb{P} \in \mathfrak{P}$, consider

$$\phi_\mathbb{P}(x) := \sup_{i \in \mathcal{I} \colon \mathbb{P} \in \mathfrak{P}_i} \frac{-u_i(-x)}{1 + c_i(\mathbb{P})}, \quad \text{for } x \in [0, \infty).$$

$\phi_\mathbb{P}$ is convex, lower semicontinuous, nondecreasing, and satisfies $\phi_\mathbb{P}(0) = 0$. We shall impose the condition that $\phi_\mathbb{P}$ is an Orlicz function. Moreover, one easily obtains

$$\phi_\mathbb{P}(1) \leq 1, \quad \text{for all } \mathbb{P} \in \mathfrak{P}.$$

Set $\Phi := (\phi_\mathbb{P})_{\mathbb{P} \in \mathfrak{P}}$ and consider the associated robust Orlicz space which satisfies $L^\infty(\mathfrak{P}) \subset L^\Phi(\mathfrak{P})$.

We claim that $L^\Phi(\mathfrak{P})$ is a canonical maximal model space to study all individual preferences $\preceq_i$, $i \in \mathcal{I}$, simultaneously. This is due to the observation that each preference relation $\preceq_i$ canonically extends to a *continuous* preference relation on $L^\Phi(\mathfrak{P})$. Indeed, note that by concave duality, for each $i \in \mathcal{I}$, there is a set $\mathbf{M}_i \ll \mathfrak{P}$ of finite measures and a non-negative function $h_i : \mathbf{M}_i \to [0, \infty)$ such that

$$\mathfrak{U}_i(X) = \inf_{\mu \in \mathbf{M}_i} \mu X + h_i(\mu), \quad \text{for all } X \in L^\infty(\mathfrak{P}).$$

Moreover, one can show that, for each $i \in \mathcal{I}$, all $X \in L^\infty(\mathfrak{P})$, and all $\mu \in \mathbf{M}_i$,

$$-1 \leq \mathfrak{U}_i \left(-\|X\|^{-1}_{L^\Phi(\mathfrak{P})} |X| \right) \leq \mu\big(- \|X\|^{-1}_{L^\Phi(\mathfrak{P})} |X| \big) + h_i(\mu).$$

From this, we infer

$$\mu|X| \leq (1 + h_i(\mu)) \|X\|_{L^\Phi(\mathfrak{P})}, \quad \text{for all } X \in L^\infty(\mathfrak{P}),\, i \in \mathcal{I}, \text{ and } \mu \in \mathbf{M}_i.$$

By monotone convergence, the same estimate holds for all $X \in L^\Phi(\mathfrak{P})$. Hence, setting

$$\mathfrak{U}_i^\sharp(X) := \inf_{\mu \in \mathbf{M}_i} \mu X + h_i(\mu), \quad \text{for all } X \in L^\Phi(\mathfrak{P}),$$

and

$$X \preceq_i Y \quad :\iff \quad \mathfrak{U}_i^\sharp(X) \le \mathfrak{U}_i^\sharp(Y), \quad \text{for } X, Y \in \; L^\Phi(\mathfrak{P}),$$

we have extended the initial preference relations to $L^\Phi(\mathfrak{P})$ in a continuous manner.
In case that all agents have the same attitude towards *risk*, i.e., the utility function u_i does not depend on i, it is straightforward to construct examples where Theorem 2.12 is applicable and we have the identity

$$L^\Phi(\mathfrak{P}) = \mathfrak{L}^\Phi(\mathfrak{P}),$$

i.e., $L^\Phi(\mathfrak{P})$ is the model space for the minimal agreement among all agents under consideration on which well-defined utility can be attached to all objects.

Appendix A. Proofs of Section 2

Proof of Proposition 2.3. The fact that $L^\Phi(\mathfrak{P})$ is an ideal of $L^0(\mathfrak{P})$ follows directly from the fact that each $\phi_{\mathbb{P}}$ is nondecreasing and convex and the fact that the supremum is subadditive. Hence, it is a Dedekind σ-complete vector lattice with respect to the $\mathfrak{P}$-q.s. order because $L^0(\mathfrak{P})$ is Dedekind σ-complete. In a similar way, it follows that $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ defines a norm on $L^\Phi(\mathfrak{P})$. Let $(X_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Notice that, since $\phi_{\mathbb{P}}$ is convex and nontrivial for all $\mathbb{P} \in \mathfrak{P}$, there exist $a_{\mathbb{P}} > 0$ and $b_{\mathbb{P}} \in \mathbb{R}$ such that

$$\phi_{\mathbb{P}}(x) \ge (a_{\mathbb{P}}x + b_{\mathbb{P}})^+, \quad \text{for all } x \in \mathbb{R}. \tag{A.1}$$

By possibly passing to a subsequence, we may assume that

$$\|X_n - X_{n+1}\|_{L^\Phi(\mathfrak{P})} < 4^{-n}, \quad \text{for all } n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$, let $\lambda_n > 0$ with $\|X_n - X_{n+1}\|_{L^\Phi(\mathfrak{P})} < \lambda_n \le 4^{-n}$. In particular, $\lambda_n^{-1}2^{-n} \ge 2^n$, i.e. we can fix $n_{\mathbb{P}} \in \mathbb{N}$ such that $a_{\mathbb{P}}\lambda_n^{-1}2^{-n} + b_{\mathbb{P}} > 0$ holds for all $n \ge n_{\mathbb{P}}$. Markov's inequality together with equation (A.1) shows, for all $\mathbb{P} \in \mathfrak{P}$,

$$\begin{aligned}
\sum_{n=n_{\mathbb{P}}}^\infty \mathbb{P}\big(|X_n - X_{n+1}| \ge 2^{-n}\big) &\le \sum_{n=n_{\mathbb{P}}}^\infty \mathbb{P}\left(\big(a_{\mathbb{P}}(\lambda_n^{-1}|X_n - X_{n+1}|) + b_{\mathbb{P}}\big)^+ \ge \big(a_{\mathbb{P}}\lambda_n^{-1}2^{-n} + b_{\mathbb{P}}\big)^+\right) \\
&\le \sum_{n=n_{\mathbb{P}}}^\infty (a_{\mathbb{P}}2^n + b_{\mathbb{P}})^{-1}\,\mathbb{E}_{\mathbb{P}}\left[\phi_{\mathbb{P}}\left(\lambda_n^{-1}|X_n - X_{n+1}|\right)\right] \\
&\le \sum_{n=n_{\mathbb{P}}}^\infty \frac{1}{a_{\mathbb{P}}2^n + b_{\mathbb{P}}} < \infty.
\end{aligned}$$

Applying the Borel-Cantelli Lemma yields that

$$\inf_{\mathbb{P}\in\mathfrak{P}} \mathbb{P}\big(|X_n - X_{n+1}| \le 2^{-n} \text{ eventually}\big) = 1.$$

Hence, the event $\Omega^* := \{\lim_{n\to\infty} X_n \text{ exists in } \mathbb{R}\} \in \mathcal{F}$ satisfies $\mathbb{P}(\Omega^*) = 1$ for all $\mathbb{P} \in \mathfrak{P}$. We set X to be (the equivalence class in $L^0(\mathfrak{P})$ induced by) $\limsup_{n\to\infty} X_n$. Now, let $\mathbb{P} \in \mathfrak{P}$ and $\alpha > 0$ be arbitrary. Choose $k \in \mathbb{N}$ such that $\sum_{i\ge k} \lambda_i\alpha \le 1$. For $l > k$, we can estimate

$$\phi_{\mathbb{P}}(\alpha|X_{n_k} - X_{n_l}|) \le \phi_{\mathbb{P}}\left(\sum_{i=k}^{l-1} \alpha|X_{n_{i+1}} - X_{n_i}|\right) \le \sum_{i=k}^{l-1} \lambda_i\alpha\phi_{\mathbb{P}}\left(\lambda_i^{-1}|X_{n_{i+1}} - X_{n_i}|\right) \le \sum_{i=k}^\infty \lambda_i\alpha.$$

Notice that the last bound is uniform in l and $\mathbb{P}$. Letting $l \to \infty$ and using lower semicontinuity of $\phi_{\mathbb{P}}$,

$$\phi_{\mathbb{P}}(\alpha|X_{n_k} - X|) \le \sum_{i=k}^\infty \lambda_i\alpha.$$

This implies

$$\lim_{k\to\infty}\sup_{\mathbb{P}\in\mathfrak{P}}\mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}\left(\alpha|X_{n_k}-X|\right)]\leq\lim_{k\to\infty}\sum_{i=k}^{\infty}\lambda_i\alpha=0.$$

As $\alpha>0$ was arbitrary, $X\in L^{\Phi}(\mathfrak{P})$ and $\lim_{k\to\infty}\|X_k-X\|_{L^{\Phi}(\mathfrak{P})}=0$ follow.
At last, let $X\in L^{\Phi}(\mathfrak{P})$. By lower semicontinuity of $\phi_{\mathbb{P}}$ and Fatou's Lemma, for all $\mathbb{P}\in\mathfrak{P}$,

$$\mathbb{E}_{\mathbb{P}}[a_{\mathbb{P}}\|X\|^{-1}_{L^{\Phi}(\mathfrak{P})}|X|+b_{\mathbb{P}}]\leq\mathbb{E}_{\mathbb{P}}\big[\phi_{\mathbb{P}}(\|X\|^{-1}_{L^{\Phi}(\mathfrak{P})}|X|)\big]\leq 1,$$

showing that $\mathbb{E}_{\mathbb{P}}[|X|]\leq\frac{1-b_{\mathbb{P}}}{a_{\mathbb{P}}}\|X\|_{L^{\Phi}(\mathfrak{P})}$, that is, (2.3). $\square$

Proof of Theorem 2.6. Suppose $L^{\infty}(\mathfrak{P})\subset L^{\Phi}(\mathfrak{P})$. Then, we can find some $\alpha>0$ such that

$$\sup_{\mathbb{P}\in\mathfrak{P}}\phi_{\mathbb{P}}(\alpha)=\sup_{\mathbb{P}\in\mathfrak{P}}\mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(\alpha\mathbf{1}_{\Omega})]\leq 1.$$

Now let $\alpha>0$ with $\phi_{\mathrm{Max}}(\alpha)=\sup_{\mathbb{P}\in\mathfrak{P}}\phi_{\mathbb{P}}(\alpha)<\infty$. Since ϕ_{Max} is convex, we may w.l.o.g. assume that $\phi_{\mathrm{Max}}(\alpha)\leq 1$. For $\mathbb{P}\in\mathfrak{P}$ and a finite measure $\mu\ll\mathbb{P}$, let

$$\|\mu\|'_{\mathbb{P}}:=\sup\big\{\mu|X|\,\big|\,\|X\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}=1\big\}.$$

Then, by [28, Theorem 2.6.9 & Corollary 2.6.6],[6]

$$\|X\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}=\sup\big\{\mu|X|\,\big|\,\mu\in\mathbf{ca}_+(\mathbb{P}),\ \|\mu\|'_{\mathbb{P}}=1\big\},\quad\text{for all }\mathbb{P}\in\mathfrak{P}\text{ and }X\in L^{\phi_{\mathbb{P}}}(\mathbb{P}).\tag{A.2}$$

Since $\sup_{\mathbb{P}\in\mathfrak{P}}\phi_{\mathbb{P}}(\alpha)\leq 1$, $\|\mathbf{1}_{\Omega}\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}\leq\alpha^{-1}$. Hence, for all $\mu\in\mathbf{ca}_+(\mathbb{P})$ with $\|\mu\|'_{\mathbb{P}}=1$,

$$\mu(\Omega)=\|\mathbf{1}_{\Omega}\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}\mu(\|\mathbf{1}_{\Omega}\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})})^{-1}\mathbf{1}_{\Omega})\leq\frac{1}{\alpha}.\tag{A.3}$$

For $\mathbb{P}\in\mathfrak{P}$, let

$$\mathfrak{Q}_{\mathbb{P}}:=\{\tfrac{1}{\mu(\Omega)}\mu\mid\mu\in\mathbf{ca}_+(\mathbb{P}),\ \|\mu\|'_{\mathbb{P}}=1\}.$$

By (2.3), $\mathbb{P}\in\mathfrak{Q}_{\mathbb{P}}$ holds for all $\mathbb{P}\in\mathfrak{P}$. We also define

$$\mathfrak{Q}:=\{\mathbb{Q}\in\mathbf{ca}^1_+(\mathfrak{P})\mid\exists\,\mathbb{P}\in\mathfrak{P}:\ \mathbb{Q}\in\mathfrak{Q}_{\mathbb{P}}\}\supset\mathfrak{P}.$$

Fix $\mathbb{Q}\in\mathfrak{Q}$, let $\mathbb{P}\in\mathfrak{P}$ such that $\mathbb{Q}\in\mathfrak{Q}_{\mathbb{P}}$, and let $\mu\in\mathbf{ca}_+(\mathbb{P})$ such that $\mathbb{Q}=\mu(\Omega)^{-1}\mu$. Then, (A.3) implies that

$$\|\mathbb{Q}\|'_{\mathbb{P}}=\mu(\Omega)^{-1}\geq\alpha.$$

The function

$$\theta(\mathbb{Q}):=\frac{1}{\inf_{\mathbb{P}\in\mathfrak{P}:\ \mathbb{Q}\in\mathfrak{Q}_{\mathbb{P}}}\|\mathbb{Q}\|'_{\mathbb{P}}},\quad\text{for }\mathbb{Q}\in\mathfrak{Q},$$

is thus bounded and takes positive values. Moreover, for $X\in L^0(\mathfrak{P})$,

$$\begin{aligned}\|X\|_{L^{\Phi}(\mathfrak{P})}&=\sup_{\mathbb{P}\in\mathfrak{P}}\|X\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}=\sup_{\mathbb{P}\in\mathfrak{P}}\sup_{\mathbb{Q}\in\mathfrak{Q}_{\mathbb{P}}}\frac{1}{\|\mathbb{Q}\|'_{\mathbb{P}}}\mathbb{E}_{\mathbb{Q}}[|X|]\\&=\frac{1}{\inf_{\mathbb{P}\in\mathfrak{P}:\ \mathbb{Q}\in\mathfrak{Q}_{\mathbb{P}}}\|\mathbb{Q}\|'_{\mathbb{P}}}\sup_{\mathbb{Q}\in\mathfrak{Q}}\mathbb{E}_{\mathbb{Q}}[|X|]=\sup_{\mathbb{Q}\in\mathfrak{Q}}\theta(\mathbb{Q})\mathbb{E}_{\mathbb{Q}}[|X|].\end{aligned}$$

This is (3).
At last, suppose that $L^{\Phi}(\mathfrak{P})$ reduces to a weighted robust L^1-space as in the assertion. From $\mathfrak{Q}\approx\mathfrak{P}$, we infer that the latter space contains $L^{\infty}(\mathfrak{Q})=L^{\infty}(\mathfrak{P})$.
For the last statement, choose $\kappa:=\sup_{\mathbb{Q}\in\mathfrak{Q}}\theta(\mathbb{Q})$ or, equivalently, $\kappa:=\|\mathbf{1}_{\Omega}\|_{L^{\Phi}(\mathfrak{P})}$. $\square$

Proof of Theorem 2.7. The separability of $\mathcal{H}$ implies that the unit ball $\big\{\ell\in\mathcal{H}^*\,\big|\,\|\ell\|_{\mathcal{H}^*}\leq 1\big\}$ endowed with the weak* topology is compact, metrisable, and thus separable, cf. [42, Theorem 3.16]. Hence, the

[6] The cases $L^{\phi_{\mathbb{P}}}(\mathbb{P})\in\{L^1(\mathbb{P}),L^{\infty}(\mathbb{P})\}$ are not treated in this reference, but equation (A.2) is well known for them.

set

$$\left\{\frac{\mathbb{P}}{\|\mathbb{P}\|_{\mathcal{H}^*}}\,\middle|\,\mathbb{P}\in\mathfrak{P}\right\}\subset\left\{\ell\in\mathcal{H}^*\,\middle|\,\|\ell\|_{\mathcal{H}^*}\le 1\right\}$$

is separable, and there exists a countable family $(\mathbb{P}_n)_{n\in\mathbb{N}}$ such that, for all $X\in\mathcal{H}$, $\sup_{n\in\mathbb{N}}\mathbb{E}_{\mathbb{P}_n}[|X|]>0$ holds if and only if $X\neq 0$. Consider the measure

$$\mu^*:=\sum_{n\in\mathbb{N}}2^{-n}\min\{1,\|\mathbb{P}_n\|_{\mathcal{H}^*}^{-1}\}\mathbb{P}_n\in\mathcal{H}^*,$$

which satisfies $\mu^*(\Omega)\le 1$. For $s>0$ appropriately chosen, the probability measure $\mathbb{P}^*:=s\mu^*\in\mathcal{H}^*$ is a countable convex combination of $(\mathbb{P}_n)_{n\in\mathbb{N}}$, and the functional $\mathbb{P}^*$ is strictly positive by construction. Hence, for $X,Y\in\mathcal{H}$, $X\preceq Y$ if and only if $\mathbb{E}_{\mathbb{P}^*}[(Y-X)^-]=0$, which immediately proves that the canonical projection $J_{\mathbb{P}^*}\colon\mathcal{H}\to L^1(\mathbb{P}^*)$ is injective. By construction, we see that $\mathbb{P}^*\in\mathfrak{P}$ if $\mathfrak{P}$ is countably convex. □

Proof of Corollary 2.8. As in the proof of the previous theorem, we see that the set

$$\left\{\theta(\mathbb{Q})\cdot\mathbb{Q}\,\middle|\,\mathbb{Q}\in\mathfrak{Q}\right\}\subset\left\{\ell\in\mathcal{H}^*\,\middle|\,\|\ell\|_{\mathcal{H}^*}\le 1\right\}$$

is separable with respect to the relative weak* topology. Hence, there exists a countable family $(\mathbb{Q}_n)_{n\in\mathbb{N}}\subset\mathfrak{Q}$ such that, for all $X\in\mathcal{H}$,

$$\sup_{n\in\mathbb{N}}\theta(\mathbb{Q}_n)\|X\|_{L^1(\mathbb{Q}_n)}=\sup_{\mathbb{Q}\in\mathfrak{Q}}\theta(\mathbb{Q})\|X\|_{L^1(\mathbb{Q})}=\|X\|_{L^\Phi(\mathfrak{P})}.$$

□

Proof of Proposition 2.11. (2) clearly implies (1). Now suppose that (1) holds. By Theorem 2.6, we have

$$\|X\|_{L^\Phi(\mathfrak{P})}=\sup_{\mu\in\mathfrak{D}}\mu|X|,$$

where $\mathfrak{D}:=\{\mu\in\mathbf{ca}_+(L^\Phi(\mathfrak{P}))\mid\|\mu\|_{L^\Phi(\mathfrak{P})^*}\le 1\}$ satisfies $\sup_{\mu\in\mathfrak{D}}\mu(\Omega)<\infty$ because of the assumption $L^\infty(\mathfrak{P})\subset L^\Phi(\mathfrak{P})$. Set

$$\begin{aligned}&\mathfrak{R}:=\{\mu(\Omega)^{-1}\mu\mid\mu\in\mathfrak{D}\}\subset\mathbf{ca}_+^1(L^\Phi(\mathfrak{P})),\\&\psi_{\mathbb{Q}}(x):=\mu(\Omega)x,\ x\ge 0,\ \text{for }\mathbb{Q}=\mu(\Omega)^{-1}\mu\in\mathfrak{R},\\&\Psi=(\psi_{\mathbb{Q}})_{\mathbb{Q}\in\mathfrak{R}}.\end{aligned}$$

Then, $L^\Phi(\mathfrak{P})\subset\mathfrak{L}^\Psi(\mathfrak{R})$ holds by construction. Suppose now that $X\in L^0(\mathfrak{P})\setminus L^\Phi(\mathfrak{P})$. Then, we must be able to find a sequence $(\mu_n)_{n\in\mathbb{N}}\subset\mathfrak{D}$ such that $\mu_n|X|\ge 2^n$, $n\in\mathbb{N}$. By the Banach space property of $L^\Phi(\mathfrak{P})^*$, $\mu^*:=\sum_{n=1}^\infty 2^{-n}\mu_n\in\mathfrak{D}$, and we observe

$$\mu^*|X|=\sum_{n=1}^\infty 2^{-n}\mu_n|X|\ge\sum_{n=1}^\infty 1=\infty.$$

This completes the proof of the identity $L^\Phi(\mathfrak{P})=\mathfrak{L}^\Psi(\mathfrak{R})$.

Consider now the special case of $\mathfrak{P}$ being countably convex and (2.5) being satisfied. Observe that, for all $\alpha>0$, $\mathbb{P}\in\mathfrak{P}$, and all $X\in L^0(\mathfrak{P})$,

$$\mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(\alpha|X|)]\le\mathbb{E}_{\mathbb{P}}[\phi_{\mathrm{Max}}(\alpha|X|)]\le\mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(\alpha c_{\mathbb{P}}|X|)].$$

If we set $\Psi=(\phi_{\mathrm{Max}})_{\mathbb{P}\in\mathfrak{P}}$, this is sufficient to prove the following chain of inclusions:

$$L^\Psi(\mathfrak{P})\subset L^\Phi(\mathfrak{P})\subset\mathfrak{L}^\Phi(\mathfrak{P})=\mathfrak{L}^\Psi(\mathfrak{P}).$$

The proof is complete if we can show $\mathfrak{L}^\Psi(\mathfrak{P}) \subset L^\Psi(\mathfrak{P})$. To this end, let $X \in L^0(\mathfrak{P}) \setminus L^\Phi(\mathfrak{P})$. Then, there exists a sequence $(\mathbb{P}_n)_{n\in\mathbb{N}} \subset \mathfrak{P}$ with

$$\|X\|_{L^{\phi_{\mathrm{Max}}(\mathbb{P}_n)}} > 2^n n \quad \text{for all } n \in \mathbb{N}.$$

Define $\mathbb{P} := \sum_{n\in\mathbb{N}} 2^{-n}\mathbb{P}_n \in \mathfrak{P}$ (because $\mathfrak{P}$ is countably convex), and let $s > 0$ be arbitrary. Then,

$$\mathbb{E}_{\mathbb{P}}[\phi_{\mathrm{Max}}(s|X|)] = \sum_{n=1}^\infty 2^{-n}\mathbb{E}_{\mathbb{P}_n}[\phi_{\mathrm{Max}}(s|X|)] \ge \sum_{n=1}^\infty \mathbb{E}_{\mathbb{P}_n}\left[\phi_{\mathrm{Max}}(2^{-n}s|X|)\right] = \infty,$$

which proves that $X \notin \mathfrak{L}^\Psi(\mathfrak{P})$. □

Proof of Theorem 2.12. Let $X \in L^0(\mathfrak{P}) \setminus L^\Phi(\mathfrak{P})$. Then, there is a sequence $(\mathbb{P}_n)_{n\in\mathbb{N}} \subset \mathfrak{P}$ such that, for all $n \in \mathbb{N}$, $\|X\|_{L^{\phi_{\mathbb{P}_n}(\mathbb{P}_n)}} > 2^{2n}$, which in particular entails

$$\mathbb{E}_{\mathbb{P}_n}[\phi(\theta(\mathbb{P}_n)2^{-n}|X|)] > 2^n(1+\gamma(\mathbb{P}_n)).$$

Fix $\mathbb{P}^* \in \mathfrak{P}$ and consider the measure

$$\mathbb{Q} := \sum_{n=1}^\infty 2^{-n}\big(\tfrac{\gamma(\mathbb{P}_n)}{1+\gamma(\mathbb{P}_n)}\mathbb{P}^* + \tfrac{1}{1+\gamma(\mathbb{P}_n)}\mathbb{P}_n\big).$$

By convexity of γ and the countable convexity of its lower level sets, $\gamma(\mathbb{Q}) \le \gamma(\mathbb{P}^*) + 1$, $n \in \mathbb{N}$. For $\alpha > 0$ arbitrary, set $I := \{n \in \mathbb{N} \mid \theta(\mathbb{Q})\alpha \ge \theta(\mathbb{P}_n)2^{-n}\}$, an infinite set. Then,

$$\mathbb{E}_{\mathbb{Q}}[\phi(\theta(\mathbb{Q}_n)\alpha|X|)] \ge \sum_{n\in I} \tfrac{1}{2^n(1+\gamma(\mathbb{P}_n))}\mathbb{E}_{\mathbb{P}_n}[\phi(\theta(\mathbb{P}_n)2^{-n}|X|)] = \infty.$$

This proves $\|X\|_{L^{\phi_{\mathbb{Q}}(\mathbb{Q})}} = \infty$, which means $X \notin \mathfrak{L}^\Phi(\mathfrak{P})$. □

Appendix B. Proofs of Section 3

Proof of Lemma 3.3. Let $\mathcal{Y}$ denote the real vector space of all σ-order continuous linear functionals on $\mathcal{H}$. As $\mathcal{H}$ is a vector lattice, $\mathcal{Y}$ is a vector lattice itself when endowed with the order

$$\ell \preceq^* \ell' \quad :\iff \quad \forall\, X \in \mathcal{H}:\ \ell(X) \le \ell'(X);$$

cf. [1, Theorem 1.57]. As such, for each $\ell \in \mathcal{Y}$ there are unique $\ell^+, \ell^- \succeq^* 0$ such that $\ell = \ell^+ - \ell^-$. We may hence assume for the moment that $\ell \succeq^* 0$.

Then, for each sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ possessing representatives $(f_n)_{n\in\mathbb{N}}$ such that $f_n \downarrow 0$ holds pointwise, $\inf_{n\in\mathbb{N}} X_n = 0$ holds in $\mathcal{H}$. Consider the vector lattice

$$\mathcal{L} := \big\{f \in \mathcal{L}^0(\Omega,\mathcal{F}) \,\big|\, [f] \in \mathcal{H}\big\}.$$

and the linear map $J\colon \mathcal{L} \to \mathcal{H}$ defined by $J(f) = [f]$, $f \in \mathcal{L}$. The linear functional $\ell_0 := \ell \circ J$ satisfies $\ell_0(f_n) \downarrow 0$ for all sequences $(f_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ such that $f_n \downarrow 0$ pointwise. Since, by our assumption on $\mathcal{H}$, $\mathcal{F} = \sigma(\mathcal{L})$, [8, Theorem 7.8.1] provides a unique finite measure μ on $(\Omega,\mathcal{F})$ such that

$$\ell_0(f) = \int f\,\mathrm{d}\mu \quad \text{for all } f \in \mathcal{L}.$$

As $|f| \in \mathcal{L}$ for all $f \in \mathcal{L}$, each $f \in \mathcal{L}$ is μ-integrable. Moreover, for all $X \in \mathcal{H}$ and $f, g \in X$,

$$\int f\,\mathrm{d}\mu = \ell_0(f) = \ell(X) = \ell_0(g) = \int g\,\mathrm{d}\mu.$$

In particular, considering that $\mathbf{1}_N \in \mathcal{L}$ for all $N \in \mathcal{F}$ satisfying $\sup_{\mathbb{P}\in\mathfrak{P}}\mathbb{P}(N) = 0$, $\mu \in \mathbf{ca}_+(\mathfrak{P})$ follows. Finally, for a general $\ell \in \mathcal{Y}$, let $\nu, \eta \in \mathbf{ca}_+(\mathfrak{P})$ be the finite measures corresponding to ℓ^+ and ℓ^-, respectively. Setting $\mu := \nu - \eta$, we obtain for all $X \in \mathcal{H}$ that

$$\ell(X) = \ell^+(X) - \ell^-(X) = \nu X - \eta X = \mu X.$$

Moreover, the total variation measure $|\mu|$ satisfies $\int |f|\,\mathrm{d}|\mu| \le \int |f|\,\mathrm{d}(\nu+\eta) < \infty$, $f \in \mathcal{L}$.
At last, suppose that the representing signed measure of $\ell \in \mathcal{Y}$ is a measure. Then, $\ell \succeq^* 0$ holds automatically, and the proof is complete. $\square$

Proof of Lemma 3.4. Let $\ell \in \mathcal{H}^*$. In order to verify $\ell \in \mathbf{ca}^\sigma(\mathcal{H})$, let $X, Y \in \mathcal{H}$. Then, all $Z \in \mathcal{H}$ with the property $X \preceq Z \preceq Y$ satisfy $|Z| \le X^- + Y^+$. We obtain

$$\sup\{\ell(Z) \mid X \preceq Z \preceq Y\} \le \|\ell\|_{\mathcal{H}^*} \sup\{\|Z\|_{L^\Phi(\mathfrak{P})} \mid X \preceq Z \preceq Y\} \le \|\ell\|_{\mathcal{H}^*}\big(\|X^-\|_{L^\Phi(\mathfrak{P})} + \|Y^+\|_{L^\Phi(\mathfrak{P})}\big) < \infty.$$

This gives condition (i) in Definition 3.2. The validity of condition (ii) is a direct consequence of the lattice norm property and σ-order continuity of $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ on $\mathcal{H}$. The inclusion $\mathcal{H}^* \subset \mathbf{ca}^\sigma(\mathcal{H})$ together with Lemma 3.3 implies $\mathcal{H}^* \subset \mathbf{ca}(\mathfrak{P})$. $\square$

Proof of Proposition 3.5. By [28, Proposition 1.2.3(ii)], the closure $(\mathcal{C}, \preceq)$ of the sublattice $(\mathcal{H}, \preceq)$ of $L^\Phi(\mathfrak{P})$ is a sublattice as well. As $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ is a lattice norm on $\mathcal{C}$, $\big(\mathcal{C}, \preceq, \|\cdot\|_{L^\Phi(\mathfrak{P})}\big)$ is a Banach lattice by construction.
The inclusion $\mathbf{ca}(\mathcal{C}) \subset \mathbf{ca}(\mathcal{H})$ is trivial. For the converse inclusion, let $\mu \in \mathbf{ca}(\mathcal{H})$, i.e., $|\mu| \in \mathcal{H}^*$. Since $\mathcal{H}$ is dense in $\mathcal{C}$, there exists a unique $\ell \in \mathcal{C}^*$ with

$$\ell(X) = |\mu|X, \quad \text{for all } X \in \mathcal{H}.$$

Let $X \in \mathcal{C} \cap L^\infty(\mathfrak{P})$. Then, by Proposition 2.3, there exists a sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{H} \cap L^\infty(\mathfrak{P})$ with $\sup_{n\in\mathbb{N}} \|X_n\|_{L^\infty(\mathfrak{P})} < \infty$, $\|X - X_n\|_{L^\Phi(\mathfrak{P})} \to 0$, and $X_n \to X$ $\mathfrak{P}$-q.s. as $n \to \infty$. Since $|\mu| \ll \mathfrak{P}$, dominated convergence implies

$$|\mu|X = \lim_{n\to\infty} |\mu|X_n = \lim_{n\to\infty} \ell(X_n) = \ell(X).$$

Now, let $X \in \mathcal{C}$ arbitrary. Then,

$$\begin{aligned} |\mu|(|X|) &\le \sup_{n\in\mathbb{N}} |\mu|\big(|X| \wedge n\mathbf{1}_\Omega\big) = \sup_{n\in\mathbb{N}} \ell\big(|X| \wedge n\mathbf{1}_\Omega\big) \\ &\le \sup_{n\in\mathbb{N}} \|\ell\|_{\mathcal{C}^*} \big\||X| \wedge n\mathbf{1}_\Omega\big\|_{L^\Phi(\mathfrak{P})} \le \|\ell\|_{\mathcal{C}^*} \|X\|_{L^\Phi(\mathfrak{P})}. \end{aligned} \tag{B.1}$$

From this observation, the equality $|\mu| = \ell$ follows, which is sufficient to prove that $\mu \in \mathbf{ca}(\mathcal{C})$.
The remaining assertions easily follow with $\mathcal{C}^* = \mathcal{H}^*$ and Lemma 3.4. $\square$

Proof of Lemma 3.6. Let $X \in L^\Phi(\mathfrak{P})$. First, notice that

$$\sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_\mathbb{P}\big[\phi_\mathbb{P}(\alpha|X|)\mathbf{1}_{\{|X|>n\}}\big] \to 0, \quad \text{as } n \to \infty$$

for all $\alpha > 0$ is equivalent to $\|X\mathbf{1}_{\{|X|>n\}}\|_{L^\Phi(\mathfrak{P})} \to 0$ as $n \to \infty$.
If $\|X\mathbf{1}_{\{|X|>n\}}\|_{L^\Phi(\mathfrak{P})} \to 0$ as $n \to \infty$, it follows that $\|X - X_n\|_{L^\Phi(\mathfrak{P})} \to 0$ as $n \to \infty$ with $X_n := X\mathbf{1}_{\{|X|\le n\}} \in L^\infty(\mathfrak{P})$ for all $n \in \mathbb{N}$.
Now, assume that $X \in \mathrm{cl}\big(L^\infty(\mathfrak{P})\big)$. Let $(Y_n)_{n\in\mathbb{N}} \subset L^\infty(\mathfrak{P})$ with $\|X - Y_n\|_{L^\Phi(\mathfrak{P})} \to 0$ as $n \to \infty$. Let $X_m := (X \wedge m\mathbf{1}_\Omega) \vee (-m\mathbf{1}_\Omega)$ for all $m \in \mathbb{N}$. Then, for all $m, n \in \mathbb{N}$ with $m \ge \|Y_n\|_{L^\infty(\mathfrak{P})}$, it follows that

$$|X - X_m| \le |X - Y_n|,$$

which implies that $\|X - X_m\|_{L^\Phi(\mathfrak{P})} \to 0$ as $m \to \infty$. Finally notice that

$$\begin{aligned} |X|\mathbf{1}_{\{|X|>2m\}} &= (|X| - m\mathbf{1}_\Omega)\mathbf{1}_{\{|X|>2m\}} + m\mathbf{1}_{\{|X|>2m\}} \le 2\big(|X| - m\mathbf{1}_\Omega\big)\mathbf{1}_{\{|X|>2m\}} \\ &\le 2\big(|X| - m\mathbf{1}_\Omega\big)\mathbf{1}_{\{|X|>m\}} = 2|X - X_m|, \end{aligned}$$

which shows that $\|X\mathbf{1}_{\{|X|>m\}}\|_{L^\Phi(\mathfrak{P})} \to 0$ as $m \to \infty$. $\square$

For the sake of clarity, we give the proofs of Lemma 3.8, Proposition 3.9, and Proposition 3.10 in advance of Theorem 3.7.

Proof of Lemma 3.8. Let $X \in \mathcal{H}$ and $c \in \mathbb{R}$. Consider $Y_k := k(X - c\mathbf{1}_\Omega)^+ \wedge \mathbf{1}_\Omega \in \mathcal{H}$, $k \in \mathbb{N}$. The sequence $(Y_k)_{k\in\mathbb{N}}$ is nondecreasing and satisfies $0 \preceq Y_k \preceq \mathbf{1}_\Omega$. By monotone convergence,

$$\mu(\{X > c\}) = \sup_{k\in\mathbb{N}} \mu Y_k = \lim_{k\to\infty} \mu Y_k \tag{B.2}$$

holds for all $\mu \in \mathbf{ca}_+(\mathfrak{P})$. Moreover, by Dedekind σ-completeness of $\mathcal{H}$, $U := \sup_{k\in\mathbb{N}} Y_k$ exists and lies in $\mathcal{H}_+$. *A priori*, $\mathbf{1}_{\{X>c\}} \preceq U$ has to hold. Moreover, one can show that $U = (nU) \wedge \mathbf{1}_\Omega$ holds for all $n \in \mathbb{N}$. Hence, there is an event $B \in \mathcal{F}$ such that $\mathbf{1}_B = U$ in $\mathcal{H}$. For each $\mu \in \mathbf{ca}_+^\sigma(\mathcal{H})$,

$$\lim_{k\to\infty} \mu Y_k = \mu(B). \tag{B.3}$$

Equations (B.2) and (B.3) together with $\mathfrak{P} \approx \mathbf{ca}_+^\sigma(\mathcal{H})$ now imply that $\mathbf{1}_{\{X>c\}} = U \in \mathcal{H}$, that is, for every $X \in \mathcal{H}$, $f \in X$, and $c \in \mathbb{R}$, the equivalence class generated by $\mathbf{1}_{\{f>c\}}$ lies in $\mathcal{H}$. At last, consider the π-system $\Pi := \{\{f > c\} \mid X \in \mathcal{H},\, f \in X,\, c \in \mathbb{R}\}$, which generates $\mathcal{F}$ and is a subset of

$$\Lambda := \{A \in \mathcal{F} \mid \mathbf{1}_A \in \mathcal{H}\}.$$

Since $\mathcal{H}$ is Dedekind σ-complete and $\mathfrak{P} \approx \mathbf{ca}_+^\sigma(\mathcal{H})$, the latter can be shown to be a λ-system. By Dynkin's Lemma, it follows that $\Lambda = \mathcal{F}$. We have thus shown that $\mathcal{H}$ contains all representatives of $\mathcal{F}$-measurable simple functions. Each $X \in L^\infty(\mathfrak{P})$ is the supremum of a countable family of simple functions in $L^\Phi(\mathfrak{P})$. As $\mathcal{H}$ is Dedekind σ-complete and $\mathfrak{P} \approx \mathbf{ca}_+^\sigma(\mathcal{H})$, we conclude that $L^\infty(\mathfrak{P}) \subset \mathcal{H}$. □

Proof of Proposition 3.9. (2) clearly implies (1). In order to see that (1) implies (3), note first that $L^\infty(\mathfrak{P}) \subset \mathcal{H}$ holds by Lemma 3.8. Now let $X \in L^\Phi(\mathfrak{P})$, $Y \in \mathcal{H}$, and assume $0 \preceq X \preceq Y$ holds. The set $\{X_n := X \wedge n\mathbf{1}_\Omega \mid n \in \mathbb{N}\} \subset \mathcal{H}$ is order bounded above by Y in $\mathcal{H}$. By Dedekind σ-completeness, $X^* := \sup_{n\in\mathbb{N}} X_n$ exists in $\mathcal{H}$ and satisfies $X \preceq X^*$ *a priori*. Arguing as in Lemma 3.8, one verifies $X = X^* \in \mathcal{H}$.
In order to see that (3) implies (2), we first show that $\mathcal{H}$ is Dedekind σ-complete. Let $\mathcal{D} \subset \mathcal{H}$ be order bounded from above and countable. Since $L^\Phi(\mathfrak{P})$ is Dedekind σ-complete, $U := \sup \mathcal{D}$ exists in $L^\Phi(\mathfrak{P})$. Let $Y \in \mathcal{H}$ be any upper bound of $\mathcal{D}$ and $X \in \mathcal{D}$. Then, $X \preceq U \preceq Y$. As $\mathcal{H}$ is an ideal in $L^\Phi(\mathfrak{P})$, $U \in \mathcal{H}$ has to hold and we have proved that $\mathcal{H}$ is Dedekind σ-complete.
Now we prove that each $\mu \in \mathbf{ca}(\mathcal{H})$ is σ-order continuous. For condition (i) in Definition 3.2, we can argue as in the proof of Lemma 3.4. For condition (ii), let $(X_n)_{n\in\mathbb{N}} \in \mathcal{H}$ be a sequence with $X_{n+1} \preceq X_n$ for all $n \in \mathbb{N}$ and $\inf_{n\in\mathbb{N}} X_n = 0$ in $\mathcal{H}$. By [1, Theorem 1.35], $\inf_{n\in\mathbb{N}} X_n = 0$ holds in $L^\Phi(\mathfrak{P})$, which is equivalent to $\inf_{\mathbb{P}\in\mathfrak{P}} \mathbb{P}(X_n \downarrow 0) = 1$. Moreover, by definition of $\mathbf{ca}(\mathcal{H})$, $|\mu| X_1 < \infty$. Dominated convergence yields

$$\lim_{n\to\infty} \mu X_n = \lim_{n\to\infty} |\mu|\big(\tfrac{\mathrm{d}\mu}{\mathrm{d}|\mu|} X_n\big) = 0.$$

Now assume that, additionally, $\mathcal{H} \subset L^\infty(\mathfrak{P})$. If (3) holds, $\mathcal{H}$ is an ideal containing the equivalence class of $\mathbf{1}_\Omega$ and must therefore also be a superset of $L^\infty(\mathfrak{P})$. Trivially, (4) implies (3), and the proof is complete. □

Proof of Proposition 3.10. The equivalence of (1)–(3) follows directly from Proposition 3.9 up to two additional observations: $\mathcal{C}$ is a Banach lattice by Proposition 3.5, and therefore each element of $\mathbf{ca}^\sigma(\mathcal{C})$ is a continuous linear functional by [28, Proposition 1.3.7]. If $\mathcal{H} \subset L^\infty(\mathfrak{P})$, $\mathcal{C} \subset \mathrm{cl}(L^\infty(\mathfrak{P}))$ must hold, and the converse inclusion is a direct consequence under (3). (4) implies (3) because $L^\infty(\mathfrak{P})$ is an ideal and norm closures of ideals in Banach lattices remain ideals ([28, Proposition 1.2.3(iii)]). □

Proof of Theorem 3.7. (1) is equivalent to (2): Theorem 2.7 provides a strictly positive linear functional in the present situation. Hence, the equivalence of (1) and (2) follows with [29, Lemma A.3].

(1) implies (3): Under assumption (1), $\mathcal{C}$ is thus a separable and Dedekind σ-complete Banach lattice. From [1, Corollary 4.52], we deduce that $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ is σ-order continuous on $\mathcal{C}$. Now, in view of Lemma 3.3 and [28, Proposition 1.3.7], σ-order continuity of the norm on $\mathcal{C}$ shows

$$\mathcal{C}^* = \mathbf{ca}(\mathcal{C}) = \mathbf{ca}^\sigma(\mathcal{C}).$$

In particular, each $\mathbb{P} \in \mathfrak{P}$ satisfies $\mathbb{P} \in \mathbf{ca}^\sigma_+(\mathcal{C})$. Lemma 3.8 implies $L^\infty(\mathfrak{P}) \subset \mathcal{C}$, which entails that, for all $X \in \mathcal{C}$, $|X|\mathbf{1}_{\{|X|\leq n\}} \uparrow |X|$ as $n \to \infty$, both in order and in norm. This proves $\mathcal{C} = \mathrm{cl}(L^\infty(\mathfrak{P}))$, which is (3).
(3) always implies (4).
(4) implies (1): This has been demonstrated already in the proof of Proposition 3.10.
(1)–(4) implies (5): Note that the equivalent assertions (1)–(4) have already been demonstrated to imply $\mathcal{C}^* = \mathbf{ca}(\mathcal{C}) \supset \mathbf{ca}(L^\Phi(\mathfrak{P})$. For the converse inclusion $\mathbf{ca}(\mathcal{C}) \subset \mathbf{ca}(L^\Phi(\mathfrak{P}))$, note that each $|\mu| \in \mathbf{ca}(\mathcal{C})$ extends to a continuous linear functional $\ell \in L^\Phi(\mathfrak{P})^*$. Arguing as in (B.1), we obtain

$$\ell(X) = |\mu|X, \quad \text{for all } X \in L^\Phi(\mathfrak{P}),$$

which means that $|\mu|$ (or equivalently, μ) lies in $\mathbf{ca}(L^\Phi(\mathfrak{P}))$. Finally, let $\mathbb{P}^* \in \mathbf{ca}\big(L^\Phi(\mathfrak{P})\big)$ as in Theorem 2.7 and let $A \in \mathcal{F}$. $\mathbf{1}_A \in \mathcal{C}$ is implied by (3), and it follows that $\mathbf{ca}(L^\Phi(\mathfrak{P})) \approx \mathfrak{P} \approx \mathbb{P}^*$.
In order to see that the densities of measures in the unit ball of $\mathbf{ca}(\mathcal{C})$ form a weakly compact subset of $L^1(\mathbb{P}^*)$, note that (B.1) admits the representation

$$\{\mu \in \mathbf{ca}(\mathcal{C}) \mid \|\mu\|_{\mathcal{C}^*} \leq 1\} = \{\mu \in \mathbf{ca}(\mathfrak{P}) \mid \forall\, X \in L^\infty(\mathbb{P}^*) :\ |\mu X| \leq \|X\|_{L^\Phi(\mathfrak{P})}\}.$$

The right-hand side is clearly weakly closed in $L^1(\mathbb{P}^*)$.
Now we consider a sequence $(A_n)_{n\in\mathbb{N}}$ such that

$$\mathbb{P}^*(A_n) \leq 2^{-n} \text{ and } \|\mathbf{1}_{A_n}\|_{L^\Phi(\mathfrak{P})} \geq \tfrac{1}{2}\sup\{\|\mathbf{1}_B\|_{L^\Phi(\mathfrak{P})} \mid B \in \mathcal{F},\, \mathbb{P}^*(B) \leq 2^{-n}\}.$$

Then, $\mathbf{1}_{A_n}$ is a sequence in $\mathcal{C}$ converging to 0 in order, and $\lim_{n\to\infty} \|\mathbf{1}_{A_n}\|_{L^\Phi(\mathfrak{P})} = 0$.[7] Set $\mathfrak{B}$ to be the set of all $\mu \in \mathbf{ca}_+(\mathcal{C})$ with $\|\mu\|_{\mathcal{C}^*} \leq 1$. We obtain

$$\sup\{\mu(B) \mid \mu \in \mathfrak{B},\, B \in \mathcal{F},\, \mathbb{P}^*(B) \leq 2^{-n}\} = \sup\{\|\mathbf{1}_B\|_{L^\Phi(\mathfrak{P})} \mid B \in \mathcal{F},\, \mathbb{P}^*(B) \leq 2^{-n}\} \to 0, \quad n \to \infty.$$

This shows that, for all $\varepsilon > 0$, there is $\delta > 0$ such that $\mathbb{P}^*(B) \leq \delta$ implies $\mu(B) \leq \varepsilon$, no matter the choice of $\mu \in \mathfrak{B}$. Moreover, $\mathfrak{B}$ is bounded in total variation. By [7, Theorem 4.7.25], $\mathfrak{B}$ and thus also the unit ball of $\mathbf{ca}(\mathcal{C})^*$ is weakly compact in $L^1(\mathbb{P}^*)$. This completes the verification of (5).
(5) implies (3): Under assumption (5), the unit ball of $\mathbf{ca}\big(L^\Phi(\mathfrak{P})\big)$, which is sufficient to determine $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ on all of $L^\Phi(\mathfrak{P})$, can be identified with a weakly compact subset $\mathcal{Z} \subset L^1(\mathbb{P}^*)$. Each $X \in L^\infty(\mathfrak{P}) = L^\infty(\mathbb{P}^*)$ can be identified with a (linear) continuous function on $\mathcal{Z}$, and if $L^\infty(\mathfrak{P}) \ni X_n \downarrow 0$ $\mathfrak{P}$-q.s. (or $\mathbb{P}^*$-a.s.), the associated sequence of functions converges pointwise to 0 on $\mathcal{Z}$. As this pointwise convergence must be uniform, σ-order continuity of $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ on $L^\infty(\mathfrak{P})$ follows.
We now observe

- For each $X \in \mathcal{C}$ and each $c \in \mathbb{R}$, the sequence $Y_k := k(X - c\mathbf{1}_\Omega)^+ \wedge \mathbf{1}_\Omega \in \mathcal{C} \cap L^\infty(\mathfrak{P})$, $k \in \mathbb{N}$, satisfies $Y_k \uparrow \mathbf{1}_{\{X>c\}}$ in $L^\infty(\mathfrak{P})$.
- For each increasing sequence $(A_n)_{n\in\mathbb{N}}$ of events in $\Lambda := \{A \in \mathcal{F} \mid \mathbf{1}_A \in \mathcal{C}\}$, $\mathbf{1}_{A_n} \uparrow \mathbf{1}_{\bigcup_{k\in\mathbb{N}} A_k}$ holds in $L^\infty(\mathfrak{P})$.

Arguing as in Lemma 3.8 and using σ-order continuity of the norm as well as closedness of $\mathcal{C}$ shows $\mathcal{C} \cap L^\infty(\mathfrak{P}) = L^\infty(\mathfrak{P})$, i.e., $\mathcal{M} := \mathrm{cl}(L^\infty(\mathfrak{P})) \subset \mathcal{C}$.

[7] More precisely, set $B_n := \bigcup_{k\geq n} A_k$, a decreasing sequence of events. As $\mathbb{P}^*(B_n) \downarrow 0$, $\mathbf{1}_{B_n} \downarrow 0$ holds w.r.t. the $\mathfrak{P}$-q.s. order in $\mathcal{C}$. It remains to note that $\mathbf{1}_{A_n} \preceq \mathbf{1}_{B_n}$, $n \in \mathbb{N}$.

Towards a contradiction, assume that we can find $X \in \mathcal{C} \setminus \mathcal{M}$. Then, there is a measure $0 \neq \mu \in \mathcal{C}^* = \mathbf{ca}(L^\Phi(\mathfrak{P}))$ such that

$$\mu|_{\mathcal{M}} \equiv 0 \quad \text{and} \quad \mu X \neq 0.$$

This however would mean $\mu|_{L^\infty(\mathfrak{P})} \equiv 0$, which is impossible. $\mathcal{C} \subset \mathcal{M}$ follows.
We have already proved (6) above.
For (7), assume that condition (3.1) holds. Then, there exist $a > 0$ and $b \leq 0$ such that $\phi_{\mathbb{P}}(x) \geq ax + b$ for all $\mathbb{P} \in \mathfrak{P}$ and $x \in [0, \infty)$. By (2.3),

$$\sup_{\mathbb{P} \in \mathfrak{P}} \|a(1-b)^{-1}\mathbb{P}\|_{L^\Phi(\mathfrak{P})^*} \leq 1. \tag{B.4}$$

The assertion follows with (5), and the proof is complete. □

Appendix C. Proofs of Section 4

Proof of Proposition 4.1. As each subset of a separable normed space is separable itself, we can w.l.o.g. consider the maximal case $\mathcal{H} = C_b$.
By Theorem 2.6, there exists some constant $\kappa > 0$ such that

$$\|X\|_{L^\Phi(\mathfrak{P})} \leq \kappa \|X\|_{L^\infty(\mathfrak{P})}, \quad \text{for all } X \in C_b.$$

Let d be a metric consistent with the topology on Ω, and $(\omega_n)_{n \in \mathbb{N}}$ dense in Ω. For $m, n \in \mathbb{N}$ and $\omega \in \Omega$, let $X_{m,n}(\omega) := d(\omega, \omega_n) \wedge m$. The algebra $\mathcal{A} \subset C_b$ generated by $\{\mathbf{1}_\Omega\} \cup \{(X_{m,n}) \mid m, n \in \mathbb{N}\}$ is separable and separates the points of each compact set $K \subset \Omega$. We show that the separable set

$$\mathcal{M} := \big\{(X_0 \wedge m\mathbf{1}_\Omega) \vee (-m\mathbf{1}_\Omega) \,\big|\, X_0 \in \mathcal{A},\, m \geq 0\big\}$$

is dense in $L^\Phi(\mathfrak{P})$. To this end, let $X \in C_b$, $\varepsilon > 0$, and $K \subset \Omega$ compact with

$$\|\mathbf{1}_{\Omega \setminus K}\|_{L^\Phi(\mathfrak{P})} < \frac{\varepsilon}{2(1 + 2\|X\|_\infty)}.$$

By the Stone-Weierstrass Theorem, there exists some $X_0 \in \mathcal{M}$ with $\|X_0\|_{L^\infty(\mathfrak{P})} \leq 1 + \|X\|_{L^\infty(\mathfrak{P})}$ and

$$\big\|(X - X_0)\mathbf{1}_K\big\|_{L^\infty(\mathfrak{P})} < \frac{\varepsilon}{2\kappa}.$$

Hence,

$$\begin{aligned} \|X - X_0\|_{L^\Phi(\mathfrak{P})} &\leq \big\|(X - X_0)\mathbf{1}_K\big\|_{L^\Phi(\mathfrak{P})} + \big\|(X - X_0)\mathbf{1}_{\Omega \setminus K}\big\|_{L^\Phi(\mathfrak{P})} \\ &\leq \kappa \big\|(X - X_0)\mathbf{1}_K\big\|_{L^\infty(\mathfrak{P})} + \big(\|X\|_{L^\infty(\mathfrak{P})} + \|X_0\|_{L^\infty(\mathfrak{P})}\big)\|\mathbf{1}_{\Omega \setminus K}\|_{L^\Phi(\mathfrak{P})} < \varepsilon. \end{aligned}$$

□

Proof of Lemma 4.2. (1) trivially implies (4.1).
Under condition (2), let $\varepsilon > 0$ and set $t := \varepsilon^{-1}$. Choose $K \subset \Omega$ compact with

$$\phi_{\mathrm{Max}}(t) \sup_{\mathbb{P} \in \mathfrak{P}_t} \mathbb{P}(\Omega \setminus K) \leq 1.$$

Then,

$$\sup_{\mathbb{P} \in \mathfrak{P}_t} \mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(t\mathbf{1}_{\Omega \setminus K})] \leq \sup_{\mathbb{P} \in \mathfrak{P}_t} \mathbb{E}_{\mathbb{P}}[\phi_{\mathrm{Max}}(t\mathbf{1}_{\Omega \setminus K})] = \phi_{\mathrm{Max}}(t) \sup_{\mathbb{P} \in \mathfrak{P}_t} \mathbb{P}(\Omega \setminus K) \leq 1.$$

Moreover,

$$\sup_{\mathbb{P} \in \mathfrak{P} \setminus \mathfrak{P}_t} \mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(t\mathbf{1}_{\Omega \setminus K})] \leq 1.$$

This entails $\|\mathbf{1}_{\Omega \setminus K}\|_{L^\Phi(\mathfrak{P})} \leq \varepsilon$.
Condition (3) is a special case of condition (2).

Suppose now that (4.1) holds and let $x_0 > 0$ be as in (3.1). By (B.4), $\mathfrak{P}$ is a bounded subset of $\mathcal{C}^*$, and we have

$$\sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_{\mathbb{P}}[|X|] \le \frac{1-b}{a}\|X\|_{L^\Phi(\mathfrak{P})} \quad \text{for all } X \in L^\Phi(\mathfrak{P}),$$

where $a > 0$ and $b \le 0$ are suitably chosen. Replacing X by $\mathbf{1}_{\Omega\setminus K}$ for suitable $K \subset \Omega$ compact immediately yields tightness of $\mathfrak{P}$. □

Proof of Corollary 4.5. Let $(X_n)_{n\in\mathbb{N}} \subset C_b$ with $X_n \downarrow 0$ as $n \to \infty$ and $\alpha > 0$. Then, $\phi_{\mathrm{Max}}(\alpha X_n) \in C_b$ for all $n \in \mathbb{N}$, and $\phi_{\mathrm{Max}}(\alpha X_n) \downarrow 0$ as $n \to \infty$. Since $\mathfrak{P}$ is weakly compact and the functions $\mathfrak{P} \ni \mathbb{P} \mapsto \mathbb{E}_{\mathbb{P}}[\Phi_{\mathrm{Max}}(\alpha X_n)]$ are weakly continuous, Dini's Theorem implies

$$\lim_{n\to\infty} \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(\alpha|X_n|)] = 0.$$

This suffices to conclude $\lim_{n\to\infty} \|X_n\|_{L^\Phi(\mathfrak{P})} = 0$. The assertion now follows from Proposition 3.5. □

Proof of Proposition 4.6. We first prove that $\mathcal{H}_X$ is a sublattice of $L^\Phi(\mathfrak{P})$. The latter space is Dedekind σ-complete and thus also uniformly complete ([28, Proposition 1.1.8]). As such, we may replicate the argument in the proof of [18, Theorem 3.1]. $\big(\mathcal{C}_X, \preceq, \|\cdot\|_{L^\Phi(\mathfrak{P})}\big)$ is a Banach lattice by Proposition 3.5. Now, the span of the countable set $\{\mathbf{1}_\Omega\} \cup \{(X - k\mathbf{1}_\Omega)^+ \mid k \text{ rational}\}$ over the rational numbers lies dense in $\mathcal{H}_X$, whence separability of $\mathcal{H}_X$ and its norm closure $\mathcal{C}_X$ follow. □

Proof of Corollary 4.7. Note that $J_{\mathbb{P}^*}(L^\Phi(\mathfrak{P})) \subset L^1(\mathbb{P}^*)$ is an ideal on which $\mathbb{P}^*$ acts as a strictly positive bounded linear functional. The assertion thus follows directly from [18, Corollary 3.2(b)]. □

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