FUNCTION SPACES UNDER MODEL UNCERTAINTY: ORDER AND AGGREGATION

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ABSTRACT. In robust finance, Knightian uncertainty is often captured by a non-dominated set of priors, probability measures on the future states of the world. This usually comes at the cost of losing tractability; in contrast to the dominated setting, advanced functional analytic tools are often not available anymore. Thus additional assumptions are made to gain a certain degree of tractability, in particular with respect to aggregation of families of consistent random variables. We investigate from a reverse perspective the implications of such assumptions.

Keywords: Robustness, quasi-sure analysis, (super) Dedekind complete function spaces, supports of measures, Hahn property

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1. Introduction

In the aftermath of the financial crisis of 2008, *Knightian uncertainty* has enjoyed increasing interest in the finance community; cf. [42, p. 45]. It describes situations in which the parameters determining the realisation of an economic outcome are ambiguous or simply impossible to be known. A fairly recent, by now established and widely studied mathematical approach to this phenomenon are *robust financial models* which do not assume the existence of a dominating probability measure on the future states of the world; see [11, 14, 16, 19, 37, 41] and the references therein. A well-known difficulty here is that the usual analytic tools which strongly rely on the Banach space and lattice properties of function spaces such as $L^\infty(\mathbb{P}) := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, are in general not available beyond a dominated framework. As Bouchard & Nutz remark in their seminal paper [14]:

> The main difficulty in our endeavor is that $\mathcal{P}$ [a class of probability measures] can be nondominated which leads to the failure of various tools of probability theory and functional analysis ... As a consequence, we have not been able to reach general results by using separation arguments in appropriate function spaces ...
> (Bouchard & Nutz [14], p. 824)

In other words, the higher the degree of Knightian uncertainty the less tractable the mathematical model tends to be. There are a number of *ad hoc* approaches to overcome these difficulties. For instance Cohen [18] requires a specific structure of the underlying measure space in relation to the occurring probabilities. It is referred to as the Hahn property and in particular posits the existence of pairwise disjoint supports of the probability measures. Another popular requirement is that the underlying set Ω of future states of the world has a product structure, which in turn admits dynamic programming and measurable selection arguments, see [6, 7, 12, 14, 15]. Yet another line of literature focuses on a particularly well-behaved set of states, the Wiener space, see [38, 39, 40, 41]. There are also some attempts to develop a robust counterpart to the theory of function spaces built over classical probability spaces, see for instance [19, 21, 33] and to some extent [11].

In contrast to the mentioned *ad hoc* approaches, this paper takes a reverse perspective on the problem. Assuming a set of desirable properties of the function spaces under ambiguity—for instance requiring the model to admit aggregation—we ask what implications these properties have on the underlying set of probability measures, and also which other characteristics of the robust function space follow. It has already been widely noted that the crucial issues are related to the so-called $\mathfrak{P}$-quasi-sure ($\mathfrak{P}$-q.s.) order on those function spaces. Here $\mathfrak{P}$ is a typically non-dominated set of probability measures on $(\Omega, \mathcal{F})$ which in the robust framework plays the role of the classically postulated single probability measure $\mathbb{P}$. Note that both $L^\infty(\mathbb{P})$ equipped with the usual $\mathbb{P}$-almost sure order and its robust counterpart $L^\infty(\mathfrak{P}) := L^\infty(\Omega, \mathcal{F}, \mathfrak{P})$ equipped with the $\mathfrak{P}$-q.s. order are Banach lattices. But apart from that, the dominated and the non-dominated case differ fundamentally because the $\mathfrak{P}$-q.s. order *interpolates* between the structure of almost sure and pointwise orders, respectively.

Our studies will revolve around the question of *aggregation* which is also in the focus of [18, 41]. Aggregation refers to the problem of aggregating a compatible family of random variables $X^{\mathbb{P}}$, $\mathbb{P} \in \mathfrak{P}$, to a single random variable X, see Definition 4.27. The other mentioned line of literature in the framework of [14] does not treat feasibility of general aggregations as an important structural property of a model. It is therefore only marginally covered by our investigation. We will demonstrate that aggregation is closely related to Dedekind completeness of $L^\infty(\mathfrak{P})$, a property we will thoroughly study throughout this paper.

We begin by studying Dedekind complete function spaces with the additional countable sup property, so-called super Dedekind complete spaces. Those are known to be structurally akin to $L^\infty(\mathbb{P})$ equipped with the $\mathbb{P}$-almost sure order, which is in particular super Dedekind complete. Hence, asking for super Dedekind completeness would be the most obvious attempt to generalise the case of a single dominating prior. However, we will prove that super Dedekind completeness of $L^\infty(\mathfrak{P})$ is in fact equivalent to $\mathfrak{P}$ being dominated. Hence, robust models which do not allow for a dominating measure cannot be super Dedekind complete but only Dedekind complete if aggregation is assumed to be possible, and these spaces pose new analytic difficulties. The characterisation of super Dedekind completeness is based on the observation that under this requirement probability measures are *supported*, not necessarily in a statewise or topological, but in an order sense; see Definition 4.1. Indeed the supports we construct are only unique up to $\mathfrak{P}$-polar sets, i.e. events $A \in \mathcal{F}$ such that $\sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{P}(A) = 0$. Another property stronger than Dedekind completeness, which has appeared in the literature, is that the dual space $\mathbf{ca}(\mathfrak{P})^*$ of the space of finite signed measures $\mathbf{ca}(\mathfrak{P})$ which are dominated by the set of priors $\mathfrak{P}$, may be identified with $L^\infty(\mathfrak{P})$, see [21, 33]. This admits the application of important techniques which rely on the Krein-Šmulian Theorem. Among other things we prove that $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ is equivalent to Dedekind completeness of $L^\infty(\mathfrak{P})$ in conjunction with supportability of every measure in $\mathbf{ca}(\mathfrak{P})$ in the order sense already mentioned. Hence, supportability again comes into play, and the conjecture seems not far-fetched that this is a consequence of asking for a certain degree of tractability.

Instead of demanding that every measure in $\mathbf{ca}(\mathfrak{P})$ is supportable, as the conditions mentioned above imply, a natural weaker condition is to consider what we call the class (S) setting. The set $\mathfrak{P}$ of priors is *of class* (S) if there is a set $\mathfrak{Q}$ of probability measures over the same measurable space as $\mathfrak{P}$ with $\mathfrak{Q} \approx \mathfrak{P}$, i.e. the $\mathfrak{P}$-q.s. and $\mathfrak{Q}$-q.s. orders coincide, and such that every element of $\mathfrak{Q}$ admits an order support. This set $\mathfrak{Q}$ is called *supportable alternative* to $\mathfrak{P}$. We will see that the prominent models in [18] and [41]—which will serve as benchmarks throughout our study—all fall under class (S). In fact, the simultaneity of the class (S) property of $\mathfrak{P}$ and Dedekind completeness of $L^\infty(\mathfrak{P})$ plays a central role throughout the paper.

Figure 1 illustrates some of our major findings. Th. XX, Lem. XX, and Rem. XX abbreviate Theorem XX, Lemma XX, or Remark XX, respectively, of this paper proving the stated relation. For the sake of simplicity, the graph outlines the results for the function space $L^\infty(\mathfrak{P})$. Many implications hold true though for any Banach lattice $\mathcal{X}$ of q.s. equivalence classes of random variables with $L^\infty(\mathfrak{P}) \subset \mathcal{X}$.

FIGURE 1: THE PAPER IN A NUTSHELL
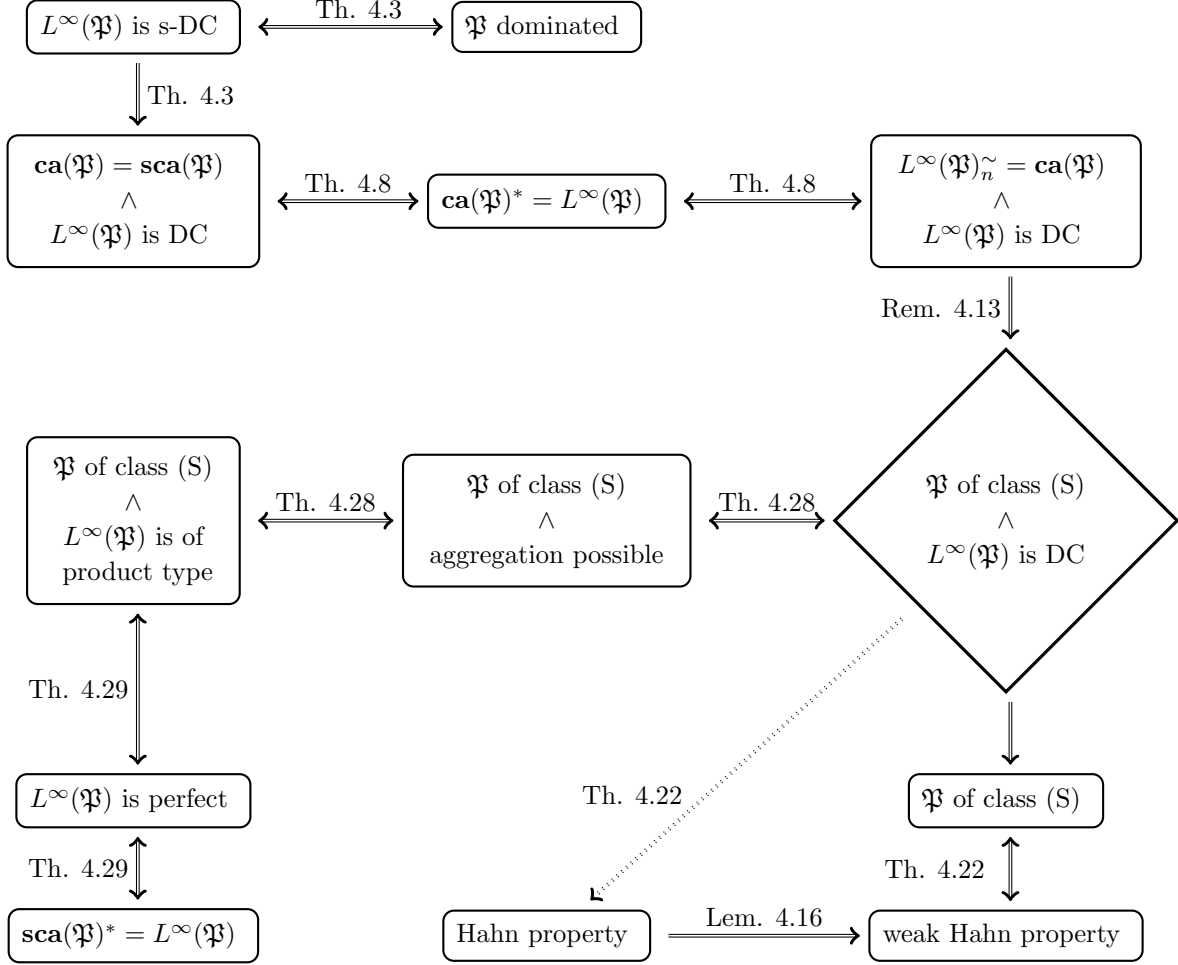
Legend:
s-DC = super Dedekind complete
DC = Dedekind complete

The upper branch of the graph concerns the already mentioned properties such as (super) Dedekind completeness in combination with a nice structure of order continuous functionals. Here $\mathbf{sca}(\mathfrak{P}) \subset \mathbf{ca}(\mathfrak{P})$ is the set of signed measures which are supportable in the order sense. $L^\infty(\mathfrak{P})_n^\sim$ denotes the order continuous dual of $L^\infty(\mathfrak{P})$. In any case $\mathfrak{P}$ must necessarily be of class (S) and $L^\infty(\mathfrak{P})$ is Dedekind complete.

Under the assumption that $\mathfrak{P}$ is of class (S), the left branch depicts that the existence of an aggregator of compatible local random variables given for each $\mathbb{Q} \in \mathfrak{Q}$ in the supportable alternative $\mathfrak{Q}$ to $\mathfrak{P}$ is indeed equivalent to Dedekind completeness of $L^\infty(\mathfrak{P})$. It is also equivalent to $L^\infty(\mathfrak{P})$ having a product structure which verifies the intuition that aggregation requires a product structure. Other equivalents here are the identity $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ and perfectness of $L^\infty(\mathfrak{P})$, i.e. $L^\infty(\mathfrak{P})$ may be identified with the order continuous dual of its order continuous dual: $(L^\infty(\mathfrak{P})_n^\sim)_n^\sim = L^\infty(\mathfrak{P})$. In both cases the class (S) property follows. As regards $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$, recall the condition $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ from the upper branch which is equivalent to Dedekind completeness and $\mathbf{ca}(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})$. Replacing the requirement $\mathbf{ca}(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})$ by class (S) thus yields the condition $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$. The latter is weaker than $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$. The question how much weaker it actually is, however, illustrates some difficulties which we will comment on below.

The last line of results quoted in Figure 1 collected in the lower right branch compare the class (S) property to the Hahn property required in [18]. The main feature of the Hahn property is asking for a supportable alternative $\mathfrak{Q}$ of $\mathfrak{P}$ with order supports which are pairwise disjoint as sets. The intuition suggests and [18] proves that under the Hahn property robust function spaces are well behaved and admit aggregation. At first glance the Hahn property is a much stronger requirement than $\mathfrak{P}$ being of class (S). The question is how much stronger this requirement is. Surprisingly we find that being of class (S) is in fact equivalent to a weak version of the Hahn property which asserts the existence of a supportable alternative $\mathfrak{Q}$ to $\mathfrak{P}$ such that the order supports are pairwise disjoint in an order sense. The weak Hahn property is of course implied by the Hahn property. Moreover, if $L^\infty(\mathfrak{P})$ is Dedekind complete, $\mathfrak{P}$ is of class (S), and there is a supportable alternative $\mathfrak{Q}$ at most equinumerous with the continuum, then in fact the Hahn property holds. This is represented by the dotted implication in Figure 1. These results are surprising as they show that a mild regularity condition like the class (S) property immediately implies versions of the Hahn property and even the Hahn property itself—under the cardinality constraint—whenever aggregation works.

There are other interesting results in this paper which are not suited to be quoted in Figure 1. For instance, we study the problem of order completing $L^\infty(\mathfrak{P})$ in case it is not Dedekind complete, and whether this so-called Dedekind completion has an interpretation as a function space. Also there is line of results which refers to another difficulty in the non-dominated framework: Many researchers in the field have noticed that the non-dominated framework to some extent tests the boundaries of ordinary mathematics within **ZFC**. This often manifests itself in situations when additional requirements are sufficient to prove certain properties of the robust function spaces, but it is not clear whether they are also necessary since counterexamples showing their necessity are not available. As an illustration, recall the conditions $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ and $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ from above. We know that $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ is weaker than $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$. Of course we would like to provide an example in which $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ but $\mathbf{ca}(\mathfrak{P})^* \neq L^\infty(\mathfrak{P})$, or equivalently, $L^\infty(\mathfrak{P})$ is Dedekind complete and $\mathfrak{P}$ is of class (S), but $\mathbf{sca}(\mathfrak{P}) \neq \mathbf{ca}(\mathfrak{P})$, so aggregation is possible even though not every

measure is supportable. In a similar vein, it would be desirable to provide an example of a measurable space $(\Omega, \mathcal{F})$ and a set $\mathfrak{P}$ of probability measures such that $L^\infty(\mathfrak{P})$ is Dedekind complete, but $\mathfrak{P}$ is *not* of class (S). We prove in Corollary 4.33 that it is impossible to give such examples. Indeed, we show that the equivalence

$$\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P}) \quad \Longleftrightarrow \quad \mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P}) \tag{1.1}$$

would hold if and only if there are no solutions to Banach's measure problem. In that case, Dedekind completeness of $L^\infty(\mathfrak{P})$ would imply both the class (S) property of $\mathfrak{P}$ and the supportability of all measures, i.e. $\mathbf{ca}(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})$; see Corollary 4.32. Banach's measure problem has been studied for generations, see [23, Chapter 10], and one cannot prove the existence of solutions in **ZFC**. Constructing any of the desired examples in **ZFC**, however, would yield such an existence proof. Such examples are therefore impossible in the framework of classical mathematics. In addition, adding the continuum hypothesis, equivalence (1.1) can be shown to hold provided there is a supportable alternative $\mathfrak{Q}$ at most equinumerous with the continuum. The latter is the case in many applications. All these results suggest that there is no alternative from a practical perspective to sets of priors which are of class (S) when aggregation in robust function spaces is assumed to be possible.

The paper is organised as follows: Section 2 collects preliminaries on vector lattices and measures relevant throughout the text. The reader firmly acquainted with these concepts may prefer to skip this section. In Section 3 we introduce the concept of quasi-sure orders with respect to a set $\mathfrak{P}$ of probability measures. All main results are collected and discussed in Section 4, their proofs, however, are delayed to Section 6. Section 5 prepares them by presenting more results on order supports which are of independent interest. Minor technical lemmas are outsourced to Appendix A.

2. Vector lattices and spaces of measures

We begin with some preliminaries on vector lattices and spaces of measures. The readers acquainted with the theory as presented in e.g. Aliprantis & Border [1] or Aliprantis & Burkinshaw [2, 3] may prefer to skip this section.

A tuple $(\mathcal{X}, \preceq)$ is a VECTOR LATTICE if $\mathcal{X}$ is a real vector space and $\preceq$ is a partial order on $\mathcal{X}$ with the following properties:

- For $x, y, z \in \mathcal{X}$ and scalars $\alpha \geq 0$, $x \preceq y$ implies $\alpha x + z \preceq \alpha y + z$.
- For all $x, y \in \mathcal{X}$ there is a least upper bound $z := x \vee y = \sup\{x, y\}$, the *maximum* of x and y, which satisfies $x \preceq z$ and $y \preceq z$ as well as $z \preceq z'$ whenever $x \preceq z'$ and $y \preceq z'$ hold.

The existence of the modulus $|x| = x \vee (-x)$, the positive part $x^+ = x \vee 0$, the negative part $x^- = (-x) \vee 0$, and the minimum $x \wedge y := \inf\{x, y\} = -\sup\{-x, -y\}$ follow. The POSITIVE CONE $\mathcal{X}_+$ is the set of all $x \in \mathcal{X}$ such that $x \succeq 0$, and $\mathcal{X}_{++} := \mathcal{X}_+ \backslash \{0\} = \{x \in \mathcal{X} \mid x \succ 0\}$. If $\mathcal{X}$ is additionally normed with a complete norm $\|\cdot\|$ which satisfies $\|x\| \leq \|y\|$ whenever $|x| \preceq |y|$, $(\mathcal{X}, \|\cdot\|, \preceq)$ is a BANACH LATTICE.

A subset $\mathcal{C} \subset \mathcal{X}$ is called ORDER BOUNDED FROM ABOVE if there is a $y \in \mathcal{X}$ such that

$$\forall\, x \in \mathcal{C}: \; x \preceq y. \tag{2.1}$$

It has a SUPREMUM if it is order bounded from above and there is a least upper bound; more precisely, there is a vector u which satisfies (2.1) as well as $u \preceq y$ for all $y \in \mathcal{X}$ satisfying (2.1). This u will be denoted by $\sup \mathcal{C}$. $\mathcal{C}$ being order bounded from below is defined in exact analogy. In that case, the INFIMUM $\inf \mathcal{C}$ is defined as $\inf \mathcal{C} := -\sup(-\mathcal{C})$, provided it exists.
Linear functionals: A linear functional $\phi : \mathcal{X} \to \mathbb{R}$ is ORDER BOUNDED if for all $x, y \in \mathcal{X}$, the set $\{\phi(z) \mid x \preceq z \preceq y\} \subset \mathbb{R}$ is bounded. $\mathcal{X}^{\sim}$ denotes the vector lattice of all order bounded linear functionals and is referred to as the ORDER DUAL of $\mathcal{X}$. Its order is given by the positive cone $\mathcal{X}^{\sim}_+$ of $\mathcal{X}^{\sim}$, and $\phi \in \mathcal{X}^{\sim}_+$ if and only if $\phi(x) \geq 0$ for all $x \in \mathcal{X}_+$. Moreover, if $\phi, \psi \in \mathcal{X}^{\sim}$ and $x \in \mathcal{X}_+$,

$$(\phi \vee \psi)(x) := \sup\{\phi(y) + \psi(z) \mid y, z \in \mathcal{X}_+, \; y + z = x\},$$
$$(\phi \wedge \psi)(x) := \inf\{\phi(y) + \psi(z) \mid y, z \in \mathcal{X}_+, \; y + z = x\}.$$

A net $(x_\alpha)_{\alpha \in I} \subset \mathcal{X}$ is order convergent to $x \in \mathcal{X}$ if there is another net $(y_\alpha)_{\alpha \in I}$ which is *decreasing* ($\alpha, \beta \in I$ and $\alpha \leq \beta$ implies $y_\beta \preceq y_\alpha$), satisfies $\inf_{\alpha \in I} y_\alpha := \inf\{y_\alpha \mid \alpha \in I\} = 0$, and for all $\alpha \in I$ it holds that $0 \preceq |x_\alpha - x| \preceq y_\alpha$. The ORDER CONTINUOUS DUAL is the space $\mathcal{X}^{\sim}_n \subset \mathcal{X}^{\sim}$ of all order bounded linear functionals ϕ which are order continuous, i.e. ϕ carries an order convergent net with limit x in $\mathcal{X}$ to a net converging to $\phi(x)$ in $\mathbb{R}$. The σ-ORDER CONTINUOUS DUAL is the space $\mathcal{X}^{\sim}_c \subset \mathcal{X}^{\sim}$ of all order bounded linear functionals ϕ which carry an order convergent sequence with limit x in $\mathcal{X}$ to a convergent sequence with limit $\phi(x)$ in $\mathbb{R}$. Obviously, $\mathcal{X}^{\sim}_n \subset \mathcal{X}^{\sim}_c$.
The order continuous dual is a vector lattice in its own right. Moreover, for $x \in \mathcal{X}$ fixed, the linear functional

$$\ell_x : \mathcal{X}^{\sim}_n \ni \phi \mapsto \phi(x) \tag{2.2}$$

is order continuous on $\mathcal{X}^{\sim}_n$. A vector lattice is called PERFECT if $(\mathcal{X}^{\sim}_n)^{\sim}_n$ may be canonically identified with $\mathcal{X}$, i.e. the map $\mathcal{X} \ni x \mapsto \ell_x$ mapping $\mathcal{X}$ to $(\mathcal{X}^{\sim}_n)^{\sim}_n$ is one-to-one and onto.
Vector sublattices, ideals and bands: Given a vector lattice $\mathcal{X}$, a subspace $\mathcal{Y} \subset \mathcal{X}$ is a VECTOR SUBLATTICE if for every $x, y \in \mathcal{Y}$, the maximum $x \vee y$ computed in $\mathcal{X}$ lies in $\mathcal{Y}$. It is ORDER DENSE in $\mathcal{X}$ if for all $0 \prec x \in \mathcal{X}$ we can find some $y \in \mathcal{Y}$ such that $0 \prec y \preceq x$. It is MAJORISING if for every $x \in \mathcal{X}$ there is $y \in \mathcal{Y}$ such that $x \preceq y$.
A vector subspace $\mathcal{B}$ of $\mathcal{X}$ with the property that $\{x \in \mathcal{X} \mid |x| \preceq |y|\} \subset \mathcal{B}$ for all $y \in \mathcal{B}$ is an IDEAL. Every ideal is a vector sublattice. An ideal is a BAND if it is order closed, i.e. the order limit of each net $(x_\alpha)_{\alpha \in I} \subset \mathcal{B}$ order converging in $\mathcal{X}$ lies in $\mathcal{B}$. The DISJOINT COMPLEMENT of an ideal $\mathcal{B}$ is defined by

$$\mathcal{B}^{\mathrm{d}} := \{x \in \mathcal{X} \mid \forall\, y \in \mathcal{B}: \; |x| \wedge |y| = 0\}.$$

It is always a band. Given $\phi \in \mathcal{X}^{\sim}$, its NULL IDEAL is the ideal $N(\phi) := \{x \in \mathcal{X} \mid |\phi|(|x|) = 0\}$, and its CARRIER is $C(\phi) := N(\phi)^{\mathrm{d}}$.
Order completeness properties: The vector lattice $(\mathcal{X}, \preceq)$ is DEDEKIND COMPLETE (or order complete) if *every* order bounded from above subset $\mathcal{C} \subset \mathcal{X}$ has a supremum. The

order dual $\mathcal{X}^{\sim}$, for instance, is always Dedekind complete. We say that $(\mathcal{X}, \preceq)$ has the COUNTABLE SUP PROPERTY if every non-empty order bounded subset $\mathcal{C} \subset \mathcal{X}$ which has a supremum admits the selection of a countable subset $\mathcal{D} \subset \mathcal{C}$ such that $\sup \mathcal{D} = \sup \mathcal{C}$. A Dedekind complete vector lattice with the countable sup property is called SUPER DEDEKIND COMPLETE (or super order complete).

We call a Dedekind complete vector lattice $(\mathcal{Y}, \unlhd)$ the DEDEKIND COMPLETION of $(\mathcal{X}, \preceq)$ if there is an order dense and majorising vector sublattice $\mathcal{L} \subset \mathcal{Y}$ lattice isomorphic to $\mathcal{X}$. A LATTICE ISOMORPHISM $J : \mathcal{X} \to \mathcal{L}$ is a linear bijection such that $x \wedge y = 0$ in $\mathcal{X}$ implies $J(x) \wedge J(y) = 0$ in $\mathcal{L}$.[1] $\mathcal{X}$ has a Dedekind completion if and only if $\mathcal{X}$ is Archimedean, i.e. $\frac{1}{n} x \xrightarrow{o} 0$, $n \to \infty$, for every $x \in \mathcal{X}_+$.

We now turn our attention to spaces of set functions over a fixed measurable space $(\Omega, \mathcal{F})$. By **ba** we denote the real vector space of all additive set functions $\mu : \mathcal{F} \to \mathbb{R}$ with bounded total variation, i.e. the quantity $TV(\mu)$ defined as the supremum of all values $\sum_{A \in \Pi} |\mu(A)|$, Π running through all finite measurable partitions of Ω, is finite. **ba** is a vector lattice when endowed with the setwise order: for $\mu, \nu \in \mathbf{ba}$, $\mu \preceq_{\mathcal{F}} \nu$ holds if, for all $A \in \mathcal{F}$, $\mu(A) \leq \nu(A)$. The triple $(\mathbf{ba}, TV, \preceq_{\mathcal{F}})$ is in fact a Banach lattice.

Given non-empty sets $\mathfrak{Q} \subset \mathbf{ba}$ and $\mathfrak{P} \subset \mathbf{ba}_+$, we say $\mathfrak{P}$ DOMINATES $\mathfrak{Q}$ ($\mathfrak{Q} \ll \mathfrak{P}$) if for all $N \in \mathcal{F}$ satisfying $\sup_{\nu \in \mathfrak{P}} \nu(N) = 0$ we have $\sup_{\mu \in \mathfrak{Q}} |\mu|(N) = 0$. Here and in the following, $|\mu| \in \mathbf{ba}_+$ denotes the modulus of μ with respect to $\preceq_{\mathcal{F}}$,

$$|\mu|(A) = \sup\{\mu(B) - \mu(A \backslash B) \mid B \in \mathcal{F}, \ B \subset A\}.$$

For the sake of brevity, we shall write $\mathfrak{Q} \ll \nu$ instead of $\mathfrak{Q} \ll \{\nu\}$ and $\mu \ll \mathfrak{P}$ instead of $\{\mu\} \ll \mathfrak{P}$. Moreover, given a non-empty set $\{0\} \neq \mathfrak{P} \subset \mathbf{ba}_+$, we set $\mathbf{ba}(\mathfrak{P}) := \{\mu \in \mathbf{ba} \mid \mu \ll \mathfrak{P}\}$. The real vector space of all countably additive signed measures, **ca**, that is, the space of all $\mu \in \mathbf{ba}$ such that, additionally, for any sequence $(A_i)_{i \in \mathbb{N}} \subset \mathcal{F}$ of pairwise disjoint events we have

$$\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i),$$

is a TV-closed ideal within **ba**. The set of all probability measures on $(\Omega, \mathcal{F})$ is denoted by $\Delta(\mathcal{F}) := \{\mu \in \mathbf{ca}_+ \mid \mu(\Omega) = 1\}$.

For $\{0\} \neq \mathfrak{P} \subset \mathbf{ba}_+$, the space $\mathbf{ca}(\mathfrak{P}) := \mathbf{ca} \cap \mathbf{ba}(\mathfrak{P})$ is a TV-closed ideal within **ba** as well. Hence, $(\mathbf{ca}(\mathfrak{P}), TV, \preceq_{\mathcal{F}})$ is a Banach lattice in its own right. Finally, the set of MEASURES on $(\Omega, \mathcal{F})$ absolutely continuous with respect to $\mathfrak{P}$ is $\mathbf{ca}(\mathfrak{P})_+$.

For $\mu \in \mathbf{ca}_+$ the set of μ-negligible sets is

$$\mathfrak{n}(\mu) := \{A \subset \Omega \mid \exists N \in \mathcal{F} : \ \mu(N) = 0, \ A \subset N\}.$$

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[1] The term "lattice isomorphism" is ambiguous in the literature. [2, Definition 1.30], for instance, replaces our assumption of bijectivity by mere injectivity. By [2, p. 16], however, two vector lattices are *lattice isomorphic* if there is a surjective lattice isomorphism between them. It is therefore worth pointing out that the results from [2] we use deal with lattice isomorphisms which are bijective.

The μ-completion of $\mathcal{F}$ is the σ-algebra $\sigma(\mathcal{F} \cup \mathfrak{n}(\mu))$. Given a subset $\Phi \subset \mathbf{ca}_+$ of countably additive finite measures, we define the Φ-COMPLETION of $\mathcal{F}$ to be the σ-algebra

$$\mathcal{F}(\Phi) := \bigcap_{\mu \in \Phi} \sigma(\mathcal{F} \cup \mathfrak{n}(\mu)).$$

A set $A \subset \Omega$ belongs to $\mathcal{F}(\Phi)$ if and only if for all $\mu \in \Phi$ there is a potentially μ-dependent $B \in \mathcal{F}$ such that $A \triangle B := (A \backslash B) \cup (B \backslash A) \in \mathfrak{n}(\mu)$. Note that each $\mu \in \Phi$ extends uniquely to a finite measure $\mu^\sharp$ on the σ-algebra $\mathcal{F}(\Phi)$, where $\mu^\sharp(A) = \mu(B)$ whenever $A \in \mathcal{F}(\Phi)$ and $B \in \mathcal{F}$ are such that $A \triangle B \in \mathfrak{n}(\mu)$.

3. Quasi-sure orders

Throughout the paper $(\Omega, \mathcal{F})$ denotes an arbitrary measurable space, and the letters $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{R}$ are used to denote non-empty sets of probability measures on $(\Omega, \mathcal{F})$.

Let $\emptyset \neq \mathfrak{P} \subset \Delta(\mathcal{F})$ be a non-empty set of probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and consider the real vector space $\mathcal{L}^0 := \mathcal{L}^0(\Omega, \mathcal{F})$ of all real-valued random variables $f : \Omega \to \mathbb{R}$. One easily sees that the $\mathfrak{P}$-QUASI-SURE ORDER ($\mathfrak{P}$-q.s. order)

$$f \preceq_{\mathfrak{P}} g \quad :\Longleftrightarrow \quad \sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{P}(f > g) = 0$$

is a vector space preorder on $\mathcal{L}^0$. Let $\sim_{\mathfrak{P}}$ denote the symmetric part of this binary relation, i.e. $f \sim_{\mathfrak{P}} g$ if $f \preceq_{\mathfrak{P}} g$ and $g \preceq_{\mathfrak{P}} f$. We usually say that $f = g$ $\mathfrak{P}$-quasi surely ($\mathfrak{P}$-q.s.) instead of $f \sim_{\mathfrak{P}} g$, the latter being equivalent to

$$\inf_{\mathbb{P} \in \mathfrak{P}} \mathbb{P}(f = g) = 1.$$

$\sim_{\mathfrak{P}}$ defines an equivalence relation, and $\preceq_{\mathfrak{P}}$ defines a vector space order on the space $L^0(\mathfrak{P}) := \mathcal{L}^0(\Omega, \mathcal{F}) / \sim_{\mathfrak{P}}$ of equivalence classes of all real-valued random variables on $(\Omega, \mathcal{F})$ up to $\mathfrak{P}$-q.s. equality in a canonical manner. The elements $f : \Omega \to \mathbb{R}$ in the equivalence class $X \in L^0(\mathfrak{P})$ are called REPRESENTATIVES and are denoted by $f \in X$. Conversely, each measurable function f induces an equivalence class which we will sometimes denote by $[f] \in L^0(\mathfrak{P})$. In order to facilitate notation we suppress the dependence of $\preceq_{\mathfrak{P}}$ on $\mathfrak{P}$ in the following and write $\preceq$ instead. $(L^0(\mathfrak{P}), \preceq)$ is a vector lattice, and for $X, Y \in L^0(\mathfrak{P})$, $f \in X$, and $g \in Y$, the minimum $X \wedge Y$ is the equivalence class $[f \wedge g]$ generated by the pointwise minimum $f \wedge g$, whereas the maximum $X \vee Y$ is the equivalence class $[f \vee g]$ generated by the pointwise maximum $f \vee g$. We will follow common practice and often not distinguish between equivalence classes $X \in L^0(\mathfrak{P})$ and their representatives $f \in X$. Sometimes, however, in order to avoid confusion, we will make this distinction, in particular in situations when different sets of underlying measures imply different equivalence classes. As a case in point, for an event $A \in \mathcal{F}$ and depending on the context, $\mathbf{1}_A$ can denote a representative and an equivalence class in $L^0(\mathfrak{P})$, respectively.

An important subspace of $L^0(\mathfrak{P})$ which we will study thoroughly is the space $L^\infty(\mathfrak{P})$ of equivalence classes of $\mathfrak{P}$-q.s. bounded random variables, i.e.

$$L^\infty(\mathfrak{P}) := \{X \in L^0(\mathfrak{P}) \mid \exists\, m > 0 : \ |X| \preceq m\mathbf{1}_\Omega\}.$$

It is an ideal within $L^0(\mathfrak{P})$ and even a Banach lattice when endowed with the norm

$$\|X\|_{\infty,\mathfrak{P}} := \inf\{m > 0 \mid |X| \preceq m\mathbf{1}_\Omega\}, \quad X \in L^\infty(\mathfrak{P}).$$

If $\mathfrak{P} = \{\mathbb{P}\}$ is given by a singleton, we write $L^0(\mathbb{P})$ and $L^\infty(\mathbb{P})$ instead of $L^0(\{\mathbb{P}\})$ and $L^\infty(\{\mathbb{P}\})$. Also, the quasi-sure order in this case is as usual called *almost sure* order, and properties hold $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.). The spaces $L^0(\mu)$ and $L^\infty(\mu)$ for general measures $\mu \in \mathbf{ca}_{++} = \mathbf{ca}_+\backslash\{0\}$ are defined analogously.

Sometimes we will not suppress the dependence of $L^\infty(\mathfrak{P})$ on the underlying measurable space and write $L^\infty(\Omega, \mathcal{F}, \mathfrak{P})$ instead of $L^\infty(\mathfrak{P})$.

Given $\mu \in \mathbf{ca}(\mathfrak{P})_{++}$ and $X \in L^0(\mathfrak{P})$, we set $j_\mu(X)$ to be the equivalence class in $L^0(\mu)$ generated by any representative $f \in X$. By the assumption of absolute continuity, this defines a function $j_\mu : L^0(\mathfrak{P}) \to L^0(\mu)$ which we shall use frequently. Note that $j_\mu(L^\infty(\mathfrak{P})) = L^\infty(\mu)$. At last, for $\emptyset \neq \mathcal{C} \subset L^0(\mathfrak{P})$ and $A \in \mathcal{F}$, we write

$$\mathbf{1}_A\mathcal{C} := \{\mathbf{1}_A X \mid X \in \mathcal{C}\}.$$

The following Example 3.1 illustrates a situation captured by the setting introduced above. It is of particular relevance for financial applications. We will get back to it in Section 4.2.

Example 3.1 (Volatility uncertainty). In continuous time models one of the most relevant sources of uncertainty is related to the estimation of the volatility of price processes. A flourishing branch of literature, which exceeds the original Black & Scholes approach, addresses this uncertainty in local volatility models or stochastic volatility models. The pioneering work of Avellaneda et al. [4] explains the drawback of choosing a specific probabilistic model:

> [O]ption prices reflect the market's expectation about the future value of the underlying asset as well as its projection of future volatility. Since this projection changes as the market reacts to new information, implied volatility fluctuates unpredictably. In these circumstances, fair option values and perfectly replicating hedges cannot be determined with certainty. The existence of so-called "volatility risk" in option trading is a concrete manifestation of market incompleteness.

We therefore consider the robust framework discussed in, for instance, [9, 18, 19, 41]. Let $\mathbb{P}_0$ be the Wiener measure on the Wiener space Ω of continuous functions $\omega : \mathbb{R}_+ \to \mathbb{R}$ with $\omega(0) = 0$. Let $B := (B_t)_{t\geq 0}$ be the canonical process, i.e. $B_t(\omega) = \omega(t)$, $t \in \mathbb{R}_+$. The process B turns out to be a standard Brownian motion under $\mathbb{P}_0$ with respect to the natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0} := (\sigma(B_s \mid 0 \leq s \leq t))_{t\geq 0}$. Ω is assumed to be endowed with the Borel-σ-algebra $\mathcal{F}$ generated by the usual metric $d : (\omega, \tilde{\omega}) \mapsto \sum_{T\in\mathbb{N}} 2^{-T} \min\{\max_{0\leq t\leq T} |\omega(t) - \tilde{\omega}(t)|, 1\}$.

However, we are interested in a situation where the canonical process can be observed and the modelling hypothesis is that its paths are a stochastic integral with respect to *a* Brownian motion. What the "right" integrand and Brownian motions are is subject to uncertainty

though. An immediate consequence of the modelling hypothesis is that the canonical process is a local martingale under the correct probabilistic point of view. In order to capture the situation with a set of probability measures, recall from Karandikar [25] that there is an $\mathbb{F}$-adapted process $\langle B \rangle$ such that under each probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ with respect to which B is a local martingale, $\langle B \rangle$ agrees with the usual $\mathbb{P}$-quadratic variation of B $\mathbb{P}$-a.s. Consider the *density process*

$$\langle B \rangle_t'(\omega) := \limsup_{\varepsilon \downarrow 0} \frac{\langle B \rangle_t - \langle B \rangle_{t-\varepsilon}}{\varepsilon}, \quad (t, \omega) \in (0, \infty) \times \Omega,$$

which allows us to capture the uncertainty in the set $\mathfrak{P}^{obs}$ of all probability measures $\mathbb{P}$ under which the canonical process is a local martingale and for which $\mathbb{P}$-a.s. $\langle B \rangle$. is absolutely continuous in t and takes positive values.

This point of view of the situation, however, does not immediately allow to describe the uncertainty we face in terms of the integrand process. To see this, consider the set $\mathcal{A}$ of processes $\sigma : \mathbb{R}_+ \times \Omega \to (0, \infty)$ which are $\mathbb{F}$-progessively measurable and satisfy $\int_0^T \sigma_s(\omega)^2 ds < \infty$ for all $T > 0$ and for each $\omega \in \Omega$. For a given $\mathbb{P} \in \mathfrak{P}^{obs}$ we could filter out the compatible processes by

$$\mathcal{A}(\mathbb{P}) := \{ \sigma \in \mathcal{A} \mid \mathbb{P}\text{-a.s.} : \ \sigma^2 = \langle B \rangle' \}.$$

The set of all observation compatible processes is then defined by

$$\mathcal{A}^{obs} := \bigcup_{\mathbb{P} \in \mathfrak{P}^{obs}} \mathcal{A}(\mathbb{P})$$

and satisfies $\mathcal{A}^{obs} \subsetneq \mathcal{A}$; cf. [41, p. 1852]. However, a given process $\sigma \in \mathcal{A}^{obs}$ might belong to more than one $\mathcal{A}(\mathbb{P})$. It might hence be impossible to infer probability laws $\mathbb{P} \in \mathcal{A}^{obs}$ from the associated volatility process $\sigma \in \mathcal{A}^{obs}$, thereby establishing a one-tow-one correspondence between processes and probability measures.

The aim is to restrict uncertainty to situations in which there is uncertainty about the *volatility process* σ. σ is then assumed to be chosen from a non-empty set $\mathcal{V} \subset \mathcal{A}$ such that the stochastic differential equation under the Wiener measure $\mathbb{P}_0$

$$dX_t = \sigma_t(X)dB_t \tag{3.1}$$

has *weak uniqueness* in the sense of [41, Definition 4.1]. Given σ, this admits the selection of a unique $\mathbb{P}^\sigma$ such that $dB_t = \sigma_t(B)dW_t^\sigma$ $\mathbb{P}^\sigma$-a.s. and W^σ is a $\mathbb{P}^\sigma$-standard Brownian motion. The set $\mathcal{V}$ could, for instance, be chosen to consist of processes

$$\sigma := \sum_{n=0}^{\infty} \sigma_n \mathbf{1}_{[\tau_n, \tau_{n+1})},$$

where $(\tau_n)_{n\in\mathbb{N}_0}$ a sequence of $\mathbb{F}$-stopping times such that $0 = \tau_0 \le \tau_n \uparrow \infty$, $n \to \infty$, and each σ_n is a positive $\mathcal{F}_{\tau_n}$-measurable random variable; cf. [41, Example 4.5].[2] Clearly, each constant process $\kappa \in (0, \infty)$ can be written in this way.

The set of probability measures $\mathfrak{P} \subset \mathfrak{P}^{obs}$ we obtain from such a set $\mathcal{V}$ is usually *non-dominated*: there is no probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ such that $\mathfrak{P} \ll \mathbb{P}^*$.

4. Main results

The aim of this paper is to elaborate the analytic problems arising from quasi-sure analysis by a reverse approach. More precisely, we would like to explore which consequences for the set $\mathfrak{P}$ of priors can be drawn if we assume nice properties of subspaces of $L^0(\mathfrak{P})$ which make them tractable. This approach also allows us to characterise the range of situations in which the feigned dichotomy between robustness and mathematical tractability can be resolved. We will mostly focus on the space $L^\infty(\mathfrak{P})$.

The proofs of all results in this section are postponed to Section 6.

4.1. **Super Dedekind completeness: the dominated case.** Our starting point is the question under which condition the space $L^\infty(\mathfrak{P})$ is super Dedekind complete, i.e. order complete in terms of countable operations. This property of the q.s. order would be particularly desirable and render easy tractability, because in this case $L^\infty(\mathfrak{P})$ behaves like the classical space $L^\infty(\mathbb{P}^*)$, where $\mathbb{P}^*$ is a single probability measure on $(\Omega, \mathcal{F})$. By [20, Theorem A.37], the latter is indeed super Dedekind complete.

First of all, we remark that by [21, Lemma 34], Dedekind completeness of $L^\infty(\mathfrak{P})$ is equivalent to Dedekind completeness of $(L^0(\mathfrak{P}), \preceq)$, which in turn is equivalent to Dedekind completeness of every ideal $\mathcal{X} \subset L^0(\mathfrak{P})$ with the property $L^\infty(\mathfrak{P}) \subset \mathcal{X}$. The same equivalences hold for super Dedekind completeness. In this respect, focusing on $L^\infty(\mathfrak{P})$ in most of our results does not limit their scope.

However, Theorem 4.3 below shows that super Dedekind completeness is incompatible with the assumption that the set of priors $\mathfrak{P}$ is non-dominated. The latter, however, is precisely the additional degree of generality considered as the main argument for the q.s. approach. We will see that, in fact, the space $L^\infty(\mathfrak{P})$ is super Dedekind complete if and only if $\mathfrak{P}$ is equivalent to a single probability measure $\mathbb{P}^*$. Hence, $L^\infty(\mathfrak{P}) = L^\infty(\mathbb{P}^*)$ must hold.

Before we present the actual theorem, we would like to emphasise that our proof of the implication

$$L^\infty(\mathfrak{P}) \text{ is super Dedekind complete} \implies \mathfrak{P} \text{ is dominated}$$

relies on the construction of *supports* for each measure $\mathbb{P} \in \mathfrak{P}$ which are minimal in the sense of the q.s. order. That is, for each $\mathbb{P} \in \mathfrak{P}$ we construct an event $S(\mathbb{P}) \in \mathcal{F}$ which has full

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[2] We set $\mathcal{F}_\infty := \sigma(\bigcup_{t\ge 0} \mathcal{F}_t)$. By [26, Exercise 2.4.2], $\mathcal{F}_\infty = \mathcal{F}$ holds. For a stopping time $\tau : \Omega \to [0, \infty]$, the stopped σ-algebra is then defined as

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty \mid \forall t \in \mathbb{R}_+ : \ A \cap \{\tau \le t\} \in \mathcal{F}_t\}.$$

measure under $\mathbb{P}$ (that is, $\mathbb{P}(S(\mathbb{P})) = 1$) and whose indicator is minimal among all indicators of $\mathbb{P}$-certain events. Consequently, each $\mathbb{P} \in \mathfrak{P}$ will turn out to be *supportable*.

Definition 4.1. Suppose $\mu \in \mathbf{ca}(\mathfrak{P})_+$ is a finite measure absolutely continuous with respect to $\mathfrak{P}$. We say μ is SUPPORTABLE if there is a measurable set $S(\mu) \in \mathcal{F}$ such that

(a) $\mu(S(\mu)) = \mu(\Omega)$;

(b) $S(\mu)$ is $\mathfrak{P}$-q.s. minimal among all μ-certain sets in that $\mathbf{1}_{S(\mu)} \preceq \mathbf{1}_A$ whenever $A \in \mathcal{F}$ satisfies $\mu(A) = \mu(\Omega)$.

The set $S(\mu)$ above is called the ($\mathfrak{P}$-q.s.) ORDER SUPPORT of μ.

A general signed measure $\mu \in \mathbf{ca}(\mathfrak{P})$ is supportable if its modulus $|\mu|$ with respect to the setwise order $\preceq_{\mathcal{F}}$ introduced in Section 2 is supportable. We set

$$\mathbf{sca}(\mathfrak{P}) := \{\mu \in \mathbf{ca}(\mathfrak{P}) \mid \mu \text{ supportable}\},$$

the real vector space of all supportable signed measures.

It is easy to verify that if two events $S, S' \in \mathcal{F}$ satisfy (a) and (b), then $\mathbf{1}_S = \mathbf{1}_{S'}$ $\mathfrak{P}$-q.s., i.e. the symmetric difference $S \triangle S'$ satisfies $\sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{P}(S \triangle S') = 0$. Although the order support $S(\mu)$ may not be unique *as an event*, it is unique up to $\mathfrak{P}$-polar sets, i.e. events $N \in \mathcal{F}$ with the property $\sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{P}(N) = 0$. Whenever we write $S(\mu)$ in the following, we therefore mean a version of the order support. We emphasise that order supports provide an order theoretic notion of supports which may not agree with the topological notion of the support of a measure; see [1, p. 441].

As illustrative example consider a set $\mathfrak{P}$ of priors which is dominated. Let $\mathbb{P}^*$ be a probability measure which is equivalent to $\mathfrak{P}$, i.e. the $\mathfrak{P}$-q.s. and the $\mathbb{P}^*$-a.s. order agree. Then each $\mathbb{P} \in \mathfrak{P}$ is supportable with $S(\mathbb{P}) = \{f > 0\}$, where $f \in \frac{d\mathbb{P}}{d\mathbb{P}^*}$ is an arbitrary version of the Radon-Nikodym derivative $\frac{d\mathbb{P}}{d\mathbb{P}^*}$. It is important, however, to emphasise that order supports provide an *order theoretic* notion of supports, a feature which becomes particularly striking if $\mathfrak{P}$ is not dominated. The order support may also not agree with the topological notion of the support of a measure; see [1, p. 441].

It is not clear at this stage if every measure $\mu \in \mathbf{ca}(\mathfrak{P})$ has an order support; see Example 4.7 in Section 4.2. However, the trivial measure $0 \in \mathbf{ca}(\mathfrak{P})$ is always supportable with order support $S(0) = \emptyset$. The following lemma complements Definition 4.1 and gives necessary and sufficient conditions for the supportability of a measure.

Lemma 4.2. *Let $\mathfrak{P}$ be a set of probability measures on $(\Omega, \mathcal{F})$. Let $\mu \in \mathbf{ca}(\mathfrak{P})_+$ and set $\mathcal{C} := \{\mathbf{1}_A \mid A \in \mathcal{F}, \mu(A) = \mu(\Omega)\} \subset L^\infty(\mathfrak{P})$.*

(1) If $\inf \mathcal{C}$ exists in $L^\infty(\mathfrak{P})$, then there is an event $S \in \mathcal{F}$ such that

$$\mathbf{1}_S = \inf \mathcal{C}.$$

(2) If $\inf \mathcal{C}$ exists, μ is supportable if and only if S constructed in (1) satisfies $\mu(S) = \mu(\Omega)$. In that case S is a version of the order support $S(\mu)$.

We now state the already advertised theorem on super Dedekind completeness.

Theorem 4.3. *Let $\mathfrak{P}$ be a set of probability measures on $(\Omega, \mathcal{F})$. Then the following are equivalent:*

(1) *$\mathfrak{P}$ is dominated, i.e. there is a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ such that $\mathfrak{P} \ll \mathbb{P}^*$.*
(2) *$L^\infty(\mathfrak{P})$ is super Dedekind complete.*
(3) *Each measure $\mu \in \mathbf{ca}(\mathfrak{P})$ is supportable, i.e. $\mathbf{sca}(\mathfrak{P}) = \mathbf{ca}(\mathfrak{P})$, and $L^\infty(\mathfrak{P})$ has the countable sup property.*
(4) *Each $\mathbb{P} \in \mathfrak{P}$ is supportable and $L^\infty(\mathfrak{P})$ has the countable sup property.*

Another quintessential content of the preceding theorem is negative: If $\mathfrak{P}$ is not dominated, the existence of order supports cannot be guaranteed by classical exhaustion arguments as in the proof of the implication $(2) \Rightarrow (3)$. This would require super Dedekind completeness of $L^\infty(\mathfrak{P})$ or other Banach lattices. In particular, one encounters this problem in almost all examples mentioned in the introduction.

4.2. **Supportable measures, order continuity, and the structure of uncertainty robust spaces.** In this section we take a wider perspective and consider Banach lattices $(\mathcal{X}, \|\cdot\|_\mathcal{X}, \preceq)$ such that $(\mathcal{X}, \preceq)$ is an ideal of $(L^0(\mathfrak{P}), \preceq)$ and contains all equivalence classes of constant random variables. *A fortiori*, $L^\infty(\mathfrak{P})$ is always a subset of $\mathcal{X}$. Let $\mathbf{ca}(\mathcal{X})$ be the space of all $\mu \in \mathbf{ca}(\mathfrak{P})$ such that the functional $\mathcal{X} \ni X \mapsto \int X \, d\mu$ is an element of the norm dual $\mathcal{X}^*$. Note that the setwise order $\preceq_\mathcal{F}$ on $\mathbf{ca}(\mathcal{X})$ agrees with the usual dual order from Section 2: for all $\mu, \nu \in \mathbf{ca}(\mathcal{X})$, $\mu \preceq_\mathcal{F} \nu$ is equivalent to $\int X \, d\mu \le \int X \, d\nu$ holding for all $X \in \mathcal{X}_+$. Denoting by $\|\cdot\|_{\mathbf{ca}(\mathcal{X})}$ the dual norm,

$$\|\mu\|_{\mathbf{ca}(\mathcal{X})} = \sup_{X \in \mathcal{X}:\, \|X\|_\mathcal{X} \le 1} \left| \int X \, d\mu \right|, \quad \mu \in \mathbf{ca}(\mathcal{X}),$$

one verifies that $(\mathbf{ca}(\mathcal{X}), \|\cdot\|_{\mathbf{ca}(\mathcal{X})}, \preceq_\mathcal{F})$ is a Banach lattice in its own right. In fact, $\mathbf{ca}(\mathcal{X})$ agrees with the σ-order continuous dual $\mathcal{X}_c^\sim$ of $\mathcal{X}$ via the embedding

$$\mathbf{ca}(\mathcal{X}) \ni \mu \mapsto \left(\phi_\mu : \mathcal{X} \to \mathbb{R}, \quad X \mapsto \int X \, d\mu \right). \tag{4.1}$$

Indeed, $\mathbf{ca}(\mathcal{X}) \subset \mathcal{X}_c^\sim$ is clear by monotone convergence. Conversely, σ-order continuity of $\phi \in \mathcal{X}_c^\sim \cap \mathcal{X}_+^\sim$ together with the Daniell-Stone Theorem [13, Theorem 7.8.1] provides a unique finite measure μ on $\mathcal{F}$ such that

$$\phi(X) = \int X \, d\mu, \quad X \in \mathcal{X}.$$

$\mu \in \mathbf{ca}(\mathcal{X})$ follows from $\mathcal{X}_c^\sim \subset \mathcal{X}^\sim = \mathcal{X}^*$ where $\mathcal{X}^\sim = \mathcal{X}^*$ is a general property of Banach lattices, see [35, Proposition 1.3.7]. For general $\phi \in \mathcal{X}_c^\sim$ we have that $\phi = \phi^+ - \phi^-$ where $\phi^+, \phi^- \in \mathcal{X}_c^\sim \cap \mathcal{X}_+^\sim$. We will hence write $X_c^\sim = \mathbf{ca}(\mathcal{X})$.

In the following we introduce and discuss a number of structural properties of the space $\mathcal{X}$ which have appeared in the literature to ensure tractability.

(A1) $\mathbf{ca}(\mathcal{X})$ separates the points of $\mathcal{X}$, that is,

$$\forall\, X \in \mathcal{X}\backslash\{0\}\ \exists\, \mu \in \mathbf{ca}(\mathcal{X}) : \ \int X\,d\mu \neq 0.$$

(A2) For all $\mu \in \mathbf{ca}(\mathcal{X})$, the functional ϕ_μ defined in (4.1) is order continuous:

$$\mathcal{X}_n^\sim = \mathcal{X}_c^\sim = \mathbf{ca}(\mathcal{X}).$$

(A3) Each measure $\mu \in \mathbf{ca}(\mathcal{X})$ is supportable:

$$\mathbf{ca}(\mathcal{X}) = \mathbf{sca}(\mathcal{X}) := \mathbf{ca}(\mathcal{X}) \cap \mathbf{sca}(\mathfrak{P}).$$

(A4) The vector lattice $(\mathcal{X}, \preceq)$ is Dedekind complete.

The last two properties concern the embedding

$$J : \mathcal{X} \to \mathbf{ca}(\mathcal{X})^*, \quad X \mapsto \left(\mu \mapsto \int X\,d\mu \right). \tag{4.2}$$

(A5) $\mathcal{X}$ and the order continuous dual $\mathbf{ca}(\mathcal{X})_n^\sim$ are lattice isomorphic via J. We write

$$\mathcal{X} = \mathbf{ca}(\mathcal{X})_n^\sim.$$

(A6) $\mathcal{X}$ and the dual space $\mathbf{ca}(\mathcal{X})^*$ of $(\mathbf{ca}(\mathcal{X}), \|\cdot\|_{\mathbf{ca}(\mathcal{X})})$ are lattice isomorphic via J. We write

$$\mathcal{X} = \mathbf{ca}(\mathcal{X})^*.$$

(A1) is a rather mild assumption which ensures that $\langle \mathcal{X}, \mathbf{ca}(\mathcal{X}) \rangle$ is a dual pair giving rise to a locally convex Hausdorff topology on $\mathcal{X}$. In particular, it is automatically satisfied if $\mathcal{X} = L^\infty(\mathfrak{P})$, as in that case $\mathbf{ca}(\mathcal{X}) = \mathbf{ca}(\mathfrak{P})$. Indeed, for each $0 \neq X \in L^\infty(\mathfrak{P})$ and a fixed representative $f \in X$, there is $\mathbb{P} \in \mathfrak{P}$ such that $\mathbb{P}(f \neq 0) > 0$. $\mu \in \mathbf{ca}(\mathfrak{P})$ defined by $\frac{d\mu}{d\mathbb{P}} = \frac{f}{|f|}\mathbf{1}_{\{f \neq 0\}}$ satisfies $\int X\,d\mu = \int f\,d\mu \neq 0$.

For **(A2)**, note first that the order continuous dual $\mathcal{X}_n^\sim$ of $\mathcal{X}$ may always be identified with a subset of $\mathbf{ca}(\mathcal{X})$ via (4.1). Moreover, in the case $\mathcal{X} = L^\infty(\mathfrak{P})$, **(A3)** is reduced to $\mathbf{ca}(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})$, an identity we derived as a consequence of super Dedekind completeness of $L^\infty(\mathfrak{P})$ in Theorem 4.3(3).

The next proposition shows that supportable measures are strongly connected to order continuous functionals and the existence of suprema, that is, **(A4)**. It already hints at the connection between properties **(A2)**–**(A4)** drawn in Theorem 4.8.

Proposition 4.4. *Let $\mathfrak{P}$ be a set of probability measures on $(\Omega, \mathcal{F})$ and let $\mathcal{X} \subset L^0(\mathfrak{P})$ be a Banach lattice containing all constants.*

    (1) *If $\mu \in \mathbf{sca}(\mathcal{X})$, the functional ϕ_μ is order continuous, i.e. $\mathbf{sca}(\mathcal{X}) \subset \mathcal{X}_n^\sim$.*

    (2) *If $\mathcal{X}$ is Dedekind complete* **(A4)**, *then $\mathbf{sca}(\mathcal{X}) = \mathcal{X}_n^\sim$.*

The motivation for **(A6)** is the following: If $\mathcal{X} = L^\infty(\mathfrak{P})$, we obtain $\mathbf{ca}(\mathcal{X}) = \mathbf{ca}(\mathfrak{P})$. Then **(A6)** is the condition $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$. Now suppose $\mathfrak{P}$ is dominated by a single probability measure $\mathbb{P}^*$. Without loss of generality we assume $\mathfrak{P} \approx \mathbb{P}^*$. Then $\mathbf{ca}(\mathfrak{P})$ may be identified with the space $L^1(\mathbb{P}^*)$ of all $\mathbb{P}^*$-integrable random variables. Thus **(A6)** is an intuitive

robust version of the classical weak* duality $\langle L^\infty(\mathbb{P}^*), L^1(\mathbb{P}^*) \rangle$, which is amenable to well-known functional analytic tools like the Krein-Šmulian Theorem and can be characterised nicely in terms of the $\mathbb{P}^*$-a.s. order as in the Grothendieck Lemma; see [33].

(A5) is a relaxation of **(A6)**. Indeed, for $X \in \mathcal{X}$ arbitrary, $J(X)$ is order continuous on $\mathbf{ca}(\mathcal{X})$ by (2.2), i.e. $J(\mathcal{X}) \subset \mathbf{ca}(\mathcal{X})_n^\sim \subset \mathbf{ca}(\mathcal{X})^\sim = \mathbf{ca}(\mathcal{X})^*$ (recall [35, Proposition 1.3.7] for the last equality). However, given an element ψ in $\mathbf{ca}(\mathcal{X})^*$ or $\mathbf{ca}(\mathcal{X})_n^\sim$, respectively, the problem how to obtain a random variable $X \in \mathcal{X}$ with $\psi(\mu) = \int X \, d\mu$ for all $\mu \in \mathbf{ca}(\mathcal{X})$ has no obvious solution. **(A5)** and **(A6)** unsurprisingly will turn out to be rather restrictive properties.

Example 4.5. Suppose that $\mathfrak{P} := \{\mathbb{P}\}$ for a fixed probability measure $\mathbb{P}$ and let $p \in [1, \infty)$. The space $\mathcal{X} = L^p(\mathbb{P})$ of all $\mathbb{P}$-a.s. equivalence classes of random variables with finite p-th moment is an example of a Banach lattice as described above. In this context, properties **(A1)**–**(A6)** are well-known. If $1 < p < \infty$, all of the above properties **(A1)**–**(A6)** hold true. In fact, $\mathbf{ca}(L^p(\mathbb{P}))$ can be identified with $L^q(\mathbb{P})$, where $q := \frac{p}{p-1} \in (1, \infty)$. Given $\mu \in \mathbf{ca}(L^p(\mathbb{P}))$ with associated $Z \in L^q(\mathbb{P})$, its order support is given by the set $S(\mu) := \{|f| > 0\}$ for any $\mathbb{P}$-representative $f \in Z$.
If $p = 1$, **(A1)**–**(A5)** hold true, but not **(A6)** if $L^1(\mathbb{P})$ is infinite dimensional. This is due to the well-known identities $\mathbf{ca}(L^1(\mathbb{P}))^* = L^\infty(\mathbb{P})^* = \mathbf{ba}(\mathbb{P})$.

Example 4.6. Consider $\mathcal{X} = L^\infty(\mathfrak{P})$ for $\mathfrak{P}$ which is dominated by a probability measure $\mathbb{P}^*$. Then $\mathbb{P}^*$ can always be chosen to satisfy $\mathfrak{P} \approx \mathbb{P}^*$ and $\mathbf{ca}(\mathcal{X})$ may hence be identified with $L^1(\mathbb{P}^*)$, so **(A1)**–**(A6)** all hold true.

In the previous examples, the space $\mathcal{X}$ is not only Dedekind complete **(A4)**, but has the much stronger property of being super Dedekind complete studied in Section 4.1. The regularity of these spaces stems from $\mathfrak{P}$ being dominated by a single probability measure. In a non-dominated setting properties **(A1)**–**(A6)** are not necessarily satisfied. This is illustrated by the following example.

Example 4.7. Consider $\Omega = [0,1]$, $\mathcal{F} = \mathbb{B}([0,1])$, the Borel-$\sigma$-algebra over $[0,1]$, and the set $\mathfrak{P} := \Delta(\mathcal{F})$ of *all* probability measures on $(\Omega, \mathcal{F})$. The $\mathfrak{P}$-q.s. order $\preceq$ then agrees with the pointwise order, and $L^\infty(\mathfrak{P})$ agrees with the space of all bounded random variables. If μ is the Lebesgue measure on $[0,1]$, it is absolutely continuous with respect to $\mathfrak{P}$, but not supportable: we have $0 = \inf\{\mathbf{1}_A \mid A \in \mathcal{F}, \mu(A) = 1\}$. Hence, **(A3)** is not met.
Moreover, consider the set $\mathcal{C} := \{\mathbf{1}_{\{\omega\}} \mid \omega \in V\}$, where V denotes the Vitali set, which is not Borel measurable. $\mathcal{C}$ is order bounded from above, but has no supremum. Indeed, let X be any upper bound of $\mathcal{C}$. Then X is non-negative and $X(\omega) \geq 1$ holds for all $\omega \in V$. However, suppose $\omega^* \in \Omega \setminus V$ satisfies $X(\omega^*) > 0$. Consider $X^* := X\mathbf{1}_{\Omega \setminus \{\omega^*\}}$ and note that it is an upper bound of $\mathcal{C}$ as well and satisfies $X^* \prec X$. Hence, $\sup \mathcal{C}$ would have to agree with $\mathbf{1}_V$. As V is not measurable, $\sup \mathcal{C}$ cannot exist. Thus **(A4)** is not satisfied either.
According to Theorem 4.8 below, in fact none of the properties **(A3)**–**(A6)** are satisfied.

Under further conditions on $\mathcal{X}$, which are for instance met by $L^\infty(\mathfrak{P})$, the following Theorem 4.8 now relates the structural properties **(A1)**–**(A6)**. This result generalises observations made in [21] beyond the case $\mathcal{X} = L^\infty(\mathfrak{P})$.

Theorem 4.8. *Let $\mathfrak{P}$ be a set of probability measures on $(\Omega, \mathcal{F})$ and $\mathcal{X} \subset L^0(\mathfrak{P})$ be a Banach lattice which contains all constant random variables.*

   (1) *If $\mathcal{X}$ is Dedekind complete, order continuity of every functional ϕ_μ, $\mu \in \mathbf{ca}(\mathcal{X})$, is equivalent to $\mathbf{ca}(\mathcal{X}) = \mathbf{sca}(\mathcal{X})$:*

$$\textbf{(A2)} \wedge \textbf{(A4)} \iff \textbf{(A3)} \wedge \textbf{(A4)}$$

   (2) *Suppose $\mathcal{X}$ is monotonically complete: for every norm bounded net $(X_\alpha)_{\alpha \in I}$ with $X_\alpha \preceq X_\beta$, $\alpha \leq \beta$, $\sup_{\alpha \in I} X_\alpha$ exists in $\mathcal{X}$. Then*

$$\textbf{(A6)} \implies \textbf{(A5)} \iff \textbf{(A1)} \wedge \textbf{(A3)} \wedge \textbf{(A4)} \iff \textbf{(A1)} \wedge \textbf{(A2)} \wedge \textbf{(A4)}.$$

     $\mathcal{X}$ is a perfect Banach lattice under each of these equivalent conditions.

   (3) *Suppose $\|\cdot\|_{\mathbf{ca}(\mathcal{X})}$ is order continuous. Then*

$$\textbf{(A5)} \iff \textbf{(A6)}.$$

Remark 4.9. (1) Recall that for a given set $\mathfrak{P}$ of priors $\mathbf{ca}(L^\infty(\mathfrak{P})) = \mathbf{ca}(\mathfrak{P})$. Hence $\|\mu\|_{\mathbf{ca}(\mathfrak{P})} = |\mu|(\Omega)$ holds for all $\mu \in \mathbf{ca}(\mathfrak{P})$, resulting in order continuity of $\|\cdot\|_{\mathbf{ca}(\mathfrak{P})}$. Theorem 4.8(3) applies. If $L^\infty(\mathfrak{P})$ is additionally Dedekind complete, it is also monotonically complete and Theorem 4.8(2) applies.

(2) Suppose $\mathcal{X}$ is Dedekind complete and there is an increasing functional $\rho : L^0(\mathfrak{P})_+ \to [0, \infty]$ with the following properties:

   (i) $\mathcal{X} = \{X \in L^0(\mathfrak{P}) \mid \rho(|X|) < \infty\}$ and $\|\cdot\|_{\mathcal{X}} = \rho(|\cdot|)$.

   (ii) for each increasing net $(X_\alpha)_{\alpha \in I} \subset L^0(\mathfrak{P})_+$ with $X := \sup_{\alpha \in I} X_\alpha$ in $L^0(\mathfrak{P})$ we have $\rho(X) \leq \sup_{\alpha \in I} \rho(X_\alpha)$. (Equality holds automatically because ρ is increasing.)

   (iii) for each increasing sequence $(X_n)_{n \in \mathbb{N}} \subset L^0(\mathfrak{P})_+$ with $\sup_{n \in \mathbb{N}} \|X_n\|_{\mathcal{X}} < \infty$ there is $X \in L^0(\mathfrak{P})$ with $X = \sup_{n \in \mathbb{N}} X_n$.

Then $\mathcal{X}$ is monotonically complete.

(3) In [33, Proposition 3.10], the authors initially asserted that for any set of probability measures $\mathfrak{P}$ and for the space $\mathcal{X} = L^\infty(\mathfrak{P})$, the properties **(A4)** and **(A6)** are equivalent; see also [34]. We will discuss this issue in Section 4.6 and show that this claim cannot be disproved with a counterexample in **ZFC**; cf. Corollary 4.33.

The following proposition provides a useful criterion to decide whether **(A2)**–**(A6)** are met. In fact we prove that for a large class of models $\mathfrak{P}$ and many spaces $\mathcal{X}$ neither **(A2)** nor **(A3)** hold, and thus in particular also **(A5)** and **(A6)** fail (Theorem 4.8). To this end we recall that a subset P of a Polish space (Σ, τ) is *perfect* if it is closed and if for every $\sigma \in P$ there is a sequence $(\sigma_n)_{n \in \mathbb{N}} \subset P \backslash \{\sigma\}$ with $\sigma_n \to \sigma$.

Proposition 4.10. *Let Ω be Polish, $\mathcal{F}$ be the Borel-σ-algebra on Ω, and $\mathfrak{P}$ be a set of priors on $(\Omega, \mathcal{F})$. Suppose there is a non-empty set $\mathfrak{R} \subset \mathbf{sca}(\mathfrak{P}) \cap \Delta(\mathcal{F})$ such that*

   • *$\mathfrak{R}$ is a perfect set in $\Delta(\mathcal{F})$;*

- *for $\mathbb{Q}, \mathbb{Q}' \in \mathfrak{R}$, $\mathbb{Q} \neq \mathbb{Q}'$ implies $\mathbf{1}_{S(\mathbb{Q})} \wedge \mathbf{1}_{S(\mathbb{Q}')} = 0$.*

Then the following holds:

(1) *There is a probability measure $\mu \in \mathbf{ca}(\mathfrak{P}) \backslash \mathbf{sca}(\mathfrak{P})$.*
(2) *If a Banach lattice $\mathcal{X} \subset L^0(\mathfrak{P})$ contains all constants and satisfies*

$$\sup_{\mathbb{Q} \in \mathfrak{R}} \|\mathbb{Q}\|_{\mathbf{ca}(\mathcal{X})} < \infty,$$

then there is $\mu \in \mathbf{ca}(\mathcal{X}) \backslash \mathbf{sca}(\mathcal{X})$, ie. $(\mathbf{A3})$ fails. Moreover, if $\sup_{\mathbb{Q} \in \mathfrak{R}} \mathbf{1}_{S(\mathbb{Q})}$ exists in $\mathcal{X}$, then the associated functional ϕ_μ satisfies $\phi_\mu \in \mathcal{X}_c^\sim \backslash \mathcal{X}_n^\sim$, i.e. $(\mathbf{A2})$ fails.

The assertions also hold if perfectness of $\mathfrak{R}$ is replaced by the assumption that $\mathfrak{R}$ is an uncountable set which is in addition either Borel or analytic.

We close this section with an illustration of Proposition 4.10 in the context of volatility uncertainty as in Example 3.1. This shows that for the practical purposes of Soner et al. [41], assumption $(\mathbf{A3})$, and *a fortiori* also $(\mathbf{A5})$ and $(\mathbf{A6})$, are too restrictive.

Example 4.11. Here we consider a set $\mathfrak{P} = \{\mathbb{P}^\sigma \mid \sigma \in \mathcal{V}\}$ of laws associated to a set of volatility processes $\mathcal{V}$ as in Example 3.1. More precisely, we shall assume $\mathcal{V}$ only contains deterministic processes, i.e. functions $\sigma. : \mathbb{R}_+ \to (0, \infty)$ which are Borel measurable and satisfy $\int_0^T \sigma_u^2 du < \infty$ for all $T > 0$. In particular, the associated function $\mathbb{R}_+ \ni t \mapsto \int_0^t \sigma_u^2 du$ is continuous. Under this assumption, for each $\sigma \in \mathcal{V}$ there is a unique local martingale measure $\mathbb{P}^\sigma \in \mathfrak{P}^{obs}$ such that $\mathbb{P}^\sigma$ is the weak solution to (3.1) with initial condition $B_0 = 0$ $\mathbb{P}^\sigma$-a.s. Here and in the following, B denotes the canonical process.

First we verify that each $\mathbb{P}^\sigma$ is supportable with respect to $\mathfrak{P}$. To this effect, let $\langle B \rangle$ denote the non-decreasing universal quadratic variation process of B introduced in Example 3.1. For $\sigma \in \mathcal{V}$ we claim

$$S(\mathbb{P}^\sigma) := \{\omega \in \Omega \mid \forall t \in \mathbb{Q}_+ : \langle B \rangle_t(\omega) = \int_0^t \sigma_s^2 ds\} \in \mathcal{F}$$

is a version of the $\mathfrak{P}$-q.s. order support of $\mathbb{P}^\sigma$. Indeed, for property (a) from Definition 4.1 we recall that $\mathbb{P}^\sigma(S(\mathbb{P}^\sigma)) = 1$ for all $\sigma \in \mathcal{V}$. For property (b), fix $\sigma, \nu \in \mathcal{V}$. From the condition $\mathbb{P}^\sigma(S(\mathbb{P}^\nu) \cap S(\mathbb{P}^\sigma)) > 0$ we may infer $\int_0^t \sigma_s^2 ds = \int_0^t \nu_s^2 ds$ for all $t \in \mathbb{R}_+$ by continuity. This implies $\sigma_t = \nu_t$ for all $t \in \mathbb{R}_+$ outside of a Borel set of Lebesgue measure 0, and hence $\mathbb{P}^\sigma = \mathbb{P}^\nu$. For $A \in \mathcal{F}$ with $\mathbb{P}^\sigma(A) = 1$, we thus obtain

$$\mathbb{P}^\nu(\mathbf{1}_A < \mathbf{1}_{S(\mathbb{P}^\sigma)}) = \mathbb{P}^\nu\left((S(\mathbb{P}^\sigma) \cap S(\mathbb{P}^\nu)) \backslash A\right) \begin{cases} = \mathbb{P}^\sigma(S(\mathbb{P}^\sigma) \backslash A) = 0, & \mathbb{P}^\sigma = \mathbb{P}^\nu, \\ \leq \mathbb{P}^\nu(S(\mathbb{P}^\sigma) \cap S(\mathbb{P}^\nu)) = 0, & \mathbb{P}^\sigma \neq \mathbb{P}^\nu. \end{cases}$$

Hence, $\mathbf{1}_{S(\mathbb{P}^\sigma)} \preceq \mathbf{1}_A$, and we have proved that $S(\mathbb{P}^\sigma)$ is a version of the order support of $\mathbb{P}^\sigma$. Suppose now that for constants $\kappa_1, \kappa_2 > 0$ and for each $\kappa_1 \leq \kappa \leq \kappa_2$ the constant function $\mathbb{R}_+ \ni t \mapsto \kappa$, representing a constant volatility process, lies in $\mathcal{V}$. One easily verifies that the set

$$\mathfrak{R} := \{\mathbb{P}^\kappa \mid \kappa \in [\kappa_1, \kappa_2]\} \subset \Delta(\mathcal{F}) \cap \mathbf{ca}(\mathfrak{P})$$

is uncountable and closed in the topology of weak convergence on $\Delta(\mathcal{F})$.

Moreover, the argument shows that for $\kappa_1 \leq \kappa < \kappa' \leq \kappa_2$, $\mathbf{1}_{S(\mathbb{P}^\kappa)} \wedge \mathbf{1}_{S(\mathbb{P}^{\kappa'})} = 0$ $\mathfrak{P}$-q.s. We thus are precisely in the situation of Proposition 4.10(1).

As for (2), suppose that a Banach lattice $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ is such that

$$\sup_{\kappa_1 \leq \kappa \leq \kappa_2} \|\mathbb{P}^\kappa\|_{\mathbf{ca}(\mathcal{X})} < \infty,$$

a condition which is always satisfied if we consider $\mathcal{X} = L^\infty(\mathfrak{P})$. Define

$$\Omega^* := \{\omega \in \Omega \mid \forall\, t \in \mathbb{Q}_+ : \tfrac{\langle B \rangle_t}{t} = \langle B \rangle_1\} \cap \{\omega \in \Omega \mid \langle B \rangle_1 \in [\kappa_1^2, \kappa_2^2]\}.$$

We claim that $\mathbf{1}_{\Omega^*} = \sup_{\kappa_1 \leq \kappa \leq \kappa_2} \mathbf{1}_{S(\mathbb{P}^\kappa)}$ in $\mathcal{X}$. Indeed, $\mathbf{1}_{\Omega^*}$ is an upper bound. Conversely, fix an upper bound $U \in \mathcal{X}$ and let $\nu \in \mathcal{V}$. Then

$$\mathbb{P}^\nu(\mathbf{1}_{\Omega^*} > U) = \mathbb{P}^\nu(\{\mathbf{1}_{\Omega^*} > U\} \cap S(\mathbb{P}^\nu)).$$

If ν is such that $\frac{1}{t} \int_0^t \nu_s^2 ds \neq \int_0^1 \nu_s^2 ds$ for some $t \in (0, \infty) \cap \mathbb{Q}$, or $\frac{1}{t} \int_0^t \nu_s^2 ds = \int_0^1 \nu_s^2 ds$ for all $t \in (0, \infty) \cap \mathbb{Q}$, but $\int_0^1 \nu_s^2 ds \notin [\kappa_1^2, \kappa_2^2]$, we obtain

$$\mathbb{P}^\nu(\mathbf{1}_{\Omega^*} > U) = \mathbb{P}^\nu(\{0 > U\} \cap S(\mathbb{P}^\nu)) = 0.$$

If ν satisfies the conditions posed by Ω^*, then there is a constant $\kappa_\nu \in (\kappa_1, \kappa_2)$ such that $\mathbb{P}^\nu = \mathbb{P}^{\kappa_\nu}$ and $S(\mathbb{P}^\nu) = S(\mathbb{P}^{\kappa_\nu})$ by the reasoning above. Hence,

$$\mathbb{P}^\nu(\mathbf{1}_{\Omega^*} > U) = \mathbb{P}^{\kappa_\nu}(\{\mathbf{1}_{\Omega^{\kappa_\nu}} > U\} \cap S(\mathbb{P}^{\kappa_\nu})) = 0.$$

In total, $\mathbf{1}_{\Omega^*} \preceq U$, and we obtain $\sup_{\kappa_1 \leq \kappa \leq \kappa_2} \mathbf{1}_{S(\mathbb{P}^\kappa)} = \mathbf{1}_{\Omega^*}$. Proposition 4.10(2) can thus be applied to obtain a continuous linear functional which is not order continuous.

For the purpose of illustration we remark that a continuous functional which is not order continuous is for instance given by

$$\phi(X) := \int_{\kappa_1}^{\kappa_2} \mathbb{E}_{\mathbb{P}^\kappa}[X] \pi(d\kappa), \quad X \in \mathcal{X},$$

where π is any non-atomic probability measure on $[\kappa_1, \kappa_2]$ (cf. proof of Proposition 4.10 for further details). ϕ can be understood in terms of smooth ambiguity from a decision-theoretic perspective; see Klibanoff et al. [28].

4.3. **The class (S) property.** Wrapping up the preceding two sections, super Dedekind completeness of $L^\infty(\mathfrak{P})$ as well as the robust equivalent to the $\langle L^\infty, L^1 \rangle$-duality, $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$, both imply that all signed measures in $\mathbf{ca}(\mathfrak{P})$ are supportable: $\mathbf{ca}(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})$. In particular, each $\mathbb{P} \in \mathfrak{P}$ is supportable. However, we have also seen in Example 4.11 that this condition would be too restrictive to encompass prominent examples from the literature. A logical relaxation would be to only assume that the space of supportable measures is "rich enough". More precisely, one is tempted to assume that there is an alternative set of priors $\mathfrak{Q}$ encoding the full information given by $\mathfrak{P}$ which consists of supportable probability measures. As such, this axiom is invariant under equivalent transformations of the set $\mathfrak{P}$.

Definition 4.12. Let $\mathfrak{P}$ be a set of probability measures on $(\Omega, \mathcal{F})$. $\mathfrak{P}$ is OF CLASS (S) if there is a set $\mathfrak{Q} \subset \mathbf{sca}(\mathfrak{P})_+$ of supportable probability measures such that $\mathfrak{P} \approx \mathfrak{Q}$. We refer to the set $\mathfrak{Q}$ as SUPPORTABLE ALTERNATIVE to $\mathfrak{P}$.

Remark 4.13. (1) Clearly, being of class (S) does not entail that each $\mathbb{P} \in \mathfrak{P}$ is supportable. In the situation of Example 4.7, for instance, the chosen set $\mathfrak{P} := \Delta\left(\mathbb{B}([0,1])\right)$ is of class (S). A supportable alternative would be the set $\mathfrak{Q} := \{\delta_\omega \mid \omega \in [0,1]\}$. We have already observed though that the Lebesgue measure λ on $[0,1]$ is not supportable.
(2) $\mathfrak{P}$ being of class (S) is necessary for properties **(A5)** and **(A6)** from Section 4.2 to hold for $L^\infty(\mathfrak{P})$. Indeed, both imply **(A3)** by Theorem 4.8, i.e. the identity $\mathbf{ca}(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})$. In particular, each measure $\mathbb{P} \in \mathfrak{P}$ is supportable.

The next example shows that Soner et al. [41] fall precisely in the class (S) setting; we also refer to [18, Example 3.8].

Example 4.14. Consider the setting described in Example 3.1 and used in Example 4.11. In the latter, we have shown that each of the probability measures $\mathbb{P}^\kappa$, $\kappa > 0$ being a constant, is supportable whenever volatility uncertainty is modelled by the set of probability measures

$$\mathfrak{P} := \{\mathbb{P}^\sigma \mid \sigma : \mathbb{R}_+ \to (0, \infty) \text{ Borel measurable}, \forall T > 0 : \int_0^T \sigma_s^2 ds < \infty\}.$$

The $\mathfrak{P}$-q.s. order support of $\mathbb{P}^\kappa$ was shown to be

$$\Omega^\kappa := \{\omega \in \Omega \mid \forall t \in \mathbb{Q}_+ : \langle B \rangle_t(\omega) = \kappa^2 t\}.$$

In fact, for $\kappa \neq \kappa'$, the order supports Ω^κ and $\Omega^{\kappa'}$ are disjoint.
This naturally leads to the question whether each probability measure $\mathbb{P}^\sigma$ is supportable when $\mathfrak{P}$ is modelled using more complex and potentially ω-dependent volatility processes σ. In each case, $\mathbb{P}^\sigma$ is concentrated on a certain region in the path space which can be characterised with the universal quadratic variation process $\langle B \rangle$ as

$$\{\omega \in \Omega \mid \forall t \in \mathbb{Q}_+ : \langle B \rangle_t(\omega) = \int_0^t \sigma_s^2(\omega) ds\}.$$

However, one eventually notices that in the general case this set is not necessarily quasi surely minimal as demanded by condition (b) in Definition 4.1. The main difficulty is that control of the intersections of such regions is tedious.
This is precisely the reason why in Soner et al. [41] the authors chose to thin out the set of admissible volatility processes by choosing a smaller set $\mathcal{V}$ of *separable diffusion coefficients* generated by a *generating class* $\mathcal{V}_0$. More precisely, they consider a set $\mathcal{V}_0 \subset \mathcal{A}^{obs}$ whose processes admit weak uniqueness in the SDE (3.1) and which have two additional properties:

- $\sigma \mathbf{1}_{[0,t)} + \nu \mathbf{1}_{[t,\infty)}$ lies in $\mathcal{V}_0$ whenever $\sigma, \nu \in \mathcal{V}_0$ and $t \in \mathbb{R}_+$.
- the stopping time $\tau^{\sigma,\nu}(\omega) := \inf\{t \geq 0 \mid \int_0^t \sigma_s^2(\omega) ds \neq \int_0^t \nu_s^2(\omega) ds\}$, $\omega \in \Omega$, is constant for all $\sigma, \nu \in \mathcal{V}_0$.

As an example for $\mathcal{V}_0$ the reader may think of the set of deterministic processes considered in Example 4.11.

Now consider the set $\mathcal{V}$ of processes of the shape

$$\sigma_t(\omega) := \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} (\sigma_i^n)_t(\omega) \mathbf{1}_{E_i^n}(\omega) \mathbf{1}_{[\tau_n(\omega), \tau_{n+1}(\omega))}(t), \quad (\omega, t) \in \Omega \times [0, T],$$

where

- $(\sigma_i^n) \in \mathcal{V}_0$, $n \in \mathbb{N}_0$, $i \in \mathbb{N}$,
- $(\tau_n)_{n \in \mathbb{N}_0}$ is a non-decreasing sequence of $\mathbb{F}$-stopping times taking countably many values with $\tau_0 = 0$, $\inf\{n \mid \tau_n(\omega) = \infty\} < \infty$, and $\tau_n(\omega) < \tau_{n+1}(\omega)$ whenever $\tau_n(\omega) < \infty$, $\omega \in \Omega$,
- $(E_i^n)_{i \in \mathbb{N}} \subset \mathcal{F}_{\tau_n}$ is a measurable partition of Ω.

Let $\mathfrak{P} := \{\mathbb{P}^\sigma \mid \sigma \in \mathcal{V}\}$. As in Example 4.11 consider

$$S(\mathbb{P}^\sigma) := \{\omega \in \Omega \mid \forall s \in \mathbb{Q}_+ : \ \langle B \rangle_s(\omega) = \int_0^s \sigma_u^2(\omega) du\}.$$

For all $\sigma \in \mathcal{V}$, $S(\mathbb{P}^\sigma)$ satisfies condition (a) in Definition 4.1, i.e. $\mathbb{P}^\sigma(S(\mathbb{P}^\sigma)) = 1$. Regarding condition (b) relative to $\mathfrak{P}$, let $\sigma, \nu \in \mathcal{V}$ and set $\Omega^{\sigma,\nu} := \{\tau^{\sigma,\nu} = \infty\}$, hence $S(\mathbb{P}^\sigma) \cap S(\mathbb{P}^\nu) \subset \Omega^{\sigma,\nu}$. Suppose now that $A \in \mathcal{F}$ is such that $\mathbb{P}^\sigma(A) = 1$, whence $\mathbb{P}^\sigma(S(\mathbb{P}^\sigma) \cap A^c) = 0$ follows. Let $\nu \in \mathcal{V}$ be arbitrary. From [41, Lemma 5.2] we infer

$$\mathbb{P}^\nu(S(\mathbb{P}^\sigma) \cap A^c) = \mathbb{P}^\nu(S(\mathbb{P}^\nu) \cap S(\mathbb{P}^\sigma) \cap A^c) \leq \mathbb{P}^\nu(\Omega^{\sigma,\nu} \cap S(\mathbb{P}^\sigma) \cap A^c) = \mathbb{P}^\sigma(\Omega^{\sigma,\nu} \cap S(\mathbb{P}^\sigma) \cap A^c) = 0.$$

This shows $\mathbf{1}_{S(\mathbb{P}^\sigma) \cap A^c} = 0$ $\mathbb{P}^\nu$-a.s. for all $\nu \in \mathcal{V}$, and $\mathfrak{P}$-q.s.

$$\mathbf{1}_{S(\mathbb{P}^\sigma)} = \mathbf{1}_{S(\mathbb{P}^\sigma) \cap A} + \mathbf{1}_{S(\mathbb{P}^\sigma) \cap A^c} = \mathbf{1}_{S(\mathbb{P}^\sigma) \cap A} \preceq \mathbf{1}_A.$$

$S(\mathbb{P}^\sigma)$ hence checks the minimality condition (b).

Both observations together imply that $S(\mathbb{P}^\sigma)$ is the $\mathfrak{P}$-q.s. order support of $\mathbb{P}^\sigma$. Therefore, $\mathfrak{P}$ is its own supportable alternative. The main setting of interest in [41] can thus be embedded in our framework of probabilities of class (S).

Another assumption on the set of priors is inspired by Cohen [18, Definition 3.2]: The following *Hahn property*—we use the same terminology as [18]—is a special instance of the class (S) property.

Definition 4.15. A set $\mathfrak{P}$ of probability measures on a measurable space $(\Omega, \mathcal{F})$ has the HAHN PROPERTY if there is another set of probability measures $\mathfrak{Q} \subset \mathbf{ca}(\mathfrak{P})$ and a family of events $(S_\mathbb{Q})_{\mathbb{Q} \in \mathfrak{Q}} \subset \mathcal{F}$ with the following properties:

(a) $\mathfrak{P} \approx \mathfrak{Q}$.
(b) The $\mathfrak{P}$- and the $\mathfrak{Q}$-completion of the σ-algebra $\mathcal{F}$ agree: $\mathcal{F}(\mathfrak{P}) = \mathcal{F}(\mathfrak{Q})$.
(c) For all $\mathbb{Q} \in \mathfrak{Q}$, $\mathbb{Q}(S_\mathbb{Q}) = 1$.
(d) If $\mathbb{Q}, \mathbb{Q}' \in \mathfrak{Q}$ and $\mathbb{Q} \neq \mathbb{Q}'$, then $S_\mathbb{Q} \cap S_{\mathbb{Q}'} = \emptyset$.

Note that we slightly alter and generalise Cohen's definition of the Hahn property, see [18, Definition 3.2].[3] Cohen demonstrates that sets of priors with the Hahn property are analytically well behaved and admit aggregation procedures of random variables. The following lemma shows that the Hahn property indeed implies the class (S) property, see also [18, Remark 3.13].

Lemma 4.16. *Suppose $\mathfrak{P}$ is a set of probability measures on $(\Omega, \mathcal{F})$ with the Hahn property. Suppose $\mathfrak{Q} \approx \mathfrak{P}$ and $(S_{\mathbb{Q}})_{\mathbb{Q} \in \mathfrak{Q}} \subset \mathcal{F}$ satisfy conditions (a)–(d) from Definition 4.15. Then each $\mathbb{Q} \in \mathfrak{Q}$ is supportable and $\mathbf{1}_{S(\mathbb{Q})} = [\mathbf{1}_{S_{\mathbb{Q}}}]$ in $L^{\infty}(\mathfrak{P})$.*

The next proposition shows that Proposition 4.4(2) holds true under the weaker assumption of class (S) instead of Dedekind completeness.

Proposition 4.17. *Suppose $\mathfrak{P}$ is of class (S) and $\mathcal{X} \subset L^{0}(\mathfrak{P})$ is a Banach lattice which contains all constants. Then $\mathcal{X}_{n}^{\sim} = \mathbf{sca}(\mathcal{X})$.*

Proposition 4.18. *Let $\mathfrak{P}$ be a set of probability measures.*
  (1) *If $\mathfrak{P}$ is of class (S), the order continuous dual $L^{\infty}(\mathfrak{P})_{n}^{\sim} = \mathbf{sca}(\mathfrak{P})$ separates the points of $L^{\infty}(\mathfrak{P})$.*
  (2) *If $L^{\infty}(\mathfrak{P})$ is Dedekind complete, $\mathfrak{P}$ is of class (S) if and only if $L^{\infty}(\mathfrak{P})_{n}^{\sim} = \mathbf{sca}(\mathfrak{P})$ separates the points of $L^{\infty}(\mathfrak{P})$.*

We conclude this section with a decomposition result for the dual of some Banach lattice $L^{\infty}(\mathfrak{P}) \subset \mathcal{X} \subset L^{0}(\mathfrak{P})$. In particular, this generalises [30, Theorem 4.12].

Theorem 4.19. *Suppose $\mathfrak{P}$ is of class (S) and let $\mathcal{X} \subset L^{0}(\mathfrak{P})$ be any Banach lattice containing all constants. Then its dual may be decomposed as*

$$\mathcal{X}^{\sim} = \mathcal{X}^{*} = \mathbf{sca}(\mathcal{X}) \oplus \mathbf{sca}(\mathcal{X})^{\perp} \oplus \mathbf{ca}(\mathcal{X})^{\mathrm{d}}. \tag{4.3}$$

Here, $\mathbf{sca}(\mathcal{X})^{\perp} \subset \mathbf{ca}(\mathcal{X})$ is the set of all ORTHOGONAL *signed measures: for each $\mu \in \mathbf{sca}(\mathcal{X})^{\perp}$ and all $\nu \in \mathbf{sca}(\mathfrak{P})$ we have $|\mu|(S(\nu)) = 0$. In order to verify whether $\mu \in \mathbf{sca}(\mathcal{X})^{\perp}$ it is sufficient to show that for all $\mathbb{Q} \in \mathfrak{Q}$ we have $|\mu|(S(\mathbb{Q})) = 0$, where $\mathfrak{Q}$ is an arbitrary supportable alternative to $\mathfrak{P}$. $\mathbf{ca}(\mathcal{X})^{\mathrm{d}}$ denotes the disjoint complement of $\mathbf{ca}(\mathcal{X})$ in $\mathcal{X}^{*}$ and consists of continuous linear functionals which do not correspond to an integral with respect to a countably additive signed measure. Moreover, all spaces appearing in (4.3) are bands, and $\mathbf{ca}(\mathcal{X}) = \mathbf{sca}(\mathcal{X}) \oplus \mathbf{sca}(\mathcal{X})^{\perp}$.*

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[3] Indeed, Cohen defines the Hahn property on a sub-σ-algebra of an ambient σ-algebra as for him conditional expectations are of foremost interest. We make no such assumption. Cohen demands that (i) $\mathcal{F}(\mathfrak{P}) = \mathcal{F}(\mathfrak{Q})$ and (ii) for all $N \in \mathcal{F}(\mathfrak{P})$ we have $\sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{P}^{\sharp}(N) = 0$ if and only if $\sup_{\mathbb{Q} \in \mathfrak{Q}} \mathbb{Q}^{\sharp}(N) = 0$. This is equivalent to the conjunction of (a) and (b) above. Also requiring that we find $(S_{\mathbb{Q}})_{\mathbb{Q} \in \mathfrak{Q}} \subset \mathcal{F}$ with properties (c) and (d) is equivalent to Cohen's requirement that a family $(S_{\mathbb{Q}})_{\mathbb{Q} \in \mathfrak{Q}} \subset \mathcal{F}(\mathfrak{P})$ satisfies (c) and (d). Finally, we omit Cohen's further condition [18, Definition 3.2(iii)] as it plays no role in our studies. However, one can prove that if $\mathfrak{P}$ has the Hahn property in the sense of Definition 4.15, then there is a set $\widehat{\mathfrak{P}}$ of probabili ty measures which is equivalent to $\mathfrak{P}$ and which has the Hahn property in the sense of Cohen [18, Definition 3.2].

Example 4.20. In the context of Example 4.7 suppose ν is an atomless probability measure on $(\Omega, \mathcal{F})$ and $\zeta := \sum_{i=1}^{\infty} \lambda_i \delta_{\omega_i}$ for some sequence $(\omega_i)_{i \in \mathbb{N}} \subset \Omega$ and $(\lambda_i)_{i \in \mathbb{N}} \subset [0,1]$ with $\sum_{i=1}^{\infty} \lambda_i = 1$. Then for any $0 \leq s \leq 1$, the orthogonal component of the measure $\mu_s := s\nu + (1-s)\zeta$ is $s\nu$, whereas the supportable component is $(1-s)\zeta$.

4.4. **The product structure of class (S) spaces.** Lemma 4.16 shows that the Hahn property is a special case of the class (S) property which demands the existence of versions of the order supports of a supportable alternative which are pairwise disjoint. This is a "pointwise property" which has no immediate order counterpart. From our perspective of focusing on the q.s. order, it is therefore sensible to introduce a relaxation of the Hahn property.

Definition 4.21. Suppose $\mathfrak{P}$ is a set of probability measures on $(\Omega, \mathcal{F})$. A set $\mathfrak{R} \subset \mathbf{sca}(\mathfrak{P})$ of supportable probability measures is called DISJOINT if

$$\forall\, \mathbb{Q}, \mathbb{Q}' \in \mathfrak{R}: \ \mathbb{Q} \neq \mathbb{Q}' \implies \mathbf{1}_{S(\mathbb{Q})} \wedge \mathbf{1}_{S(\mathbb{Q}')} = \mathbf{1}_{S(\mathbb{Q}) \cap S(\mathbb{Q}')} = 0 \text{ in } L^{\infty}(\mathfrak{P}).$$

$\mathfrak{P}$ has the WEAK HAHN PROPERTY if it is of class (S) and there is a supportable alternative $\mathfrak{Q} \approx \mathfrak{P}$ which is disjoint. We call $\mathfrak{Q}$ a disjoint supportable alternative to $\mathfrak{P}$.

Instead of demanding disjointness of certain versions of the order supports as events, the weak Hahn property demands disjointness in terms of the q.s. order.
At first sight it may seem like the class (S) property is weaker than the weak Hahn property. The next theorem shows that this is not the case, however.

Theorem 4.22. *Let $\mathfrak{P}$ be a set of probability measures on $(\Omega, \mathcal{F})$. Then the following are equivalent:*

    (1) *$\mathfrak{P}$ is of class (S).*
    (2) *$\mathfrak{P}$ has the weak Hahn property.*

Moreover, if $L^{\infty}(\mathfrak{P})$ is Dedekind complete and each disjoint set of supportable probability measures is at most equinumerous with the continuum, then (1) and (2) are equivalent to:

    (3) *There is a disjoint supportable alternative $\mathfrak{Q}$ to $\mathfrak{P}$ and a family of pairwise disjoint events $(S_{\mathbb{Q}})_{\mathbb{Q} \in \mathfrak{Q}}$ such that $[\mathbf{1}_{S_{\mathbb{Q}}}] = \mathbf{1}_{S(\mathbb{Q})}$, $\mathbb{Q} \in \mathfrak{Q}$.*

The message of Theorem 4.22 is twofold. Firstly, recall the discussion about class (S) from the introduction. We will prove below that it is impossible to construct concrete models which admit aggregation—the property of $L^{\infty}(\mathfrak{P})$ to be Dedekind complete—and which are not of class (S). Hence, in concrete models aggregation requires a disjoint supportable alternative and it even turns out that the stronger Hahn property is in some sense "almost optimal". If the cardinality of the disjoint supportable alternative is not too large, the order supports admit pairwise disjoint versions as in the Hahn property. In fact, Theorem 4.22 together with Theorem 4.39 below will allow us to prove the equivalence of the class (S) property and the Hahn property under certain additional conditions. Secondly, Theorem 4.22 shows that we cannot expect $\mathfrak{P}$ to be of class (S) if sets of full measure of probability measures $\mathbb{Q} \ll \mathfrak{P}$ are

too intertwined. The reader may think here of the phenomenon discussed in Example 4.14 which led Soner et al. [41] to their notion of separable diffusion coefficients.

For clarification of the conditions of Theorem 4.22, we note that:

Proposition 4.23. *Suppose $\mathfrak{P}$ is of class* (S).

(1) *There is a countable disjoint supportable alternative if and only if each disjoint set of supportable probability measures is countable, which is equivalent to $\mathfrak{P}$ being dominated.*

(2) *There is a supportable alternative $\mathfrak{Q}$ at most equinumerous with the continuum if and only if there is a disjoint supportable alternative $\mathfrak{Q}$ at most equinumerous with the continuum. This is equivalent to each disjoint set of supportable probability measures being at most equinumerous with the continuum.*

The next result shows that the class (S) property entails that $L^\infty(\mathfrak{P})$ has a product structure provided it is Dedekind complete. If not, then the Dedekind completion of $L^\infty(\mathfrak{P})$, which always exists since $L^\infty(\mathfrak{P})$ is Archimedean, equals a product space (up to isomorphism).

Proposition 4.24. *Suppose $\mathfrak{P}$ is of class* (S). *For any disjoint supportable alternative $\mathfrak{Q}$ to $\mathfrak{P}$, the Dedekind completion of $L^\infty(\mathfrak{P})$ may be identified with the space*

$$\mathcal{Y} := \Big\{ \mathbf{X} \in \prod_{\mathbb{Q} \in \mathfrak{Q}} L^\infty(\mathbb{Q}) \mid \exists\, m > 0 \,\forall\, \mathbb{Q} \in \mathfrak{Q} : \ |X_\mathbb{Q}| \le m \ \mathbb{Q}\text{-a.s.} \Big\}.$$

Gao & Munari [21] and Maggis et al. [33] work under the assumption that $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ **(A6)**. This assumption implies Dedekind completeness of $L^\infty(\mathfrak{P})$ and the class (S) property of $\mathfrak{P}$; see Theorem 4.8 and Remark 4.13(2). Proposition 4.24 shows that they *necessarily* work with product spaces. This observation is in line with examples provided in these papers. Nevertheless one should note that the Dedekind completion $\mathcal{Y}$ is built on a (generally) uncountable number of copies of $(\Omega, \mathcal{F})$, each coordinate evaluated with a different probability measure. *A priori* it is therefore questionable whether an element in the Dedekind completion has an interpretation as an equivalence class of functions mapping Ω to $\mathbb{R}$, as would be necessary for aggregation procedures to be possible. There are however cases where this is possible as discussed in the following remark. We will also come back to this topic in Section 4.7.

Remark 4.25. Suppose $\mathfrak{P}$ is of class (S) and we can find a disjoint supportable alternative $\mathfrak{Q}$ and a family $(S_\mathbb{Q})_{\mathbb{Q} \in \mathfrak{Q}} \subset \mathcal{F}$ with properties (c) and (d) from Definition 4.15, i.e. the sets $S_\mathbb{Q}$, $\mathbb{Q} \in \mathfrak{Q}$, are pairwise disjoint versions of the respective order supports. In addition to the Dedekind completion $\mathcal{Y}$ constructed in Proposition 4.24 consider the $\mathfrak{Q}$-completion $\mathcal{F}(\mathfrak{Q})$ of $\mathcal{F}$ and the set $\mathfrak{Q}^\sharp = \{\mathbb{Q}^\sharp \mid \mathbb{Q} \in \mathfrak{Q}\}$ of extensions to $\mathcal{F}(\mathfrak{Q})$ as defined in Section 2. Given $(X_\mathbb{Q})_{\mathbb{Q} \in \mathfrak{Q}} \in \mathcal{Y}$ and any choice $(f^\mathbb{Q})_{\mathbb{Q} \in \mathfrak{Q}}$ of uniformly bounded $\mathbb{Q}$-representatives $f^\mathbb{Q} \in X_\mathbb{Q}$, the function

$$g(\omega) := \sum_{\mathbb{Q} \in \mathfrak{Q}} f^\mathbb{Q}(\omega) \mathbf{1}_{S_\mathbb{Q}}(\omega), \quad \omega \in \Omega,$$

is well-defined and $\mathcal{F}(\mathfrak{Q})$-measurable. This procedure yields a map $J_1 : \mathcal{Y} \to L^\infty(\mathfrak{Q}^\sharp) = L^\infty(\Omega, \mathcal{F}(\mathfrak{Q}), \mathfrak{Q}^\sharp)$. Conversely, given $Y \in L^\infty(\mathfrak{Q}^\sharp)$ and a bounded $\mathcal{F}(\mathfrak{Q})$-measurable $g \in Y$, we can find for all $\mathbb{Q} \in \mathfrak{Q}$ an $\mathcal{F}$-measurable $f^\mathbb{Q}$ such that $\{g \neq f^\mathbb{Q}\} \in \mathfrak{n}(\mathbb{Q})$. If $X_\mathbb{Q}$ denotes the equivalence class generated by $f^\mathbb{Q}$ in $L^\infty(\mathbb{Q})$, $(X_\mathbb{Q})_{\mathbb{Q} \in \mathfrak{Q}}$ lies in $\mathcal{Y}$. This procedure gives rise to a map $J_2 : L^\infty(\mathfrak{Q}^\sharp) \to \mathcal{Y}$. It is straightforward to prove that J_1, J_2 are lattice homomorphisms which satisfy $J_1 \circ J_2 = id_{L^\infty(\mathfrak{Q}^\sharp)}$ and $J_2 \circ J_1 = id_\mathcal{Y}$. Hence, $L^\infty(\mathfrak{Q}^\sharp)$ and $\mathcal{Y}$ are lattice isomorphic, and $L^\infty(\Omega, \mathcal{F}(\mathfrak{Q}), \mathfrak{Q}^\sharp)$ is the Dedekind completion of $L^\infty(\mathfrak{P})$ as well. In particular, this observation applies to sets of priors with the Hahn property and is in line with [18, Theorem 3.16].

At last, one may wonder if Theorem 4.22(3) holds without the cardinality condition. This question remains open to us, see Question 4.41 below. However, the class (S) property implies the Hahn property under a suitable change of the underlying measurable space.

Proposition 4.26. *Suppose $\mathfrak{P}$ is a set of priors on $(\Omega, \mathcal{F})$ which is of class* (S)*. Then there is a measurable space $(\widehat{\Omega}, \widehat{\mathcal{F}})$ and a set of priors $\widehat{\mathfrak{Q}}$ which has the Hahn property such that $L^\infty(\Omega, \mathcal{F}, \mathfrak{P})$ is lattice isomorphic to an order dense and majorising sublattice of the Dedekind completion $L^\infty(\widehat{\Omega}, \widehat{\mathcal{F}}(\widehat{\mathfrak{Q}}), \widehat{\mathfrak{Q}}^\sharp)$ of $L^\infty(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathfrak{Q}})$.*

4.5. **Dedekind completeness and aggregation.** The aim of this section is to provide a precise statement on the equivalence between Dedekind completeness and the feasibility of aggregating suitably compatible random variables as is common practice in quasi-sure analysis.

Definition 4.27. Let $\mathfrak{P}$ be of class (S) and let $\mathfrak{Q}$ be a supportable alternative. A family $(X^\mathbb{Q})_{\mathbb{Q} \in \mathfrak{Q}} \subset L^\infty(\mathfrak{P})$ such that $X^\mathbb{Q} \in \mathbf{1}_{S(\mathbb{Q})} L^\infty(\mathfrak{P})$, $\mathbb{Q} \in \mathfrak{Q}$, is called COMPATIBLE if it is order bounded above and

$$X^\mathbb{Q} \mathbf{1}_{S(\mathbb{Q}) \cap S(\tilde{\mathbb{Q}})} = X^{\tilde{\mathbb{Q}}} \mathbf{1}_{S(\mathbb{Q}) \cap S(\tilde{\mathbb{Q}})} \text{ in } L^\infty(\mathfrak{P})$$

for all $\mathbb{Q}, \tilde{\mathbb{Q}} \in \mathfrak{Q}$. $\mathfrak{Q}$ has the AGGREGATION PROPERTY if for every compatible family $(X^\mathbb{Q})_{\mathbb{Q} \in \mathfrak{Q}}$ there is an equivalence class $Y \in L^\infty(\mathfrak{P})$—called the AGGREGATOR of $(X^\mathbb{Q})_{\mathbb{Q} \in \mathfrak{Q}}$—such that $Y \mathbf{1}_{S(\mathbb{Q})} = X^\mathbb{Q} \mathbf{1}_{S(\mathbb{Q})}$, $\mathbb{Q} \in \mathfrak{Q}$.

The existence of aggregations as in the preceding definition is prominently studied by Cohen [18] and Soner et al. [41], which fall in the class (S) setting as demonstrated in Section 4.3. The next result shows that [18, 41] essentially prove Dedekind completeness of a suitable space of random variables.

Theorem 4.28. *Suppose $\mathfrak{P}$ is of class* (S)*. Then the following are equivalent:*
(1) *$L^\infty(\mathfrak{P})$ and $\mathcal{Y} := \{\mathbf{X} \in \prod_{\mathbb{Q} \in \mathfrak{Q}} L^\infty(\mathbb{Q}) \mid \exists m > 0 \forall \mathbb{Q} \in \mathfrak{Q} : |X_\mathbb{Q}| \leq m \text{ } \mathbb{Q}\text{-a.s.}\}$ are lattice isomorphic. Here, $\mathfrak{Q}$ is a disjoint supportable alternative to $\mathfrak{P}$.*
(2) *$L^\infty(\mathfrak{P})$ is Dedekind complete.*
(3) *Every supportable alternative $\mathfrak{Q}$ of $\mathfrak{P}$ has the aggregation property.*
(4) *$\mathfrak{P}$ has a supportable alternative $\mathfrak{Q}$ with the aggregation property.*

We can also link Dedekind completeness in conjunction with the class (S) property to perfectness of the space $L^\infty(\mathfrak{P})$.

Theorem 4.29. *Let $\mathfrak{P}$ be a set of probability measures on $(\Omega, \mathcal{F})$. Then the following are equivalent:*

(1) *$L^\infty(\mathfrak{P})$ is perfect, i.e. $(L^\infty(\mathfrak{P})_n^\sim)_n^\sim = L^\infty(\mathfrak{P})$ via the embedding (2.2).*
(2) *$\mathfrak{P}$ is of class (S) and $L^\infty(\mathfrak{P})$ is Dedekind complete.*
(3) *$\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ as in (4.2).*

Theorem 4.29(3) weakens the rather restrictive assumption **(A6)** in the case $\mathcal{X} = L^\infty(\mathfrak{P})$, namely that $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$, to $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$. Clearly, both assumptions coincide in case $\mathbf{sca}(\mathfrak{P}) = \mathbf{ca}(\mathfrak{P})$. The question is how much weaker $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ or equivalently $\mathfrak{P}$ being of class (S) and $L^\infty(\mathfrak{P})$ being Dedekind complete is compared to **(A6)**. The next section shows that the answer to the aforementioned question how much weaker exactly $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ is compared to **(A6)** will lead us to the fringes of classical mathematics.

4.6. **Dedekind completeness and the limits of ZFC.** It will turn out that the relation of the condition $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ with the condition $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ is intimately linked to *Banach's measure problem* [5].

Definition 4.30. A non-empty set S is said to admit solutions to Banach's measure problem if there is a probability measure $\pi : 2^S \to [0,1]$ on the power set 2^S with the property $\pi(\{s\}) = 0$ for all $s \in S$.

If S admits a solution to Banach's measure problem, its cardinality $\kappa := |S|$ plays a fundamental role in logic and set theory; see [23, Chapter 10]. It may qualify for properties such as being *measurable*, *real-valued measurable*, or *inaccessible*. These properties are very hard to grasp in the classical framework of mathematics, **ZFC**, which is for instance consistent with the assumption that inaccessible cardinals do not exist. However, such large and measurable cardinals play a fundamental role in various branches of mathematics; cf. [24, Chapter 1.2].

Theorem 4.31. *The following are equivalent:*

(1) *$\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$, i.e. **(A6)** holds for $L^\infty(\mathfrak{P})$.*
(2) *$\mathfrak{P}$ is of class (S), $L^\infty(\mathfrak{P})$ is Dedekind complete—or equivalently: $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$— and no disjoint supportable alternative $\mathfrak{Q}$ admits a solution to Banach's measure problem.*

The following corollary embeds class (S) spaces in an important observation due to Luxemburg [31]. We particularly emphasise the equivalence of (2) and (3) which means that the robust spaces $L^\infty(\mathfrak{P})$ are in some sense of universal relevance for vector lattices.

Corollary 4.32. *Consider the following statements:*

(1) *Banach's measure problem has no solution.*
(2) *Each Dedekind complete vector lattice $\mathcal{X}$ satisfies $\mathcal{X}_c^\sim = \mathcal{X}_n^\sim$.*

(3) *For each set $\mathfrak{P}$ of probability measures on a measurable space $(\Omega, \mathcal{F})$, $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$—i.e. the space $L^\infty(\mathfrak{P})$ has property* **(A6)**—*if and only if $L^\infty(\mathfrak{P})$ is Dedekind complete* **(A4)**.

(4) *For each set $\mathfrak{P}$ of probability measures on a measurable space $(\Omega, \mathcal{F})$, $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$—i.e. the space $L^\infty(\mathfrak{P})$ has property* **(A6)**—*if and only if $\mathfrak{P}$ is of class* (S) *and $L^\infty(\mathfrak{P})$ is Dedekind complete* **(A4)**.

(5) *For each set $\mathfrak{P}$ of probability measures on a measurable space $(\Omega, \mathcal{F})$, $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$—i.e. the space $L^\infty(\mathfrak{P})$ has property* **(A6)**—*if and only if $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$.*

(6) *$L^\infty(\mathfrak{P})$ is Dedekind complete only if $\mathfrak{P}$ is of class* (S).

Then (1)–(5) *are equivalent, and they all imply* (6).

Corollary 4.33. *It is impossible in* **ZFC** *to construct an example of a measurable space $(\Omega, \mathcal{F})$ and a set $\mathfrak{P}$ of probability measures such that*

(1) *$L^\infty(\mathfrak{P})$ is Dedekind complete, but $\mathbf{ca}(\mathfrak{P})^* \neq L^\infty(\mathfrak{P})$, or*

(2) *$L^\infty(\mathfrak{P})$ is Dedekind complete, but $\mathfrak{P}$ is not of class* (S).

At last, Theorem 4.31 admits to show that the equivalence of $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ with the seemingly weaker condition $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ is consistent with **ZFC**, provided the cardinality of the set of measures is not too large. To this end, we recall that the continuum hypothesis **CH** is consistent with **ZFC**; see Gödel [22].[4]

Corollary 4.34. *In* **ZFC+CH** *let $\mathfrak{P}$ be a set of probability measures on a measurable space $(\Omega, \mathcal{F})$ and suppose that any disjoint set $\mathfrak{R} \subset \mathbf{sca}(\mathfrak{P})$ of supportable probability measures is at most equinumerous with the continuum. Then the following are equivalent:*

(1) *$\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$, i.e. $L^\infty(\mathfrak{P})$ has property* **(A6)**.

(2) *$\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$, i.e. $L^\infty(\mathfrak{P})$ has property* **(A5)**.

(3) *$\mathfrak{P}$ is of class* (S) *and $L^\infty(\mathfrak{P})$ is Dedekind complete, i.e. it has property* **(A4)**.

Under any of the conditions (1)–(3) *we have $\mathbf{ca}(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})$, that is,* **(A3)**.

This family of results implies that within **ZFC** it is impossible to decide whether $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$ is weaker than $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$. Corollary 4.34 shows the equivalence of those properties in **ZFC+CH**, provided any disjoint set of supportable probability measures is at most equinumerous with the continuum. The latter is automatically satisfied if, for instance, $(\Omega, \mathcal{F})$ is Polish—a situation encountered regularly—because then $\Delta(\mathcal{F})$ itself is at most equinumerous with the continuum.

Example 4.35. Recall the situation of Example 4.11. There is an obvious surjection

$$\widehat{\vartheta} : L^2 := L^2\left([0, T], \mathbb{B}([0, T]), \lambda\right) \to \mathfrak{P},$$

---

[4] The continuum hypothesis states that there is no set M such that $\aleph_0 < |M| < 2^{\aleph_0}$. Here, $\aleph_0$ denotes the cardinality of $\mathbb{N}$ and $2^{\aleph_0}$ the cardinality of the power set $2^{\mathbb{N}}$, which is precisely the cardinality of the continuum.

where λ denotes the Lebesgue measure on $([0,T],\mathbb{B}([0,T]))$. $L^2([0,T],\mathbb{B}([0,T]),\lambda)$ is Polish, hence $|L^2| = |\mathbb{R}|$, and the disjoint set $\mathfrak{P}$ of probability measures is equinumerous with the continuum. By Theorem 4.22 the order supports of measures in $\mathfrak{P}$ may be chosen such that they are pairwise disjoint. By Remark 4.25, the Dedekind completion of $L^\infty(\mathfrak{P})$ is the space $L^\infty(\mathfrak{P}^\sharp)$, where $\mathfrak{P}^\sharp = \{\mathbb{P}^\sharp \mid \mathbb{P} \in \mathfrak{P}\}$ denotes the set of extensions of the probability measures in $\mathfrak{P}$ to the $\mathfrak{P}$-completion $\mathcal{F}(\mathfrak{P})$ of $\mathcal{F}$. Moreover, $\mathfrak{P}^\sharp$ is of class (S) and its own supportable alternative; see Theorem 4.38 below.

Corollary 4.34 shows the identity $\mathbf{sca}(\mathfrak{P}^\sharp) = \mathbf{ca}(\mathfrak{P}^\sharp)$ in **ZFC+CH**. By Proposition 4.10 and Example 4.11, however, $\mathbf{sca}(\mathfrak{P}) \subsetneq \mathbf{ca}(\mathfrak{P})$ holds for the non-Dedekind complete space $L^\infty(\Omega, \mathcal{F}, \mathfrak{P})$. Hence, the pathological measures and functionals defined by (6.2) in the proof of Proposition 4.10 can in particular not be extended to the larger σ-algebra $\mathcal{F}(\mathfrak{P})$ on Ω. We will come back to this issue in Corollary 4.40.

4.7. **Enlargement of σ-algebras.** So far we have discussed necessary and sufficient conditions for and consequences of Dedekind completeness of the space $L^\infty(\mathfrak{P})$. Proposition 4.24 shows that the Dedekind completion of $L^\infty(\mathfrak{P})$ is a product space under the class (S) property. Remark 4.25 proved that Dedekind completion can also be obtained by completing the underlying σ-algebra along a disjoint supportable alternative in case the latter admits the selection of pairwise disjoint versions of its order supports. Proposition 4.26 illustrated that finding these can necessitate changing the underlying measurable space.

In this final section, we therefore address the more specific question under which conditions this change is not necessary. That is, the Dedekind completion of $L^\infty(\mathfrak{P})$ can be obtained by appropriately enlarging the underlying σ-algebra $\mathcal{F}$ on Ω—thereby weakening the notion of measurability—and changing to a new set of priors on the larger σ-algebra.

Definition 4.36. Given a measurable space $(\Omega, \mathcal{F})$ endowed with a set of priors $\mathfrak{P}$, an ENLARGEMENT of $(\Omega, \mathcal{F}, \mathfrak{P})$ is a tuple $(\mathcal{G}, \widehat{\mathfrak{P}})$, where $\mathcal{F} \subset \mathcal{G} \subset 2^\Omega$ is a σ-algebra on Ω, and $\widehat{\mathfrak{P}}$ is a set of priors on $(\Omega, \mathcal{G})$ such that, for all $N \in \mathcal{F}$, $\sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{P}(N) = 0$ is equivalent to $\sup_{\widehat{\mathbb{P}} \in \widehat{\mathfrak{P}}} \widehat{\mathbb{P}}(N) = 0$, that is, $\widehat{\mathfrak{P}}|_\mathcal{F} \approx \mathfrak{P}$. We define the embedding

$$\iota_\mathcal{G} : L^\infty(\Omega, \mathcal{F}, \mathfrak{P}) \to L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}}), \quad X = [f] \mapsto \langle f \rangle, \tag{4.4}$$

where $\langle f \rangle$ denotes the equivalence class generated by an $\mathcal{F}$-measurable random variable f in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$. An enlargement $(\mathcal{G}, \widehat{\mathfrak{P}})$ COMPLETES $L^\infty(\mathfrak{P})$ if $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$ is the Dedekind completion of $L^\infty(\mathfrak{P})$ and $\iota_\mathcal{G}(L^\infty(\mathfrak{P}))$ is the order dense majorising copy of $L^\infty(\mathfrak{P})$ in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$.

Note that the condition $\widehat{\mathfrak{P}}|_\mathcal{F} \approx \mathfrak{P}$ ensures that $\iota_\mathcal{G}$ is well-defined and one-to-one, which is necessary if $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$ completes $L^\infty(\mathfrak{P})$. Let us introduce some candidates for completing enlargements we will consider in the following:

  Universal enlargement: Recall that $\Delta(\mathcal{F})$ denotes the set of all probability measures on $(\Omega, \mathcal{F})$. Consider the enlargement $(\mathcal{H}, \mathfrak{P}^\mathcal{H})$, where $\mathcal{H} := \mathcal{F}(\Delta(\mathcal{F}))$ is the *universal completion* of $\mathcal{F}$, and $\mathfrak{P}^\mathcal{H} := \{\mathbb{P}^\mathcal{H} \mid \mathbb{P} \in \mathfrak{P}\}$ is the set of unique extensions of the initial priors to $(\Omega, \mathcal{H})$; cf. Section 2. Each $\mu \in \mathbf{ca}$ has a unique extension to $\mathcal{H}$ which we

will denote by $\mu^{\mathcal{H}}$. The $\mathfrak{P}^{\mathcal{H}}$-q.s. order on $L^\infty(\mathfrak{P}^{\mathcal{H}}) := L^\infty(\Omega, \mathcal{H}, \mathfrak{P}^{\mathcal{H}})$ will be denoted by $\preceq^{\mathcal{H}}$. Universal completions play an important role in [14, 16, 38], e.g. in the proof of the measurability of so-called "arbitrage aggregators".

$\mathfrak{P}$-universal enlargement: This refers to $(\mathcal{A}, \mathfrak{P}^{\mathcal{A}})$, where $\mathcal{A} := \mathcal{F}(\mathbf{ca}(\mathfrak{P})_+)$ denotes the completion along all (probability) measures absolutely continuous with respect to $\mathfrak{P}$, and $\mathfrak{P}^{\mathcal{A}} := \{\mathbb{P}^{\mathcal{A}} \mid \mathbb{P} \in \mathfrak{P}\}$ denotes the set of extensions of priors $\mathbb{P} \in \mathfrak{P}$ to $\mathcal{A}$. We set $L^\infty(\mathfrak{P}^{\mathcal{A}}) := L^\infty(\Omega, \mathcal{A}, \mathfrak{P}^{\mathcal{A}})$.

Supportable enlargement: Suppose $\mathfrak{P}$ is of class (S). We will then consider the enlargement $(\mathcal{S}, \mathfrak{Q}^{\sharp})$, where $\mathcal{S} := \mathcal{F}(\mathbf{sca}(\mathfrak{P})_+)$ is the completion along all supportable (probability) measures, and $\mathfrak{Q}^{\sharp} := \{\mathbb{Q}^{\sharp} \mid \mathbb{Q} \in \mathfrak{Q}\}$ is the set of extensions of a supportable alternative $\mathfrak{Q}$ of $\mathfrak{P}$ to $\mathcal{S}$. This leads to the space $L^\infty(\mathfrak{Q}^{\sharp}) := L^\infty(\Omega, \mathcal{S}, \mathfrak{Q}^{\sharp})$ endowed with the $\mathfrak{Q}^{\sharp}$-q.s. order which we denote by $\preceq^{\sharp}$. Note that $L^\infty(\mathfrak{Q}^{\sharp})$ does not depend on the particular choice of the supportable alternative $\mathfrak{Q}$, but any choice produces the same space. A bounded $\mathcal{S}$-measurable function h induces an equivalence class $[h]_{\sharp} \in L^\infty(\mathfrak{Q}^{\sharp})$. Completions along a particular set of priors are crucial in [41, Section 7] and throughout [18] to construct a conditional version of sublinear expectations. We also refer to [17, 36, 38].

A priori it is clear that $\mathcal{F} \subset \mathcal{H} \subset \mathcal{A}$, and $\mathcal{A} \subset \mathcal{S}$ whenever $\mathfrak{P}$ is of class (S). The next proposition shows that $\mathcal{S}$ can equivalently obtained by completing $\mathcal{F}$ along an arbitrary supportable alternative.

Proposition 4.37. *Suppose $\mathfrak{P}$ is of class* (S) *and $\mathfrak{Q}$ is a supportable alternative to $\mathfrak{P}$. Then $\mathcal{S} = \mathcal{F}(\mathfrak{Q})$.*

Theorem 4.38. *Let $\mathfrak{P}$ be of class* (S) *and let $\mathfrak{Q}$ be a supportable alternative to $\mathfrak{P}$.*

(1) *Suppose that $L^\infty(\mathfrak{P})$ is Dedekind complete. Then*

$$\iota_{\mathcal{H}}(L^\infty(\mathfrak{P})) = L^\infty(\mathfrak{P}^{\mathcal{H}}), \quad \iota_{\mathcal{A}}(L^\infty(\mathfrak{P})) = L^\infty(\mathfrak{P}^{\mathcal{A}}), \quad \iota_{\mathcal{S}}(L^\infty(\mathfrak{P})) = L^\infty(\mathfrak{Q}^{\sharp}).$$

Hence, in this case we may assume $\mathcal{F} \in \{\mathcal{H}, \mathcal{A}, \mathcal{S}\}$ without altering the structure of the corresponding L^∞-space up to lattice isomorphisms.

(2) *Suppose an enlargement $(\mathcal{G}, \widehat{\mathfrak{P}})$ completes $L^\infty(\mathfrak{P})$. Then each $\mu \in \mathbf{sca}(\mathfrak{P})$ extends uniquely to a $\widehat{\mu} \in \mathbf{sca}(\widehat{\mathfrak{P}})$. In particular, $\mathfrak{Q}$ extends to a supportable alternative $\widehat{\mathfrak{Q}} := \{\widehat{\mathbb{Q}} \mid \mathbb{Q} \in \mathfrak{Q}\} \approx \widehat{\mathfrak{P}}$ and we may assume that $\mathcal{G} = \mathcal{G}(\mathbf{sca}(\widehat{\mathfrak{P}})_+)$ and*

$$\mathcal{F} \subset \mathcal{H} \subset \mathcal{A} \subset \mathcal{S} \subset \mathcal{G}.$$

Hence, if Dedekind completion of $L^\infty(\Omega, \mathcal{F}, \mathfrak{P})$ is obtained by an enlargement $(\mathcal{G}, \widehat{\mathfrak{P}})$, $\mathcal{G}$ should be at least as large as the supportable completion $\mathcal{S}$ of $\mathcal{F}$. Therefore, neither $L^\infty(\mathfrak{P}^{\mathcal{H}})$ nor $L^\infty(\mathfrak{P}^{\mathcal{A}})$ should be Dedekind complete. We will make this intuition precise in many cases of an underlying Polish Borel structure, see Corollary 4.40 below. In these cases, we know already that the Dedekind completion is constructed by completing along any supportable alternative $\mathfrak{Q}$. Hence, the ($\mathfrak{P}$-)universal enlargement is not a sensible choice if the aim is

aggregation. Note that the class (S) assumption does not pose a severe restriction here, as class (S) cannot be disproved under Dedekind completeness.

Next we focus on the situation when the supportable enlargement completes $L^\infty(\mathfrak{P})$. Our result again elaborates the connection to the Hahn property and complements Theorem 4.22 and Proposition 4.26.

Theorem 4.39. *Suppose that $\mathfrak{P}$ is of class* (S) *and admits a supportable alternative at most equinumerous with the continuum. Then the following are equivalent:*

(1) *There is a set of probability measures $\mathfrak{P}^* \approx \mathfrak{P}$ such that $\mathfrak{P}^*$ has the Hahn property.*

(2) *There exists an enlargement $(\mathcal{G}, \widehat{\mathfrak{P}})$ which completes $L^\infty(\mathfrak{P})$ in such a way that there exists a supportable alternative $\mathfrak{O} \approx \mathfrak{P}$ such that for all $\mathbb{O} \in \mathfrak{O}$ and all $B \in \mathcal{G}$ there is $A \in \mathcal{F}$ with $A \subset B$ such that $\mathbb{O}(A) = \widehat{\mathbb{O}}(B)$. Here $\widehat{\mathbb{O}}$ denotes the unique extension of $\mathbb{O}$ to $\mathcal{G}$; see Theorem 4.38(2).*

(3) *The supportable enlargement $(\mathcal{S}, \mathfrak{Q}^\sharp)$ completes $L^\infty(\mathfrak{P})$ for any choice of a supportable alternative $\mathfrak{Q} \approx \mathfrak{P}$.*

At last, in light of Theorem 4.38(1), it seems unlikely that $L^\infty(\mathfrak{P}^\mathcal{H})$ or $L^\infty(\mathfrak{P}^\mathcal{A})$ are Dedekind complete. The next corollary provides many typical situations in which this intuition holds true.

Corollary 4.40. *In* **ZFC+CH** *assume that that $\mathfrak{P}$ is of class* (S) *and admits a supportable alternative at most equinumerous with the continuum. If $\mathbf{ca}(\mathfrak{P}) \backslash \mathbf{sca}(\mathfrak{P})$ is non-empty, then neither $L^\infty(\mathfrak{P}^\mathcal{H})$ nor $L^\infty(\mathfrak{P}^\mathcal{A})$ are Dedekind complete. In particular, this assertion holds if Ω is Polish, $\mathcal{F}$ is the Borel-σ-algebra on Ω, $\mathfrak{P}$ is a set of Borel priors which is of class* (S), *and a disjoint supportable alternative $\mathfrak{Q}$ contains a perfect or uncountable analytic or Borel subset $\mathfrak{R}$ of $\Delta(\mathcal{F})$.*

Corollary 4.40 for instance applies in the situation of Example 4.11.

4.8. **A caveat.** Our investigations naturally lead to a problem which remains open to us:

Question 4.41. Is there an example of a measurable space $(\Omega, \mathcal{F})$, a non-dominated set $\mathfrak{P}$ of priors which is of class (S), and a disjoint supportable alternative $\mathfrak{Q}$ such that no selection $(S_\mathbb{Q})_{\mathbb{Q} \in \mathfrak{Q}}$ of *pairwise disjoint* versions of the order supports of $\mathfrak{Q}$ exists? If yes, can an example be given such that $L^\infty(\mathfrak{P})$ is Dedekind complete?

However, a negative answer to Question 4.41 or a proof that no such examples can be found in **ZFC** would sharpen many of our results. For instance, by Remark 4.25, Dedekind completion would then always be given by the supportable enlargement.

5. Properties of order supports

In this section we state and prove a few properties of order supports.

In some aspects, the following lemma reflects observations made by Cohen in the context of the Hahn property. We refer to [18, Lemmas 3.5 & 3.14].

Lemma 5.1. *Let $\mathfrak{P}$ be a set of probability measures on a measurable space $(\Omega, \mathcal{F})$ and suppose $\mu, \nu \in \mathbf{sca}(\mathfrak{P})_{++}$.*

(1) *If $A \in \mathcal{F}$ with $A \subset S(\mu)$, then either $\mu(A) > 0$ or $\mathbf{1}_A = 0$ in $L^0(\mathfrak{P})$.*

(2) *If $S := S(\mu) \cap S(\nu)$ satisfies $\mathbf{1}_S \neq 0$, then $\mu(\cdot|S) \approx \nu(\cdot|S)$.[5]*

(3) *Suppose $\mathcal{C} \subset L^0(\mathfrak{P})$ is order bounded from above, let $S = S(\mu) \cap S(\nu)$, and assume $\mathbf{1}_S \neq 0$. Let f and g be measurable with f being a μ-representative of $\sup j_\mu(\mathcal{C})$ in $L^0(\mu)$ and g being a ν-representative of $\sup j_\nu(\mathcal{C})$ in $L^0(\nu)$. Then $[f\mathbf{1}_S] = [g\mathbf{1}_S]$.*

Proof. (1) Suppose $A \in \mathcal{F}$, $A \subset S(\mu)$, and $\mu(A) = 0$. Then $\mu(S(\mu)\backslash A) = \mu(\Omega)$ and $\mathbf{1}_{S(\mu)} = \mathbf{1}_{S(\mu)\backslash A}$ by Definition 4.1(b). We infer that in $L^0(\mathfrak{P})$

$$\mathbf{1}_A = \mathbf{1}_{S(\mu)} - \mathbf{1}_{S(\mu)\backslash A} = 0.$$

(2) Suppose $N \in \mathcal{F}$ is a $\mu(\cdot|S)$-null set, i.e. $\mu(N \cap S) = 0$. As $N \cap S \subset S(\mu)$, (1) implies $\mathbf{1}_{N \cap S} = 0$ q.s. and hence

$$\nu(N|S) = \frac{\nu(N \cap S)}{\nu(S)} = 0.$$

The symmetry of the argument yields the equivalence of $\mu(N|S) = 0$ and $\nu(N|S) = 0$ for all $N \in \mathcal{F}$.

(3) First of all, both f and g exist because the spaces $L^0(\mu)$ and $L^0(\nu)$ are super Dedekind complete by [20, Theorem A.37] and $j_\mu(\mathcal{C})$ and $j_\nu(\mathcal{C})$ are order bounded from above in $L^0(\mu)$ and $L^0(\nu)$, respectively. For any $X \in \mathcal{C}$ and any representative $h \in X$, we have

$$\mu(\{h > f\} \cap S) = 0,$$

because f is a μ-representative of $\sup j_\mu(\mathcal{C})$. As $\{h > f\} \cap S \subset S(\mu)$, $\mathbf{1}_{\{h>f\}\cap S} = 0$ by (1). Since $\nu \ll \mathfrak{P}$ we also have

$$\nu(\{h > f\} \cap S) = 0,$$

and thus

$$\nu(\{g > f\} \cap S) = 0.$$

Again by (1) we obtain $\mathbf{1}_{\{g>f\}\cap S} = 0$. A symmetric argument shows $\mathbf{1}_{\{f>g\}\cap S} = 0$. We infer $f\mathbf{1}_S = g\mathbf{1}_S$ q.s. □

Lemma 5.2. *Let $\mathfrak{P}$ be a set of probability measures on a measurable space $(\Omega, \mathcal{F})$. Let $\mathcal{X} \subset L^0(\mathfrak{P})$ be a Banach lattice which contains all constant random variables, let $\mu \in \mathbf{sca}(\mathcal{X})$, and let $\phi_\mu \in \mathcal{X}_c^\sim$ be the functional on $\mathcal{X}$ associated to μ. Then the carrier of ϕ_μ satisfies $C(\phi_\mu) = \mathbf{1}_{S(\mu)}\mathcal{X}$, whereas the null ideal of ϕ_μ satisfies $N(\phi_\mu) = \mathbf{1}_{S(\mu)^c}\mathcal{X}$. Both are bands in $\mathcal{X}$ which may be decomposed as the direct sum*

$$\mathcal{X} = (\mathbf{1}_{S(\mu)}\mathcal{X}) \oplus (\mathbf{1}_{S(\mu)^c}\mathcal{X}) = C(\phi_\mu) \oplus N(\phi_\mu). \tag{5.1}$$

---

[5] The latter mean the measures conditioned on the set S; more precisely, $\mu(\cdot|S)$ is defined by $\mu(A|S) := \frac{\mu(A \cap S)}{\mu(S)}$, $A \in \mathcal{F}$.

Proof. Suppose $\mu \in \mathbf{sca}(\mathcal{X})$. We may assume without loss of generality that μ is positive. Then $\mathbf{1}_{S(\mu)^c}\mathcal{X}$ is the null ideal $N(\phi_\mu)$ of the functional ϕ_μ. Clearly, as $\mu(S(\mu)^c) = 0$, for all $X \in \mathcal{X}$ we have

$$\phi_\mu(|X\mathbf{1}_{S(\mu)^c}|) = \int |X|\mathbf{1}_{S(\mu)^c}\, d\mu = 0.$$

Conversely, suppose $X \in \mathcal{X}$ satisfies $\phi_\mu(|X|) = 0$. Then for all $f \in X$

$$0 = \phi_\mu(|X|) \geq \phi_\mu(|X|\mathbf{1}_{S(\mu)}) = \int |f|\mathbf{1}_{S(\mu)}d\mu \geq 0.$$

This implies $|f|\mathbf{1}_{S(\mu)} = 0$ q.s. by Lemma 5.1(1) and $X = X\mathbf{1}_{S(\mu)^c}$. At last,

$$C(\phi_\mu) = N(\phi_\mu)^{\mathrm{d}} = (\mathbf{1}_{S(\mu)^c}\mathcal{X})^{\mathrm{d}} = \mathbf{1}_{S(\mu)}\mathcal{X}.$$

By Lemma A.1, both summands appearing in (5.1) are indeed bands, and the equation is a decomposition of $\mathcal{X}$ as a direct sum. $\square$

A consequence of the next important observation is that the order of a class (S) space is locally super Dedekind complete, one of the main reasons why the class (S) property makes robustness tractable.

Proposition 5.3. *Suppose $\mu \in \mathbf{sca}(\mathfrak{P})_{++}$, $A \subset S(\mu)$ is measurable, and $\mu(A) > 0$. Set $\mathcal{B} := \mathbf{1}_A L^0(\mathfrak{P})$. Then:*

(1) *For all $X, Y \in \mathcal{B}$:*

$$X \preceq Y \iff j_\mu(X) \leq j_\mu(Y) \text{ in } L^0(\mu). \tag{5.2}$$

(2) *$\mathcal{B}$ is a super Dedekind complete band.*

(3) *μ dominates $\{\nu(\cdot \cap A) \mid \nu \in \mathbf{ca}(\mathfrak{P}), |\nu|(A) > 0\}$.*

Proof. (1) Two elements $X, Y \in \mathcal{B}$ satisfy $X \preceq Y$ if and only if $\sup_{k \in \mathbb{N}}(X - Y)^+ \wedge k = 0$. As $Z_k := (X - Y)^+ \wedge k \in C(\phi_\mu)$ by Lemma 5.2 and ϕ_μ is strictly positive on its carrier (see the remark preceding [2, Lemma 1.80]), this is equivalent to

$$\sup_{k \in \mathbb{N}} \phi_\mu((X - Y)^+ \wedge k) = 0.$$

However, we have for all $k \in \mathbb{N}$ that

$$\phi_\mu((X - Y)^+ \wedge k) = \int (j_\mu(X) - j_\mu(Y))^+ \wedge k\, d\mu.$$

Hence, $X \preceq Y$ is equivalent to $\int (j_\mu(X) - j_\mu(Y))^+\, d\mu = 0$, which is itself equivalent to $j_\mu(X) \leq j_\mu(Y)$ in $L^0(\mu)$. We have proved (5.2).

(2) The space $\mathcal{B}$ is a band by Lemma A.1 and has the countable sup property by Lemma 5.2 and [2, Lemma 1.80]. Dedekind completeness of $\mathcal{B}$ follows from (5.2) and the super Dedekind completeness of $L^0(\mu)$.

(3) This is a direct consequence of Lemma 5.1(1).

$\square$

Corollary 5.4. *Let $\mathfrak{P}$ be a set of probability measures on $(\Omega, \mathcal{F})$.*

(1) *Suppose $\mu \in \mathbf{ca}(\mathfrak{P})$ and a sequence $(\mu_n)_{n \in \mathbb{N}} \subset \mathbf{sca}(\mathfrak{P})_{++}$ satisfy $\mu \ll \{\mu_n \mid n \in \mathbb{N}\}$. Then $\mu \in \mathbf{sca}(\mathfrak{P})$.*

(2) *$\mathbf{sca}(\mathfrak{P})$ is an ideal of $\mathbf{ca}(\mathfrak{P})$ which is closed with respect to convergence in total variation.*

Proof. (1) Define
$$\nu := \sum_{n \in \mathbb{N}} \frac{1}{2^n \mu_n(\Omega)} \mu_n \in \mathbf{ca}(\mathfrak{P}).$$

We claim that ν is supportable. For versions $S(\mu_n)$ of the order supports, define $S := \bigcup_{n \in \mathbb{N}} S(\mu_n)$. Clearly, $\nu(S) = \nu(\Omega)$. Let now $A \in \mathcal{F}$ be arbitrary with the property $\nu(A) = \nu(\Omega)$. Then $\mu_n(A) = \mu_n(\Omega)$ has to hold for all $n \in \mathbb{N}$. This implies $\mathbf{1}_{S(\mu_n)} \preceq \mathbf{1}_A$ for all $n \in \mathbb{N}$, and hence $\mathbf{1}_S = \sup_{n \in \mathbb{N}} \mathbf{1}_{S(\mu_n)} \preceq \mathbf{1}_A$. Combining both observations, $\mathbf{1}_S = \mathbf{1}_{S(\nu)}$. As $|\mu|(S^c) = 0$, Proposition 5.3(3) yields that $|\mu| \ll \nu$. Hence, there is a ν-density f of $|\mu|$. One easily verifies $S(\mu) = S(|\mu|) = \{f > 0\} \cap S$.

(2) In order to show that $\mathbf{sca}(\mathfrak{P})$ is an ideal of $\mathbf{ca}(\mathfrak{P})$, it suffices to show that for all $\mu \in \mathbf{sca}(\mathfrak{P})_{++}$ and all $0 \preceq_{\mathcal{F}} \nu \preceq_{\mathcal{F}} \mu$ we have $\nu \in \mathbf{sca}(\mathfrak{P})_+$. As $\nu(S(\mu)^c) = 0$, Proposition 5.3(3) implies $\nu \ll \mu$. Let $f : \Omega \to \mathbb{R}$ be any version of $\frac{d\nu}{d\mu}$ and note that $\{f > 0\} \cap S(\mu)$ is the order support of ν.

For closedness, suppose $(\mu_n)_{n \in \mathbb{N}} \subset \mathbf{sca}(\mathfrak{P})$ satisfies $\lim_{n \to \infty} TV(\mu_n - \mu) = 0$. Let $N \in \mathcal{F}$ be such that $\sup_{n \in \mathbb{N}} |\mu_n|(N) = 0$. Then
$$|\mu|(N) = \lim_{n \to \infty} |\mu_n|(N) = 0.$$

Hence, $\mu \ll \{|\mu_n| \mid n \in \mathbb{N}\}$ and is thus supportable by (1).

$\square$

At last, we provide a local variant of the Radon-Nikodym Theorem. Given a family $\mathfrak{P}$ of probability measures, the σ-CONVEX HULL of $\mathfrak{P}$ is the set $\mathrm{co}_\sigma(\mathfrak{P})$ of all countable convex combinations
$$\sum_{n=1}^{\infty} \lambda_n \mathbb{P}_n, \quad (\mathbb{P}_n)_{n \in \mathbb{N}} \subset \mathfrak{P}, \ (\lambda_n)_{n \in \mathbb{N}} \subset [0,1], \ \sum_{n=1}^{\infty} \lambda_n = 1. \tag{5.3}$$

For a set $\mathfrak{P}$ of probability measures on $(\Omega, \mathcal{F})$ we introduce the space $\mathbf{ca}(\mathfrak{P})^\infty$ as the linear span of all signed measures
$$\mu_{\mathbb{P},g} : \mathcal{F} \ni A \mapsto \mathbb{E}_{\mathbb{P}}[g\mathbf{1}_A],$$

where g is measurable and bounded, and $\mathbb{P} \in \mathfrak{P}$. This space has been introduced by Gao & Munari [21], where it is denoted by $\mathbf{ca}_c^\infty$, as a simple and tractable ideal in $L^\infty(\mathfrak{P})^\sim$.

Corollary 5.5. *Let $\mathfrak{P}$ be a set of probability measures.*

(1) *$\mu \in \mathbf{ca}(\mathfrak{P})$ lies in the closure of $\mathbf{ca}(\mathfrak{P})^\infty$ with respect to the total variation norm TV if and only if there is $\mathbb{P}^* \in \mathrm{co}_\sigma(\mathfrak{P})$ and $h : \Omega \to \mathbb{R}$ measurable and $\mathbb{P}^*$-integrable such that*
$$\forall A \in \mathcal{F} : \ \mu(A) = \mathbb{E}_{\mathbb{P}^*}[h\mathbf{1}_A]. \tag{5.4}$$

(2) *If $\mathfrak{P}$ is of class* (S) *and $\mathfrak{Q}$ is a supportable alternative to $\mathfrak{P}$, $\mathbf{sca}(\mathfrak{P})$ is the closure of $\mathbf{ca}(\mathfrak{Q})^\infty$ with respect to TV. Hence, $\mu \in \mathbf{sca}(\mathfrak{P})$ if and only if there is $\mathbb{Q}^* \in \mathrm{co}_\sigma(\mathfrak{Q})$ and $\mathbb{Q}^*$-integrable $h : \Omega \to \mathbb{R}$ such that $\mu(A) = \mathbb{E}_{\mathbb{Q}^*}[h\mathbf{1}_A]$, $A \in \mathcal{F}$.*

Proof. (1) Assume first $\mu \in \mathbf{ca}(\mathfrak{P})$ lies in the closure of $\mathbf{ca}(\mathfrak{P})^\infty$ with respect to TV, i.e. there is a sequence $(\mu_k)_{k \in \mathbb{N}} \subset \mathbf{ca}(\mathfrak{P})^\infty$ such that $TV(\mu_k - \mu) \to 0$, $k \to \infty$. For each $k \in \mathbb{N}$ let $\mathfrak{P}_k \subset \mathfrak{P}$ be finite such that $|\mu_k| \ll \mathfrak{P}_k$. Let

$$\mathbb{P}^* := \sum_{k=1}^\infty 2^{-k} \sum_{\mathbb{P} \in \mathfrak{P}_k} \frac{1}{|\mathfrak{P}_k|} \mathbb{P} \in \mathrm{co}_\sigma(\mathfrak{P}),$$

where $|\mathfrak{P}_k|$ denotes the cardinality of $\mathfrak{P}_k$. Suppose $N \in \mathcal{F}$ satisfies $\mathbb{P}^*(N) = 0$. Then

$$|\mu|(N) \le |\mu - \mu_k|(N) + |\mu_k|(N) = |\mu - \mu_k|(N) \le TV(\mu - \mu_k).$$

Letting $k \to \infty$ on the right-hand side implies $|\mu|(N) = 0$. By virtue of the Radon-Nikodym Theorem, there is $h : \Omega \to \mathbb{R}$ measurable and $\mathbb{P}^*$-integrable such that (5.4) is satisfied.

Now suppose μ is of shape (5.4). Let $g_k := h\mathbf{1}_{\{|h| \le k\}}$, $k \in \mathbb{N}$, and define $\nu_k \in \mathbf{ca}(\mathfrak{P})$ by $\nu_k(A) := \mathbb{E}_{\mathbb{P}^*}[g_k \mathbf{1}_A]$. Then

$$TV(\mu - \nu_k) = \mathbb{E}_{\mathbb{P}^*}[|h|\mathbf{1}_{\{|h| > k\}}] \to 0, \quad k \to \infty,$$

where we used dominated convergence. Now suppose $(\mathbb{P}_n)_{n \in \mathbb{N}} \subset \mathfrak{P}$ and $(\lambda_n)_{n \in \mathbb{N}} \subset (0, 1]$ are sequences such that

$$\mathbb{P}^* = \sum_{n=1}^\infty \lambda_n \mathbb{P}_n.$$

Fix $k \in \mathbb{N}$ and let $\mu_N := \sum_{n=1}^N \lambda_n \mathbb{E}_{\mathbb{P}_n}[g_k \mathbf{1}.] \in \mathbf{ca}(\mathfrak{P})^\infty$, where $N \in \mathbb{N}$. Then

$$TV(\mu_N - \nu_k) = \sum_{n=N+1}^\infty \lambda_n \mathbb{E}_{\mathbb{P}_n}[|g_k|] \le k \sum_{n=N+1}^\infty \lambda_n \to 0, \quad N \to \infty.$$

This shows that ν_k lies in the closure of $\mathbf{ca}(\mathfrak{P})^\infty$, and *a fortiori* the same holds for μ.

(2) Let $\mu \in \mathbf{sca}(\mathfrak{P})_+$ without loss of generality and let $\mathfrak{Q}$ be a supportable alternative to $\mathfrak{P}$. First of all, $\mathbf{1}_{S(\mu)} = \sup_{\mathbb{Q} \in \mathfrak{Q}} \mathbf{1}_{S(\mu)} \mathbf{1}_{S(\mathbb{Q})} = \sup_{\mathbb{Q} \in \mathfrak{Q}} \mathbf{1}_{S(\mu) \cap S(\mathbb{Q})}$ by Lemma A.4. The band $\mathbf{1}_{S(\mu)} L^\infty(\mathfrak{P})$ is super Dedekind complete by Proposition 5.3(2). Hence, there is a sequence $(\mathbb{Q}_n)_{n \in \mathbb{N}} \subset \mathfrak{Q}$ such that

$$\sup_{n \in \mathbb{N}} \mathbf{1}_{S(\mathbb{Q}_n) \cap S(\mu)} = \mathbf{1}_{S(\mu)}.$$

Consider $\mathbb{Q}^* := \sum_{n \in \mathbb{N}} 2^{-n} \mathbb{Q}_n$, which is supportable by Corollary 5.4(1), and $S(\mathbb{Q}^*) = \bigcup_{n \in \mathbb{N}} S(\mathbb{Q}_n)$. By Proposition 5.3(3), $\mu \ll \mathbb{Q}^*$. The Radon-Nikodym Theorem yields a $\mathbb{Q}^*$-integrable $h : \Omega \to \mathbb{R}$ such that $\mu(A) = \mathbb{E}_{\mathbb{Q}^*}[h\mathbf{1}_A]$ for all $A \in \mathcal{F}$. The assertion follows with (1).

$\square$

6. Proofs of the main results

Throughout this section we assume the reader to be familiar with the results from Section 5.

6.1. **Proofs of Sections 4.1 and 4.2.**

Proof of Lemma 4.2. (1) This is a direct application of Lemma A.3.

(2) Suppose that $\mathbf{1}_S = \inf \mathcal{C}$ exists as in (1). If μ is supportable and $S(\mu)$ is its $\mathfrak{P}$-q.s. order support, then $\mathbf{1}_{S(\mu)} \preceq \mathbf{1}_S = \inf\{\mathbf{1}_A \mid A \in \mathcal{F}, \ \mu(A) = \mu(\Omega)\}$ holds by condition (b) in Definition 4.1. As $\mathbf{1}_{S(\mu)} \in \mathcal{C}$, we also have $\mathbf{1}_S \preceq \mathbf{1}_{S(\mu)}$. Hence, $\mathbf{1}_S = \mathbf{1}_{S(\mu)}$ has to hold. Conversely, if $\mu(S) = \mu(\Omega)$, conditions (a) and (b) from Definition 4.1 are met. Hence, S is the $\mathfrak{P}$-q.s. order support of μ.

$\square$

Proof of Theorem 4.3. (1) implies (2): If $\mathfrak{P}$ is dominated by some probability measure $\mathbb{P}^*$, we can apply [20, Theorem 1.61] and assume $\mathbb{P}^* \in \mathrm{co}_\sigma(\mathfrak{P})$ and $\mathbb{P}^* \approx \mathfrak{P}$, where $\mathrm{co}_\sigma(\mathfrak{P})$ is defined in (5.3). As a consequence, $L^\infty(\mathfrak{P})$ is lattice isomorphic to $L^\infty(\mathbb{P}^*)$, and the latter is super Dedekind complete by [20, Theorem A.37].

(2) implies (3): (2) already entails the countable sup property by definition. Let now $\mu \in \mathbf{ca}(\mathfrak{P})_+$ be arbitrary. Consider the set $\mathcal{C} := \{\mathbf{1}_A \mid A \in \mathcal{F}, \ \mu(A) = \mu(\Omega)\}$, which is order bounded below by $0 \in L^\infty(\mathfrak{P})$. By super Dedekind completeness of the latter space, $\inf \mathcal{C}$ exists and there is a sequence $(A_n)_{n \in \mathbb{N}} \subset \mathcal{F}$ such that $\inf \mathcal{C} = \inf_{n \in \mathbb{N}} \mathbf{1}_{A_n}$. Set $S := \bigcap_{n \in \mathbb{N}} A_n$ and note that $\mu(S) = \mu(\Omega)$. Moreover, $\mathbf{1}_S = \inf_{n \in \mathbb{N}} \mathbf{1}_{A_n} = \inf \mathcal{C}$ is readily verified. By Lemma 4.2, S is the order support of μ.

(3) clearly implies (4).

(4) implies (1): The assumption of (4) implies that $\mathfrak{P}$ has the class (S) property from Definition 4.12 and is its own supportable alternative. From Lemma A.4, we infer $\mathbf{1}_\Omega = \sup_{\mathbb{P} \in \mathfrak{P}} \mathbf{1}_{S(\mathbb{P})}$. By the countable sup property this supremum is attained by a countable subfamily $(\mathbb{P}_n)_{n \in \mathbb{N}} \subset \mathfrak{P}$, i.e. $\mathbf{1}_\Omega = \sup_{n \in \mathbb{N}} \mathbf{1}_{S(\mathbb{P}_n)}$. However, $\sup_{n \in \mathbb{N}} \mathbf{1}_{S(\mathbb{P}_n)} = \mathbf{1}_S$, where

$$S := \bigcup_{n \in \mathbb{N}} S(\mathbb{P}_n).$$

From this observation $\mathbf{1}_\Omega = \mathbf{1}_S$ follows. For all $N \in \mathcal{F}$, we thus have

$$\mathbf{1}_N = \mathbf{1}_N \wedge \mathbf{1}_\Omega = \sup_{n \in \mathbb{N}} \mathbf{1}_N \wedge \mathbf{1}_{S(\mathbb{P}_n)} = \sup_{n \in \mathbb{N}} \mathbf{1}_{N \cap S(\mathbb{P}_n)}, \tag{6.1}$$

where we have used [2, Lemma 1.5]. Suppose $0 = \mathbb{P}_n(N) = \mathbb{P}_n(N \cap S(\mathbb{P}_n))$. By Lemma 5.1(1), $\mathbf{1}_{N \cap S(\mathbb{P}_n)} = 0$ q.s. By (6.1), $\sup_{n \in \mathbb{N}} \mathbb{P}_n(N) = 0$ implies $\mathbf{1}_N = 0$ q.s. The converse implication holds *a priori*. Hence,

$$\mathfrak{P} \approx \{\mathbb{P}_n \mid n \in \mathbb{N}\} \approx \sum_{n=1}^{\infty} 2^{-n} \mathbb{P}_n =: \mathbb{P}^*.$$

This is (1). $\square$

Proof of Proposition 4.4. Throughout the proof, we may assume $\mu \in \mathbf{ca}(\mathfrak{P})_{++}$.

(1) Let $(X_\alpha)_{\alpha \in I}$ be a net such that $X_\alpha \downarrow 0$. Then $X_\alpha \mathbf{1}_{S(\mu)} \downarrow 0$. By Lemma 5.2, that $\mathbf{1}_{S(\mu)} \mathcal{X} = C(\phi_\mu)$ for the associated functional $\phi_\mu \in \mathcal{X}_c^\sim$. By Proposition 5.3(2), $\mathbf{1}_{S(\mu)} \mathcal{X}$ equipped with the $\mathfrak{P}$-q.s. order $\preceq$ has the countable sup property. Hence, there is a countable subnet $(\alpha_n)_{n \in \mathbb{N}}$ such that $X_{\alpha_n} \mathbf{1}_{S(\mu)} \downarrow 0$ in order. By monotone convergence,

$$0 \le \inf_\alpha \phi_\mu(X_\alpha) = \inf_\alpha \phi_\mu(X_\alpha \mathbf{1}_{S(\mu)}) \le \inf_{n \in \mathbb{N}} \phi_\mu(X_{\alpha_n} \mathbf{1}_{S(\mu)}) = \inf_{n \in \mathbb{N}} \int X_{\alpha_n} \mathbf{1}_{S(\mu)} \, d\mu = 0,$$

whence order continuity of ϕ_μ follows.

(2) Let ϕ_μ be order continuous and $\mathcal{X}$ be Dedekind complete. Consider the net

$$(\mathbf{1}_A)_{A \in \mathcal{F}, \, \mu(A) = \mu(\Omega)},$$

where the index set is ordered by set inclusion. Then $\mathbf{1}_A \downarrow \mathbf{1}_S$ for some $S \in \mathcal{F}$ by Lemma 4.2(1), and $\mu(S) = \phi_\mu(\mathbf{1}_S) = \inf\{\phi_\mu(\mathbf{1}_A) \mid A \in \mathcal{F}, \mu(A) = \mu(\Omega)\} = \mu(\Omega)$ by order continuity. μ is supportable by Lemma 4.2(2).

$\square$

Proof of Theorem 4.8. (1) This is an immediate consequence of Proposition 4.4.

(2) We have already observed that **(A6)** implies **(A5)**. Moreover, the second equivalence follows from (1).

For the first equivalence, suppose first $\mathcal{X}$ has properties **(A1)**—$\mathbf{ca}(\mathcal{X})$ separates the points of $\mathcal{X}$—**(A2)**, and **(A4)**. As $\mathcal{X}$ is monotonically complete, [35, Theorem 2.4.22] yields $\mathcal{X} = (\mathcal{X}_n^\sim)_n^\sim$, i.e. $\mathcal{X}$ is perfect. **(A2)** yields $(\mathcal{X}_n^\sim)_n^\sim = \mathbf{ca}(\mathcal{X})_n^\sim$ Both identities together provide **(A5)**.

Conversely, assume $\mathcal{X}$ has property **(A5)**. As $\mathbf{ca}(\mathcal{X})$ always separates the points of its order continuous dual, **(A1)** is an immediate consequence of **(A5)**. Now, as $\mathbb{R}$ is Dedekind complete, $\mathbf{ca}(\mathcal{X})^\sim$ is Dedekind complete by the Riesz-Kantorovich Theorem [2, Theorem 1.67], and $\mathbf{ca}(\mathcal{X})_n^\sim$ is a band in $\mathbf{ca}(\mathcal{X})^\sim$ by Ogasawara's Theorem [3, Theorem 1.57]. $\mathbf{ca}(\mathcal{X})_n^\sim$ is therefore Dedekind complete in its own right. This implies that $\mathcal{X} = \mathbf{ca}(\mathcal{X})_n^\sim$ is Dedekind complete **(A4)**. Last, for **(A2)**, [35, Theorem 2.4.22] would imply

$$\mathbf{ca}(\mathcal{X}) = (\mathbf{ca}(\mathcal{X})_n^\sim)_n^\sim = \mathcal{X}_n^\sim,$$

provided $\mathbf{ca}(\mathcal{X})$ is Dedekind complete, *monotonically complete* in the sense of [35, Definition 2.4.18], and $\mathbf{ca}(\mathcal{X})_n^\sim$ separates the points of $\mathbf{ca}(\mathcal{X})$. For Dedekind completeness, combine the Riesz-Kantorovich Theorem, Ogasawara's Theorem, and (4.1) to see that $\mathbf{ca}(\mathcal{X}) = \mathcal{X}_c^\sim$ is a band in the Dedekind complete order dual $\mathcal{X}^\sim$ and therefore Dedekind complete. $\mathbf{ca}(\mathcal{X}) = \mathcal{X}_c^\sim$ is monotonically complete by [35, Proposition 2.4.19(ii)]. At last, $\mathbf{ca}(\mathcal{X})_n^\sim = \mathcal{X}$ separates the points of $\mathbf{ca}(\mathcal{X})$ by definition.

Perfectness of $\mathcal{X}$ has been shown above under the assumption **(A1)**$\wedge$**(A2)**$\wedge$**(A4)**.

(3) This is an immediate consequence of [1, Theorem 9.22].

$\square$

Proof of Proposition 4.10. (1) By [1, Theorem 15.15], $\Delta(\mathcal{F})$ is Polish. Let $\mathfrak{R}$ be the mentioned subset of $\Delta(\mathcal{F}) \cap \mathbf{sca}(\mathfrak{P})$. Then there is a continuous injective map $\mathbb{Q}_\bullet : \{0,1\}^{\mathbb{N}} \to$

$\Delta(\mathcal{F})$ with $\{\mathbb{Q}_\sigma \mid \sigma \in \{0,1\}^{\mathbb{N}}\} \subset \mathfrak{R}$; cf. [27, Theorem 6.2]. The Cantor space $\{0,1\}^{\mathbb{N}}$ is tacitly assumed to be endowed with its Polish topology, the discrete product topology. For any $E \in \mathcal{F}$, the function $\{0,1\}^{\mathbb{N}} \ni \sigma \mapsto \mathbb{Q}_\sigma(E)$, is Borel measurable as a composition of the continuous map $\mathbb{Q}_\bullet$ and the Borel measurable function $\Delta(\mathcal{F}) \ni \mu \to \mu(E)$ ([1, Lemma 15.16]). Let π be any non-atomic Borel probability measure on $\{0,1\}^{\mathbb{N}}$, whose existence follows from [1, Theorem 12.22]. Consider

$$\mu : \mathcal{F} \to [0,1], \quad E \mapsto \int_{\{0,1\}^{\mathbb{N}}} \mathbb{Q}_\sigma(E)\pi(d\sigma). \tag{6.2}$$

μ is a probability measure in $\mathbf{ca}(\mathfrak{P})_{++}$. Assume for contradiction that μ is supportable. Let $\sigma \in \{0,1\}^{\mathbb{N}}$ be arbitrary. For all $\{0,1\}^{\mathbb{N}} \ni \sigma' \neq \sigma$, we have $\mathbf{1}_{S(\mathbb{Q}_\sigma)} \wedge \mathbf{1}_{S(\mathbb{Q}_{\sigma'})} = 0$ in $L^\infty(\mathfrak{P})$, whence $\mathbb{Q}_{\sigma'}(S(\mathbb{Q}_\sigma)) = 0$ follows. By (6.2),

$$\mu(S(\mu) \cap S(\mathbb{Q}_\sigma)) = \mathbb{Q}_\sigma\left(S(\mu) \cap S(\mathbb{Q}_\sigma)\right)\pi(\{\sigma\}) = 0.$$

Lemma 5.1(1) shows that $\mathbf{1}_{S(\mu)\cap S(\mathbb{Q}_\sigma)} = 0$ in $L^\infty(\mathfrak{P})$. As σ was chosen arbitrarily, $\mathbb{Q}_\sigma(S(\mu)) = \mathbb{Q}_\sigma(S(\mu) \cap S(\mathbb{Q}_\sigma)) = 0$ for all $\sigma \in \{0,1\}^{\mathbb{N}}$, contradicting $\mu(S(\mu)) = 1$.

(2) Let μ be defined by (6.2) and suppose $\sup_{\mathbb{Q}\in\mathfrak{R}} \|\mathbb{Q}\|_{\mathbf{ca}(\mathcal{X})} < \infty$. Then all $X \in \mathcal{X}$ satisfy

$$\int |X|\, d\mu = \int_{\{0,1\}^{\mathbb{N}}} \mathbb{E}_{\mathbb{Q}_\sigma}[|X|]\pi(d\sigma) \leq \sup_{\mathbb{Q}\in\mathfrak{R}} \|\mathbb{Q}\|_{\mathbf{ca}(\mathcal{X})}\|X\|_{\mathcal{X}}.$$

Hence, $\mu \in \mathbf{ca}(\mathcal{X})$. Now assume $\sup_{\mathbb{Q}\in\mathfrak{R}} \mathbf{1}_{S(\mathbb{Q})}$ exists. By Lemma A.3, there is a set $S \in \mathcal{F}$ such that $\mathbf{1}_S = \sup_{\mathbb{Q}\in\mathfrak{R}} \mathbf{1}_{S(\mathbb{Q})}$ in $\mathcal{X}$. Also, $\mathbb{Q}(S) = 1$, $\mathbb{Q} \in \mathfrak{R}$, which entails $\phi_\mu(\mathbf{1}_S) = \mu(S) = 1$. However, for any $\mathfrak{F} \subset \mathfrak{R}$ finite, define $Y_{\mathfrak{F}} := \max_{\mathbb{Q}\in\mathfrak{F}} \mathbf{1}_{S(\mathbb{Q})} = \mathbf{1}_{\bigcup_{\mathbb{Q}\in\mathfrak{F}} S(\mathbb{Q})}$ and note that

$$\phi_\mu(Y_{\mathfrak{F}}) = 0.$$

Moreover, the net $Y_{\mathfrak{F}}$, $\mathfrak{F} \subset \mathfrak{R}$ finite, where the index set is ordered by inclusion, converges to $\mathbf{1}_S$ in order. Hence,

$$\lim_{\mathfrak{F}\subset\mathfrak{R}\text{ finite}} \phi_\mu(Y_{\mathfrak{F}}) = 0 < 1 = \phi_\mu\left(\lim_{\mathfrak{F}\subset\mathfrak{R}\text{ finite}} Y_{\mathfrak{F}}\right) = \phi_\mu(\mathbf{1}_S),$$

so ϕ_μ cannot be order continuous.

At last, any uncountable Borel or analytic subset of $\Delta(\mathcal{F})$ contains a non-empty perfect set; see [27, Theorems 13.6 & 29.1]. $\qquad\square$

6.2. **Proofs of Section 4.3.**

Proof of Lemma 4.16. By Definition 4.15(a), $\mathfrak{Q} \approx \mathfrak{P}$, $L^\infty(\mathfrak{P}) = L^\infty(\mathfrak{Q})$, and the $\mathfrak{P}$-q.s. order agrees with the $\mathfrak{Q}$-q.s. order. By Definition 4.15(c), $\mathbb{Q}(S_{\mathbb{Q}}) = 1$ for all $\mathbb{Q} \in \mathfrak{Q}$. This shows property (a) from Definition 4.1. As for Definition 4.1(b), let $\mathbb{Q}^* \in \mathfrak{Q}$ be arbitrary and suppose an event $A \in \mathcal{F}$ satisfies $\mathbb{Q}^*(A) = 1$. For all $\mathbb{Q} \in \mathfrak{Q}\setminus\{\mathbb{Q}^*\}$,

$$\begin{aligned}
\mathbb{Q}(\mathbf{1}_A < \mathbf{1}_{S_{\mathbb{Q}^*}}) &= \mathbb{Q}\left(\{\mathbf{1}_A < \mathbf{1}_{S_{\mathbb{Q}}^*}\} \cap S_{\mathbb{Q}}\right) \\
&= \mathbb{Q}((S_{\mathbb{Q}^*}\setminus A) \cap S_{\mathbb{Q}}) \\
&\leq \mathbb{Q}(S_{\mathbb{Q}^*} \cap S_{\mathbb{Q}}) = 0.
\end{aligned}$$

Moreover, $\mathbb{Q}^*(\mathbf{1}_A < \mathbf{1}_{S_\mathbb{Q}^*}) = \mathbb{Q}^*(S_{\mathbb{Q}^*} \setminus A) = 0$ as well. Hence $\mathbf{1}_{S_\mathbb{Q}} \preceq \mathbf{1}_A$.   □

Proof of Proposition 4.17. The inclusion $\mathbf{sca}(\mathcal{X}) \subset \mathcal{X}_n^\sim$ is Proposition 4.4(1). For the converse inclusion, let ϕ be any order continuous functional. As $\mathcal{X}_n^\sim \subset \mathcal{X}_c^\sim = \mathbf{ca}(\mathcal{X})$, there is a $\mu \in \mathbf{ca}(\mathcal{X})$ such that $\phi = \phi_\mu$ as in (4.1). Since $\mathcal{X}_n^\sim$ is a band in $\mathcal{X}^\sim = \mathcal{X}^*$ by [35, Proposition 1.3.9], we can assume without loss of generality that ϕ is positive. Let $\mathbb{F}$ be the set of all finite subsets of a supportable alternative $\mathfrak{Q} \approx \mathfrak{P}$ ordered by set inclusion. Consider the net

$$\left(\mathbf{1}_{\bigcup_{\mathbb{Q} \in \mathfrak{F}} S(\mathbb{Q})}\right)_{\mathfrak{F} \in \mathbb{F}} = \left(\max_{\mathbb{Q} \in \mathfrak{F}} \mathbf{1}_{S(\mathbb{Q})}\right)_{\mathfrak{F} \in \mathbb{F}}.$$

Then $\mathbf{1}_{\bigcup_{\mathbb{Q} \in \mathfrak{F}} S(\mathbb{Q})} \uparrow \mathbf{1}_\Omega$ by Lemma A.4. Since ϕ is order continuous, we obtain

$$\mu\left(\bigcup_{\mathbb{Q} \in \mathfrak{F}} S(\mathbb{Q})\right) = \phi_\mu(\mathbf{1}_{\bigcup_{\mathbb{Q} \in \mathfrak{F}} S(\mathbb{Q})}) \uparrow \phi_\mu(\mathbf{1}_\Omega) = \mu(\Omega).$$

As $\mathbb{R}$ is super Dedekind complete, we can choose a sequence $(\mathbb{Q}_n)_{n \in \mathbb{N}} \subset \mathfrak{Q}$ such that

$$\mu\left(\bigcup_{n \in \mathbb{N}} S(\mathbb{Q}_n)\right) = \mu(\Omega).$$

We obtain $\mu \ll \{\mathbb{Q}_n \mid n \in \mathbb{N}\}$, and $\mu \in \mathbf{sca}(\mathfrak{P})$ follows with Corollary 5.4(1).   □

Proof of Proposition 4.18. (1) As $\mathfrak{P}$ is of class (S), $\mathbf{sca}(\mathfrak{P}) \approx \mathfrak{P}$, and $\mathbf{sca}(\mathfrak{P}) = L^\infty(\mathfrak{P})_n^\sim$ by Proposition 4.17. Hence, for all $0 \neq X \in L^\infty(\mathfrak{P})$, there is $\mu \in \mathbf{sca}(\mathfrak{P})_+$ such that $\int |X| \, d\mu > 0$. For $f \in X$ arbitrary, define $\nu \in \mathbf{ca}(\mathfrak{P})$ by

$$\nu(A) = \int_A \frac{f}{|f|} \mathbf{1}_{\{|f|>0\}} \, d\mu, \quad A \in \mathcal{F}.$$

We immediately verify $\nu \in \mathbf{sca}(\mathfrak{P})$ and $S(\nu) = \{|f| > 0\} \cap S(\mu)$. Moreover,

$$\int X \, d\nu = \int |X| \, d\mu > 0,$$

whence we infer that $\mathbf{sca}(\mathfrak{P}) = L^\infty(\mathfrak{P})_n^\sim$ separates the points of $L^\infty(\mathfrak{P})$.

(2) Suppose that $L^\infty(\mathfrak{P})$ is Dedekind complete and that $L^\infty(\mathfrak{P})_n^\sim$ separates the points of $L^\infty(\mathfrak{P})$. Then the set of probability measures

$$\mathfrak{Q} := \{\mathbb{Q}_\phi \mid \phi \in (L^\infty(\mathfrak{P})_n^\sim)_{++}\}$$

defined by

$$\mathbb{Q}_\phi(A) := \frac{\phi(\mathbf{1}_A)}{\phi(\mathbf{1}_\Omega)}, \quad A \in \mathcal{F},$$

satisfies $\mathfrak{Q} \approx \mathfrak{P}$. In order to see that $\mathfrak{Q}$ is a supportable alternative to $\mathfrak{P}$, note that for each $\phi \in (L^\infty(\mathfrak{P})_n^\sim)_{++}$ the null ideal $N(\phi)$ is a band. [21, Lemma 43] implies the existence of some $E_\phi \in \mathcal{F}$ such that $N(\phi) = \mathbf{1}_{E_\phi} L^\infty(\mathfrak{P})$. Therefore $S(\mathbb{Q}_\phi) := E_\phi^c$ is the order support of $\mathbb{Q}_\phi$.

□

Proof of Theorem 4.19. $\mathbf{ca}(\mathcal{X})$ agrees with the σ-order continuous dual $\mathcal{X}_c^\sim$, and $\mathbf{sca}(\mathcal{X})$ corresponds to the order continuous dual $\mathcal{X}_n^\sim$ by Proposition 4.17. By [35, Proposition 1.3.7], $\mathcal{X}^\sim = \mathcal{X}^*$. As $\mathbb{R}$ is Dedekind complete, both $\mathbf{ca}(\mathcal{X})$ and $\mathbf{sca}(\mathcal{X})$ are bands in $\mathcal{X}^\sim$ by Ogasawara's Theorem [3, Theorem 1.57].

Moreover, $\mathcal{X}^*$ is Dedekind complete. By the Freudenthal Spectral Theorem [2, Theorem 1.59] it has the projection property, i.e. for every band $\mathcal{B} \subset \mathcal{X}^*$ we have $\mathcal{X}^* = \mathcal{B} \oplus \mathcal{B}^{\mathrm{d}}$. Applying this to $\mathcal{B} = \mathbf{ca}(\mathcal{X})$, we obtain $\mathcal{X}^* = \mathbf{ca}(\mathcal{X}) \oplus \mathbf{ca}(\mathcal{X})^{\mathrm{d}}$. Next, note that $\mathbf{ca}(\mathcal{X})$ is a band and therefore a Dedekind complete vector lattice if we restrict the order to $\mathbf{ca}(\mathcal{X}) \times \mathbf{ca}(\mathcal{X})$. $\mathbf{sca}(\mathcal{X})$ being a band in $\mathcal{X}^\sim$ implies that it is a band in $\mathbf{ca}(\mathcal{X})$, as well. Another application of the projection property involving $\mathcal{B} := \mathbf{sca}(\mathcal{X})$ yields

$$\mathbf{ca}(\mathcal{X}) = \mathbf{sca}(\mathcal{X}) \oplus \mathbf{sca}(\mathcal{X})^{\mathrm{d}},$$

the disjoint complement taken in $\mathbf{ca}(\mathcal{X})$.

It remains to prove that $\mathbf{sca}(\mathcal{X})^{\mathrm{d}} = \mathbf{sca}(\mathcal{X})^{\perp}$. As the spaces are ideals, it suffices to prove that a positive measure $\mu \in \mathbf{ca}(\mathcal{X})_{++}$ belongs to $\mathbf{sca}(\mathcal{X})^{\mathrm{d}}$ if and only if it belongs to $\mathbf{sca}(\mathcal{X})^{\perp}$. To this end, note that by Corollary 5.5(2) $\mu \in \mathbf{sca}(\mathfrak{P})^{\perp}$ if and only if $|\mu|(S(\mathbb{Q})) = 0$ for all $\mathbb{Q} \in \mathfrak{Q}$ where $\mathfrak{Q}$ is a supportable alternative to $\mathfrak{P}$. Hence, let $\mu \in \mathbf{sca}(\mathcal{X})^{\mathrm{d}} \cap \mathbf{ca}(\mathcal{X})_{++}$ and fix an arbitrary $\mathbb{Q} \in \mathfrak{Q}$. Define $\zeta := \mu + \mathbb{Q} \in \mathbf{ca}(\mathfrak{P})_{++}$, fix representatives f and g of $\frac{d\mu}{d\zeta}$ and $\frac{d\mathbb{Q}}{d\zeta}$, respectively, and note that

$$(\mu \wedge \mathbb{Q})(A) = \int_A f \wedge g \, d\zeta, \quad A \in \mathcal{F}.$$

We obtain that $f \wedge g = 0$ ζ-almost everywhere. In particular, $f\mathbf{1}_{S(\mathbb{Q}) \cap \{g > 0\}} = 0$ ζ-a.e. Moreover, $\mathbb{Q}(\{g = 0\} \cap S(\mathbb{Q})) = 0$. From Lemma 5.1(1), we infer $\mu(\{g = 0\} \cap S(\mathbb{Q})) = 0$. In total, we obtain

$$\mu(S(\mathbb{Q})) = \mu(\{g = 0\} \cap S(\mathbb{Q})) + \int_{\{g > 0\} \cap S(\mathbb{Q})} f \, d\zeta = 0.$$

Hence $\mu(S(\mathbb{Q})) = 0$, $\mathbb{Q} \in \mathfrak{Q}$, and $\mu \in \mathbf{sca}(\mathcal{X})^{\perp}$. Conversely, assume $\mu \in \mathbf{sca}(\mathcal{X})^{\perp} \cap \mathbf{ca}(\mathcal{X})_{++}$. Let $\nu \in \mathbf{sca}(\mathfrak{P})$ and let $(\mathbb{Q}_n)_{n \in \mathbb{N}} \subset \mathfrak{Q}$ be a sequence as constructed in the proof of Corollary 5.5 which satisfies $S(|\nu|) \subset \bigcup_{n \in \mathbb{N}} S(\mathbb{Q}_n)$. For all $A \in \mathcal{F}$, we obtain

$$\mu(S(|\nu|) \cap A) + |\nu|(A \backslash S(|\nu|)) = \mu(S(|\nu|) \cap A) \leq \mu\left(\bigcup_{n \in \mathbb{N}} S(\mathbb{Q}_n)\right)$$

$$\leq \sum_{n=1}^{\infty} \mu(S(\mathbb{Q}_n)) = 0.$$

From [1, Theorem 9.52], we infer $\mu \wedge |\nu| = 0$, and $\mu \in \mathbf{sca}(\mathcal{X})^{\mathrm{d}}$. $\square$

6.3. **Proofs from Section 4.4.** For the proof of Theorem 4.22 we need the notion of a MAXIMAL DISJOINT SYSTEM in a vector lattice $(\mathcal{X}, \preceq)$, that is, a subset $\mathfrak{E} \subset \mathcal{X}_+$ of mutually singular vectors—i.e. $x \wedge y = 0$ for all $x, y \in \mathfrak{E}$, $x \neq y$—with the property that $0 \notin \mathfrak{E}$ and

$\sup_{x \in \mathfrak{E}} x \wedge |z| = 0$ implies $z = 0$. Each non-trivial vector lattice admits a maximal disjoint system by [32, Theorem 4.28.5].

Proof of Theorem 4.22. (2) obviously implies (1).

(1) implies (2): As $\mathfrak{P}$ is of class (S), the vector lattice $(\mathbf{sca}(\mathfrak{P}), \preceq_{\mathcal{F}})$ is non-trivial. Consider a maximal disjoint system $\mathfrak{E} \subset \mathbf{sca}(\mathfrak{P})_+$. Assume for contradiction that $\mathfrak{E}$ is not equivalent to $\mathfrak{P}$, that is, we can find $A \in \mathcal{F}$ such that $0 \prec \mathbf{1}_A$, but

$$\sup_{\mu \in \mathfrak{E}} \mu(A) = 0.$$

Let $\widetilde{\mathfrak{Q}}$ be a supportable alternative to $\mathfrak{P}$. By Lemma A.4, $\mathbf{1}_A = \sup_{\mathbb{Q} \in \widetilde{\mathfrak{Q}}} \mathbf{1}_{A \cap S(\mathbb{Q})}$, hence there is a $\mathbb{Q} \in \widetilde{\mathfrak{Q}}$ such that $\mathbf{1}_{A \cap S(\mathbb{Q})} \succ 0$. By

$$\xi : \mathcal{F} \to [0, \infty), \quad B \mapsto \mathbb{Q}(B \cap A),$$

we define an element $\xi \in \mathbf{sca}(\mathfrak{P})_{++}$ whose order support is $S(\xi) = A \cap S(\mathbb{Q})$. Let $\mu \in \mathfrak{E}$ and $B \in \mathcal{F}$ be arbitrary. Then

$$\begin{aligned}
(\xi \wedge \mu)(B) &\leq \xi(B \cap S(\xi)^c) + \mu(B \cap S(\xi)) \\
&= \mu(B \cap A \cap S(\mathbb{Q})) \\
&\leq \mu(A) = 0.
\end{aligned}$$

We infer $\xi \wedge \mu = 0$ for all $\mu \in \mathfrak{E}$. That implies $\xi = 0$, or equivalently $\mathbf{1}_{A \cap S(\mathbb{Q})} = 0$ q.s. This is a CONTRADICTION. Hence, $\mathfrak{E} \approx \mathfrak{P}$ must hold.

A disjoint supportable alternative is now given by the set

$$\mathfrak{Q} = \{ \tfrac{1}{\mu(\Omega)} \mu \mid \mu \in \mathfrak{E} \},$$

provided we can show that for $\mu, \nu \in \mathfrak{E}$, $\mu \neq \nu$, we have $\mathbf{1}_{S(\mu)} \wedge \mathbf{1}_{S(\nu)} = 0$ q.s. To see this, let $\zeta := \mu + \nu$ and let f and g be ζ-versions of $\frac{d\mu}{d\zeta}$ and $\frac{d\nu}{d\zeta}$, respectively. Note that $\mathbf{1}_{S(\mu)} = \mathbf{1}_{\{f > 0\} \cap S(\zeta)}$ and $\mathbf{1}_{S(\nu)} = \mathbf{1}_{\{g > 0\} \cap S(\zeta)}$ in $L^\infty(\mathfrak{P})$. Moreover, it is not difficult to show that

$$0 = \frac{d(\mu \wedge \nu)}{d\zeta} = f \wedge g \quad \zeta\text{-a.e.}$$

This shows

$$\mathbf{1}_{\{f > 0\} \cap \{g > 0\} \cap S(\zeta)} = \mathbf{1}_{\{f \wedge g > 0\} \cap S(\zeta)} = 0 \quad \zeta\text{-a.e.}$$

By Lemma 5.1(1),

$$0 = \mathbf{1}_{\{f > 0\} \cap \{g > 0\} \cap S(\zeta)} = \mathbf{1}_{S(\mu) \cap S(\nu)} = \mathbf{1}_{S(\mu)} \wedge \mathbf{1}_{S(\nu)} \quad \text{in } L^\infty(\mathfrak{P}).$$

Now assume that $L^\infty(\mathfrak{P})$ is Dedekind complete and that any disjoint set of probability measures in $\mathbf{ca}(\mathfrak{P})$ is at most equinumerous with the continuum. Clearly, (3) implies (2). Supposing that (2) holds, let $\mathfrak{Q}$ be a disjoint supportable alternative. Let $g : \mathfrak{Q} \to (0, 1)$ be an injective function. The family

$$\mathcal{C} := \{ g(\mathbb{Q}) \mathbf{1}_{S(\mathbb{Q})} \mid \mathbb{Q} \in \mathfrak{Q} \}$$

is order bounded from above and admits a supremum $U \in L^\infty(\mathfrak{P})$ by Dedekind completeness. For all $\mathbb{Q} \in \mathfrak{Q}$, $U\mathbf{1}_{S(\mathbb{Q})^c} + g(\mathbb{Q})\mathbf{1}_{S(\mathbb{Q})}$ is an upper bound of $\mathcal{C}$ as well since $\mathbf{1}_{S(\mathbb{Q})} \wedge \mathbf{1}_{S(\mathbb{Q}')} = 0$ for all $\mathbb{Q}' \in \mathfrak{Q}\backslash\{\mathbb{Q}\}$. As *a priori*

$$U\mathbf{1}_{S(\mathbb{Q})^c} + g(\mathbb{Q})\mathbf{1}_{S(\mathbb{Q})} \preceq U$$

we obtain $U = U\mathbf{1}_{S(\mathbb{Q})^c} + g(\mathbb{Q})\mathbf{1}_{S(\mathbb{Q})}$ and hence $U\mathbf{1}_{S(\mathbb{Q})} = g(\mathbb{Q})\mathbf{1}_{S(\mathbb{Q})}$. Let $u \in U$ be any representative. Define

$$S_\mathbb{Q} := \{\omega \in \Omega \mid u(\omega) = g(\mathbb{Q})\} \in \mathcal{F}, \quad \mathbb{Q} \in \mathfrak{Q}.$$

Then $\mathbb{Q}(S_\mathbb{Q}) = \mathbb{Q}(S(\mathbb{Q})) = 1$ and by injectivity of g, $S_\mathbb{Q} \cap S_{\mathbb{Q}'} = \emptyset$ whenever $\mathbb{Q} \neq \mathbb{Q}'$. It remains to show that $\mathbf{1}_{S_\mathbb{Q}} \preceq \mathbf{1}_{S(\mathbb{Q})}$ q.s. To this end, note that for all $\mathbb{Q}' \in \mathfrak{Q}\backslash\{\mathbb{Q}\}$

$$\mathbb{Q}'(\mathbf{1}_{S_\mathbb{Q}} > \mathbf{1}_{S(\mathbb{Q})}) = \mathbb{Q}'\left((S_\mathbb{Q}\backslash S(\mathbb{Q})) \cap S_{\mathbb{Q}'}\right)$$
$$\leq \mathbb{Q}'(\{u = g(\mathbb{Q})\} \cap \{u = g(\mathbb{Q}')\}) = 0.$$

Hence, $\mathbf{1}_{S_\mathbb{Q}} \preceq \mathbf{1}_{S(\mathbb{Q})}$. By definition, $\mathbf{1}_{S_\mathbb{Q}} = \mathbf{1}_{S(\mathbb{Q})}$ has to hold. $\square$

Proof of Proposition 4.23. (1) It is clear that the existence of a countable disjoint supportable alternative $\mathfrak{Q}$ implies that $\mathfrak{P}$ is dominated. Conversely, if $\mathfrak{P}$ is dominated, each disjoint set $\mathfrak{R}$ of supportable probability measures is countable. Indeed, if $\mathbb{P}^*$ is a probability measure on $(\Omega, \mathcal{F})$ such that $\mathfrak{P} \approx \mathbb{P}^*$, and $f^\mathbb{Q}$ is a representative of $\frac{d\mathbb{Q}}{d\mathbb{P}^*}$, $\mathbb{Q} \in \mathfrak{R}$, then $(\{f^\mathbb{Q} > 0\})_{\mathbb{Q} \in \mathfrak{R}} \subset \mathcal{F}$ is a family of versions of the order supports of $\mathfrak{Q}$ which are pairwise disjoint up to a $\mathbb{P}^*$-null set and each have positive $\mathbb{P}^*$-probability. This clearly entails countability of that family, and *a fortiori* countability of $\mathfrak{R}$.

(2) Suppose $\mathfrak{Q}$ is a supportable alternative to $\mathfrak{P}$ which is at most equinumerous with the continuum. Let $\mathfrak{R} \subset \mathbf{sca}(\mathfrak{P})$ be any disjoint set of supportable probability measures. By Corollary 5.5(2), for each $\mathbb{Q} \in \mathfrak{R}$ we can select a sequence $(\mathbb{Q}_n^*)_{n\in\mathbb{N}} \in \mathfrak{Q}^\mathbb{N}$ such that $\mathbb{Q} \ll \{\mathbb{Q}_n^* \mid n \in \mathbb{N}\}$. This gives rise to a map $J_1 : \mathfrak{R} \to \mathfrak{Q}^\mathbb{N}$. Fix now $(\mathbb{Q}_n^*)_{n\in\mathbb{N}} \in \mathfrak{Q}^\mathbb{N}$ and let $S := \bigcup_{n\in\mathbb{N}} S(\mathbb{Q}_n^*)$ and $\mathbb{Q}^* := \sum_{n\in\mathbb{N}} \frac{1}{2^n}\mathbb{Q}_n^*$. Any $\mathbb{Q} \in \mathfrak{R}$ such that $\mathbb{Q} \ll \{\mathbb{Q}_n^* \mid n \in \mathbb{N}\}$ must satisfy $\mathbb{Q}^*(S(\mathbb{Q}) \cap S) > 0$. By disjointness of the set $\mathfrak{R}$ there can only be countably many $\mathbb{Q} \in \mathfrak{R}$ with the property $\mathbb{Q}^*(S(\mathbb{Q}) \cap S) > 0$ and thus only countably many $\mathbb{Q} \in \mathfrak{R}$ with $\mathbb{Q} \ll \{\mathbb{Q}_n^* \mid n \in \mathbb{N}\}$. Hence, $J^{-1}(\{(\mathbb{Q}_n^*)_{n\in\mathbb{N}}\})$ is countable for all $(\mathbb{Q}_n^*)_{n\in\mathbb{N}} \in \mathfrak{Q}^\mathbb{N}$. This admits to define an injective map $J_2 : \mathfrak{R} \to \mathfrak{Q}^{\mathbb{N}\times\mathbb{N}}$, which at last proves

$$|\mathfrak{R}| \leq |\mathfrak{Q}^{\mathbb{N}\times\mathbb{N}}| \leq |\mathbb{R}^{\mathbb{N}^2}| = |\{0,1\}^{\mathbb{N}^3}| = |\{0,1\}^\mathbb{N}| = |\mathbb{R}|.$$

This proves the only nontrivial assertion.

$\square$

Proof of Proposition 4.24. We denote the natural order on $\mathcal{Y}$ by $\trianglelefteq$, i.e. $\mathbf{X} = (X_\mathbb{Q})_{\mathbb{Q}\in\mathfrak{Q}}, \mathbf{Y} = (\mathbf{Y}_\mathbb{Q})_{\mathbb{Q}\in\mathfrak{Q}} \in \mathcal{Y}$ satisfy $\mathbf{X} \trianglelefteq \mathbf{Y}$ if and only if $X_\mathbb{Q} \leq Y_\mathbb{Q}$ $\mathbb{Q}$-a.s., $\mathbb{Q} \in \mathfrak{Q}$. Consider the sublattice

$$\mathcal{L} := \{(j_\mathbb{Q}(X))_{\mathbb{Q}\in\mathfrak{Q}} \mid X \in L^\infty(\mathfrak{P})\},$$

which is lattice isomorphic to $L^\infty(\mathfrak{P})$ by Proposition 5.3.

$\mathcal{L}$ is order dense in $\mathcal{Y}$, the notion being introduced in Section 2. To see this, suppose $\mathbf{0} \lhd \mathbf{X} = (X_{\mathbb{Q}})_{\mathbb{Q} \in \mathfrak{Q}} \in \mathcal{Y}$. Then there is $\mathbb{Q}^* \in \mathfrak{Q}$ such that $X_{\mathbb{Q}^*} \in L^\infty(\mathbb{Q}^*)_{++}$. Let $f \in X_{\mathbb{Q}^*}$ be a $\mathbb{Q}^*$-representative and set

$$\mathbf{Y} := \left(j_{\mathbb{Q}} \left([f \mathbf{1}_{S(\mathbb{Q}^*)}] \right) \right)_{\mathbb{Q} \in \mathfrak{Q}} \in \mathcal{L}.$$

Then $\mathbf{0} \lhd \mathbf{Y} \trianglelefteq \mathbf{X}$.

Moreover, $\mathcal{L}$ is majorising; cf. Section 2. This is due to $\{(j_{\mathbb{Q}}(m\mathbf{1}_\Omega))_{\mathbb{Q} \in \mathfrak{Q}} \mid m \in \mathbb{R}\} \subset \mathcal{L}$ being majorising.

At last, $\mathcal{Y}$ is Dedekind complete because each factor $L^\infty(\mathbb{Q})$ is Dedekind complete. Conclude with the Nakano-Judin Theorem [2, Theorem 1.41]. $\qquad\square$

Proof of Proposition 4.26. Let $\mathfrak{Q}$ be a disjoint supportable alternative to $\mathfrak{P}$. Define $\widehat{\Omega}$ as the disjoint union of Ω over $\mathfrak{Q}$, that is,

$$\widehat{\Omega} = \bigsqcup_{\mathbb{Q} \in \mathfrak{Q}} \Omega := \{(\omega, \mathbb{Q}) \mid \omega \in \Omega,\ \mathbb{Q} \in \mathfrak{Q}\}.$$

On $\widehat{\Omega}$ we consider the σ-algebra

$$\widehat{\mathcal{F}} = \sigma \left(\{A \times \{\mathbb{Q}\} \mid A \in \mathcal{F},\ \mathbb{Q} \in \mathfrak{Q}\} \right).$$

It satisfies $\widehat{\mathcal{F}} = \sigma(\mathcal{E})$, where $\mathcal{E}$ is the set of all sets $B \subset \widehat{\Omega}$ such that there are $\emptyset \neq \mathfrak{F} \subset \mathfrak{Q}$ finite and $(A_{\mathbb{Q}})_{\mathbb{Q} \in \mathfrak{F}} \subset \mathcal{F}$ with the property $B = \{(\omega, \mathbb{Q}) \mid \mathbb{Q} \in \mathfrak{F},\ \omega \in A_{\mathbb{Q}}\}$. $\mathcal{E}$ is in fact a π-system. Consider $\mathbf{p} : \widehat{\Omega} \to \Omega$ to be the projection onto the first coordinate. Moreover, define

$$\Lambda := \{B \in \widehat{\mathcal{F}} \mid \forall \mathbb{Q} \in \mathfrak{Q} :\ \mathbf{p}(B \cap (\Omega \times \{\mathbb{Q}\})) \in \mathcal{F}\}.$$

Then Λ is a λ-system which contains $\mathcal{E}$. By Dynkin's π-λ-Theorem, $\widehat{\mathcal{F}} = \Lambda$. This admits the definition of "versions" of the measures $\mathbb{Q} \in \mathfrak{Q}$ on $(\widehat{\Omega}, \widehat{\mathcal{F}})$ in the following way:

$$\widehat{\mathbb{Q}}(B) := \mathbb{Q} \left(\mathbf{p}(B \cap (\Omega \times \{\mathbb{Q}\})) \right), \quad B \in \widehat{\mathcal{F}}.$$

Consider the set of priors $\widehat{\mathfrak{Q}} := \{\widehat{\mathbb{Q}} \mid \mathbb{Q} \in \mathfrak{Q}\}$. Each of them is supportable in the $\widehat{\mathfrak{Q}}$-q.s. order on $L^\infty(\widehat{\mathfrak{Q}})$, and the $\widehat{\mathfrak{Q}}$-q.s. order supports may be chosen as

$$S_{\widehat{\mathbb{Q}}} := S_{\mathbb{Q}} \times \{\mathbb{Q}\}, \quad \widehat{\mathbb{Q}} \in \widehat{\mathfrak{Q}},$$

where $S_{\mathbb{Q}} \in \mathcal{F}$ is an arbitrary version of the $\mathfrak{P}$-q.s. order support of $\mathbb{Q}$. Hence, the $\widehat{\mathfrak{Q}}$-q.s. order supports admit a family of pairwise disjoint versions, which means that $\widehat{\mathfrak{Q}}$ has the Hahn property. Its Dedekind completion is now given by $\widehat{\mathcal{Y}} := L^\infty(\widehat{\Omega}, \widehat{\mathcal{F}}(\widehat{\mathfrak{Q}}), \widehat{\mathfrak{Q}}^\sharp)$; cf. Remark 4.25. Consider the map $F : L^\infty(\mathfrak{P}) \to \widehat{\mathcal{Y}}$, where the equivalence class $X = [f] \in L^\infty(\mathfrak{P})$ is mapped to the equivalence class generated by the function

$$\widehat{f} : \widehat{\Omega} \ni (\omega, \mathbb{Q}) \mapsto f(\omega) \mathbf{1}_{S_{\mathbb{Q}}}(\omega).$$

First, we observe that $\widehat{f}$ is indeed $\widehat{\mathcal{F}}(\widehat{\mathfrak{Q}})$-measurable, i.e. $\widehat{f}$ is $\widehat{\mathcal{F}}(\{\widehat{\mathbb{Q}}^*\})$-measurable for any fixed $\widehat{\mathbb{Q}}^* \in \widehat{\mathfrak{Q}}$. To this aim consider

$$g : \widehat{\Omega} \ni (\omega, \mathbb{Q}) \mapsto f(\omega) \mathbf{1}_{S_{\mathbb{Q}^*} \times \{\mathbb{Q}^*\}}(\omega, \mathbb{Q}^*).$$

Then $\{g \neq \widehat{f}\} \subset \{(\omega, \mathbb{Q}) \mid \omega \in \Omega, \mathbb{Q} \neq \mathbb{Q}^*\} = \widehat{\Omega} \backslash (\Omega \times \{\mathbb{Q}\}) \in \mathfrak{n}(\widehat{\mathbb{Q}^*})$. Moreover, g is $\widehat{\mathcal{F}}$-measurable by the characterisation of $\widehat{\mathcal{F}}$ above. Secondly, whenever $f, h : \Omega \to \mathbb{R}$ are bounded, measurable, and satisfy $[f] = [h]$, $\widehat{f}$ generates the same equivalence class in $\widehat{\mathcal{Y}}$ as $\widehat{h}$. Indeed, $[f] = [h]$ holds if and only if $\sup_{\mathbb{Q} \in \mathfrak{Q}} \mathbb{Q}(\{\omega \in S_{\mathbb{Q}} \mid f(\omega) \neq h(\omega)\}) = 0$. We thus obtain for all $\mathbb{Q} \in \mathfrak{Q}$ that

$$\widehat{\mathbb{Q}}(\widehat{f} \neq \widehat{h}) = \widehat{\mathbb{Q}}(\{\omega \in S_{\mathbb{Q}} \mid f(\omega) \neq h(\omega)\} \times \{\mathbb{Q}\}) = \mathbb{Q}(\{\omega \in S_{\mathbb{Q}} \mid f(\omega) \neq h(\omega)\}) = 0.$$

Hence, the map F is well-defined and $F(L^\infty(\mathfrak{P})) \subset \widehat{\mathcal{Y}}$ holds. Moreover, F is an injective lattice homomorphism. It remains to show that $F(L^\infty(\mathfrak{P}))$ is order dense and majorising in $\widehat{\mathcal{Y}}$. For the latter, note that $F([m\mathbf{1}_\Omega])$ generates the same equivalence class as $m\mathbf{1}_{\widehat{\Omega}}$, $m \in \mathbb{R}$. For order density, suppose $Y \in \widehat{\mathcal{Y}}_{++}$. Choose $\mathbb{Q}^* \in \mathfrak{Q}$ such that $Y\mathbf{1}_{S_{\widehat{\mathbb{Q}}^*}}$ is strictly positive. Let $g \in Y$ be any representative. Then we can find an $\widehat{\mathcal{F}}$-measurable and bounded g^* such that

$$\{g\mathbf{1}_{S_{\widehat{\mathbb{Q}}^*}} \neq g^*\mathbf{1}_{S_{\widehat{\mathbb{Q}}^*}}\} \in \mathfrak{n}(\widehat{\mathbb{Q}^*}).$$

In particular, the function

$$f(\omega) := g^*(\omega, \mathbb{Q}^*)\mathbf{1}_{S_{\mathbb{Q}^*}}(\omega), \quad \omega \in \Omega,$$

is $\mathcal{F}$-measurable and bounded by our characterisation of $\widehat{\mathcal{F}}$ above. Moreover, if we set $X := [f]$, then $F(X)$ posesses the representative defined by

$$\widehat{f}(\omega, \mathbb{Q}) = g^*(\omega, \mathbb{Q}^*)\mathbf{1}_{S_{\mathbb{Q}^*} \cap S_{\mathbb{Q}}}(\omega), \quad (\omega, \mathbb{Q}) \in \widehat{\Omega},$$

which satisfies

$$\{\widehat{f} \neq 0\} \subset \left(\widehat{\Omega} \backslash (\Omega \times \{\mathbb{Q}\})\right) \cup ((S_{\mathbb{Q}} \cap S_{\mathbb{Q}^*}) \times \{\mathbb{Q}\}) \in \mathfrak{n}(\widehat{\mathbb{Q}}), \quad \mathbb{Q} \in \mathfrak{Q} \backslash \{\mathbb{Q}^*\},$$

and $\{\widehat{f} \neq g^*\} \in \mathfrak{n}(\widehat{\mathbb{Q}^*})$. Hence, $F(X) = Y\mathbf{1}_{S_{\widehat{\mathbb{Q}}^*}}$. This proves that $F(L^\infty(\mathfrak{P}))$ is order dense. $\qquad\square$

6.4. **Proofs from Section 4.5.**

Proof of Theorem 4.28. (1) is equivalent to (2): As Dedekind completeness is preserved under lattice isomorphism, the equivalence follows from Proposition 4.24.

(2) implies (3): Let $\mathfrak{Q}$ be any supportable alternative and let $(X^\mathbb{Q})_{\mathbb{Q} \in \mathfrak{Q}} \subset L^\infty(\mathfrak{P})$ be a compatible family. Then $\mathcal{C} := \{X^\mathbb{Q} \mid \mathbb{Q} \in \mathfrak{Q}\}$ is order bounded from above and thus admits a supremum $U \in L^\infty(\mathfrak{P})$. Let $\mathbb{Q} \in \mathfrak{Q}$ be arbitrary. By compatibility, $U^\mathbb{Q} := X^\mathbb{Q} + U\mathbf{1}_{S(\mathbb{Q})^c} \preceq U$ is an upper bound of $\mathcal{C}$ as well. Hence, $U = U^\mathbb{Q}$, which implies $U\mathbf{1}_{S(\mathbb{Q})} = X^\mathbb{Q}\mathbf{1}_{S(\mathbb{Q})} = X^\mathbb{Q}$, $\mathbb{Q} \in \mathfrak{Q}$.

(3) obviously implies (4).

(4) implies (2): Let $\emptyset \neq \mathcal{C} \subset L^\infty(\mathfrak{P})$ be order bounded from above and let $\mathfrak{Q} \approx \mathfrak{P}$ be a supportable alternative with the aggregation property. For each $\mathbb{Q} \in \mathfrak{Q}$, the set $\mathbf{1}_{S(\mathbb{Q})}\mathcal{C}$ is order bounded from above as well and a subset of the super Dedekind complete band $\mathbf{1}_{S(\mathbb{Q})}L^\infty(\mathfrak{P})$; cf. Proposition 5.3(2). Hence, $U^\mathbb{Q} := \sup \mathbf{1}_{S(\mathbb{Q})}\mathcal{C}$ exists in $\mathbf{1}_{S(\mathbb{Q})}L^\infty(\mathfrak{P})$. The family $(U^\mathbb{Q})_{\mathbb{Q} \in \mathfrak{Q}}$ is compatible by Lemma 5.1(3). The aggregation property of $\mathfrak{Q}$ implies the

existence of some $U \in L^\infty(\mathfrak{P})$ such that, for all $\mathbb{Q} \in \mathfrak{Q}$, $U\mathbf{1}_{S(\mathbb{Q})} = U^\mathbb{Q}$. One verifies $U = \sup \mathcal{C}$. As $\mathcal{C}$ is arbitrarily chosen, we have proved Dedekind completeness of $L^\infty(\mathfrak{P})$. $\qquad \square$

Proof of Theorem 4.29. (1) implies (2): As a perfect vector lattice, $L^\infty(\mathfrak{P})$ is Dedekind complete and $L^\infty(\mathfrak{P})^\sim_n$ separates the points of $L^\infty(\mathfrak{P})$; cf. [3, Theorem 1.71]. We conclude that $\mathfrak{P}$ is of class (S) with Proposition 4.18(2).

(2) implies (3): As $\mathfrak{P}$ is of class (S), $\mathbf{sca}(\mathfrak{P}) = L^\infty(\mathfrak{P})^\sim_n$ separates the points of $L^\infty(\mathfrak{P})$ by Proposition 4.18(1), so the embedding $J : L^\infty(\mathfrak{P}) \to \mathbf{sca}(\mathfrak{P})^*$ as in (4.2) is one-to one. Thus we only have to prove that it is also onto, that is, for arbitrary $\psi \in \mathbf{sca}(\mathfrak{P})^*$ there is a $Y \in L^\infty(\mathfrak{P})$ such that $\psi = J(Y)$. To this end, fix a disjoint supportable alternative $\mathfrak{Q} \approx \mathfrak{P}$. For $\mathbb{Q} \in \mathfrak{Q}$ consider the map

$$J_\mathbb{Q} : L^1(\mathbb{Q}) \to \mathbf{sca}(\mathfrak{P}),$$

which maps $Z \in L^1(\mathbb{Q})$ to the signed measures defined by $\mathcal{F} \ni A \mapsto \mathbb{E}_\mathbb{Q}[Z\mathbf{1}_{S(\mathbb{Q})}\mathbf{1}_A]$. By Proposition 5.3(3), $J_\mathbb{Q}$ is a lattice isomorphism onto its image, which is the set

$$\mathcal{B}_\mathbb{Q} := \{\mu \in \mathbf{sca}(\mathfrak{P}) \mid |\mu| \ll \mathbb{Q}\}.$$

As $\psi \circ J_\mathbb{Q} \in L^1(\mathbb{Q})^*$ and $L^1(\mathbb{Q})^* = L^\infty(\mathbb{Q})$, there is a unique $Y^\mathbb{Q} \in L^\infty(\mathbb{Q})$ such that for all $Z \in L^1(\mathbb{Q})$:

$$\psi(J_\mathbb{Q}(Z)) = \mathbb{E}_\mathbb{Q}\left[ZY^\mathbb{Q}\mathbf{1}_{S(\mathbb{Q})}\right].$$

As $\mathfrak{Q}$ is a disjoint supportable alternative, the family $(Y^\mathbb{Q}\mathbf{1}_{S(\mathbb{Q})})_{\mathbb{Q}\in\mathfrak{Q}}$—now seen as a subset of $L^\infty(\mathfrak{P})$—is compatible. By the aggregation property of $\mathfrak{Q}$—see Theorem 4.28—, there is $Y \in L^\infty(\mathfrak{P})$ such that

$$\forall\, \mathbb{Q} \in \mathfrak{Q}\, \forall\, \mu \in \mathcal{B}_\mathbb{Q} : \ \psi(\mu) = \int Y \, d\mu.$$

At last, let $\mu \in \mathbf{sca}(\mathfrak{P})$ be arbitrary. By Corollary 5.5(2), there is $\widehat{\mathbb{Q}} = \sum_{n=1}^\infty \lambda_n \mathbb{Q}_n \in \mathrm{co}_\sigma(\mathfrak{Q})$ and $Z \in L^1(\widehat{\mathbb{Q}})$ such that

$$\mu = \mathbb{E}_{\widehat{\mathbb{Q}}}[Z\mathbf{1}.] = \sum_{n=1}^\infty \mathbb{E}_{\mathbb{Q}_n}[\lambda_n Z\mathbf{1}_{S(\mathbb{Q}_n)}\mathbf{1}.] = \sum_{n=1}^\infty \mu_n,$$

where $\mu_n(A) := \mathbb{E}_{\mathbb{Q}_n}[\lambda_n Z\mathbf{1}_{S(\mathbb{Q}_n)}\mathbf{1}_A]$, $A \in \mathcal{F}$, $n \in \mathbb{N}$. The latter sum is absolutely convergent with respect to the total variation norm TV. Moreover, each μ_n lies in $\mathcal{B}_{\mathbb{Q}_n}$. We obtain that

$$\psi(\mu) = \sum_{n=1}^\infty \psi(\mu_n) = \sum_{n=1}^\infty \int Y \, d\mu_n = \int Y \, d\mu.$$

(3) implies (1): $(\mathbf{sca}(\mathfrak{P}), \preceq_\mathcal{F}, TV)$ is a Banach lattice, hence its order dual agrees with its dual $\mathbf{sca}(\mathfrak{P})^*$, cf. [35, Proposition 1.3.7]. Apply the Riesz-Kantorovich Theorem [2, Theorem 1.67] to deduce that $\mathbf{sca}(\mathfrak{P})^*$, and by assumption $L^\infty(\mathfrak{P})$, is Dedekind complete. Moreover, combining with Proposition 4.4(2) yields $L^\infty(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})^* = (L^\infty(\mathfrak{P})^\sim_n)^*$. Using (2.2) for the last inclusion, we at last obtain

$$L^\infty(\mathfrak{P}) = (L^\infty(\mathfrak{P})^\sim_n)^* \supset (L^\infty(\mathfrak{P})^\sim_n)^\sim_n \supset L^\infty(\mathfrak{P}),$$

so $L^\infty(\mathfrak{P})$ is perfect. $\qquad \square$

6.5. **Proofs from Section 4.6.** Preparing the proof of Theorem 4.31, recall for the next lemma that the σ-order continuous dual $L^\infty(\mathfrak{P})^\sim_c$ may be identified with $\mathbf{ca}(\mathfrak{P})$.

Lemma 6.1. *Suppose $\mathfrak{P}$ is of class* (S) *with disjoint supportable alternative $\mathfrak{Q}$. Assume furthermore that $L^\infty(\mathfrak{P})$ is Dedekind complete. Let $\phi \in (L^\infty(\mathfrak{P})^\sim_c)_+$.*

(1) *ϕ induces a countably additive finite measure π_ϕ on $(\mathfrak{Q}, 2^\mathfrak{Q})$ via*

$$\pi_\phi(\mathfrak{R}) = \phi\left(\sup_{\mathbb{Q} \in \mathfrak{R}} \mathbf{1}_{S(\mathbb{Q})}\right).$$

(2) *$\phi = \phi_\mu$ for some $\mu \in \mathbf{sca}(\mathfrak{P})_+$ if and only if there is a countable subset $\mathfrak{R} \subset \mathfrak{Q}$ such that $\pi_\phi(\mathfrak{R}) = \pi_\phi(\mathfrak{Q})$.*

(3) *$\phi = \phi_\mu$ for some $\mu \in \mathbf{sca}(\mathfrak{P})^\perp_+$ if and only if $\pi_\phi(\{\mathbb{Q}\}) = 0$ for all $\mathbb{Q} \in \mathfrak{Q}$.*

Proof. As $L^\infty(\mathfrak{P})$ is Dedekind complete, for all $\mathfrak{R} \subset \mathfrak{Q}$ there is some event $A(\mathfrak{R}) \in \mathcal{F}$ such that $\sup_{\mathbb{Q} \in \mathfrak{R}} \mathbf{1}_{S(\mathbb{Q})} = \mathbf{1}_{A(\mathfrak{R})}$ (Lemma A.3). Clearly, $\mathbb{Q}(A(\mathfrak{R})) = 1$ for all $\mathbb{Q} \in \mathfrak{R}$.

(1) If $\mathfrak{R}, \mathfrak{R}' \subset \mathfrak{Q}$ are disjoint, one easily verifies the identity

$$\mathbf{1}_{A(\mathfrak{R} \cup \mathfrak{R}')} = \mathbf{1}_{A(\mathfrak{R})} \vee \mathbf{1}_{A(\mathfrak{R}')} = \mathbf{1}_{A(\mathfrak{R})} + \mathbf{1}_{A(\mathfrak{R}')}.$$

Hence,

$$\pi_\phi(\mathfrak{R} \cup \mathfrak{R}') = \phi(\mathbf{1}_{A(\mathfrak{R})}) + \phi(\mathbf{1}_{A(\mathfrak{R}')}) = \pi_\phi(\mathfrak{R}) + \pi_\phi(\mathfrak{R}'),$$

and π_ϕ is additive. Let now $(\mathfrak{R}_n)_{n \in \mathbb{N}} \subset 2^\mathfrak{Q}$ be a sequence of subsets of $\mathfrak{Q}$ such that $\mathfrak{R}_n \downarrow \emptyset$. Then $\mathbf{1}_{A(\mathfrak{R}_n)} \downarrow 0$. Indeed, let $V := \inf_{n \in \mathbb{N}} \mathbf{1}_{A(\mathfrak{R}_n)}$ and let $\mathbb{Q}^* \in \mathfrak{Q}$ be arbitrary. Choose $n \in \mathbb{N}$ large enough such that $\mathbb{Q}^* \notin \mathfrak{R}_n$. Then

$$0 \preceq V \mathbf{1}_{S(\mathbb{Q}^*)} \preceq \mathbf{1}_{A(\mathfrak{R}_n)} \mathbf{1}_{S(\mathbb{Q}^*)} = \sup_{\mathbb{Q} \in \mathfrak{R}_n} \mathbf{1}_{S(\mathbb{Q}) \cap S(\mathbb{Q}^*)} = 0,$$

where we have used that $\mathfrak{Q}$ is a disjoint supportable alternative in the last step. By Lemma A.4, $V = \sup_{\mathbb{Q} \in \mathfrak{Q}} V \mathbf{1}_{S(\mathbb{Q})} = 0$. As $\phi \in L^\infty(\mathfrak{P})^\sim_c$, we obtain

$$\inf_{n \in \mathbb{N}} \pi_\phi(\mathfrak{R}_n) = \inf_{n \in \mathbb{N}} \phi(\mathbf{1}_{A(\mathfrak{R}_n)}) = \phi(0) = 0.$$

(2) Let $\mu \in \mathbf{ca}(\mathfrak{P})_+$ with associated $\phi_\mu \in (L^\infty(\mathfrak{P})^\sim_c)_+$. Assume that there is a countable subset $\mathfrak{R} \subset \mathfrak{Q}$ with

$$\mu(\Omega) = \pi_{\phi_\mu}(\mathfrak{Q}) = \pi_{\phi_\mu}(\mathfrak{R}) = \phi_\mu(\sup_{\mathbb{Q} \in \mathfrak{R}} \mathbf{1}_{S(\mathbb{Q})}).$$

Let $S := \bigcup_{\mathbb{Q} \in \mathfrak{R}} S(\mathbb{Q})$ and note that for all $X \in L^\infty(\mathfrak{P})$, we have

$$0 \leq \phi_\mu(|X| \mathbf{1}_{S^c}) \leq \|X\|_{\infty, \mathfrak{P}} \phi_\mu(\mathbf{1}_{S^c}) = 0.$$

Hence, $\phi_\mu(X) = \phi_\mu(X \mathbf{1}_S)$ for all $X \in L^\infty(\mathfrak{P})$. It follows that $\mu \ll \mathfrak{R}$ and thus μ is supportable by Corollary 5.4(1). Conversely, if $\mu \in \mathbf{sca}(\mathfrak{P})_+$, then by Corollary 5.5(2) there is a countable set $\mathfrak{R} \subset \mathfrak{Q}$ such that $\mu \ll \mathfrak{R}$ and thus $\mathbf{1}_{S(\mu)} \preceq \sup_{\mathbb{Q} \in \mathfrak{R}} \mathbf{1}_{S(\mathbb{Q})} = \mathbf{1}_S$. Then

$$\pi_{\phi_\mu}(\mathfrak{Q}) \geq \pi_{\phi_\mu}(\mathfrak{R}) = \phi_\mu(\mathbf{1}_S) \geq \phi_\mu(\mathbf{1}_{S(\mu)}) = \phi_\mu(\mathbf{1}_\Omega) = \pi_{\phi_\mu}(\mathfrak{Q}).$$

(3) For all $\mu \in \mathbf{ca}(\mathfrak{P})_+$ with associated $\phi_\mu \in L^\infty(\mathfrak{P})_c^\sim$ we compute

$$\pi_{\phi_\mu}(\{\mathbb{Q}\}) = \phi_\mu(\mathbf{1}_{S(\mathbb{Q})}) = \int \mathbf{1}_{S(\mathbb{Q})}\, d\mu = \mu(S(\mathbb{Q})).$$

Hence, by Theorem 4.19, $\pi_{\phi_\mu}(\{\mathbb{Q}\}) = 0$ holds for all $\mathbb{Q} \in \mathfrak{Q}$ if and only if $\mu \in \mathbf{sca}(\mathfrak{P})_+^\perp$.
□

Proof of Theorem 4.31. (1) implies (2): If $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$, $L^\infty(\mathfrak{P})$ is Dedekind complete and $\mathbf{sca}(\mathfrak{P}) = \mathbf{ca}(\mathfrak{P})$ by Theorem 4.8. In particular, $\mathfrak{P}$ is of class (S). Let $\mathfrak{Q} \approx \mathfrak{P}$ be a disjoint supportable alternative and consider any probability measure π on $(\mathfrak{Q}, 2^{\mathfrak{Q}})$. One verifies that

$$\phi : L^\infty(\mathfrak{P}) \ni X \mapsto \int_{\mathfrak{Q}} \mathbb{E}_{\mathbb{Q}}[X\mathbf{1}_{S(\mathbb{Q})}]\pi(d\mathbb{Q}),$$

defines an element of $L^\infty(\mathfrak{P})_c^\sim = \mathbf{ca}(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})$. Also, $\pi = \pi_\phi$ (adopting the notation from Lemma 6.1). To see this, let $\mathfrak{R} \subset \mathfrak{Q}$ be arbitrary and note that

$$\mathbf{1}_{A(\mathfrak{R})}\mathbf{1}_{S(\mathbb{Q})} = \begin{cases} \mathbf{1}_{S(\mathbb{Q})} & \text{if } \mathbb{Q} \in \mathfrak{R} \\ 0 & \text{otherwise} \end{cases}.$$

Hence,

$$\pi_\phi(\mathfrak{R}) = \phi(\mathbf{1}_{A(\mathfrak{R})}) = \int_{\mathfrak{Q}} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{A(\mathfrak{R})}\mathbf{1}_{S(\mathbb{Q})}]\pi(d\mathbb{Q}) = \int_{\mathfrak{R}} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{S(\mathbb{Q})}]\pi(d\mathbb{Q}) = \int_{\mathfrak{R}} \pi(d\mathbb{Q}) = \pi(\mathfrak{R}).$$

By Lemma 6.1(2), as $\phi = \phi_\mu$ for some $\mu \in \mathbf{sca}(\mathfrak{P})$, there is a countable subset $\mathfrak{R} \subset \mathfrak{Q}$ such that $\pi(\mathfrak{R}) = \pi(\mathfrak{Q})$. Hence, $\mathfrak{Q}$ does not admit any solution to Banach's measure problem.

(2) implies (1): Under assumption (2), Theorem 4.29 implies that $\mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$. Let $\phi \in (L^\infty(\mathfrak{P})_c^\sim)_{++}$. By Lemma 6.1(3), the normalised measure $\bar{\pi}_\phi := \frac{1}{\phi(\mathbf{1}_\Omega)}\pi_\phi$, where π_ϕ is as constructed in Lemma 6.1(1), solves Banach's measure problem on $\mathfrak{Q}$ if and only if $\phi = \phi_\mu$ for some $\mu \in \mathbf{sca}(\mathfrak{P})_+^\perp$. If no such measure π exists on $(\mathfrak{Q}, 2^{\mathfrak{Q}})$, $\mathbf{sca}(\mathfrak{P})^\perp$ must be trivial. Hence, $\mathbf{ca}(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})$ by Theorem 4.19. We finally infer

$$\mathbf{ca}(\mathfrak{P})^* = \mathbf{sca}(\mathfrak{P})^* = L^\infty(\mathfrak{P}).$$

□

Proof of Corollary 4.32. (1) implies (2): This is [31, Theorem 3].

(2) implies (3): If $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$, Dedekind completeness of the latter space follows from Theorem 4.8. Conversely, assume $L^\infty(\mathfrak{P})$ is Dedekind complete. By (2), for each $\mu \in \mathbf{ca}(\mathfrak{P})$ the associated functional ϕ_μ is order continuous. By Proposition 4.4(2), $\mathbf{ca}(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})$, and $\mathfrak{P}$ is of class (S). Theorem 4.29 yields the canonical identity $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$.

(3) implies (4): By (3), Dedekind completeness of $L^\infty(\mathfrak{P})$ is equivalent to $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$, which in turn implies Dedekind completeness in conjunction with $\mathfrak{P}$ being of class (S) (Theorem 4.8 and Remark 4.13(2)).

(4) and (5) are equivalent by Theorem 4.29.

(4) implies (1): Let Ω be an arbitrary non-empty set. Consider the measurable space $(\Omega, 2^\Omega)$ equipped with the set of priors $\mathfrak{P} = \{\delta_\omega \mid \omega \in \Omega\}$ which is of class (S). As $L^\infty(\mathfrak{P})$ agrees

with the space of all bounded functions $f : \Omega \to \mathbb{R}$ and this space is Dedekind complete, (4) implies $\mathbf{ca}(\mathfrak{P})^* = L^\infty(\mathfrak{P})$, and Theorem 4.31 implies that $\mathfrak{P}$ does not admit a solution to Banach's measure problem, so neither does Ω.

Now assume that any of the statements (1)–(5) holds. The proof of the implication (3) $\Rightarrow$ (4) has shown that Dedekind completeness entails that $\mathfrak{P}$ is of class (S). $\qquad\square$

Proof of Corollary 4.33. Suppose the construction of such an example as described in (1) or (2) is possible in **ZFC**. According to Corollary 4.32, Banach's measure problem must have a solution. Let $\kappa = |S|$ be the least cardinal which admits such a solution π. By [23, Corollaries 10.7 & 10.15] κ is weakly inaccessible. As such, the construction would provide a proof of the existence of weakly inaccessible cardinals in **ZFC**, and such a proof is known to be impossible; cf. [24, p. 16]. $\qquad\square$

Proof of Corollary 4.34. (2) is equivalent to (3) in **ZFC** according to Theorem 4.29.
(1) implies (3): This is Theorem 4.8 and Remark 4.13(2) and holds in **ZFC**.
(3) implies (1): Let $(\Omega, \mathcal{F})$ be any measurable space, and $\mathfrak{P}$ be a set of priors on $(\Omega, \mathcal{F})$ which is of class (S) such that $L^\infty(\mathfrak{P})$ is Dedekind complete. Also let $\mathfrak{Q}$ be a disjoint supportable alternative to $\mathfrak{P}$. If $\mathfrak{Q}$ can be chosen to be at most equinumerous with $\mathbb{N}$, then $\mathfrak{P}$ is dominated, and the assertion is well-known and holds in **ZFC**. Assume that $\aleph_0 < |\mathfrak{Q}|$. By assumption $|\mathfrak{Q}| \leq 2^{\aleph_0}$, so under the continuum hypothesis $|\mathfrak{Q}| = 2^{\aleph_0}$. Again the continuum hypothesis implies that $\mathfrak{Q}$ does not admit any solution to Banach's measure problem by [13, Corollary 1.12.41], so we may conclude with Theorem 4.31.
Finally, (1) implies $\mathbf{ca}(\mathfrak{P}) = \mathbf{sca}(\mathfrak{P})$ by Theorem 4.8. $\qquad\square$

6.6. **Proofs from Section 4.7.**

Proof of Proposition 4.37. It suffices to show that $\mathcal{F}(\mathfrak{Q}) \subset \mathcal{F}(\widetilde{\mathfrak{Q}})$ whenever $\mathfrak{Q}$ and $\widetilde{\mathfrak{Q}}$ are supportable alternatives to $\mathfrak{P}$. First of all, $B \in \mathcal{F}(\mathfrak{Q})$ if and only if $B \cap S(\mathbb{Q}) \in \mathcal{F}(\{\mathbb{Q}\})$, $\mathbb{Q} \in \mathfrak{Q}$, because $B \cap (\Omega \setminus S(\mathbb{Q})) \in \mathfrak{n}(\mathbb{Q})$. The same holds for $\mathcal{F}(\widetilde{\mathfrak{Q}})$. Hence, for fixed $B \in \mathcal{F}(\mathfrak{Q})$, it suffices to show that $B \cap S(\widetilde{\mathbb{Q}}) \in \mathcal{F}(\{\widetilde{\mathbb{Q}}\})$, $\widetilde{\mathbb{Q}} \in \widetilde{\mathfrak{Q}}$. Fix an arbitrary $\widetilde{\mathbb{Q}} \in \widetilde{\mathfrak{Q}}$. By Corollary 5.5(2) we can find a countable set $\mathfrak{R} \subset \mathfrak{Q}$ such that $\widetilde{\mathbb{Q}} \ll \mathfrak{R}$. Fix $\mathcal{F}$-measurable versions $S(\widetilde{\mathbb{Q}})$, $(S(\mathbb{Q}))_{\mathbb{Q} \in \mathfrak{R}}$ of the order supports such that $S(\widetilde{\mathbb{Q}}) \subset \bigcup_{\mathbb{Q} \in \mathfrak{R}} S(\mathbb{Q})$. By assumption, $B \cap S(\mathbb{Q}) \in \mathcal{F}(\{\mathbb{Q}\})$ for all $\mathbb{Q} \in \mathfrak{R}$, hence also $B \cap S(\mathbb{Q}) \cap S(\widetilde{\mathbb{Q}}) \in \mathcal{F}(\{\mathbb{Q}\})$. Thus there are events $A_{\mathbb{Q}}, N_{\mathbb{Q}} \in \mathcal{F}$ such that $A_{\mathbb{Q}} \subset B \cap S(\mathbb{Q}) \cap S(\widetilde{\mathbb{Q}})$, $\mathbb{Q}(N_{\mathbb{Q}}) = 0$, and $(B \cap S(\mathbb{Q}) \cap S(\widetilde{\mathbb{Q}})) \setminus A_{\mathbb{Q}} \subset N_{\mathbb{Q}}$. We obtain

$$B \cap S(\widetilde{\mathbb{Q}}) = \bigcup_{\mathbb{Q} \in \mathfrak{R}} B \cap S(\widetilde{\mathbb{Q}}) \cap S(\mathbb{Q}) \subset \bigcup_{\mathbb{Q} \in \mathfrak{R}} A_{\mathbb{Q}} \cup \bigcup_{\mathbb{Q} \in \mathfrak{R}} N_{\mathbb{Q}} \cap S(\widetilde{\mathbb{Q}}) \cap S(\mathbb{Q}).$$

For each $\mathbb{Q} \in \mathfrak{R}$, $\mathbf{1}_{N_{\mathbb{Q}} \cap S(\widetilde{\mathbb{Q}}) \cap S(\mathbb{Q})} = 0$ $\mathfrak{P}$-q.s. by Lemma 5.1(1). Hence,

$$\widetilde{\mathbb{Q}}\Big(\bigcup_{\mathbb{Q} \in \mathfrak{R}} N_{\mathbb{Q}} \cap S(\widetilde{\mathbb{Q}}) \cap S(\mathbb{Q}) \Big) = 0.$$

It remains to observe that $A_\infty := \bigcup_{\mathbb{Q} \in \mathfrak{R}} A_{\mathbb{Q}} \in \mathcal{F}$ and satisfies $(B \cap S(\widetilde{\mathbb{Q}})) \triangle A_\infty \in \mathfrak{n}(\widetilde{\mathbb{Q}})$, which implies $B \cap S(\widetilde{\mathbb{Q}}) \in \mathcal{F}(\{\widetilde{\mathbb{Q}}\})$. $\qquad\square$

The next lemma follows directly from the definition of an enlargement.

Lemma 6.2. *Suppose* $(\mathcal{G}, \widehat{\mathfrak{P}})$ *is an enlargement of* $(\Omega, \mathcal{F}, \mathfrak{P})$. *Then* $\iota_{\mathcal{G}}$ *defined as in* (4.4) *is a lattice isomorphism onto its image. In particular,* $\iota_{\mathcal{G}}$ *is strictly positive.*

Lemma 6.3. *Let* $\mathfrak{P}$ *be a set of priors on* $(\Omega, \mathcal{F})$. *Then for any of the following enlargements* $(\mathcal{G}, \widehat{\mathfrak{P}})$ *of* $(\Omega, \mathcal{F}, \mathfrak{P})$, $\iota_{\mathcal{G}}(L^{\infty}(\mathfrak{P}))$ *is order dense and majorising in* $L^{\infty}(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$:

(1) *The universal enlargement* $(\mathcal{H}, \mathfrak{P}^{\mathcal{H}})$.
(2) *The* $\mathfrak{P}$-*universal enlargement* $(\mathcal{A}, \mathfrak{P}^{\mathcal{A}})$.
(3) *The supportable enlargement* $(\mathcal{S}, \mathfrak{Q}^{\sharp})$, *if* $\mathfrak{P}$ *is of class* (S) *and* $\mathfrak{Q} \approx \mathfrak{P}$ *is a supportable alternative.*

Proof. In all cases (1)–(3), the respective spaces are majorising because they contain equivalence classes of constant random variables. It thus remains to prove order density.

(1) Suppose $Y \in L^{\infty}(\mathfrak{P}^{\mathcal{H}})_{++}$ and let $g \in Y$ be a universally measurable representative such that $g = g^{+}$. Then there is $\mathbb{P}_{*} \in \mathfrak{P}$ and $\varepsilon > 0$ such that $\mathbb{P}_{*}^{\mathcal{H}}(g \geq \varepsilon) > 0$. As $\{g \geq \varepsilon\} \in \mathcal{F}(\{\mathbb{P}_{*}\})$ we can find $A \in \mathcal{F}$ such that $A \subset \{g \geq \varepsilon\}$ and $\{g \geq \varepsilon\}\backslash A \in \mathfrak{n}(\mathbb{P}_{*})$. Define $X = [\varepsilon\mathbf{1}_{A}]$ and note that $X \neq 0$, which implies $\iota_{\mathcal{H}}(X) \neq 0$ by Lemma 6.2. Moreover, for all $\mathbb{P} \in \mathfrak{P}$, we have

$$\mathbb{P}^{\mathcal{H}}(\varepsilon\mathbf{1}_{A} \leq g) = \mathbb{P}^{\mathcal{H}}(\{g \geq \varepsilon\} \cap A) + \mathbb{P}(A^{c}) = 1.$$

Hence, $0 \prec^{\mathcal{H}} \iota_{\mathcal{H}}(X) \preceq^{\mathcal{H}} Y$.

(2) This is proved in analogy to (1).

(3) Let $0 \neq Y \in L^{\infty}(\mathfrak{Q}^{\sharp})_{+}$ and $g \in Y$ be a $\mathcal{F}(\mathfrak{Q})$-measurable representative such that $g = g^{+}$. For each $\mathbb{Q} \in \mathfrak{Q}$ we find an $\mathcal{F}$-measurable function $f^{\mathbb{Q}}$ such that $f^{\mathbb{Q}} = (f^{\mathbb{Q}})^{+}$ and $\{f^{\mathbb{Q}} \neq g\} \in \mathfrak{n}(\mathbb{Q})$. Let $X^{\mathbb{Q}} := [f^{\mathbb{Q}}\mathbf{1}_{S(\mathbb{Q})}]_{\sharp}$ and note that

$$X^{\mathbb{Q}} \preceq^{\sharp} Y, \quad \mathbb{Q} \in \mathfrak{Q}.$$

Moreover, $Y = \sup\{X^{\mathbb{Q}} \mid \mathbb{Q} \in \mathfrak{Q}\}$. Indeed, let $U \in L^{\infty}(\mathfrak{Q}^{\sharp})$ be any upper bound and let u be a representative. Then

$$\sup_{\mathbb{Q} \in \mathfrak{Q}} \mathbb{Q}^{\sharp}(u < g) = \sup_{\mathbb{Q} \in \mathfrak{Q}} \mathbb{Q}^{\sharp}(\{u < g\} \cap S(\mathbb{Q}) \cap \{f^{\mathbb{Q}} = g\}) = \sup_{\mathbb{Q} \in \mathfrak{Q}} \mathbb{Q}^{\sharp}(u < f^{\mathbb{Q}}\mathbf{1}_{S(\mathbb{Q})}) = 0.$$

Hence, $Y \preceq^{\sharp} U$. By choosing $\mathbb{Q}^{*} \in \mathfrak{Q}$ appropriately, we obtain

$$0 \prec^{\sharp} \iota_{\mathcal{S}}([f^{\mathbb{Q}^{*}}\mathbf{1}_{S(\mathbb{Q}^{*})}]) = [f^{\mathbb{Q}^{*}}\mathbf{1}_{S(\mathbb{Q}^{*})}]_{\sharp} \preceq^{\sharp} Y.$$

$\square$

For a general enlargement $(\mathcal{G}, \widehat{\mathfrak{P}})$ of $(\Omega, \mathcal{F}, \mathfrak{P})$, we will in the following denote the $\widehat{\mathfrak{P}}$-q.s. order by $\trianglelefteq$.

Lemma 6.4. *Suppose* $\mathfrak{P}$ *is of class* (S) *and that an enlargement* $(\mathcal{G}, \widehat{\mathfrak{P}})$ *completes* $L^{\infty}(\mathfrak{P})$. *Then the following assertions hold:*

(1) $\mathbf{sca}(\mathfrak{P})$ and $\mathbf{sca}(\widehat{\mathfrak{P}})$ are isomorphic in the sense that each $\mu \in \mathbf{sca}(\mathfrak{P})_+$ extends uniquely to a $\widehat{\mu} \in \mathbf{sca}(\widehat{\mathfrak{P}})_+$ given by

$$\widehat{\mu}(B) := \sup\{\mu(A) \mid A \in \mathcal{F}, \mathbf{1}_A \trianglelefteq \mathbf{1}_B\}, \quad B \in \mathcal{G}.^6$$

Moreover, for all $\nu \in \mathbf{sca}(\widehat{\mathfrak{P}})$ we have $\nu|_{\mathcal{F}} \in \mathbf{sca}(\mathfrak{P})$ and $\widehat{\nu|_{\mathcal{F}}} = \nu$.
(2) $\widehat{\mathfrak{P}}$ is of class (S).
(3) For all $B \in \mathcal{G}$ and all $\mu \in \mathbf{sca}(\mathfrak{P})$ there is $A_\mu \in \mathcal{F}$ such that $\mathbf{1}_{B \cap S(\mu)} = \mathbf{1}_{A_\mu}$ in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$. Moreover, there is $R_1 \in \mathfrak{n}(\mu)$ and $R_2 \in \mathcal{G}$ with $\mathbf{1}_{R_2} = 0$ in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$ such that $B \triangle A_\mu = R_1 \cup R_2$.
(4) For each $\mu \in \mathbf{sca}(\mathfrak{P})$ and associated $\widehat{\mu} \in \mathbf{sca}(\widehat{\mathfrak{P}})$ we have $\mathbf{1}_{S(\mu)} = \mathbf{1}_{S(\widehat{\mu})}$ in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$ where $S(\mu) \in \mathcal{F}$ denotes the $\mathfrak{P}$-q.s. order support of μ and $S(\widehat{\mu}) \in \mathcal{G}$ is the $\widehat{\mathfrak{P}}$-q.s. order support. Hence, $S(\mu)$ also serves as $\widehat{\mathfrak{P}}$-q.s. order support of $\widehat{\mu}$.

Proof. (1) By [2, Theorem 1.84] and the proof of this result, $L^\infty(\mathfrak{P})^\sim_n$ and $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})^\sim_n$ are lattice isomorphic via the bijection given on positive functionals by $(L^\infty(\mathfrak{P})^\sim_n)_+ \ni \phi \mapsto \widehat{\phi} \in (L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})^\sim_n)_+$, where

$$\widehat{\phi} : L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}}) \to \mathbb{R}, \quad Y \mapsto \sup\{\phi(X) \mid X \in L^\infty(\mathfrak{P}), X \trianglelefteq Y\}. \tag{6.3}$$

Note that $\widehat{\phi}(\iota_\mathcal{G}(X)) = \phi(X)$ for all $X \in L^\infty(\mathfrak{P})$. By Proposition 4.17, $L^\infty(\mathfrak{P})^\sim_n = \mathbf{sca}(\mathfrak{P})$, and by Proposition 4.4(2) we have $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})^\sim_n = \mathbf{sca}(\widehat{\mathfrak{P}})$. Hence, for $\phi \in (L^\infty(\mathfrak{P})^\sim_n)_+$ and associated $\widehat{\phi} \in (L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})^\sim_n)_+$ let $\mu \in \mathbf{sca}(\mathfrak{P})_+$ and $\widehat{\mu} \in \mathbf{sca}(\widehat{\mathfrak{P}})_+$ be such that $\phi = \int \cdot \, d\mu$ and $\widehat{\phi} = \int \cdot \, d\widehat{\mu}$. Then by monotonicity of the integral we obtain for $B \in \mathcal{G}$:

$$\begin{aligned} \widehat{\mu}(B) = \widehat{\phi}(\mathbf{1}_B) &= \sup\{\phi(X) \mid X \in L^\infty(\mathfrak{P}), X \trianglelefteq \mathbf{1}_B\} \\ &= \sup\{\mu(A) \mid A \in \mathcal{F}, \mathbf{1}_A \trianglelefteq \mathbf{1}_B\}. \end{aligned} \tag{6.4}$$

In particular $\widehat{\mu}|_{\mathcal{F}} = \mu$.
(2) By Proposition 4.18(2) it suffices to show that $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})^\sim_n$ separates the points of $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$. To this end, let $Y \in L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})_{++}$. Choose $X \in L^\infty(\mathfrak{P})$ such that $0 \triangleleft X \trianglelefteq Y$ (order density of $\iota_\mathcal{G}(L^\infty(\mathfrak{P}))$ in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$). Then

$$\sup_{\widehat{\phi} \in (L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})^\sim_n)_+} \widehat{\phi}(Y) \geq \sup_{\widehat{\phi} \in (L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})^\sim_n)_+} \widehat{\phi}(\iota_\mathcal{G}(X)) = \sup_{\phi \in (L^\infty(\mathfrak{P})^\sim_n)_+} \phi(X) > 0,$$

where the equality is due to the isomorphism (6.3). The last inequality is due to $X \in L^\infty(\mathfrak{P})_{++}$, since $0 \triangleleft X$ and $\iota_\mathcal{G}$ is strictly positive, and the fact that $L^\infty(\mathfrak{P})^\sim_n$ separates the points of $L^\infty(\mathfrak{P})$ (Proposition 4.18(1)). Hence, $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})^\sim_n$ separates the points of $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$.
(3) Suppose without loss of generality that $\mu \in \mathbf{sca}(\mathfrak{P})_+$ and let $B \in \mathcal{G}$. By (6.4),

$$\widehat{\mu}(B \cap S(\mu)) = \sup\{\mu(C \cap S(\mu)) \mid C \in \mathcal{F}, \mathbf{1}_{C \cap S(\mu)} \trianglelefteq \mathbf{1}_{B \cap S(\mu)}\}.$$

---

^{6} Here and in the following we abuse notation slightly and write $X \trianglelefteq Y$ for $X \in L^\infty(\mathfrak{P})$ and $Y \in L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$ instead of $\iota_\mathcal{G}(X) \trianglelefteq Y$.

As $\mathbf{1}_{S(\mu)}L^\infty(\mathfrak{P})$ is super Dedekind complete, there is an increasing sequence of events $(C_n)_{n\in\mathbb{N}}$ such that $C_\infty := \bigcup_{n\in\mathbb{N}} C_n$ satisfies

$$\mathbf{1}_{C_\infty \cap S(\mu)} = \sup\{\mathbf{1}_{C\cap S(\mu)} \mid C \in \mathcal{F}, \ \mathbf{1}_{C\cap S(\mu)} \trianglelefteq \mathbf{1}_{B\cap S(\mu)}\} \text{ in } L^\infty(\mathfrak{P}),$$

and $\mathbf{1}_{C_\infty \cap S(\mu)} \trianglelefteq \mathbf{1}_{B\cap S(\mu)}$. Note that since $L^\infty(\mathfrak{P})$ is order dense in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$, for any $D \in \mathcal{G}$ such that $\mathbf{1}_D \neq 0$ in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$, there is $E \in \mathcal{F}$ such that $0 \triangleleft \mathbf{1}_E \trianglelefteq \mathbf{1}_D$. Hence, in fact $\mathbf{1}_{C_\infty \cap S(\mu)} = \mathbf{1}_{B\cap S(\mu)}$ in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$. Set $A_\mu := C_\infty \cap S(\mu)$. Then $R_2 := (B\triangle A_\mu) \cap S(\mu) = (B \cap S(\mu))\triangle A_\mu$ satisfies $\mathbf{1}_{R_2} = 0$ in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$. Finally, let $R_1 := (B\triangle A_\mu) \cap S(\mu)^c$ and note that $R_1 \in \mathfrak{n}(\mu)$. In total, we obtain $B\triangle A_\mu = R_1 \cup R_2$. This proves (3).

(4) In the situation of the proof of (3), if $B \in \mathcal{G}$ satisfies $\widehat{\mu}(B) = \widehat{\mu}(\Omega) = \mu(\Omega)$, then $\mu(A_\mu) = \mu(\Omega)$. By minimality of $S(\mu)$ and since $\widehat{\mathfrak{P}}|_\mathcal{F} \approx \mathfrak{P}$ we have

$$\mathbf{1}_{S(\mu)} \trianglelefteq \mathbf{1}_{A_\mu} = \mathbf{1}_{B\cap S(\mu)} \trianglelefteq \mathbf{1}_B.$$

Hence, $\mathbf{1}_{S(\mu)} = \mathbf{1}_{S(\widehat{\mu})}$ in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$.

<div align="right">□</div>

Proof of Theorem 4.38. (1) Suppose $L^\infty(\mathfrak{P})$ is Dedekind complete. Consider the enlargements $(\mathcal{H}, \mathfrak{P}^\mathcal{H})$, $(\mathcal{A}, \mathfrak{P}^\mathcal{A})$, and $(\mathcal{S}, \mathfrak{Q}^\sharp)$, where $\mathfrak{Q} \approx \mathfrak{P}$ is any supportable alternative. $\iota_\mathcal{H}(L^\infty(\mathfrak{P}))$, $\iota_\mathcal{A}(L^\infty(\mathfrak{P}))$, and $\iota_\mathcal{S}(L^\infty(\mathfrak{P}))$ are order dense and majorising Dedekind complete sublattices in the respective spaces $L^\infty(\mathfrak{P}^\mathcal{H})$, $L^\infty(\mathfrak{P}^\mathcal{A})$, and $L^\infty(\mathfrak{Q}^\sharp)$ by Lemma 6.3. By [2, Theorem 1.40] they are thus majorising ideals. This implies the claimed identities.

(2) Lemma 6.4(1) proves the first assertion. $\widehat{\mathfrak{Q}} \ll \widehat{\mathfrak{P}}$ holds *a priori*. Conversely, assume $A \in \mathcal{G}$ satisfies $0 \triangleleft \mathbf{1}_A$. As $\iota_\mathcal{G}(L^\infty(\mathfrak{P}))$ is order dense in $L^\infty(\Omega, \mathcal{G}, \widehat{\mathfrak{P}})$, we may choose $X \in L^\infty(\mathfrak{P})_+$ and $\mathbb{Q}^* \in \mathfrak{Q}$ such that $0 \triangleleft X\mathbf{1}_{S(\mathbb{Q}^*)} \trianglelefteq \mathbf{1}_A$. Hence, recalling (6.4),

$$\sup_{\mathbb{Q}\in\mathfrak{Q}} \widehat{\mathbb{Q}}(A) \geq \widehat{\mathbb{Q}}^*(A) \geq \mathbb{E}_{\mathbb{Q}^*}[X\mathbf{1}_{S(\mathbb{Q}^*)}] > 0.$$

Consequently, $\widehat{\mathfrak{Q}} \approx \widehat{\mathfrak{P}}$. By (1) we can assume $\mathcal{G} = \mathcal{G}\left(\mathbf{sca}(\widehat{\mathfrak{P}})_+\right)$ and thus $\mathcal{F} \subset \mathcal{H} \subset \mathcal{A} \subset \mathcal{S} \subset \mathcal{G}$.

<div align="right">□</div>

Proof of Theorem 4.39. (1) implies (2): Let $\mathfrak{Q}$ be a disjoint supportable alternative to $\mathfrak{P}^*$ with the properties outlined in Definition 4.15. By Remark 4.25, the enlargement $(\mathcal{S}, \mathfrak{Q}^\sharp)$ completes $L^\infty(\mathfrak{P}) = L^\infty(\mathfrak{P}^*)$. Moreover, if we set $\mathfrak{O} = \mathfrak{Q}$, the claimed properties follow from the definition of $\mathcal{F}(\mathfrak{Q})$ and Proposition 4.37.

(2) implies (3): Let $\mathfrak{Q}$ be any disjoint supportable alternative to $\mathfrak{P}$. We prove first that we can find pairwise disjoint $\mathcal{F}$-measurable supports. By Theorem 4.38(2) and Lemma 6.4(4), $\widehat{\mathfrak{Q}}$ is a disjoint supportable alternative to $\widehat{\mathfrak{P}}$. Hence, by Theorem 4.22(3), there are pairwise disjoint versions $(S_{\widehat{\mathbb{Q}}})_{\mathbb{Q}\in\mathfrak{Q}} \subset \mathcal{G}$ of the $\widehat{\mathfrak{P}}$-q.s. order supports of $\widehat{\mathfrak{Q}}$.

Let $\mathfrak{O}$ be as in (2). Again by Theorem 4.38(2) we have $\widehat{\mathfrak{O}} \approx \widehat{\mathfrak{Q}}$. Fix $\mathbb{Q} \in \mathfrak{Q}$ and select an at most countable set $\mathfrak{R} \subset \mathfrak{O}$ such that $\widehat{\mathbb{Q}} \ll \widehat{\mathfrak{R}}$ (Corollary 5.5(2)). By assumption, for all

$\mathbb{O} \in \mathfrak{R}$ there is $A_{\mathbb{O}} \in \mathcal{F}$ such that $A_{\mathbb{O}} \subset S_{\widehat{\mathbb{Q}}}$ and $\mathbb{O}(A_{\mathbb{O}}) = \widehat{\mathbb{O}}(S_{\widehat{\mathbb{Q}}})$. Define

$$S_{\mathbb{Q}} = \bigcup_{\mathbb{O} \in \mathfrak{R}} A_{\mathbb{O}} \in \mathcal{F}.$$

Then $S_{\mathbb{Q}} \subset S_{\widehat{\mathbb{Q}}}$ and one verifies that $\mathbb{Q}(S_{\mathbb{Q}}) = 1$. Hence, $\mathbf{1}_{S(\mathbb{Q})} = [\mathbf{1}_{S_{\mathbb{Q}}}]$ in $L^{\infty}(\mathfrak{P})$. The family $(S_{\mathbb{Q}})_{\mathbb{Q} \in \mathfrak{Q}}$ inherits pairwise disjointness from the family $(S_{\widehat{\mathbb{Q}}})_{\mathbb{Q} \in \mathfrak{Q}}$. Remark 4.25 proves that the enlargement $(\mathcal{S}, \mathfrak{Q}^{\sharp})$ completes $L^{\infty}(\mathfrak{P})$. Finally, as $L^{\infty}(\mathfrak{Q}^{\sharp})$ is invariant under the choice of the original supportable alternative $\mathfrak{Q}$, assertion (3) follows.

(3) implies (1): Without loss of generality we can assume $\mathfrak{Q}$ is a disjoint supportable alternative. By Propoistion 4.23(2), $\mathfrak{Q}$ is at most equinumerous with the continuum. By Theorem 4.38(2), the set of priors $\mathfrak{Q}^{\sharp}$ on $(\Omega, \mathcal{S})$ is of class (S) and its own supportable alternative. Theorem 4.22(3) yields a family $(S_{\mathbb{Q}^{\sharp}})_{\mathbb{Q} \in \mathfrak{Q}^{\sharp}} \subset \mathcal{S}$ of pairwise disjoint events with the property that

$$\mathbf{1}_{S(\mathbb{Q}^{\sharp})} = [\mathbf{1}_{S_{\mathbb{Q}^{\sharp}}}]_{\sharp}, \quad \mathbb{Q} \in \mathfrak{Q}.$$

As $\mathcal{S} = \mathcal{F}(\mathfrak{Q})$ by Proposition 4.37, for all $\mathbb{Q} \in \mathfrak{Q}$ there is $S_{\mathbb{Q}} \in \mathcal{F}$ such that $S_{\mathbb{Q}} \subset S_{\mathbb{Q}^{\sharp}}$ and $S_{\mathbb{Q}^{\sharp}} \backslash S_{\mathbb{Q}} \in \mathfrak{n}(\mathbb{Q})$. Hence, $\mathbf{1}_{S_{\mathbb{Q}}} = \mathbf{1}_{S(\mathbb{Q})}$ in $L^{\infty}(\mathfrak{P})$ and $\mathbf{1}_{S_{\mathbb{Q}}} = \mathbf{1}_{S(\mathbb{Q})} = \mathbf{1}_{S(\mathbb{Q}^{\sharp})}$ in $L^{\infty}(\mathfrak{Q}^{\sharp})$, respectively. Clearly, $\mathfrak{Q} \approx \mathfrak{P}$ has the Hahn property. $\square$

Lemma 6.5. *Suppose $\mathfrak{P}$ is of class (S) and consider the enlargements $(\mathcal{H}, \mathfrak{P}^{\mathcal{H}})$ and $(\mathcal{A}, \mathfrak{P}^{\mathcal{A}})$. Then:*

(1) *$\mathfrak{P}^{\mathcal{H}}$ is of class (S), each $\mu \in \mathbf{ca}(\mathfrak{P})$ satisfies $\mu^{\mathcal{H}} \in \mathbf{ca}(\mathfrak{P}^{\mathcal{H}})$, and each supportable alternative $\mathfrak{Q} \approx \mathfrak{P}$ extends to a supportable alternative $\mathfrak{Q}^{\mathcal{H}} \approx \mathfrak{P}^{\mathcal{H}}$ which is disjoint in case $\mathfrak{Q}$ is disjoint.*

(2) *The assertions of (1) hold true if $\mathcal{A}$ replaces $\mathcal{H}$ and $\mu^{\mathcal{A}}$ denotes the extension of $\mu \in \mathbf{ca}(\mathfrak{P})$ to $\mathcal{A}$.*

Proof. For (1), suppose first $\mu \in \mathbf{ca}(\mathfrak{P})_{+}$. Then μ extends uniquely to a finite measure $\mu^{\mathcal{H}}$ on $(\Omega, \mathcal{H})$. Suppose $N \in \mathcal{H}$ satisfies $\sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{P}^{\mathcal{H}}(N) = 0$. As $\mathcal{H} \subset \mathcal{F}(\{\mu\})$, we can choose $A \in \mathcal{F}$, $A \subset N$, such that $N \backslash A \in \mathfrak{n}(\mu)$. This entails $\sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{P}(A) = 0$, and together with $\mu \ll \mathfrak{P}$, we obtain $\mu^{\mathcal{H}}(N) = \mu(A) = 0$. Now let $\mathfrak{Q} \approx \mathfrak{P}$ be a supportable alternative. A symmetric application of the preceding argument proves that the set of extensions $\mathfrak{Q}^{\mathcal{H}} = \{\mathbb{Q}^{\mathcal{H}} \mid \mathbb{Q} \in \mathfrak{Q}\}$ is equivalent to $\mathfrak{P}^{\mathcal{H}}$. For supportability, let $\mathbb{Q} \in \mathfrak{Q}$ be arbitrary and suppose $B \in \mathcal{H}$ satisfies $\mathbb{Q}^{\mathcal{H}}(B) = 1$. As $\mathcal{H} \subset \mathcal{F}(\{\mathbb{Q}\})$, there is $A \in \mathcal{F}$, $A \subset B$, such that $B \backslash A \in \mathfrak{n}(\mathbb{Q})$. Hence, for any version $S(\mathbb{Q}) \in \mathcal{F}$ of the $\mathfrak{P}$-q.s. order support, $\mathbf{1}_{S(\mathbb{Q})} \preceq^{\mathcal{H}} \mathbf{1}_A \preceq^{\mathcal{H}} \mathbf{1}_B$, and $S(\mathbb{Q})$ is also the $\mathfrak{P}^{\mathcal{H}}$-q.s. order support of $\mathbb{Q}^{\mathcal{H}}$.

(2) is proved in complete analogy. $\square$

Proof of Corollary 4.40. First, $\mathfrak{P}^{\mathcal{H}}$ is of class (S) by Lemma 6.5(1). The same result in conjunction with Proposition 4.23(2) shows that each disjoint supportable alternative to $\mathfrak{P}^{\mathcal{H}}$ is at most equinumerous with the continuum. Second, the same assertions hold for $\mathfrak{P}^{\mathcal{A}}$ by Lemma 6.5(2). Let $\mu \in \mathbf{ca}(\mathfrak{P}) \backslash \mathbf{sca}(\mathfrak{P})$. Without loss of generality, we may assume that μ

is a measure. $\mu^{\mathcal{H}} \in \mathbf{ca}(\mathfrak{P}^{\mathcal{H}})$ and $\mu^{\mathcal{A}} \in \mathbf{ca}(\mathfrak{P}^{\mathcal{A}})$ follows with Lemma 6.5. Moreover, neither $\mu^{\mathcal{H}}$ nor $\mu^{\mathcal{A}}$ can be supportable as this would force supportability of μ. However, as the disjoint supportable alternatives are at most equinumerous with the continuum and neither $\mathbf{ca}(\mathfrak{P}^{\mathcal{H}})^* = L^\infty(\mathfrak{P}^{\mathcal{H}})$ nor $\mathbf{ca}(\mathfrak{P}^{\mathcal{A}})^* = L^\infty(\mathfrak{P}^{\mathcal{A}})$ can hold by Theorem 4.8(2), Corollary 4.34 implies that neither $L^\infty(\mathfrak{P}^{\mathcal{H}})$ nor $L^\infty(\mathfrak{P}^{\mathcal{A}})$ is Dedekind complete.

Consider now the more specific requirement that $(\Omega, \mathcal{F})$ is a Polish Borel space and $\mathfrak{P}$ is a set of Borel priors such that some disjoint supportable alternative contains a perfect or uncountable analytic set. We first observe that $\Delta(\mathcal{F})$ is Polish by [1, Theorem 15.15] and its cardinality does not exceed the cardinality of the continuum. Hence, each supportable alternative to $\mathfrak{P}$ is at most equinumerous with the continuum. Apply Proposition 4.10 and use the more general case above. $\square$

Appendix A. Some technical results

Lemma A.1. *Let $\mathcal{X} \subset L^0(\mathfrak{P})$ be an ideal and let $A \in \mathcal{F}$ be arbitrary. Then the space $\mathcal{B} := \mathbf{1}_A \mathcal{X}$ is a band in $\mathcal{X}$.*

Proof. $\mathcal{B}$ is clearly an ideal. If $(X_\alpha)_{\alpha \in I} \subset \mathcal{B}$ is a net which converges in order to $X \in \mathcal{X}$, we also have $X_\alpha \mathbf{1}_A \to X\mathbf{1}_A$ in order. However, $X_\alpha \mathbf{1}_A = X_\alpha$, and order limits are unique. Hence, $X\mathbf{1}_A = X$, which means precisely that $X \in \mathcal{B}$. $\square$

Lemma A.2. *Suppose $\mathcal{X} \subset L^0(\mathfrak{P})$ is an ideal and let $\mathcal{C} \subset \mathcal{X}$ be non-empty. For $V \in L^0(\mathfrak{P})$, the following are equivalent:*

  (1) *$V = \inf \mathcal{C}$ in $L^0(\mathfrak{P})$ and $V \in \mathcal{X}$.*
  (2) *$V = \inf \mathcal{C}$ in $\mathcal{X}$.*

The analogous result holds for suprema.

Proof. This is a straightforward application of [3, Theorem 1.35]. $\square$

Lemma A.3. *Suppose that for a set of events $\mathcal{A} \subset \mathcal{F}$ the supremum $\sup_{A \in \mathcal{A}} \mathbf{1}_A$ exists in $L^0(\mathfrak{P})$. Then there is an event $B \in \mathcal{F}$ such that*

$$\mathbf{1}_B = \sup_{A \in \mathcal{A}} \mathbf{1}_A. \tag{A.1}$$

Analogously, if $\inf_{A \in \mathcal{A}} \mathbf{1}_A$ exists in $L^0(\mathfrak{P})$, there is an event $C \in \mathcal{F}$ such that

$$\mathbf{1}_C = \inf_{A \in \mathcal{A}} \mathbf{1}_A. \tag{A.2}$$

Proof. For (A.1), suppose $U := \sup_{A \in \mathcal{A}} \mathbf{1}_A$ exists. In particular, $U = U^+$ and $U \preceq \mathbf{1}_\Omega$ has to hold, i.e. $0 \preceq U \preceq \mathbf{1}_\Omega$. As for all $n \in \mathbb{N}$ the identity $\{\mathbf{1}_A \mid A \in \mathcal{A}\} = \{(n\mathbf{1}_A) \wedge \mathbf{1}_\Omega \mid A \in \mathcal{A}\}$ holds, we obtain from [3, Theorem 1.8]

$$U = \sup_{A \in \mathcal{A}} ((n\mathbf{1}_A) \wedge \mathbf{1}_\Omega) = nU \wedge \mathbf{1}_\Omega, \quad n \in \mathbb{N}.$$

Note that $\sup_{n \in \mathbb{N}}(nU \wedge \mathbf{1}_\Omega) = \mathbf{1}_{\{u > 0\}}$, where $u \in U$ is an arbitrary representative. Hence, we may set $B := \{u > 0\}$. (A.2) follows from (A.1) as $\inf_{A \in \mathcal{A}} \mathbf{1}_A = \mathbf{1}_\Omega - \sup_{A \in \mathcal{A}} \mathbf{1}_{A^c}$ by [2, Lemma 1.4]. $\square$

Lemma A.4. *Suppose $\mathfrak{P}$ is of class* (S) *with supportable alternative $\mathfrak{Q}$. For all $X \in L^0(\mathfrak{P})_+$,* $\sup_{\mathbb{Q} \in \mathfrak{Q}} X\mathbf{1}_{S(\mathbb{Q})}$ *exists and is given by X.*

Proof. Consider the set $\mathcal{C} := \{X\mathbf{1}_{S(\mathbb{Q})} \mid \mathbb{Q} \in \mathfrak{Q}\}$ which is order bounded from above by X. Moreover, X is indeed the least upper bound of $\mathcal{C}$. In order to prove this, consider any upper bound Y and let $f \in X$ and $g \in Y$ be representatives. For all $\mathbb{Q} \in \mathfrak{Q}$ we have

$$\mathbb{Q}(g < f) = \mathbb{Q}(\{g < f\} \cap S(\mathbb{Q})) = \mathbb{Q}(\{g < f\mathbf{1}_{S(\mathbb{Q})}\} \cap S(\mathbb{Q})) = 0,$$

because $X\mathbf{1}_{S(\mathbb{Q})} \preceq Y$. We infer $\sup_{\mathbb{Q} \in \mathfrak{Q}} \mathbb{Q}(\{g < f\}) = 0$ and hence $\sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{P}(g < f) = 0$ because $\mathfrak{P} \approx \mathfrak{Q}$. Equivalently, $X \preceq Y$. $\qquad\square$

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