

# RISK SHARING WITH MULTIDIMENSIONAL SECURITY MARKETS

FELIX-BENEDIKT LIEBRICH      GREGOR SVINDLAND

*Department of Mathematics, University of Munich, Germany*

ABSTRACT. We consider the risk sharing problem for capital requirements induced by capital adequacy tests and security markets. The agents involved in the sharing procedure may be heterogeneous in that they apply varying capital adequacy tests and have access to different security markets. We discuss conditions under which there exists a representative agent. Thereafter, we study two frameworks of capital adequacy more closely, polyhedral constraints and distribution based constraints. We prove existence of optimal risk allocations and equilibria within these frameworks and elaborate on their robustness.

KEYWORDS: capital requirements, polyhedral acceptance sets, law-invariant acceptance sets, multidimensional security spaces, Pareto-optimal risk allocations, equilibria, robustness of optimal allocations.

JEL CLASSIFICATION: D52, D53, G12

2010 MATHEMATICS SUBJECT CLASSIFICATION: 91B16, 91B30, 91B32, 91B50

---

*E-mail address:* `liebrich@math.lmu.de`, `svindla@math.lmu.de`.

*Date:* August 2019.

## 1. INTRODUCTION

In this paper we consider the risk sharing problem for capital requirements. Optimal capital and risk allocation among economic agents, or business units, has for decades been a predominant subject in the respective academic and industrial research areas. Measuring financial risks with capital requirements goes back to the seminal paper by Artzner et al. [6]. There, risk measures are *by definition* capital requirements determined by two primitives: the *acceptance set* and the *security market*.

The acceptance set, a subset of an ambient space of losses, corresponds to a *capital adequacy test*. A loss is deemed adequately capitalised if it belongs to the acceptance set, and inadequately capitalised otherwise. If a loss does not pass the capital adequacy test, the agent has to take prespecified remedial actions: she can raise capital in order to buy a portfolio of securities in the security market which, when combined with the loss profile in question, results in an adequately capitalised secured loss.

Suppose the security market only consists of one numéraire asset, liquidly traded at arbitrary quantities. After discounting, one obtains a so-called *monetary risk measure*, which is characterised by satisfying the *cash-additivity property*, that is  $\rho(X + a) = \rho(X) + a$ . Here,  $\rho$  denotes the monetary risk measure,  $X$  is a loss, and  $a \in \mathbb{R}$  is a capital amount which is added to or withdrawn from the loss. Monetary risk measures have been widely studied, see Föllmer & Schied [27, Chap. 4] and the references therein. As observed in Farkas et al. [22, 23, 24] and Munari [35, Chap. 1], there are good reasons for revisiting the original approach to risk measures of Artzner et al. [6]:

- (1) Typically, more than one asset is available in the security market. It is also less costly for the agent to invest in a portfolio of securities designed to secure a specific loss rather than restricting the remedial action to investing in a single asset independent of the loss profile.
- (2) Even if securitisation is constrained to buying a single asset, discounting with this asset may be impossible because it is not a numéraire; cf. Farkas et al. [23]. Also, as risk is measured *after discounting*, the discounting procedure is implicitly assumed not to add additional risk, which is questionable in view of risk factors such as uncertain future interest rates. For a thorough discussion of this issue see El Karoui & Ravenelli [21]. Often, risk is determined purely in terms of the *distribution* of a risky position, a paradigm we discuss in detail below. Therefore, instability of this crucial *law-invariance property* of a risk measure under discounting is another objection. If the security is not riskless (i.e., is an amount of cash added or withdrawn), losses which originally were identically distributed may not share the same distribution any longer after discounting, while losses that originally display different laws may become identically distributed.
- (3) Without discounting, if only a single asset is available in the security market, cash-additivity requires the security to be riskless, and it is questionable whether such a

security is realistically available, at least for longer time horizons. This is a particularly nagging issue in the insurance context.

In this paper we will follow the original ideas in [6] and study the risk sharing problem for risk measures induced by acceptance sets and possibly multidimensional security spaces. We consider a one-period market populated by a finite number  $n \geq 2$  of agents who seek to secure losses occurring at a fixed future date, say tomorrow. We attribute to each agent  $i \in \{1, \dots, n\}$  an ordered vector space  $\mathcal{X}_i$  of *losses net of gains* she may incur, an acceptance set  $\mathcal{A}_i \subseteq \mathcal{X}_i$  as capital adequacy test, and a security market consisting of a subspace  $\mathcal{S}_i \subseteq \mathcal{X}_i$  of security portfolios as well as observable prices of these securities given by a linear functional  $\mathbf{p}_i : \mathcal{S}_i \rightarrow \mathbb{R}$ . As the securities in  $\mathcal{S}_i$  are deemed suited for hedging, the linearity assumptions on  $\mathcal{S}_i$  and  $\mathbf{p}_i$  reflect that they are liquidly traded and their bid-ask spread is zero. The risk attitudes of agent  $i$  are fully captured by the resulting *risk measure*

$$\rho_i(X) := \inf\{\mathbf{p}_i(Z) : Z \in \mathcal{S}_i, X - Z \in \mathcal{A}_i\}, \quad X \in \mathcal{X}_i, \quad (1.1)$$

that is the minimal capital required to secure  $X$  with securities in  $\mathcal{S}_i$ .

The problem we consider is how to reduce the aggregated risk in the system by means of redistribution. Formally, we assume that each individual space  $\mathcal{X}_i$  of losses net of gains is a subspace of a larger ambient ordered vector space  $\mathcal{X}$ . This space models the losses the system in total incurs if  $\mathcal{X} = \sum_{i=1}^n \mathcal{X}_i$ , which we shall assume *a priori*. Given such a market loss  $X \in \mathcal{X}$ , we need to solve the optimisation problem

$$\sum_{i=1}^n \rho_i(X_i) \rightarrow \min \quad \text{subject to } X_i \in \mathcal{X}_i \text{ and } X_1 + \dots + X_n = X. \quad (1.2)$$

A vector  $\mathbf{X} = (X_1, \dots, X_n)$ , a so-called allocation of  $X$ , which solves the optimisation problem and yields a finite optimal value is *Pareto-optimal*. However, this resembles centralised redistribution which attributes to each agent a certain portion of the aggregate loss in an overall optimal way without considering individual well-being. Redistribution by agents trading portions of losses at a certain price while adhering to individual rationality constraints leads to the notion of *equilibrium allocations* and *equilibrium prices*, a variant of the risk sharing problem above.

Special instances of this general problem have been extensively studied in the literature. Borch [10], Arrow [5] and Wilson [44] consider the problem for expected utilities. More recent are studies for convex monetary risk measures, starting with Barrieu & El Karoui [8] and Filipović & Kupper [25]. A key assumption which allows to prove existence of optimal risk sharing for convex monetary risk measures is *law-invariance*, i.e., the measured risk is the same for all losses which share the same distribution under a benchmark probability model, see Jouini et al. [31], Filipović & Svindland [26], Acciaio [1], and Acciaio & Svindland [2]. For a thorough discussion of the existing literature on risk sharing with monetary risk measures we refer to Embrechts et al. [19].

Another related line of literature is General Equilibrium Theory in economics. For a survey we refer to Mas Colell & Zame [34] and Aliprantis & Burkinshaw [4, Chap. 8]. A major

difference though is that the agents we consider have risk preferences over a vector space of losses net of gains, whereas [34] considers agents with preferences over consumption sets which are bounded from below. Hence, our methods to tackle the problem are very different from the classical ones presented in [34]. More closely related are the contributions of Dana & Le Van [15] and Dana et al. [16], even though they consider different classes of preferences. In [16] consumption sets are unbounded from below like in our work, however the authors assume a finite-dimensional economy. Dana & Le Van [15] allow an infinite dimensional economy, but assume the consumption sets to be bounded from below. As the unbounded infinite-dimensional case is the most relevant in finance and insurance applications, we do not ask for any of those restrictions.

In the following we summarise our main contributions.

**Representative agent formulation.** We prove a representative agent formulation of the risk sharing problem: the behaviour of the interacting agents in the market is, under mild assumptions, captured by a *market capital requirement* of type (1.1), namely

$$\Lambda(X) = \inf\{\pi(Z) : Z \in \mathcal{M}, X - Z \in \mathcal{A}_+\},$$

where  $\Lambda(X)$  is the infimal level of aggregated risk realised by a redistribution of  $X$  as in (1.2),  $\mathcal{A}_+$  is a market acceptance set, and  $(\mathcal{M}, \pi)$  is a global security market. This allows deriving useful conditions ensuring the existence of optimal risk allocations.

**Existence of optimal risk allocations in two case studies.** Based on the representative agent formulation, we study two prominent cases, mostly characterised by the involved notions of acceptability, for which we prove that the risk sharing problem (1.2), including the quest for equilibria, admits solutions. In the first instance, individual losses are — in the widest sense — contingent on scenarios of the future state of the economy. A loss is deemed acceptable if certain capital thresholds are not exceeded under a fixed finite set of linear aggregation rules which may vary from agent to agent. The reader may think of a combination of finitely many valuation and stress test rules as studied in Carr et al. [11], see also [27, Sect. 4.8]. The resulting acceptance sets will thus be *polyhedral*.

In the second class of acceptance sets under consideration, whether or not a given loss is deemed adequately capitalised only depends on its distributional properties under a fixed reference probability measure, not on scenariowise considerations: acceptability is a statistical notion. More precisely, losses are modelled as random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and the respective individual acceptance sets  $\mathcal{A}_i$ ,  $i \in \{1, \dots, n\}$ , will be *law-invariant*: whether or not a loss  $X$  belongs to  $\mathcal{A}_i$  only depends on its distribution under the probabilistic reference model  $\mathbb{P}$ . However, we will not assume that the security spaces  $\mathcal{S}_i$  are law-invariant. Hence, securitisation depends on the potentially varying joint distribution of the loss and the security and is thus statewise rather than distributional. This both reflects the practitioner's reality and is mathematically interesting as the resulting capital requirements  $\rho_i$  are far from law-invariant. In fact, for non-trivial law-invariant  $\mathcal{A}_i$ ,  $\rho_i$  is law-invariant only if the security space is trivial in the sense of being spanned by the cash asset, i.e.,  $\mathcal{S}_i = \mathbb{R}$ . For such risk measures, the risk sharing problem has been solved, cf. [26, 31].

We utilise these results, but should like to emphasise that reducing the general problem for non-trivial  $\mathcal{S}_i$  to the law-invariant cash-additive case is impossible.

**Robustness of optimal allocations.** As a third contribution, we carefully study continuity properties of the set-valued map assigning to an aggregated loss its optimal risk allocations in the mentioned polyhedral and law-invariant acceptability frameworks. These reflect different types of robustness under misspecification of the input. If the map is *upper hemicontinuous*, approximating a complex loss with simpler losses and calculating optimal risk allocations for these will yield an optimal risk allocation for the complex loss as a limit point. It is therefore a useful property from a numerical point of view. *Lower hemicontinuity*, on the other hand, guarantees that *any* given optimal risk allocation stays close to optimal under a slight perturbation of the underlying aggregated loss.

**Existence of optimal portfolio splits.** At last, we study optimal splitting problems in the spirit of Tsanakas [42] and Wang [43]. The question here is whether, under the presence of market frictions such as transaction costs, a financial institution can split an aggregated loss optimally by introducing subsidiaries subject to potentially varying regulatory regimes having access to potentially varying security markets. Applying our previous results, we will show that this problem admits solutions in our framework.

**Structure of the paper.** In Sect. 2 we rigorously introduce risk measurement in terms of capital requirements, agent systems, optimal allocations, and equilibria. Sect. 3 presents the representative agent formulation of the risk sharing problem and proves useful meta results. These are key to the discussion of risk sharing involving polyhedral acceptance sets in Sect. 4 and law-invariant acceptance sets in Sect. 5, as well as optimal portfolio splits in Sect. 6. For the convenience of the reader and better accessibility, Sects. 3–5 first present their main results and the discussion thereof. Ancillary results and the proofs of the main results follow in a separate subsection. Technical supplements are relegated to the appendix.

## 2. AGENT SYSTEMS AND OPTIMAL ALLOCATIONS

**2.1. Risk measurement regimes.** In a first step of modelling, we assume that the attitude of individual agents towards risk is given by a *risk measurement regime* and corresponding *risk measure*.

**Definition 2.1.** Let  $(\mathcal{X}, \preceq)$  be an ordered vector space,  $\mathcal{X}_+$  be its positive cone, i.e.,  $\mathcal{X}_+ := \{X \in \mathcal{X} : 0 \preceq X\}$ , and  $\mathcal{X}_{++} := \mathcal{X}_+ \setminus \{0\}$ .

- An *acceptance set* is a nonempty proper and convex subset  $\mathcal{A}$  of  $\mathcal{X}$  which is monotone, i.e.  $\mathcal{A} - \mathcal{X}_+ \subseteq \mathcal{A}$ .<sup>1</sup>
- A *security market* is a pair  $(\mathcal{S}, \mathbf{p})$  consisting of a finite-dimensional linear subspace  $\mathcal{S} \subseteq \mathcal{X}$  and a positive linear functional  $\mathbf{p} : \mathcal{S} \rightarrow \mathbb{R}$  such that there is  $U \in \mathcal{S} \cap \mathcal{X}_{++}$  with

---

<sup>1</sup>Here and in the following, given subsets  $A$  and  $B$  of a vector space  $\mathcal{X}$ ,  $A + B$  denotes their Minkowski sum  $\{a + b : a \in A, b \in B\}$ , and  $A - B := A + (-B)$ .

$\mathbf{p}(U) = 1$ . The elements  $Z \in \mathcal{S}$  are called security portfolios or simply securities, and  $\mathcal{S}$  is the *security space*, whereas  $\mathbf{p}$  is called *pricing functional*.

- A triple  $\mathcal{R} := (\mathcal{A}, \mathcal{S}, \mathbf{p})$  is a *risk measurement regime* if  $\mathcal{A}$  is an acceptance set and  $(\mathcal{S}, \mathbf{p})$  is a security market such that the following no-arbitrage condition holds:

$$\forall X \in \mathcal{X} : \sup\{\mathbf{p}(Z) : Z \in \mathcal{S}, X + Z \in \mathcal{A}\} < \infty. \quad (2.1)$$

- The *risk measure* associated to a risk measurement regime  $\mathcal{R}$  is the functional

$$\rho_{\mathcal{R}} : \mathcal{X} \rightarrow (-\infty, \infty], \quad X \mapsto \inf\{\mathbf{p}(Z) : Z \in \mathcal{S}, X - Z \in \mathcal{A}\}. \quad (2.2)$$

Risk measure  $\rho_{\mathcal{R}}$  is *normalised* if  $\rho_{\mathcal{R}}(0) = 0$ , or equivalently  $\sup_{Z \in \mathcal{A} \cap \mathcal{S}} \mathbf{p}(Z) = 0$ . It is *lower semicontinuous* (l.s.c.) with respect to some vector space topology  $\tau$  on  $\mathcal{X}$  provided every lower level set  $\{X \in \mathcal{X} : \rho_{\mathcal{R}}(X) \leq c\}$ ,  $c \in \mathbb{R}$ , is  $\tau$ -closed.

Immediate consequences of the definition of  $\rho_{\mathcal{R}}$  are the following properties:

- $\rho_{\mathcal{R}}$  is a proper function<sup>2</sup> by (2.1) and  $\rho_{\mathcal{R}}(Y) \leq 0$  for any choice of  $Y \in \mathcal{A}$ . Moreover, it is convex, i.e.,  $\rho_{\mathcal{R}}(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_{\mathcal{R}}(X) + (1 - \lambda) \rho_{\mathcal{R}}(Y)$  holds for all choices of  $\lambda \in [0, 1]$  and  $X, Y \in \mathcal{X}$ ;
- $\preceq$ -*monotonicity*, i.e.,  $X \preceq Y$  implies  $\rho_{\mathcal{R}}(X) \leq \rho_{\mathcal{R}}(Y)$ ;
- $\mathcal{S}$ -*additivity*, i.e.,  $\rho_{\mathcal{R}}(X + Z) = \rho_{\mathcal{R}}(X) + \mathbf{p}(Z)$  for all  $X \in \mathcal{X}$  and all  $Z \in \mathcal{S}$ .

Note that risk measures as in (2.2) evaluate the risk of *losses net of gains*  $X \in \mathcal{X}$ . The positive cone  $\mathcal{X}_+$  corresponds to pure losses. Therefore,  $\rho_{\mathcal{R}}$  is nondecreasing with respect to  $\preceq$ , not nonincreasing as in most of the literature on risk measures where the risk of *gains net of losses* is measured. The appropriate generalisation of convex risk measures in the usual monotonicity would therefore be the functional  $\tilde{\rho}$  defined by  $\tilde{\rho}(X) = \rho_{\mathcal{R}}(-X)$ ,  $X \in \mathcal{X}$ . In the same vein, the functional  $\mathfrak{U}$  defined by  $\mathfrak{U}(X) = -\rho_{\mathcal{R}}(-X)$ ,  $X \in \mathcal{X}$ , generalises monetary utility functions; cf. Delbaen [18].

In the security market, however, we consider the usual monotonicity, i.e., a security  $Z^* \in \mathcal{S}$  is better than  $Z \in \mathcal{S}$  if  $Z \preceq Z^*$ . This also explains positivity of the pricing functional  $\mathbf{p} : \mathcal{S} \rightarrow \mathbb{R}$ . Combining these two viewpoints, the impact of a security  $Z \in \mathcal{S}$  on a loss profile  $X \in \mathcal{S}$  is given by  $X - Z$ , and  $\rho_{\mathcal{R}}(X)$  is the infimal price that has to be paid for a security  $Z$  in the security market with loss profile  $-Z$  in order to reduce the risk of  $X$  to an acceptable level. The no-arbitrage condition (2.1) means that one cannot short arbitrarily valuable securities and stay acceptable.

There is a close connection between capital requirements defined by (2.2) and superhedging. Given a risk measurement regime  $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathbf{p})$  on an ordered vector space  $(\mathcal{X}, \preceq)$ , let  $\ker(\mathbf{p}) := \{N \in \mathcal{S} : \mathbf{p}(N) = 0\}$  denote the kernel of the pricing functional, i.e. the set of fully leveraged security portfolios available at zero cost. Moreover, fix an arbitrary  $U \in \mathcal{S} \cap \mathcal{X}_+$  whose price is given by  $\mathbf{p}(U) = 1$ . Each  $Z \in \mathcal{S}$  can be written as  $Z = \mathbf{p}(Z)U + (Z - \mathbf{p}(Z)U)$ , and  $Z - \mathbf{p}(Z)U \in \ker(\mathbf{p})$ . Hence,  $X \in \mathcal{X}$  and  $Z \in \mathcal{S}$  satisfy  $X - Z \in \mathcal{A}$  if and only if for

<sup>2</sup>Given a nonempty set  $M$ , a function  $f : M \rightarrow [-\infty, \infty]$  is proper if  $f^{-1}(\{-\infty\}) = \emptyset$  and  $f \not\equiv \infty$ .

$r := \mathbf{p}(Z) \in \mathbb{R}$  we can find  $N \in \ker(\mathbf{p})$  such that

$$rU + N + (-X) \in -\mathcal{A}.$$

The risk  $\rho_{\mathcal{R}}(X)$  may thus be expressed as

$$\begin{aligned} \rho_{\mathcal{R}}(X) &= \inf\{\mathbf{p}(Z) : Z \in \mathcal{S}, X - Z \in \mathcal{A}\} \\ &= \inf\{r \in \mathbb{R} : \exists N \in \ker(\mathbf{p}) \text{ such that } N + rU + (-X) \in -\mathcal{A}\}, \end{aligned}$$

The set  $-\mathcal{A}$  is the set of acceptable *gains net of losses*, and  $-X$  is the payoff associated to the loss profile  $X$ . The elements in  $\ker(\mathbf{p})$  are *zero cost investment opportunities*. If we conservatively choose the acceptance set  $\mathcal{A} = -\mathcal{X}_+$ ,

$$\rho_{\mathcal{R}}(X) = \inf\{r \in \mathbb{R} : \exists N \in \ker(\mathbf{p}) \text{ s.t. } N + rU + (-X) \succeq 0\},$$

that is we recover by  $\rho_{\mathcal{R}}(X)$  the superhedging price of the payoff  $-X$ . A general risk measurement regime thus leads to a superhedging functional involving the relaxed notion of superhedging  $N + rU + (-X) \in -\mathcal{A}$ . In the terminology of superhedging theory,  $\rho_{\mathcal{R}}(X)$  is the infimal amount of cash that needs to be invested in the security  $U$  such that  $X$  can be superhedged when combined with a suitable zero cost trade in the (security) market. Such relaxed superhedging functionals have been recently studied by, e.g., Cheridito et al. [13]. The separation between  $U$  and  $\ker(\mathbf{p})$  introduced above will be useful throughout the paper. Let us give a classical example for a risk measurement regime:

**Example 2.2.** Consider risky future monetary losses net of gains modelled by (equivalence classes) of integrable random variables on an atomless probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . In other words,  $\mathcal{X} := L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$ . A classical capital adequacy test is given by the *Average Value at Risk* at some level  $\beta \in (0, 1)$ ; that is,  $X \in L^1$  belongs to the acceptance set  $\mathcal{A}$  and thus passes the capital adequacy test if and only if

$$\sup_{Q \in \mathcal{Q}} \mathbb{E}[QX] \leq 0,$$

where  $\mathcal{Q}$  is the set of all densities  $Q \in L^1_+$  such that  $\mathbb{E}[Q] = 1$  and  $Q \leq \frac{1}{1-\beta}$   $\mathbb{P}$ -almost surely. For the sake of simplicity we will assume that interest rates are trivial. The security market may consist of a defaultable bond, i.e.  $\mathbf{1}_A$  for some  $A \in \mathcal{F}$  with  $0 < \mathbb{P}(A) < 1$ , a finite number of assets  $\mathcal{X}$  and a finite number of call and put options on these assets. For the latter, we assume for each  $X \in \mathcal{X}$  a set of strike prices  $\mathbb{K}^X$  to be given, and each of the calls  $(X - k)^+$ , and puts  $(k - X)^-$ ,  $k \in \mathbb{K}^X$ , lies in  $\mathcal{S}$ . Suppose now that  $Q^* \in L^1_+$  satisfies

$$0 < \delta \leq Q^* \leq \frac{1-\delta}{1-\beta} + \delta$$

for some  $\delta \in (0, 1)$  and  $\mathbb{E}[Q^*] = 1$ . We will then see in Sect. 5 that if securities in  $\mathcal{S}$  are priced by  $\mathbf{p}(Z) = \mathbb{E}[Q^*Z]$ , then  $(\mathcal{A}, \mathcal{S}, \mathbf{p})$  is a risk measurement regime.

We also refer to [24] for more examples of risk measurement regimes in our sense.

**2.2. Agent systems.** In order to introduce the risk sharing problem in precise terms, a notion of the interplay of the individual agents and their respective capital requirements is required; for terminology concerning ordered vector spaces, we refer to [3, Chaps. 8–9]. We consider an abstract one-period market which incurs aggregated losses net of gains modelled by a Riesz space  $(\mathcal{X}, \preceq)$ . The market comprises  $n \geq 2$  agents, and throughout the paper we identify each individual agent with a natural number  $i$  in the set  $\{1, \dots, n\}$ , which we shall denote by  $[n]$  for the sake of brevity. The agents might have rather heterogeneous assessments of risks. This is firstly reflected by the assumption that each agent operates on an (*order*) *ideal*<sup>3</sup>  $\mathcal{X}_i \subseteq \mathcal{X}$ ,  $i \in [n]$ , which may be a proper subset of  $\mathcal{X}$ . Without loss of generality we shall impose  $\mathcal{X} = \mathcal{X}_1 + \dots + \mathcal{X}_n$ . Within each ideal, and thus for each agent, adequately capitalised losses are encoded by an acceptance set  $\mathcal{A}_i \subseteq \mathcal{X}_i$ . Agent  $i \in [n]$  is allowed to secure losses she may incur with securities from a security market  $(\mathcal{S}_i, \mathbf{p}_i)$ , where  $\mathcal{S}_i \subsetneq \mathcal{X}_i$ . We shall impose that each  $\mathcal{R}_i := (\mathcal{A}_i, \mathcal{S}_i, \mathbf{p}_i)$  is a risk measurement regime on  $(\mathcal{X}_i, \preceq|_{\mathcal{X}_i \times \mathcal{X}_i})$ ,  $i \in [n]$ . In sum, the individual risk assessments are fully captured by the  $n$ -tuple of risk measurement regimes  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ .

**Definition 2.3.** An  $n$ -tuple  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ , where, for each  $i \in [n]$ ,  $\mathcal{R}_i$  is a risk measurement regime on  $\mathcal{X}_i$ , is called an *agent system* if

- ( $\star$ ) For all  $i, j \in [n]$ , the pricing functionals  $\mathbf{p}_i$  and  $\mathbf{p}_j$  agree on  $\mathcal{S}_i \cap \mathcal{S}_j$ . Moreover, if we set  $i \sim j$  if  $i \neq j$  and  $\mathbf{p}_i$  is non-trivial on  $\mathcal{S}_i \cap \mathcal{S}_j$ , the resulting graph

$$G = ([n], \{\{i, j\} \subseteq [n] : i \sim j\})$$

is connected.<sup>4</sup>

Axiom ( $\star$ ) clarifies the nature of the interaction of the involved agents: prices for securities accepted by more than one agent have to agree, and any two agents may interact and exchange securities by potentially invoking other agents as intermediaries. Throughout this paper we will assume that the agents  $[n]$  form an agent system. Such a situation is not too far-fetched:

**Definition 2.4.** The space of *jointly accepted securities* is  $\check{\mathcal{S}} := \bigcap_{i=1}^n \mathcal{S}_i$ . The *global security space* is  $\mathcal{M} := \mathcal{S}_1 + \dots + \mathcal{S}_n$ .

If, besides agreement of prices,  $\check{\mathcal{S}} \neq \{0\}$  and  $\mathbf{p}_i|_{\check{\mathcal{S}}} \neq 0$  for some and thus all  $i \in [n]$ , then assumption ( $\star$ ) is met. The resulting graph is the complete graph on  $n$  vertices. Moreover, if all agents operate on one and the same space  $\mathcal{X}_i = \mathcal{X}$ ,  $i \in [n]$ , and the available security markets are identical and given by  $\mathcal{S}_i = \mathbb{R} \cdot U$ ,  $i \in [n]$ , for some  $U \in \mathcal{X}_{++}$  and  $\mathbf{p}_i(rU) = r$ ,  $r \in \mathbb{R}$ ,  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$  is an agent system. If we further specify  $\mathcal{X}$  to be a sufficiently rich space of random variables and  $U = \mathbf{1}$  is the constant random variable with value 1, the results for

<sup>3</sup>An ideal  $\mathcal{Y}$  of a Riesz space  $(\mathcal{X}, \preceq)$  is a subspace in which the inclusion  $\{Z \in \mathcal{X} : |Z| \preceq |Y|\} \subseteq \mathcal{Y}$  holds for all  $Y \in \mathcal{Y}$ .

<sup>4</sup>That is, between any two vertices  $i, j \in [n]$ ,  $i \neq j$ , we can find a connecting path over edges of the graph, meaning that either  $i \sim j$  or we can find  $i_1, \dots, i_m \in [n]$  for a suitable  $m \in \mathbb{N}$  such that  $i \sim i_1$ ,  $i_1 \sim i_2$ ,  $\dots$ ,  $i_{m-1} \sim i_m$ , and  $i_m \sim j$ . This will for instance be needed in the proof of Proposition 3.6.

risk sharing with convex monetary risk measures can be embedded in our setting of agent systems; cf. [1, 2, 26, 31].

In the following we write  $\rho_i$  instead of  $\rho_{\mathcal{R}_i}$  for the sake of brevity. Aggregated losses in  $\mathcal{X}$  will be denoted by  $X, Y$  or  $W$ , securities by  $Z, U$  or  $N$  throughout the paper.

**2.3. The risk sharing problem and its solutions.** In order to introduce the risk sharing associated to an agent system  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ , we need the notion of attainable and security allocations:

**Definition 2.5.** A vector  $\mathbf{X} = (X_1, \dots, X_n) \in \prod_{i=1}^n \mathcal{X}_i$  is an *attainable allocation* of an aggregated loss  $W \in \mathcal{X}$  if  $W = X_1 + \dots + X_n$ . We denote the set of all attainable allocations of  $W$  by  $\mathbb{A}_W$ .

Given a global security  $Z \in \mathcal{M}$ , we denote by  $\mathbb{A}_Z^s := \mathbb{A}_Z \cap \prod_{i=1}^n \mathcal{S}_i$  the set of *security allocations* of  $Z$ .

Given a set  $S \neq \emptyset$  and a function  $f : S \rightarrow [-\infty, \infty]$ , we set

$$\text{dom}(f) := \{s \in S : f(s) < \infty\}$$

to be the *effective domain* of  $f$ . We will also abbreviate its lower level sets by  $\mathcal{L}_c(f) := \{s \in S : f(s) \leq c\}$ ,  $c \in \mathbb{R}$ .

We are now prepared to introduce the risk sharing problem. Its objective is to minimise the aggregated risk within the system. The allowed remedial action is reallocating an aggregated loss  $W \in \mathcal{X}$  among the agents involved; so we study

$$\sum_{i=1}^n \rho_i(X_i) \rightarrow \min \quad \text{subject to } \mathbf{X} \in \mathbb{A}_W. \quad (2.3)$$

The optimal value in (2.3) is less than  $+\infty$  if and only if  $W \in \sum_{i=1}^n \text{dom}(\rho_i)$ . It is furthermore well known that (2.3) is closely related to certain notions of economically optimal allocations which we define in the following.

**Definition 2.6.** Let  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$  be an agent system on an ordered vector space  $(\mathcal{X}, \preceq)$ , let  $W \in \mathcal{X}$  be an aggregated loss, and let  $\mathbf{W} \in \prod_{i=1}^n \mathcal{X}_i$  be a vector of initial loss endowments.

- (1) An attainable allocation  $\mathbf{X} \in \mathbb{A}_W$  is *Pareto-optimal* if  $\rho_i(X_i) < \infty$ ,  $i \in [n]$ , and for any  $\mathbf{Y} \in \mathbb{A}_W$  with the property  $\rho_i(Y_i) \leq \rho_i(X_i)$ ,  $i \in [n]$ , in fact  $\rho_i(X_i) = \rho_i(Y_i)$  has to hold for all  $i \in [n]$ .
- (2) Suppose  $(\mathcal{X}, \preceq, \tau)$  is a *topological Riesz space*, i.e.  $\mathcal{X}$  carries a *vector space topology*  $\tau$ . A tuple  $(\mathbf{X}, \phi)$  is an *equilibrium* of  $\mathbf{W}$  if

- $\mathbf{X} \in \mathbb{A}_{W_1 + \dots + W_n}$ ,
- $\phi \in \mathcal{X}^*$  is positive with  $\phi|_{\mathcal{S}_i} = \mathbf{p}_i$ ,  $i \in [n]$ ,
- the budget constraints  $\phi(-X_i) \leq \phi(-W_i)$ ,  $i \in [n]$ , hold,<sup>5</sup>
- and  $\rho_i(X_i) = \inf\{\rho_i(Y) : Y \in \mathcal{X}_i, \phi(-Y) \leq \phi(-W_i)\}$  for all  $i \in [n]$ .

<sup>5</sup>Note that the minus sign in the budget constraints is due to the fact that the elements in  $\mathcal{X}_i$  model losses, whereas  $\phi$  prices payoffs.

In that case,  $\mathbf{X}$  is called *equilibrium allocation* and  $\phi$  *equilibrium price*.

Now that all pieces are in place, we close this section by commenting on the static nature of the model introduced here. Indeed, we study risk sharing in a generalised one-period framework. The very general notion of market spaces underlying our definitions provide the possibility that loss profiles capture dynamics themselves, being, for instance, trajectories of the evolution of the value of a good over time. However, extending capital requirements to a dynamic multi-period framework poses some difficulties. For instance, in such an extension finite-dimensionality of security spaces might be lost. As we will see, the finite-dimensionality of the security spaces is crucial for important results in this paper. Generalising capital requirements to a multi-period framework will therefore be an interesting topic for future research.

### 3. INFIMAL CONVOLUTIONS AND THE REPRESENTATIVE AGENT

This section comprises the formal mathematical treatment of the risk sharing problem on ideals of a Riesz space as introduced in Sect. 2. We shall link risk sharing to the infimal convolution of the individual risk measures, prove its representation as a capital requirement for the market, i.e., for a *representative agent*, and find powerful sufficient conditions for the existence of optimal payoffs, Pareto-optimal allocations, and equilibria. Similar approaches have been undertaken by, e.g., [1, 2, 8, 25, 26, 31, 32].

Beforehand, however, we need to introduce further axioms that an agent system may satisfy in addition to  $(\star)$ . We shall refer to them at various stages of the paper, they are however not assumed to be met throughout. For  $n \geq 2$  let  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$  be an agent system.

(A1) *No security arbitrage.* For some  $j \in [n]$  it holds that

$$\left( \sum_{i \neq j} \ker(\mathbf{p}_i) \right) \cap \mathcal{S}_j \subseteq \ker(\mathbf{p}_j);$$

(A2) *Non-redundance of the joint security market.* There is  $Z \in \tilde{\mathcal{S}}$  and  $\mathbf{Z} \in \mathbb{A}_Z^s$  such that  $\sum_{i=1}^n \mathbf{p}_i(Z_i) \neq 0$ .

(A3) *Supportability.* The underlying space  $\mathcal{X}$  carries a locally convex Hausdorff topology  $\tau$  with dual space  $\mathcal{X}^*$ . Moreover, there is some  $\phi_0 \in \mathcal{X}_+^*$  and a constant  $\gamma \in \mathbb{R}$  such that

(i) for all  $\mathbf{Z} \in \prod_{i=1}^n \mathcal{S}_i$  with  $\sum_{i=1}^n \mathbf{p}_i(Z_i) = 0$  we have

$$\phi_0(Z_1 + \dots + Z_n) = 0,$$

and for some  $\tilde{\mathbf{Z}} \in \prod_{i=1}^n \mathcal{S}_i$  with  $\sum_{i=1}^n \mathbf{p}_i(\tilde{Z}_i) \neq 0$  we have

$$\phi_0(\tilde{Z}_1 + \dots + \tilde{Z}_n) \neq 0;$$

(ii) for all  $\mathbf{Y} \in \prod_{i=1}^n \mathcal{A}_i$  we have  $\phi_0(Y_1 + \dots + Y_n) \leq \gamma$ .

(A4) *Infinite supportability.*  $(\mathcal{R}_i)_{i \in \mathbb{N}}$  is a sequence of risk measurement regimes on a common locally convex Hausdorff topological Riesz space  $(\mathcal{X}, \preceq, \tau)$  such that  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$  satisfies  $(\star)$  for all  $n \in \mathbb{N}$  and such that there is some  $\phi_0 \in \mathcal{X}^*$  with

$$\sum_{i \in \mathbb{N}} \sup_{Y \in \mathcal{A}_i} \phi_0(Y) < \infty \text{ and } \phi_0|_{\mathcal{S}_i} = \mathbf{p}_i, \quad i \in \mathbb{N}.$$

Condition (A1) is violated if each agent would be able to obtain arbitrarily valuable securitisation from the other agents, who can provide it at zero individual cost. That would reveal a mismatch of security markets leading to hypothetical infinite wealth for all agents. Non-redundance of the joint security market is in particular satisfied if there is  $Z \in \tilde{\mathcal{S}}$  such that  $\mathbf{p}_i(Z) \neq 0$  for some  $i \in [n]$  and thus for all  $i \in [n]$  by the defining property  $(\star)$  of an agent system and the agreement of prices. Hence, under (A2) there is a jointly accepted security valuable for the market. Regarding condition (A3), think of  $\phi_0$  as a pricing functional. (i) is a consistency requirement between  $\phi_0$  and the individual prices  $\mathbf{p}_i$ . (ii) reads as the impossibility to decompose a loss  $X$  acceptably for all agents if  $X$  is sufficiently poor, that is the value  $\phi_0(-X)$  of the corresponding payoff  $-X$  under  $\phi_0$  is less than a certain level  $-\gamma$ . Condition (A4) is a strengthening of (A3) for all finite subsystems of  $(\mathcal{R}_i)_{i \in \mathbb{N}}$ .

**3.1. Main results.** According to Proposition 3.6 below, an allocation  $\mathbf{X} \in \mathbb{A}_X$  of  $X \in \mathcal{X}$  is Pareto-optimal if and only if the *risk sharing functional*

$$\Lambda : \mathcal{X} \rightarrow [-\infty, \infty], \quad Y \mapsto \inf_{\mathbf{Y} \in \mathbb{A}_Y} \sum_{i=1}^n \rho_i(Y_i),$$

is exact at  $X$ , that is,  $\Lambda(X) = \sum_{i=1}^n \rho_i(X_i) \in \mathbb{R}$ .  $\Lambda$  corresponds to the so-called *infimal convolution* of the risk measures  $\rho_1, \dots, \rho_n$ , and thus inherits properties like  $\preceq$ -monotonicity and convexity. We refer to Appendix A.2, in particular Lemma A.3, for a brief summary of these facts.

Our next result implies that, if proper,  $\Lambda$  is again a risk measure of type (2.2): the shared risk level is the minimal price the market has to pay for a cumulated security that ensures market acceptability. Thus, market behaviour may be seen as the behaviour of a *representative agent* operating on  $\mathcal{X}$ . Recall from Definition 2.5 that  $\mathbb{A}_Z^s$  denotes the set of security allocations of  $Z \in \mathcal{M}$ .

**Theorem 3.1.** Define  $\pi(Z) := \inf_{\mathbf{Z} \in \mathbb{A}_Z^s} \sum_{i=1}^n \mathbf{p}_i(Z_i)$ ,  $Z \in \mathcal{M}$ .

- (1) For any  $Z \in \mathcal{M}$  and arbitrary  $\mathbf{Z} \in \mathbb{A}_Z^s$ ,  $\pi(Z)$  may be represented as

$$\pi(Z) = \sum_{i=1}^n \mathbf{p}_i(Z_i) + \pi(0).$$

Either  $\pi(0) = 0$  or  $\pi(0) = -\infty$ .  $\pi(0) = 0$  is equivalent to (A1), and in that case  $\pi$  is real-valued, linear, and satisfies  $\pi|_{\mathcal{S}_i} = \mathbf{p}_i$ ,  $i \in [n]$ . Otherwise  $\pi \equiv -\infty$ .

- (2)  $\Lambda$  can be represented as

$$\Lambda(X) = \inf \{ \pi(Z) : Z \in \mathcal{M}, X - Z \in \mathcal{B} \}, \quad X \in \mathcal{X},$$

for any monotone and convex set  $\mathcal{B} \subseteq \mathcal{X}$  satisfying  $\mathcal{A}_+ \subseteq \mathcal{B} \subseteq \mathcal{L}_0(\Lambda)$ . Here,

$$\mathcal{A}_+ := \sum_{i=1}^n \mathcal{A}_i$$

denotes the market acceptance set.

- (3) If (A1) and (A3) hold,  $\Lambda$  is proper.  
(4) If  $\Lambda$  is proper, then (A1) holds, i.e.,  $\pi(0) = 0$ , and  $\pi$  is positive. In that case,  $(\mathcal{A}_+, \mathcal{M}, \pi)$  is a risk measurement regime on  $\mathcal{X}$  and  $\Lambda$  is the associated risk measure.

In the situation of Theorem 3.1(4), the behaviour of the representative agent is given by the risk measurement regime  $(\mathcal{A}_+, \mathcal{M}, \pi)$ . The risk sharing functional is the *market capital requirement* associated to the market acceptance set  $\mathcal{A}_+$  and the global security market  $(\mathcal{M}, \pi)$ .

The preceding theorem also offers a more geometric perspective on the assumption (A3). Suppose the agent system operates on a locally convex Hausdorff topological Riesz space  $(\mathcal{X}, \preceq, \tau)$  and satisfies (A1). Moreover, assume we can find a security  $Z^* \in \mathcal{M}$  such that

$$Z^* \notin \text{cl}_\tau(\mathcal{A}_+ + \ker(\pi)), \quad (3.1)$$

where here and in the the following  $\text{cl}_\tau(\cdot)$  denotes the closure of a set with respect to the topology  $\tau$ . Then (A3) means  $Z^*$  is a security which comes at a true cost for the market; it can be strictly separated from  $\mathcal{A}_+ + \ker(\pi)$  using a linear functional  $\phi_0 \in \mathcal{X}_+^*$ , and this functional is exactly as described in (A3).

We now turn our attention to Pareto-optimal allocations of a loss  $W \in \mathcal{X}$ . We will see that their existence is closely related to the existence of a market security  $Z^W \in \mathcal{M}$  which renders market acceptability  $W - Z^W \in \mathcal{A}_+$  at the minimal price  $\pi(Z^W) = \Lambda(W)$ . Such *optimal payoffs* have recently been studied by Baes et al. [7].

**Definition 3.2.**  $W \in \mathcal{X}$  admits an *optimal payoff*  $Z^W \in \mathcal{M}$  if  $W - Z^W \in \mathcal{A}_+$  and  $\pi(Z^W) = \Lambda(W)$ .

**Theorem 3.3.** *Suppose that  $\Lambda$  is proper. If  $X \in \mathcal{X}$  admits an optimal payoff  $Z^X \in \mathcal{M}$ , then  $X \in \text{dom}(\Lambda)$  and  $\Lambda$  is exact at  $X$ . In particular, for any  $Y_i \in \mathcal{A}_i$ ,  $i \in [n]$ , such that  $\sum_{i=1}^n Y_i = X - Z^X$ , and any  $\mathbf{Z} \in \mathbb{A}_{Z^X}^s$ , the allocation  $(Y_i + Z_i)_{i \in [n]} \in \mathbb{A}_X$  is Pareto-optimal. If moreover  $\mathcal{L}_0(\rho_i) = \mathcal{A}_i + \ker(\mathfrak{p}_i)$ ,  $i \in [n]$ , then  $\Lambda$  is exact at  $X \in \text{dom}(\Lambda)$  if, and only if  $X$  admits an optimal payoff.*

In a topological setting, the existence of optimal payoffs is intimately connected to the Minkowski sum  $\mathcal{A}_+ + \ker(\pi)$  being closed:

**Proposition 3.4.** *Suppose  $(\mathcal{X}, \preceq, \tau)$  is a topological Riesz space and  $\Lambda$  is proper. Then the following are equivalent:*

- (1)  $\mathcal{A}_+ + \ker(\pi)$  is closed.
- (2)  $\Lambda$  is l.s.c. and every  $X \in \text{dom}(\Lambda)$  admits an optimal payoff.

Proposition 3.4 is related to [7, Proposition 4.1]. Together with Theorem 3.3, it is a powerful sufficient condition for the existence of Pareto optima which we shall apply in Sects. 4 and 5. The only non-trivial steps will be to verify the properness of  $\Lambda$  and closedness of  $\mathcal{A}_+ + \ker(\pi)$ . We proceed with the discussion of equilibria in the very general case when market losses are modelled by a *Fréchet lattice*  $(\mathcal{X}, \preceq, \tau)$ . As this notion is ambiguous in the literature,

we emphasise that a Fréchet lattice is a locally convex-solid<sup>6</sup> topological Riesz space whose topology is completely metrisable.

In particular, *Banach lattices* are Fréchet lattices. As a more general example, one may consider the Wiener space  $C([0, \infty))$  of all continuous functions on the non-negative half-line with the pointwise order  $\leq$  and the topology  $\tau_D$  arising from the metric

$$D(f, g) := \sum_{k=1}^{\infty} 2^{-k} \frac{\max_{0 \leq r \leq k} |f(r) - g(r)|}{1 + \max_{0 \leq r \leq k} |f(r) - g(r)|}, \quad f, g \in C([0, \infty)).$$

Clearly,  $(C([0, \infty)), \leq, \tau_D)$  is not a Banach lattice, but a Fréchet lattice. Its choice as model space is justified if the primitives in question are continuous trajectories of, e.g., the net value of some good over time.

Recall the definition of the jointly accepted securities,  $\check{\mathcal{S}} := \bigcap_{i=1}^n \mathcal{S}_i$ . Moreover, we set here and in the following  $\text{int dom}(\Lambda)$  to be the  $\tau$ -interior of the effective domain of the risk sharing functional  $\Lambda$ .

**Theorem 3.5.** *Suppose  $\mathcal{X}$  is a Fréchet lattice and that  $\Lambda$  is l.s.c. and proper. Moreover, let (A2) be satisfied, i.e., there is a  $\tilde{Z} \in \check{\mathcal{S}}$  with  $\pi(\tilde{Z}) \neq 0$ . If a vector of loss endowments  $\mathbf{W} \in \prod_{i=1}^n \mathcal{X}_i$  satisfies  $W := W_1 + \dots + W_n \in \text{int dom}(\Lambda)$  and there exists a Pareto-optimal allocation of  $W$ , there is an equilibrium  $(\mathbf{X}, \phi)$  of  $\mathbf{W}$ .*

**3.2. Ancillary results and proofs.** Our first result links Pareto optima, equilibria, and solutions to the risk sharing problem (2.3). Proposition 3.6(2) is indeed the first fundamental theorem of welfare economics adapted to our setting.

**Proposition 3.6.** *Let  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$  be an agent system on an ordered vector space  $(\mathcal{X}, \preceq)$ , let  $W \in \mathcal{X}$  be an aggregated loss, and let  $\mathbf{W} \in \prod_{i=1}^n \mathcal{X}_i$  be a vector of initial loss endowments. The following statements hold true:*

- (1) *If  $W \in \sum_{i=1}^n \text{dom}(\rho_i)$ , then  $\mathbf{X} \in \mathbb{A}_W$  is a Pareto-optimal attainable allocation of  $W$  if and only if*

$$\sum_{i=1}^n \rho_i(X_i) = \inf_{\mathbf{Y} \in \mathbb{A}_W} \sum_{i=1}^n \rho_i(Y_i). \quad (3.2)$$

- (2) *If  $(\mathcal{X}, \preceq, \tau)$  is a topological Riesz space and  $\mathbf{W}$  satisfies  $W_1 + \dots + W_n \in \sum_{i=1}^n \text{dom}(\rho_i)$ , any equilibrium allocation of  $\mathbf{W}$  is Pareto-optimal.*

The proof requires the following well-known characterisation of Pareto optima; see, e.g., [37, Proposition 3.2].

**Lemma 3.7.** *Let  $W \in \sum_{i=1}^n \text{dom}(\rho_i)$ . If  $\mathbf{X}$  is a Pareto-optimal attainable allocation of  $W$ , there are so-called Negishi weights  $\lambda_i \geq 0$ ,  $i \in [n]$ , not all equal to zero, such that*

$$\sum_{i=1}^n \lambda_i \rho_i(X_i) = \inf_{\mathbf{Y} \in \mathbb{A}_W} \sum_{i=1}^n \lambda_i \rho_i(Y_i). \quad (3.3)$$

<sup>6</sup>This means that the topology has a neighbourhood base at 0 consisting of convex and solid sets; cf. [4, Sect. 2.3].

Conversely, if  $\mathbf{X} \in \mathbb{A}_X$  satisfies (3.3) for a set of strictly positive weights  $\lambda_i > 0$ ,  $i \in [n]$ , then  $\mathbf{X}$  is a Pareto-optimal attainable allocation.

The proof of Proposition 3.6 shows that the agent system property  $(\star)$  dictates the values of the Negishi weights.

*Proof of Proposition 3.6.* (1) By Lemma 3.7 any solution to (3.2) is Pareto-optimal. Conversely, let  $W \in \sum_{i=1}^n \text{dom}(\rho_i)$  and let  $\mathbf{X} \in \mathbb{A}_W$  be a Pareto-optimal attainable allocation. Let  $\lambda \in \mathbb{R}_{++}^n$  be any vector of Negishi weights such that  $\mathbf{X}$  is a solution to (3.3). Recall the symmetric relation  $\sim$  in  $(\star)$  and consider  $j, k \in [n]$  such that  $j \sim k$ . By definition, we find  $Z \in \mathcal{S}_j \cap \mathcal{S}_k$  such that  $p := \mathbf{p}_j(Z) = \mathbf{p}_k(Z) \neq 0$ . For  $t \in \mathbb{R}$  let

$$\mathbf{X}^t := \mathbf{X} + \frac{t}{p} Z e_j - \frac{t}{p} Z e_k \in \mathbb{A}_X.$$

Here,  $Z e_j$  is the vector whose  $j$ th entry is  $Z$ , whereas all other entries are 0. Analogously, we define  $Z e_k$ . By the  $\mathcal{S}_i$ -additivity of all the  $\rho_i$ , we infer

$$-\infty < \sum_{i=1}^n \lambda_i \rho_i(X_i) \leq \inf_{t \in \mathbb{R}} \sum_{i=1}^n \lambda_i \rho_i(X_i^t) = \sum_{i=1}^n \lambda_i \rho_i(X_i) + \inf_{t \in \mathbb{R}} t(\lambda_j - \lambda_k).$$

This is only possible if  $\lambda_j = \lambda_k$ . Using that the graph  $G$  in  $(\star)$  is connected, one inductively shows  $\lambda_1 = \dots = \lambda_n$ . As not all the  $\lambda_i$  equal 0, this implies that they all have to be positive. Dividing both sides of (3.3) by  $\lambda_1$  yields

$$\sum_{i=1}^n \rho_i(X_i) = \inf_{\mathbf{Y} \in \mathbb{A}_X} \sum_{i=1}^n \rho_i(Y_i).$$

(2) Suppose that  $\mathbf{W}$  is an initial loss endowment with associated equilibrium  $(\mathbf{X}, \phi)$ . The equality  $\phi(X_i) = \phi(W_i)$  holds for all  $i \in [n]$  because  $\sum_{i=1}^n X_i = \sum_{i=1}^n W_i$ ,  $\phi$  is linear, and  $\phi(-X_i) \leq \phi(-W_i)$  holds for all  $i \in [n]$ . Given  $Z_i \in \mathcal{S}_i$  such that  $\mathbf{p}_i(Z_i) = 1$  and arbitrary  $Y_i \in \mathcal{X}_i$ ,

$$\phi(Y_i + (\phi(X_i) - \phi(Y_i))Z_i) = \phi(X_i) = \phi(W_i)$$

holds as  $\phi = \mathbf{p}_i$  on  $\mathcal{S}_i$ . Thus the budget constraint is satisfied, and hence

$$\rho_i(X_i) \leq \rho_i(Y_i + \phi(X_i - Y_i)Z_i) = \rho_i(Y_i) + \phi(X_i) - \phi(Y_i).$$

If we set  $W := W_1 + \dots + W_n$ , for any other allocation  $\mathbf{Y} \in \mathbb{A}_W$  we obtain

$$\sum_{i=1}^n \rho_i(X_i) \leq \sum_{i=1}^n \rho_i(Y_i) + \phi(X_i) - \phi(Y_i) = \sum_{i=1}^n \rho_i(Y_i)$$

since  $\sum_{i=1}^n \phi(X_i) = \sum_{i=1}^n \phi(Y_i) = \phi(W)$ . By (1),  $\mathbf{X}$  is Pareto-optimal.  $\square$

*Proof of Theorem 3.1.* (1) Let  $Z \in \mathcal{M}$  and let  $\mathbf{Z} \in \mathbb{A}_Z^s$  be arbitrary, but fixed. The identity  $\mathbb{A}_Z^s = \mathbf{Z} + \mathbb{A}_0^s$  implies

$$\pi(Z) = \sum_{i=1}^n \mathbf{p}_i(Z_i) + \inf_{\mathbf{N} \in \mathbb{A}_0^s} \sum_{i=1}^n \mathbf{p}_i(N_i) = \sum_{i=1}^n \mathbf{p}_i(Z_i) + \pi(0).$$

Consider  $\mathcal{V} := \{(\mathbf{p}_i(N_i))_{i \in [n]} : \mathbf{N} \in \mathbb{A}_0^s\}$ , which is a subspace of  $\mathbb{R}^n$ . In the following, we denote by  $e_l$  the  $l$ th unit vector of  $\mathbb{R}^n$ . We claim  $\pi(0) = 0$  if and only if  $\dim(\mathcal{V}) < n$ . Indeed, let  $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^n$  and observe that  $\pi(0) = \inf_{x \in \mathcal{V}} \langle \mathbf{1}, x \rangle$  which is  $-\infty$  in case  $\dim(\mathcal{V}) = n$ . Suppose that  $\dim(\mathcal{V}) < n$ , i.e.  $\mathcal{V}^\perp \neq \{0\}$ , and let  $0 \neq \lambda \in \mathcal{V}^\perp$ . As in the proof of Proposition 3.6(1),  $e_j - e_k \in \mathcal{V}$  holds for all  $j, k \in [n]$  such that  $j \sim k$ , which implies  $\lambda_j = \lambda_k$ . As the relation  $\sim$  induces a connected graph,  $\lambda \in \text{span}\{\mathbf{1}\} = \mathcal{V}^\perp$ . Hence, we obtain that  $\langle \mathbf{1}, x \rangle = 0$  for all  $x \in \mathcal{V}$  which implies  $\pi(0) = 0$ , so we have proved equivalence of  $\pi(0) = 0$  and  $\dim(\mathcal{V}) < n$ . But  $\dim(\mathcal{V}) < n$  is equivalent to the fact that there is a  $j \in [n]$  such that  $e_j \notin \mathcal{V}$ , which in turn is equivalent to (A1): whenever  $Z \in \mathcal{S}_j$  lies in the Minkowski sum  $\sum_{i \neq j} \ker(\mathbf{p}_i)$ ,  $\mathbf{p}_j(Z) = 0$  has to hold.

(2) We first note that  $\mathcal{A}_+$  is convex and monotone. Indeed, let  $X, Y \in \mathcal{X}$  such that  $Y \in \mathcal{A}_+$  and  $X \preceq Y$ . Fix  $\mathbf{Y} \in \mathbb{A}_Y$  such that  $Y_i \in \mathcal{A}_i$ ,  $i \in [n]$ , and  $\mathbf{X} \in \mathbb{A}_X$  arbitrary. By the Riesz Decomposition Property (cf. [3, Sect. 8.5]), there are  $W_1, \dots, W_n \in \mathcal{X}_+$  such that  $Y - X = \sum_{i=1}^n W_i$  and  $W_i \preceq |Y_i - X_i|$ , which means  $\mathbf{W} \in \mathbb{A}_{Y-X}$ . Hence, for all  $i \in [n]$ , we obtain  $Y_i - W_i \in \mathcal{A}_i$  by the monotonicity of  $\mathcal{A}_i$ , and thus  $X = \sum_{i=1}^n Y_i - W_i \in \mathcal{A}_+$ . Moreover,  $\mathcal{L}_0(\Lambda)$  is monotone and convex as well, which follows from the corresponding properties of  $\Lambda$ . For  $\mathcal{B} \subseteq \mathcal{B}'$ , we have

$$\inf \{\pi(Z) : Z \in \mathcal{M}, X - Z \in \mathcal{B}\} \geq \inf \{\pi(Z) : Z \in \mathcal{M}, X - Z \in \mathcal{B}'\}.$$

As  $\mathcal{A}_+ \subseteq \mathcal{L}_0(\Lambda)$ , (2) is proved if for arbitrary  $X \in \mathcal{X}$  we can show the two estimates

$$\Lambda(X) \geq \inf \{\pi(Z) : Z \in \mathcal{M}, X - Z \in \mathcal{A}_+\}, \quad (3.4)$$

and

$$\Lambda(X) \leq \inf \{\pi(Z) : Z \in \mathcal{M}, X - Z \in \mathcal{L}_0(\Lambda)\}. \quad (3.5)$$

The first assertion trivially holds if  $\Lambda(X) = \infty$ . If  $X \in \text{dom}(\Lambda) = \sum_{i=1}^n \text{dom}(\rho_i)$ , choose  $\mathbf{X} \in \mathbb{A}_X$  such that  $\rho_i(X_i) < \infty$ ,  $i \in [n]$ , and  $\varepsilon > 0$  arbitrary. Suppose  $\mathbf{Z} \in \prod_{i=1}^n \mathcal{S}_i$  is such that  $\mathbf{p}_i(Z_i) \leq \rho_i(X_i) + \frac{\varepsilon}{n}$  and  $X_i - Z_i \in \mathcal{A}_i$ ,  $i \in [n]$ . Set  $Z^* := Z_1 + \dots + Z_n$  and observe  $X - Z^* \in \mathcal{A}_+$  as well as

$$\sum_{i=1}^n \rho_i(X_i) + \varepsilon \geq \sum_{i=1}^n \mathbf{p}_i(Z_i) \geq \pi(Z^*) \geq \inf \{\pi(Z) : Z \in \mathcal{M}, X - Z \in \mathcal{A}_+\}.$$

This proves (3.4). We now turn to (3.5). If  $\Lambda(X) = \infty$ , assume for contradiction there is some  $Z \in \mathcal{M}$  such that  $X - Z \in \mathcal{L}_0(\Lambda) \subseteq \sum_{i=1}^n \text{dom}(\rho_i)$ . Choose  $\mathbf{Y} \in \mathbb{A}_{X-Z}$  such that  $Y_i \in \text{dom}(\rho_i)$  for all  $i$ , and let  $\mathbf{Z} \in \mathbb{A}_Z^s$  be arbitrary. Then

$$\Lambda(X) \leq \sum_{i=1}^n \rho_i(Y_i + Z_i) = \sum_{i=1}^n \rho_i(Y_i) + \sum_{i=1}^n \mathbf{p}_i(Z_i) < \infty.$$

This is a *contradiction* and no such  $Z \in \mathcal{M}$  can exist. (3.5) holds in this case. Now assume  $X \in \text{dom}(\Lambda)$  and suppose  $Z \in \mathcal{M}$  satisfies  $X - Z \in \mathcal{L}_0(\Lambda)$ . Hence, for arbitrary  $\varepsilon > 0$  there

is  $\mathbf{Y} \in \mathbb{A}_{X-Z}$  such that  $\sum_{i=1}^n \rho_i(Y_i) \leq \varepsilon$ . As  $\mathbf{Y} + \mathbf{Z} \in \mathbb{A}_X$  for all  $\mathbf{Z} \in \mathbb{A}_Z^s$ ,

$$\Lambda(X) \leq \inf_{\mathbf{Z} \in \mathbb{A}_Z^s} \sum_{i=1}^n \rho_i(Y_i + Z_i) = \sum_{i=1}^n \rho_i(Y_i) + \pi(Z) \leq \varepsilon + \pi(Z).$$

As  $\varepsilon > 0$  was chosen arbitrarily, we obtain (3.5).

(3) Assume (A1) and (A3) are fulfilled, let  $\phi_0 \in \mathcal{X}_+^*$  as described in (A3), and note that  $\pi$  is linear by (1). We shall prove that  $\phi_0|_{\mathcal{M}} = \kappa\pi$  for some  $\kappa > 0$ , so by rescaling  $\phi_0|_{\mathcal{M}} = \pi$  may be assumed without loss of generality. To this end, we restate requirement (A3)(ii) as  $\sup_{Y \in \mathcal{A}_+} \phi_0(Y) < \infty$ . Condition (A3)(i) means in particular that  $\phi_0|_{\ker(\pi)} \equiv 0$ . For each  $i \in [n]$  fix  $U_i \in \mathcal{S}_i \cap \mathcal{X}_{++}$  such that  $U := \sum_{i=1}^n U_i$  satisfies

$$\pi(U) = \sum_{i=1}^n \mathbf{p}_i(U_i) = 1.$$

As  $Z - \pi(Z)U \in \ker(\pi)$  holds for all  $Z \in \mathcal{M}$  we infer  $\phi_0(Z - \pi(Z)U) = 0$ , or equivalently  $\phi_0 = \phi_0(U)\pi$  on  $\mathcal{M}$ . By the second part of (A3)(i),  $\phi_0(\tilde{Z}) \neq 0$  for some  $\tilde{Z} \in \mathcal{M}$  with  $\pi(\tilde{Z}) \neq 0$ . Using positivity of  $\phi_0$ , we obtain

$$0 < \frac{\phi_0(\tilde{Z})}{\pi(\tilde{Z})} = \phi_0(U),$$

hence we may set  $\kappa := \phi_0(U)$ . Finally, if  $\kappa = 1$ ,  $X \in \mathcal{X}$  is arbitrary, and  $Z \in \mathcal{M}$  is such that  $X - Z \in \mathcal{A}_+$ ,

$$\pi(Z) = \phi_0(Z) = \phi_0(X) - \phi_0(X - Z) \geq \phi_0(X) - \sup_{Y \in \mathcal{A}_+} \phi_0(Y) > -\infty.$$

The bound on the right-hand side is independent of  $Z$ . Using the representation of  $\Lambda$  in (2), properness follows.

(4) Note that  $\Lambda$  is  $\mathcal{M}$ -additive by (2). Since  $\Lambda$  is proper, we cannot have  $\pi \equiv -\infty$ , hence  $\pi(0) = 0$ , i.e., (A1) holds by (1). As regards the positivity of  $\pi$ , choose  $Y \in \mathcal{X}$  with  $\Lambda(Y) \in \mathbb{R}$ . For  $Z \in \mathcal{M} \cap \mathcal{X}_+$ , the monotonicity of  $\Lambda$  then shows  $\Lambda(Y) \leq \Lambda(Y + Z) = \Lambda(Y) + \pi(Z)$ , which entails  $\pi(Z) \geq 0$ . It follows that  $(\mathcal{A}_+, \mathcal{M}, \pi)$  is a risk measurement regime.  $\square$

*Proof of Theorem 3.3.* As  $\Lambda$  is proper, we have that  $\pi$  is linear, finite valued, and  $\pi|_{\mathcal{S}_i} = \mathbf{p}_i$ ,  $i \in [n]$ , by Theorem 3.1. Assume  $X \in \mathcal{X}$  and  $Z = Z^X \in \mathcal{M}$  are such that  $\Lambda(X) = \pi(Z)$  and  $X - Z \in \mathcal{A}_+$ . As  $\pi(Z) \in \mathbb{R}$  and  $\Lambda|_{\mathcal{A}_+} \leq 0$ ,  $X \in \text{dom}(\Lambda)$ . Choose  $Y_i \in \mathcal{A}_i$ ,  $i \in [n]$ , such that  $X - Z = \sum_{i=1}^n Y_i$ . For any  $\mathbf{Z} \in \mathbb{A}_Z^s$  we thus have  $X = \sum_{i=1}^n Y_i + Z_i$  and

$$\Lambda(X) \leq \sum_{i=1}^n \rho_i(Y_i + Z_i) = \sum_{i=1}^n \rho_i(Y_i) + \sum_{i=1}^n \mathbf{p}_i(Z_i) \leq \pi(Z) = \Lambda(X),$$

where we have used  $\rho_i(Y_i) \leq 0$  and  $\pi(Z) = \sum_{i=1}^n \mathbf{p}_i(Z_i)$  (Theorem 3.1). This shows the exactness of  $\Lambda$  at  $X$ .

Now assume  $\mathcal{L}_0(\rho_i) = \mathcal{A}_i + \ker(\mathbf{p}_i)$ ,  $i \in [n]$ . Let  $X \in \text{dom}(\Lambda)$  and  $\mathbf{X} \in \mathbb{A}_X$  such that  $\Lambda(X) = \sum_{i=1}^n \rho_i(X_i)$ . Further, let  $U_i \in \mathcal{S}_i$  with  $\mathbf{p}_i(U_i) = 1$ . As  $X_i - \rho_i(X_i)U_i \in \mathcal{A}_i + \ker(\mathbf{p}_i)$ ,  $i \in [n]$ , by assumption, we may find a vector  $\mathbf{N} \in \prod_{i=1}^n \ker(\mathbf{p}_i)$  such that  $X_i - \rho_i(X_i)U_i + N_i \in \mathcal{A}_i$

for every  $i \in [n]$ . The fact that  $\sum_{i=1}^n \rho_i(X_i)U_i - N_i$  is an optimal payoff for  $X$  is immediately verified.  $\square$

*Proof of Proposition 3.4.* Suppose first that  $\mathcal{A}_+ + \ker(\pi)$  is closed. For lower semicontinuity, we have to establish that  $\mathcal{L}_c(\Lambda)$  is closed for every  $c \in \mathbb{R}$ . To this end, let  $U_i \in \mathcal{S}_i \cap \mathcal{X}_{++}$  such that  $\mathbf{p}_i(U_i) > 0$  and set  $U := \sum_{i=1}^n U_i$ . Without loss of generality, we may assume  $\pi(U) = 1$ . We will show that

$$\mathcal{L}_c(\Lambda) = \{cU\} + \mathcal{A}_+ + \ker(\pi), \quad (3.6)$$

which is closed whenever  $\mathcal{A}_+ + \ker(\pi)$  is closed. The right-hand set in (3.6) is included in the left-hand set by the  $\mathcal{M}$ -additivity of  $\Lambda$ . For the converse inclusion, let  $X \in \mathcal{L}_c(\Lambda)$ . For every  $s > c$ , there is a  $Z_s \in \mathcal{M}$  such that  $c \leq \pi(Z_s) \leq s$  and  $X - Z_s \in \mathcal{A}_+$ . Consider the decomposition

$$X - sU = X - Z_s + (\pi(Z_s) - s)U + Z_s - \pi(Z_s)U.$$

As  $X - Z_s + (\pi(Z_s) - s)U \preceq X - Z_s \in \mathcal{A}_+$  and  $Z_s - \pi(Z_s)U \in \ker(\pi)$ , the monotonicity of  $\mathcal{A}_+$  shows  $X - sU \in \mathcal{A}_+ + \ker(\pi)$ . Thus,

$$X - cU = \lim_{s \downarrow c} X - sU \in \text{cl}_\tau(\mathcal{A}_+ + \ker(\pi)) = \mathcal{A}_+ + \ker(\pi),$$

and (3.6) is proved. Setting  $c = 0$  in (3.6) shows  $\mathcal{L}_0(\Lambda) = \mathcal{A}_+ + \ker(\pi)$ . Hence,  $X - \Lambda(X)U \in \mathcal{A}_+ + \ker(\pi)$  for all  $X \in \text{dom}(\Lambda)$ , and for a suitable  $N \in \ker(\pi)$  depending on  $X$  we have  $X - \Lambda(X)U + N \in \mathcal{A}_+$  and  $\pi(\Lambda(X)U - N) = \Lambda(X)$ . Therefore, an optimal payoff for  $X$  is given by  $\Lambda(X)U - N \in \mathcal{M}$ .

Assume now that  $\Lambda$  is l.s.c. and that every  $X \in \text{dom}(\Lambda)$  allows for an optimal payoff. Let  $(X_i)_{i \in I}$  be a net in  $\mathcal{A}_+ + \ker(\pi)$  converging to  $X \in \mathcal{X}$ . Then  $\Lambda(X) \leq 0$  by the lower semicontinuity of  $\Lambda$ . Let  $Z^X \in \mathcal{M}$  be an optimal payoff for  $X$ , so that  $\pi(Z^X) = \Lambda(X) \leq 0$ . For  $U$  as above and  $Y := X - Z^X \in \mathcal{A}_+$  we obtain  $Y + \pi(Z^X)U \in \mathcal{A}_+$  by the monotonicity of  $\mathcal{A}_+$ . Also  $Z^X - \pi(Z^X)U \in \ker(\pi)$ . Thus  $X = (Y + \pi(Z^X)U) + (Z^X - \pi(Z^X)U) \in \mathcal{A}_+ + \ker(\pi)$ .  $\square$

For the proof of Theorem 3.5, we need the notion of the *convex conjugate* of a proper function  $f : \mathcal{X} \rightarrow (-\infty, \infty]$  on a locally convex Hausdorff topological vector space, which is the function  $f^* : \mathcal{X}^* \rightarrow (-\infty, \infty]$  defined by

$$f^*(\phi) = \sup_{X \in \mathcal{X}} \phi(X) - f(X).$$

Given  $X \in \text{dom}(f)$ ,  $\phi \in \mathcal{X}^*$  is a *subgradient* of  $f$  at  $X$  if  $f(X) = \phi(X) - f^*(\phi)$ .

*Proof of Theorem 3.5.* Fix a vector  $\mathbf{W} \in \prod_{i=1}^n W_i$  with the property

$$W := W_1 + \cdots + W_n \in \text{int dom}(\Lambda).$$

As a Fréchet lattice is a barrelled space,  $\Lambda$  is subdifferentiable at  $W$  by [20, Corollary 2.5 & Proposition 5.2], i.e. there is a subgradient  $\phi \in \mathcal{X}^*$  of  $\Lambda$  at  $W$  satisfying  $\Lambda(W) = \phi(W) -$

$\Lambda^*(\phi)$ . As  $\Lambda$  is monotone,  $\phi \in \mathcal{X}_+^*$ , and by Lemma A.4

$$\Lambda^*(\phi) = \sum_{i=1}^n \rho_i^*(\phi|_{\mathcal{X}_i}).$$

Let  $\mathbf{Y}$  be any Pareto-optimal allocation of  $W$ . As  $\Lambda(W) = \sum_{i=1}^n \rho_i(Y_i) \in \mathbb{R}$ ,  $\Lambda(W)$ ,  $\Lambda^*(\phi)$  and  $\rho_i^*(\phi|_{\mathcal{X}_i})$ ,  $i \in [n]$ , are all real numbers. Also, as

$$\infty > \rho_i^*(\phi|_{\mathcal{X}_i}) \geq \sup_{Z \in \mathcal{S}_i} \phi(Y_i + Z) - \rho_i(Y_i + Z) = \phi(Y_i) - \rho_i(Y_i) + \sup_{Z \in \mathcal{S}_i} \phi(Z) - \mathfrak{p}_i(Z),$$

$\phi|_{\mathcal{S}_i} = \mathfrak{p}_i$ ,  $i \in [n]$ , has to hold. This in turn implies  $\phi|_{\mathcal{M}} = \pi$  by the linearity of  $\pi$  and Theorem 3.1. By (A2), we may fix  $\tilde{Z} \in \tilde{\mathcal{S}}$  such that  $\pi(\tilde{Z}) = 1 = \mathfrak{p}_i(\tilde{Z})$ ,  $i \in [n]$ . Let

$$X_i := Y_i + \phi(W_i - Y_i)\tilde{Z}, \quad i \in [n].$$

As  $\sum_{i=1}^n W_i = \sum_{i=1}^n Y_i = W$ ,  $\sum_{i=1}^n X_i = W$  holds and  $\mathbf{X} \in \mathbb{A}_W$ . Moreover,  $\mathbf{X}$  is Pareto-optimal:

$$\begin{aligned} \sum_{i=1}^n \rho_i(X_i) &= \sum_{i=1}^n \rho_i(Y_i) + \phi(W_i - Y_i)\pi(\tilde{Z}) = \sum_{i=1}^n \rho_i(Y_i) + \phi(W - W) \\ &= \sum_{i=1}^n \rho_i(Y_i) = \Lambda(W). \end{aligned}$$

Also, as  $\phi(X_i) - \rho_i^*(\phi|_{\mathcal{X}_i}) \leq \rho_i(X_i)$  for all  $i \in [n]$  and

$$\sum_{i=1}^n \rho_i(X_i) = \Lambda(W) = \phi(W) - \Lambda^*(W) = \sum_{i=1}^n \phi(X_i) - \rho_i^*(\phi|_{\mathcal{X}_i}),$$

$\rho_i(X_i) = \phi(X_i) - \rho_i^*(\phi|_{\mathcal{X}_i})$  has to hold for all  $i \in [n]$ . We claim that  $(\mathbf{X}, \phi)$  is an equilibrium. Indeed, as  $\phi(-X_i) = \phi(-W_i)$  holds for all  $i \in [n]$ , the budget constraints are satisfied. Moreover, if  $i \in [n]$  and  $Y \in \mathcal{X}_i$  satisfies  $\phi(-Y) \leq \phi(-W_i) = \phi(-X_i)$ , we obtain

$$\rho_i(Y) \geq \phi(Y) - \rho_i^*(\phi|_{\mathcal{X}_i}) \geq \phi(X_i) - \rho_i^*(\phi|_{\mathcal{X}_i}) = \rho_i(X_i).$$

□

#### 4. POLYHEDRAL AGENT SYSTEMS

In the next two sections, we will study two instances of the model introduced in Sect. 2 and employ the methodology discussed in Sect. 3 to find optimal payoffs for the market, Pareto-optimal allocations, and equilibria. Additionally, we will study their robustness. In this section, we shall focus on *polyhedral agent systems*.

**4.1. The setting.** Throughout this section we assume that the agent system  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$  operates on a market space  $\mathcal{X}$  given by a Fréchet lattice. Each agent  $i \in [n]$  operates on a closed ideal  $\mathcal{X}_i \subseteq \mathcal{X}$ , and  $\mathcal{X}_1 + \dots + \mathcal{X}_n = \mathcal{X}$ . The assumption of closedness implies that  $(\mathcal{X}_i, \preceq, \tau \cap \mathcal{X}_i)$  is a Fréchet lattice in its own right. We will assume that each acceptance set  $\mathcal{A}_i \subseteq \mathcal{X}_i$  is *polyhedral*.

**Definition 4.1.** Let  $(\mathcal{X}, \tau)$  be a locally convex topological vector space. A convex set  $\mathcal{C} \subseteq \mathcal{X}$  is called *polyhedral* if there is a finite set  $\mathcal{J} \subseteq \mathcal{X}^*$  and  $\beta \in \mathbb{R}^{\mathcal{J}}$  such that

$$\mathcal{A} = \{X \in \mathcal{X} : \phi(X) \leq \beta(\phi), \forall \phi \in \mathcal{J}\}.$$

If  $(\mathcal{X}, \preceq, \tau)$  is a Fréchet lattice, an agent system  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$  on  $\mathcal{X}$  is *polyhedral* if it has properties (A1) and (A3), and each acceptance set  $\mathcal{A}_i$ ,  $i \in [n]$ , is polyhedral. That is, for each  $i \in [n]$  there is a finite set  $\mathcal{J}_i \subseteq \mathcal{X}_i^*$  and  $\beta_i \in \mathbb{R}^{\mathcal{J}_i}$  such that

$$\mathcal{A}_i = \{X \in \mathcal{X}_i : \forall \phi \in \mathcal{J}_i (\phi(X) \leq \beta_i(\phi))\}.$$

The polyhedrality of a set  $\mathcal{C}$  is equivalent to the existence of some  $m \in \mathbb{N}$ , a continuous linear operator  $T : \mathcal{X} \rightarrow \mathbb{R}^m$ , and  $\beta \in \mathbb{R}^m$  such that

$$\mathcal{C} = \{X \in \mathcal{X} : T(X) \leq \beta\},$$

where the defining inequality is understood coordinatewise. In case of an acceptance set, the representing linear operator can be chosen to be positive. Risk measures with polyhedral acceptance sets play a prominent role in Baes et al. [7], where the set of optimal payoffs for a single such risk measure is studied.

**Example 4.2.** Suppose  $\Omega \neq \emptyset$  is a nonempty set of scenarios for the future state of the economy, either suggested by the internal risk management or a regulatory authority. Moreover, suppose  $\emptyset \neq \Omega_i \subsetneq \Omega$ ,  $i \in [n]$ , are such that  $\Omega = \bigcup_{i=1}^n \Omega_i$ .  $\Omega_i$  denotes the set of scenarios relevant for agent  $i \in [n]$ , whereas  $\Omega_i \cap \Omega_j$  is the (possibly empty) set of jointly relevant scenarios for agents  $i$  and  $j$ . Note that we do not assume the  $\Omega_i$  to be pairwise disjoint. While  $\Omega$  collects the scenarios relevant to the whole system, it is both individually and systemically rational of an agent to demand that her stake in the sharing of a market loss is *neutral* in scenarios  $\omega \in \Omega \setminus \Omega_i$ . The canonical choice of the model spaces is in consequence  $\mathcal{X} := \{X \in \mathbb{R}^\Omega : \sup_{\omega \in \Omega} |X(\omega)| < \infty\}$  endowed with the supremum norm and

$$\mathcal{X}_i := \{X \in \mathcal{X} : X(\omega) = 0, \omega \notin \Omega_i\}, \quad i \in [n].$$

Consider individual capital adequacy tests defined in terms of scenariowise loss constraints on a prespecified finite set of test scenarios  $\Omega^* \subseteq \Omega$ . We need to assume  $\Omega^* \cap \Omega_i \cap \Omega_j \neq \emptyset$  whenever  $\Omega_i \cap \Omega_j \neq \emptyset$ , and that  $\Omega_i^* := \Omega^* \cap \Omega_i$ ,  $i \in [n]$ , is not empty either. The polyhedral acceptance set of agent  $i \in [n]$  is then defined by

$$\mathcal{A}_i := \{X \in \mathcal{X}_i : \forall \omega \in \Omega_i^* (X(\omega) \leq \beta_i(\omega))\}, \quad i \in [n],$$

where  $\beta_i \in \mathbb{R}^{\Omega_i^*}$  is an arbitrary, but fixed vector of individual loss constraints.

Concerning the individual security spaces, let  $\Pi$  be the set of all subsets  $A \subseteq \Omega$  which have the shape  $A = \Omega_i \cap \Omega_j$  or  $A = \Omega_i \setminus \Omega_j$  for some  $i \neq j$  and are nonempty. The security space of agent  $i \in [n]$  is then set to be

$$\mathcal{S}_i := \text{span}\{\mathbf{1}_A : A \in \Pi, A \subseteq \Omega_i\}.$$

At last, for a collection  $(\sigma_\omega)_{\omega \in \Omega^*}$  of positive weights we define the linear functional

$$\pi : \mathcal{M} \rightarrow \mathbb{R}, \quad Z \mapsto \sum_{\omega \in \Omega^*} \sigma_\omega Z(\omega),$$

and the individual prices  $\mathbf{p}_i := \pi|_{\mathcal{S}_i}$ ,  $i \in [n]$ . We additionally assume that the family of intersections  $\Omega_i \cap \Omega_j$ ,  $i, j \in [n]$ ,  $i \neq j$ , which are nonempty is rich enough such that the family  $(\mathcal{R}_i)_{i \in [n]} := ((\mathcal{A}_i, \mathcal{S}_i, \mathbf{p}_i))_{i \in [n]}$  of risk measurement regimes satisfies  $(\star)$ .

In total, if the situation is as described,  $(\mathcal{R}_i)_{i \in [n]}$  is a polyhedral agent system.

**4.2. Main results.** We first turn to the existence of optimal risk allocations in polyhedral agent systems. By definition, such an agent system satisfies (A1) and (A3). The resulting risk sharing functional  $\Lambda$  is proper by Theorem 3.1(3). By Theorem 3.3 and Proposition 3.4, the existence of Pareto-optimal allocations would be proved if the closedness of  $\mathcal{A}_+ + \ker(\pi)$  can be established.

The two main results on optimal risk allocations are the following:

**Theorem 4.3.** *Let  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$  be a polyhedral agent system on a Fréchet lattice  $\mathcal{X}$ . Then the set  $\mathcal{A}_+ + \ker(\pi)$  is proper, polyhedral, and closed,  $\Lambda$  is l.s.c., and every  $X \in \text{dom}(\Lambda)$  admits an optimal payoff  $Z^X \in \mathcal{M}$ , and can thus be allocated Pareto-optimally as in Theorem 3.3.*

Theorem 4.3 is illustrated by an example in Sect. 4.4.

**Remark 4.4.** The proof of the preceding theorem shows that for each agent  $i \in [n]$ , the Minkowski sum  $\mathcal{A}_i + \ker(\mathbf{p}_i)$  is proper, polyhedral, and closed. Moreover, for all  $X_i \in \text{dom}(\rho_i)$  we can find an optimal payoff  $Z^{X_i} \in \mathcal{S}_i$ , i.e.,  $\mathbf{p}_i(Z^{X_i}) = \rho_i(X_i)$  and  $X_i - Z^{X_i} \in \mathcal{A}_i$ .

**Corollary 4.5.** *If a polyhedral agent system  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$  on a Fréchet lattice  $\mathcal{X}$  satisfies (A2), for every  $\mathbf{W} \in \prod_{i=1}^n \mathcal{X}_i$  such that  $W_1 + \dots + W_n \in \text{int dom}(\Lambda)$  there is an equilibrium  $(\mathbf{X}, \phi)$ .*

By Theorem 4.3, the correspondence  $\mathcal{P}$  mapping  $X \in \text{dom}(\Lambda)$  to the set of its Pareto-optimal allocations  $\mathbf{X} \in \mathbb{A}_X$  takes nonempty subsets of  $\mathbb{A}_X$  as values. Invoking Proposition 3.6, we can represent

$$\mathcal{P}(X) = \left\{ \mathbf{X} \in \mathbb{A}_X : \Lambda(X) = \sum_{i=1}^n \rho_i(X_i) \right\}. \quad (4.1)$$

For a brief summary of the terminology concerning correspondences (or set-valued maps) and their properties, we refer to Appendix A.3. Theorem 4.9, the main result on the robustness of  $\mathcal{P}$  in the case of a polyhedral agent system, asserts that  $\mathcal{P}$  can be shown to be lower hemicontinuous. This requires some technical assumptions though.

Suppose  $(\mathcal{R}_i)_{i \in [n]}$  is a polyhedral agent system on a market space  $\mathcal{X}$ . Then for each  $i \in [n]$ , there is a positive continuous linear operator  $T_i : \mathcal{X}_i \rightarrow \mathbb{R}^{m_i}$  for suitable  $m_i \in \mathbb{R}$  and a vector  $\beta_i \in \mathbb{R}^{m_i}$  such that

$$\mathcal{A}_i := \{Y \in \mathcal{X}_i : T_i(Y) \leq \beta_i\}.$$

In case  $\mathcal{X}$  is infinite-dimensional, we will need the following assumptions:

**Assumption 4.6.** For each  $i \in [n]$ ,  $\mathcal{X}_i$  is complemented in  $\mathcal{X}$  by a closed subspace  $\mathcal{Y}_i$ , that is  $\mathcal{X} = \mathcal{X}_i \oplus \mathcal{Y}_i$ , where  $\oplus$  denotes the direct sum of two vector spaces.

Hence, for each  $X \in \mathcal{X}$  there is a unique decomposition  $X = \tilde{X} + \tilde{Y}$ , where  $\tilde{X} \in \mathcal{X}_i$  and  $\tilde{Y} \in \mathcal{Y}_i$ . Moreover, by the closed graph theorem [30, Chap. 3, Theorem 5], the projection  $\delta_i : X \mapsto \tilde{X}$  is continuous and  $\mathcal{Y}_i = \ker(\delta_i)$ .

Let us define  $\tilde{T}_i := T_i \circ \delta_i$  and  $\mathcal{Z} := \bigcap_{i=1}^n \ker(\tilde{T}_i)$ . Hence,  $\mathcal{Y}$  is complemented in  $\mathcal{X}$  by a finite-dimensional closed subspace  $\mathcal{Z}$ , i.e.,  $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$ . The projections  $\gamma_1 : \mathcal{X} \rightarrow \mathcal{Y}$  and  $\gamma_2 : \mathcal{X} \rightarrow \mathcal{Z}$  are continuous.

**Assumption 4.7.** For each  $i \in [n]$ , we have  $\gamma_1(\mathcal{X}_i) \subseteq \mathcal{X}_i$ .

At last, we will assume

**Assumption 4.8.** For each  $i \in [n]$  there is a continuous linear  $P_i : \mathcal{Y} \rightarrow \gamma_1(\mathcal{X}_i)$  such that  $\sum_{i=1}^n P_i = id_{\mathcal{Y}}$ .

**Theorem 4.9.** Consider a polyhedral agent system  $(\mathcal{R}_i)_{i \in [n]}$  on a market space  $\mathcal{X}$ . Suppose that Assumptions 4.6–4.8 hold in case  $\mathcal{X}$  is infinite-dimensional. Then the correspondence  $\mathcal{P}$  is lower hemicontinuous on  $\text{dom}(\Lambda)$  and admits a continuous selection on  $\text{dom}(\Lambda)$ .

**Example 4.10.** (1) Assumptions 4.6–4.8 are automatically satisfied if all individual ideals  $\mathcal{X}_i$  agree with the market space  $\mathcal{X}$ . Indeed,  $\mathcal{Y}_i = \{0\}$  can be chosen in Assumption 4.6,  $\gamma_1(\mathcal{X}_i) = \gamma_1(\mathcal{X}) \subseteq \mathcal{X} = \mathcal{X}_i$  gives Assumption 4.7, and  $P_i := \frac{1}{n} id_{\mathcal{X}}$  is possible in Assumption 4.8.

(2) In the situation described by Example 4.2 Assumptions 4.6–4.8 are satisfied. Indeed, the complementing subspaces  $\mathcal{Y}_i$  of  $\mathcal{X}_i$  in Assumption 4.6 are given by

$$\mathcal{Y}_i := \{X \in \mathcal{X} : X(\omega) = 0, \forall \omega \in \Omega_i\}.$$

Recall that  $\Omega^*$  is the set of scenarios relevant for market acceptability and note that

$$\begin{aligned} \mathcal{Y} &= \{X \in \mathcal{X} : X(\omega) = 0, \forall \omega \in \Omega^*\}, \\ \mathcal{Z} &= \{X \in \mathcal{X} : X(\omega) = 0, \forall \omega \in \Omega \setminus \Omega^*\}. \end{aligned}$$

As  $\gamma_1(X) := X \mathbf{1}_{\Omega \setminus \Omega^*}$  and  $\gamma_2(X) = X \mathbf{1}_{\Omega^*}$ ,  $X \in \mathcal{X}$ , we verify easily that  $\gamma_1(\mathcal{X}_i) \subseteq \mathcal{X}_i$ ,  $i \in [n]$ , i.e., Assumption 4.7 is met. Regarding the existence of the mappings  $(P_i)_{i \in [n]}$  satisfying Assumption 4.8, we can define them consecutively by

$$P_1(X) := X \mathbf{1}_{\Omega_1}, \quad P_{i+1}(X) := X \mathbf{1}_{\Omega_{i+1} \setminus \bigcup_{j=1}^i \Omega_j}, \quad i \in [n-1], \quad X \in \mathcal{Y}.$$

Similar examples can be constructed for other function spaces such as the Hilbert space  $L^2(\Omega, \mathcal{F}, \mathbb{P})$ , the space of all equivalence classes with respect to almost sure equality of square-integrable random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

One may wonder whether the correspondence  $\mathcal{E} : \prod_{i=1}^n \mathcal{X}_i \rightarrow \prod_{i=1}^n \mathcal{X}_i \times \mathcal{X}^*$  mapping an initial loss endowment  $\mathbf{W}$  to all its equilibrium allocations such that  $\mathbf{X} \in \mathcal{P}(W_1 + \dots + W_n)$  and  $\phi$  is a subgradient of  $\Lambda$  at  $W_1 + \dots + W_n$  — as in the proof of Theorem 3.5 — is lower hemicontinuous under suitable conditions. This, however, is not the case. Suppose  $\mathcal{X}$  admits two positive functionals  $\phi, \psi \in \mathcal{X}_+^*$  such that  $\ker(\phi) \setminus \ker(\psi) \neq \emptyset$ . We assume  $n = 1$  and consider an agent system  $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathbf{p})$  such that  $\rho_{\mathcal{R}}(X) = \max\{\phi(X), \psi(X)\}$ ,  $X \in \mathcal{X}$ . Let  $W \in \mathcal{X}$  such that  $\phi(W) = 0 < \psi(W)$ . Thus, for all  $n \in \mathbb{N}$ , the equilibrium price at  $\frac{1}{n}W$  would be  $\psi$ , whereas any element of the convex hull of  $\{\phi, \psi\}$  could be chosen as equilibrium price at 0.  $\mathcal{E}$  is *not* lower hemicontinuous in this case.

**4.3. Ancillary results and proofs.** For the following lemma, recall that a Fréchet space is a completely metrisable locally convex topological vector space. In particular, every Fréchet lattice is a Fréchet space.

**Lemma 4.11.** *Let  $\mathcal{X}$  be a Fréchet space.*

- (1) *A subset  $\mathcal{C} \subseteq \mathcal{X}$  is polyhedral if and only if there are closed subspaces  $\mathcal{X}^1, \mathcal{X}^2 \subseteq \mathcal{X}$  such that  $\mathcal{X} = \mathcal{X}^1 \oplus \mathcal{X}^2$ ,  $\dim(\mathcal{X}^2) < \infty$ , and  $\mathcal{C} = \mathcal{X}^1 + \mathcal{C}'$  for a polyhedral subset  $\mathcal{C}' \subseteq \mathcal{X}^2$ .*
- (2) *Suppose  $\mathcal{Y}$  and  $\mathcal{X}$  are Fréchet spaces,  $\mathcal{C} \subseteq \mathcal{Y}$  is polyhedral, and  $T : \mathcal{Y} \rightarrow \mathcal{X}$  is a surjective linear operator. Then  $T(\mathcal{C})$  is polyhedral in  $\mathcal{X}$ .*

*Proof.* (1) Combine the proof of [46, Corollary 2.1] with the closed graph theorem [30, Chap. 3, Theorem 5].

(2) By (1), there are two closed subspaces  $\mathcal{Y}^1, \mathcal{Y}^2 \subseteq \mathcal{Y}$  such that  $\dim(\mathcal{Y}^2)$  is finite,  $\mathcal{Y} = \mathcal{Y}^1 \oplus \mathcal{Y}^2$ , and  $\mathcal{C} = \mathcal{Y}^1 + \mathcal{C}'$  for a polyhedral subset  $\mathcal{C}'$  of  $\mathcal{Y}^2$ . Define  $\mathcal{X}^2 := T(\mathcal{Y}^2)$ , which is finite-dimensional. Every finite-dimensional subspace of a Fréchet space is complemented by a closed subspace. Thus  $\mathcal{X} = \mathcal{X}^1 \oplus \mathcal{X}^2$  for a closed subspace  $\mathcal{X}^1$ . Clearly,  $T(\mathcal{C}') \subseteq \mathcal{X}^2$  is polyhedral; see [38, Theorem 19.3]. Moreover, denoting by  $\gamma_i : \mathcal{X} \rightarrow \mathcal{X}^i$  the projection in  $\mathcal{X}$  onto the linear subspaces  $\mathcal{X}^i$ , surjectivity of  $T$  implies  $\mathcal{X}^1 = \gamma_1(\mathcal{X}) = \gamma_1(T(\mathcal{Y}^1)) + \gamma_1(T(\mathcal{Y}^2)) = \gamma_1(T(\mathcal{Y}^1))$ . Moreover,

$$T(\mathcal{C}) = T(\mathcal{Y}^1) + T(\mathcal{C}') = \mathcal{X}^1 + \gamma_2(T(\mathcal{Y}^1)) + T(\mathcal{C}').$$

$\gamma_2(T(\mathcal{Y}^1))$  is polyhedral as subspace of the finite-dimensional space  $\mathcal{X}^2$ , and so is the sum  $\gamma_2(T(\mathcal{Y}^1)) + T(\mathcal{C}')$  of two polyhedral sets. Conclude with (1).  $\square$

The preceding lemma enables us to prove Theorem 4.3.

*Proof of Theorem 4.3.* The set  $\mathcal{A}_+ + \ker(\pi)$  is proper by assumption (A3). Moreover, it is polyhedral: consider the Fréchet space  $\mathcal{Y} := (\prod_{i=1}^n \mathcal{X}_i) \times \ker(\pi)$ .<sup>7</sup> By assumption, the set  $\mathcal{C} :=$

<sup>7</sup>Space  $\mathcal{Y}$  is not a Fréchet lattice, hence the necessity for the above formulation of Lemma 4.11.

$(\prod_{i=1}^n \mathcal{A}_i) \times \ker(\pi)$  is polyhedral, and  $T : \mathcal{Y} \rightarrow \mathcal{X}$  defined by  $T(X_1, \dots, X_n, N) = \sum_{i=1}^n X_i + N$  is surjective and linear. As  $\mathcal{X}$  is a Fréchet space, Lemma 4.11(2) yields the polyhedrality of  $T(\mathcal{C}) = \mathcal{A}_+ + \ker(\pi)$ . Being a polyhedral set, it is automatically closed. Since  $\Lambda$  is proper, it is l.s.c. and optimal payoffs exist for every  $X \in \text{dom}(\Lambda)$  by Proposition 3.4.  $\square$

*Proof of Corollary 4.5.* Combine Theorems 3.5 and 4.3.  $\square$

We now turn our attention to the lower hemicontinuity of the correspondence  $\mathcal{P}$  as stated in Theorem 4.9. Its proof requires the following highly technical Lemmas 4.12 and 4.13 whose proofs imitate in parts a technique from Baes et al. [7]. Note that in analogy with Theorem 4.3,  $\mathcal{A}_+$  is closed.

**Lemma 4.12.** *Suppose  $\mathcal{X}$  is a finite-dimensional locally convex Hausdorff topological vector space,  $\mathcal{X}_i \subseteq \mathcal{X}$ ,  $i \in [n]$ , are finite-dimensional subspaces such that  $\sum_{i=1}^n \mathcal{X}_i = \mathcal{X}$ , and  $\mathcal{A}_i \subseteq \mathcal{X}_i$ ,  $i \in [n]$ , are polyhedral sets. Set  $\mathcal{A}_+ := \sum_{i=1}^n \mathcal{A}_i$  and define the correspondence*

$$\Gamma : \mathcal{A}_+ \ni X \rightarrow \left\{ \mathbf{X} \in \prod_{i=1}^n \mathcal{A}_i : X_1 + \dots + X_n = X \right\}.$$

*Then  $\Gamma$  is lower hemicontinuous.*

*Proof.* As in Definition 4.1, for each  $i \in [n]$  we fix  $m_i \in \mathbb{N}$ , a linear and continuous operator  $T_i : \mathcal{X} \rightarrow \mathbb{R}^{m_i}$ , and vectors  $\beta_i \in \mathbb{R}^{m_i}$ , such that

$$\mathcal{A}_i = \{Y \in \mathcal{X}_i : T_i(Y) \leq \beta_i\}.$$

*Step 1.* For fixed  $X \in \mathcal{A}_+$  we decompose  $\Gamma(X)$  as the sum of a universal and an  $X$ -dependent component. Recall from Appendix A.1 that the recession cone of  $\Gamma(X)$  is given by

$$0^+ \Gamma(X) := \{ \mathbf{Y} \in \prod_{i=1}^n \mathcal{X}_i : \mathbf{X} + k\mathbf{Y} \in \Gamma(X), \forall \mathbf{X} \in \Gamma(X), \forall k > 0 \}.$$

The lineality space of  $\Gamma(X)$  is

$$0^+ \Gamma(X) \cap (-0^+ \Gamma(X)) = \{ \mathbf{Y} \in \mathbb{A}_0 : T_i(Y_i) = 0, \forall i \in [n] \},$$

a subspace independent of  $X$ . By virtue of Lemma A.2, there is a  $X$ -independent subspace  $\mathcal{V} \subseteq \prod_{i=1}^n \mathcal{X}_i$  such that

$$\Gamma(X) = \alpha(X) + 0^+ \Gamma(X), \quad \alpha(X) := \text{co}(\text{ext}(\Gamma(X) \cap \mathcal{V})),$$

where  $\text{co}(\cdot)$  denotes the convex hull operator and  $\text{ext}(\Gamma(X) \cap \mathcal{V})$  the set of extreme points of  $\Gamma(X) \cap \mathcal{V}$ .

*Step 2.* In this step, we prove that the correspondence  $\alpha : \mathcal{A}_+ \rightarrow \prod_{i=1}^n \mathcal{A}_i$  maps sets which are bounded with respect to some norm on  $\mathcal{X}$  to sets which are bounded with respect to some norm on  $\mathcal{V}$ . To this end, let  $D := \dim(\mathcal{X}) = \dim(\mathcal{X}^*)$  and choose a basis  $\psi_1, \dots, \psi_D$  of  $\mathcal{X}^*$ . Note that  $\mathbf{X} \in \Gamma(X) \cap \mathcal{V}$  if and only if

- $\mathbf{X}$  is an allocation of  $X$ , i.e.  $\psi_j(X_1 + \dots + X_n) = \psi_j(X)$  for all  $j \in [D]$ , or equivalently  $\psi_j(X_1 + \dots + X_n) \leq \psi_j(X)$  and  $(-\psi_j)(X_1 + \dots + X_n) \leq (-\psi_j)(X)$ ;
- each  $X_i$  lies in  $\mathcal{A}_i$ , i.e.  $T_i(X_i) \leq \beta_i$ ;

- $\mathbf{X} \in \mathcal{V}$ .

Clearly, the properties listed above describe a polyhedral subset of  $\mathcal{V}$ ; more precisely, for  $m := \sum_{i=1}^n m_i + 2D$ ,  $m_i$  defined above, we may find a continuous linear operator  $\mathbf{S} : \mathcal{V} \rightarrow \mathbb{R}^m$  and an affine function  $\mathbf{f} : \mathcal{X} \rightarrow \mathbb{R}^m$  such that

$$\Gamma(X) \cap \mathcal{V} = \{\mathbf{X} \in \mathcal{V} : \mathbf{S}(\mathbf{X}) \leq \mathbf{f}(X)\}.$$

Every ‘‘row’’  $\mathbf{S}_i$  of  $\mathbf{S}$  corresponds to an element of  $\mathcal{V}^*$ . By [9, Theorem II.4.2], for every extreme point  $\mathbf{X} \in \Gamma(X) \cap \mathcal{V}$  the set

$$I(\mathbf{X}) = \{i \in [m] : \mathbf{S}_i(\mathbf{X}) = \mathbf{f}_i(X)\},$$

whose cardinality is at least  $\dim(\mathcal{V})$ , satisfies  $\text{span}\{\mathbf{S}_i : i \in I(\mathbf{X})\} = \mathcal{V}^*$ . Let  $\mathbb{F}(X) := \{I(\mathbf{X}) : \mathbf{X} \in \text{ext}(\Gamma(X) \cap \mathcal{V})\}$  be the collection of all such  $I(\mathbf{X})$  corresponding to an extreme point. Its cardinality is bounded by the number of subsets of  $[m]$  with cardinality at least  $\dim(\mathcal{V})$ . Moreover, for each  $I \in \mathbb{F}(X)$ , the linear operator  $\mathbf{S}_I : \mathcal{V} \ni \mathbf{Y} \mapsto (\mathbf{S}_i(\mathbf{Y}))_{i \in I}$  is injective and thus invertible on its image. We have shown that  $(\mathbf{S}_I)_{I \in \mathbb{F}(X)}$  is a finite family of operators with full rank whose cardinality depends on  $\dim(\mathcal{V})$  and  $m$  only. Let  $\mathcal{B} \subseteq \mathcal{A}_+$  be a bounded set. For each  $X \in \mathcal{B}$  and  $I \in \mathbb{F}(X)$ ,  $\mathbf{f}_I$  is affine and thus maps  $\mathcal{B}$  to a bounded set. Also,  $\mathbf{S}_I^{-1}$  is continuous by the closed graph theorem [30, Chap. 3, Theorem 5], whence boundedness of  $\{\mathbf{S}_I^{-1}(\mathbf{f}_I(Y)) : Y \in \mathcal{B}, I \in \mathbb{F}(Y)\}$  follows. Recall that  $\bigcup_{X \in \mathcal{B}} \mathbb{F}(X)$  is finite. Using Carathéodory’s theorem [38, Theorem 17.1],  $\text{co}\{\mathbf{S}_I^{-1}(\mathbf{f}_I(X)) : X \in \mathcal{B}, I \in \mathbb{F}(X)\}$  is bounded. As

$$\begin{aligned} \bigcup_{X \in \mathcal{B}} \alpha(X) &= \bigcup_{X \in \mathcal{B}} \text{co}\{\mathbf{S}_I^{-1}(\mathbf{f}_I(X)) : I \in \mathbb{F}(X)\} \\ &\subseteq \text{co}\{\mathbf{S}_I^{-1}(\mathbf{f}_I(X)) : X \in \mathcal{B}, I \in \mathbb{F}(X)\}, \end{aligned}$$

it has to be bounded as well and Step 2 is proved.

*Step 3.*  $\Gamma$  is lower hemicontinuous. Let  $(X^k)_{k \in \mathbb{N}} \subseteq \mathcal{A}_+$  be convergent to  $X \in \mathcal{A}_+$  and let  $\mathbf{X} \in \Gamma(X)$ . We have to show that there is a subsequence  $(k_\lambda)_{\lambda \in \mathbb{N}}$  and  $\mathbf{X}^\lambda \in \Gamma(X^{k_\lambda})$  such that  $\mathbf{X}^\lambda \rightarrow \mathbf{X}$ ; cf. Appendix A.3. To this end, let first  $\mathbf{Y}^k \in \alpha(X^k)$ ,  $k \in \mathbb{N}$ , which is a bounded sequence by Step 2. After passing to a subsequence  $(k_\lambda)_{\lambda \in \mathbb{N}}$ , we may assume  $\mathbf{Y}^{k_\lambda} \rightarrow \mathbf{Y} \in \Gamma(X)$  (as  $\mathcal{A}_i$  is closed,  $i \in [n]$ ). If  $\Gamma(X)$  is a singleton,  $\mathbf{Y} = \mathbf{X}$  has to hold and we may choose  $\mathbf{X}^\lambda := \mathbf{Y}^{k_\lambda}$ . Otherwise, suppose first that  $\mathbf{X}$  lies in the *relative interior* of  $\Gamma(X)$ , i.e., there is an  $\varepsilon > 0$  such that  $\mathbf{X} + \varepsilon(\mathbf{X} - \mathbf{Y}) \in \Gamma(X)$ , as well. Recall the definition of the linear operators  $T_i$ ,  $i \in [n]$ , above and fix  $i \in [n]$ . Let  $1 \leq j \leq m_i$  be arbitrary. We denote by  $T_i^j(W)$  the  $j$ th entry of  $T_i(W)$ .

Case 1.  $T_i^j(X_i) = \beta_j$ . From  $Y_i \in \mathcal{A}_i$ , we infer

$$0 \geq T_i^j(X_i + \varepsilon(X_i - Y_i)) - \beta_j = \varepsilon(\beta_j - T_i^j(Y_i)) \geq 0,$$

which means  $T_i^j(Y_i) = \beta_j$ , as well. Set  $\lambda(i, j) = 1$ .

**Case 2.**  $T_i^j(X_i) < \beta_j$ . As  $Y_i^{k\lambda} \rightarrow Y_i$  for  $\lambda \rightarrow \infty$ , there must be a  $\lambda(i, j) \in \mathbb{N}$  such that for all  $\lambda \geq \lambda(i, j)$

$$T_i^j(Y_i^{k\lambda} - Y_i + X_i) \leq \beta_j.$$

Hence for all  $\lambda \geq \max_{i \in [n], 1 \leq j \leq m_i} \lambda(i, j)$ , one obtains

$$\mathbf{X}^\lambda := \mathbf{Y}^{k\lambda} - \mathbf{Y} + \mathbf{X} \in \prod_{i=1}^n \mathcal{A}_i \cap \mathbb{A}_{X^{k\lambda}} = \Gamma(X^{k\lambda}),$$

and  $\mathbf{X}^\lambda \rightarrow \mathbf{X}$ . It remains to notice that each  $\mathbf{X} \in \Gamma(X)$  may be approximated with a sequence in the relative interior of  $\Gamma(X)$ , cf. [38, Theorem 6.3]. The assertion is proved.  $\square$

**Lemma 4.13.** *Suppose the polyhedral agent system  $(\mathcal{R}_i)_{i \in [n]}$  and the infinite-dimensional market space  $\mathcal{X}$  conform to Assumptions 4.6–4.8. Then the correspondence  $\Gamma : \mathcal{A}_+ \ni X \mapsto \mathbb{A}_X \cap \prod_{i=1}^n \mathcal{A}_i$  is lower hemicontinuous.*

*Proof.* We will use the terminology introduced in Assumptions 4.6–4.8. In particular,

$$\mathcal{Y} := \bigcap_{i=1}^n \ker(T_i \circ \delta_i),$$

which is complemented in  $\mathcal{X}$  by a finite-dimensional closed subspace  $\mathcal{Z}$ , and the projections  $\gamma_1 : \mathcal{X} \rightarrow \mathcal{Y}$  and  $\gamma_2 : \mathcal{X} \rightarrow \mathcal{Z}$  are continuous.

We now consider the ambient space  $\mathcal{Z}$  which may be written as

$$\mathcal{Z} = \gamma_2(\mathcal{X}) = \sum_{i=1}^n \gamma_2(\mathcal{X}_i).$$

We will apply Lemma 4.12 to the sets

$$\mathcal{B}_i := \{Y \in \gamma_2(\mathcal{X}_i) : \tilde{T}_i(Y) \leq \beta_i\}, \quad i \in [n].$$

Suppose  $(X^k)_{k \in \mathbb{N}} \subseteq \mathcal{A}_+$  is a sequence which converges to  $X \in \mathcal{A}_+$ . Let  $\mathbf{X} \in \Gamma(X)$ . We need to find a subsequence  $(k_\lambda)_{\lambda \in \mathbb{N}}$  and a sequence of allocations  $\mathbf{X}^{k_\lambda} \in \Gamma(X^{k_\lambda})$  such that  $\mathbf{X}^{k_\lambda} \rightarrow \mathbf{X}$ .

To this end note that  $\gamma_2(X^k) \rightarrow \gamma_2(X)$  in  $\mathcal{Z}$  as  $k \rightarrow \infty$ . Moreover,  $(\gamma_2(X_i))_{i \in [n]}$  satisfies  $\gamma_2(X_i) \in \gamma_2(\mathcal{X}_i)$ ,  $i \in [n]$ ,  $\sum_{i=1}^n \gamma_2(X_i) = \gamma_2(X)$ , and by construction of  $\mathcal{Z}$ ,  $\tilde{T}_i(\gamma_2(X_i)) = \tilde{T}_i(X_i) \leq \beta_i$ . Hence, we may apply Lemma 4.12 to obtain a subsequence  $(k_\lambda)_{\lambda \in \mathbb{N}}$  and  $\mathbf{Y}^{k_\lambda} \in \prod_{i=1}^n \mathcal{B}_i$  such that the identity  $\sum_{i=1}^n Y_i^{k_\lambda} = \gamma_2(X_{k_\lambda})$  holds for all  $\lambda \in \mathbb{N}$  and  $\mathbf{Y}^{k_\lambda} \rightarrow (\gamma_2(X_i))_{i \in [n]}$ ,  $\lambda \rightarrow \infty$ .

Note that by Assumption 4.7,  $\gamma_1(X_i) \in \mathcal{X}_i$  for each  $i \in [n]$ . Let  $P_i$  be the continuous linear operators defined in Assumption 4.8. Consider

$$X_i^{k_\lambda} := \gamma_1(X_i) + P_i \left( \gamma_1(X^{k_\lambda} - X) \right) + Y_i^{k_\lambda} \in \gamma_1(\mathcal{X}_i) + \gamma_2(\mathcal{X}_i) = \mathcal{X}_i, \quad i \in [n], \quad \lambda \in \mathbb{N}.$$

Then

$$\sum_{i=1}^n X_i^{k_\lambda} = \gamma_1(X^{k_\lambda}) + \gamma_2(X^{k_\lambda}) = X^{k_\lambda}.$$

Moreover, for each  $i \in [n]$ , we have

$$T_i(X_i^{k_\lambda}) = T_i(\gamma_2(X_i^{k_\lambda})) = T_i(Y_i^{k_\lambda}) \leq \beta_i.$$

Hence,  $X_i^{k_\lambda} \in \mathcal{A}_i$  for each  $i \in [n]$ . These observations combined yield that  $\mathbf{X}^{k_\lambda} \in \Gamma(X^{k_\lambda})$ . By the continuity of  $P_i$  and  $\gamma_1$ , we obtain

$$\forall i \in [n] : X_i^{k_\lambda} \rightarrow \gamma_1(X_i) + \gamma_2(X_i) = X_i.$$

This finishes the proof.  $\square$

At last we can prove Theorem 4.9.

*Proof of Theorem 4.9.* In addition to the correspondence  $\mathcal{P}$  defined by (4.1) consider the following three correspondences:

- $\Gamma_1 : \text{dom}(\Lambda) \rightarrow \mathcal{A}_+ \times \mathcal{M}$ ,  $X \mapsto \{(X - Z, Z) : Z \in \mathcal{M}, X - Z \in \mathcal{A}_+, \Lambda(X) = \pi(Z)\}$ , which is lower hemicontinuous on  $\text{dom}(\Lambda)$  by virtue of the polyhedrality of  $\mathcal{A}_+$  and [7, Theorem 5.11].
- $\Gamma_2 : \mathcal{A}_+ \rightarrow \prod_{i=1}^n \mathcal{A}_i$ ,  $X \mapsto \mathbb{A}_X \cap \prod_{i=1}^n \mathcal{A}_i$ , which is lower hemicontinuous by Lemma 4.12, if  $\mathcal{X}$  is finite-dimensional, or Lemma 4.13, in case  $\mathcal{X}$  is infinite-dimensional.
- $\Gamma_3 : \mathcal{M} \rightarrow \prod_{i=1}^n \mathcal{S}_i$ ,  $Z \mapsto \mathbb{A}_Z^s$ , which is lower hemicontinuous by Lemma A.5.

Applying [3, Theorem 17.23],

$$\Gamma : \text{dom}(\Lambda) \ni X \mapsto \bigcup_{(X-Z, Z) \in \Gamma_1(X)} (\Gamma_2(X - Z) + \Gamma_3(Z))$$

is lower hemicontinuous as well.

In fact,  $\Gamma = \mathcal{P}$  holds. To see this, let  $X \in \text{dom}(\Lambda)$  be arbitrary. From the proof of Theorem 3.3,  $\Gamma(X) \subseteq \mathcal{P}(X)$  follows. For the converse inclusion, let  $\mathbf{X} \in \mathcal{P}(X)$  be arbitrary. Choose  $Z_i \in \mathcal{S}_i$ ,  $i \in [n]$ , such that  $X_i - Z_i \in \mathcal{A}_i$  and  $\rho_i(X_i) = \mathbf{p}_i(Z_i)$ , which is possible by Theorem 4.3 in the case  $n = 1$ ; see Remark 4.4. Let  $Z = Z_1 + \dots + Z_n$  and note that

$$\pi(Z) = \sum_{i=1}^n \mathbf{p}_i(Z_i) = \sum_{i=1}^n \rho_i(X_i) = \Lambda(X),$$

i.e.  $(X - Z, Z) \in \Gamma_1(X)$ . Moreover, as  $\mathbf{X} - \mathbf{Z} \in \Gamma_2(X - Z)$ , it only remains to note that  $\mathbf{X} = (\mathbf{X} - \mathbf{Z}) + \mathbf{Z} \in \Gamma_2(X - Z) + \Gamma_3(Z)$ . Equality of sets is established.

Finally,  $\text{dom}(\Lambda)$  is metrisable and therefore paracompact; cf. [39]. Moreover,  $\prod_{i=1}^n \mathcal{X}_i$  is a Fréchet space, and as  $\mathcal{P} : \text{dom}(\Lambda) \rightarrow \prod_{i=1}^n \mathcal{X}_i$  has nonempty closed convex values, a continuous selection for  $\mathcal{P}$  exists by the Michael selection theorem [3, Theorem 17.66].  $\square$

**4.4. An example.** We close this section by showing how Pareto optima can be computed in the situation of Example 4.2. For the sake of simplicity, we assume that  $n = 2$ ,  $\Omega$  is a finite set, and  $A := \Omega_1 \setminus \Omega_2$ ,  $B := \Omega_1 \cap \Omega_2$  and  $C := \Omega_2 \setminus \Omega_1$  are all nonempty. The specifications of Example 4.2 lead to the individual security spaces  $\mathcal{S}_1 = \text{span}\{\mathbf{1}_A, \mathbf{1}_B\}$  and  $\mathcal{S}_2 = \text{span}\{\mathbf{1}_B, \mathbf{1}_C\}$ . Let us assume that  $\Omega^* = \Omega$ . Hence, the individual acceptance sets are of the shape

$$\mathcal{A}_i := \{X \in \mathcal{X}_i : \forall \omega \in \Omega_i (X(\omega) \leq \beta_i(\omega))\}, \quad i = 1, 2,$$

where  $\beta_i \in \mathbb{R}^{\Omega_i}$  is arbitrary, but fixed. For convenience, we assume the set of weights  $(\sigma_\omega)_{\omega \in \Omega^*}$  appearing in the definition of the pricing functionals  $\pi$  and  $\mathbf{p}_i$ ,  $i \in [n]$ , to be such that  $\pi(\mathbf{1}_A) = \pi(\mathbf{1}_B) = \pi(\mathbf{1}_C) = 1$ .

Note that for  $x, y \in \mathbb{R}$ , we have

$$X - x\mathbf{1}_A - y\mathbf{1}_B \in \mathcal{A}_1 \iff \max_{a \in A} X(a) - \beta_1(a) \leq x \text{ and } \max_{b \in B} X(b) - \beta_1(b) \leq y.$$

Consequently,

$$\rho_1(X) := \rho_{\mathcal{R}_1}(X) = \max_{a \in A} X(a) - \beta_1(a) + \max_{b \in B} X(b) - \beta_1(b), \quad X \in \mathcal{X}_1,$$

and it only takes finite values. An analogous computation shows

$$\rho_2(X) := \rho_{\mathcal{R}_2}(X) = \max_{b \in B} X(b) - \beta_2(b) + \max_{c \in C} X(c) - \beta_2(c), \quad X \in \mathcal{X}_2.$$

which also takes only finite values. Set  $\tilde{\beta} := \beta_1\mathbf{1}_A + (\beta_1 + \beta_2)\mathbf{1}_B + \beta_2\mathbf{1}_C$ . The representative agent of this polyhedral agent system is given by

$$\begin{aligned} \mathcal{A}_+ &= \mathcal{A}_1 + \mathcal{A}_2 = \{X \in \mathcal{X} : X \leq \tilde{\beta}\}, \quad \mathcal{M} = \text{span}\{\mathbf{1}_A, \mathbf{1}_B, \mathbf{1}_C\}, \\ \pi(x\mathbf{1}_A + y\mathbf{1}_B + z\mathbf{1}_C) &= x + y + z, \quad x, y, z \in \mathbb{R}. \end{aligned}$$

Furthermore

$$\ker(\pi) = \{N_{x,y} := x\mathbf{1}_A - (x+y)\mathbf{1}_B + y\mathbf{1}_C : x, y \in \mathbb{R}\}.$$

We now aim to compute the associated risk sharing functional  $\Lambda$  and Pareto-optimal allocations. To this end, for  $X \in \mathcal{X}$ , we introduce the notation

$$\begin{aligned} \rho^A(X) &:= \max_{a \in A} X(a) - \beta_1(a), \quad \rho^B(X) := \max_{b \in B} X(b) - \tilde{\beta}(b), \\ \rho^C(X) &:= \max_{c \in C} X(c) - \beta_2(c). \end{aligned}$$

Using the characterisation of  $\mathcal{A}_+$ , one obtains

$$\mathcal{A}_+ + \ker(\pi) = \{X \in \mathcal{X} : \rho^B(X) \leq -\rho^A(X) - \rho^C(X)\}.$$

A straightforward computation yields

$$\Lambda(X) = \inf\{r \in \mathbb{R} : X - r\mathbf{1}_B \in \mathcal{A}_+ + \ker(\pi)\} = \rho^A(X) + \rho^B(X) + \rho^C(X).$$

Note that  $X - \Lambda(X)\mathbf{1}_B - N_{\rho^A(X), \rho^C(X)} \in \mathcal{A}_+$ , since

$$((X - \rho^A(X))\mathbf{1}_A + \beta_1\mathbf{1}_B, (X - \rho^B(X) - \beta_1)\mathbf{1}_B + (X - \rho^C(X))\mathbf{1}_C)$$

is an allocation of  $X - \Lambda(X)\mathbf{1}_B - N_{\rho^A(X), \rho^C(X)}$  which lies in  $\mathcal{A}_1 \times \mathcal{A}_2$ . For every  $\zeta \in \mathbb{R}$ , the allocation  $(X_1(\zeta), X_2(\zeta))$  given by

$$\begin{aligned} X_1(\zeta) &= X\mathbf{1}_A + (\beta_1 - \rho^A(X) + \zeta\Lambda(X))\mathbf{1}_B, \\ X_2(\zeta) &= (X - \rho^B(X) - \rho^C(X) - \beta_1 - (\zeta - 1)\Lambda(X))\mathbf{1}_B + X\mathbf{1}_C, \end{aligned}$$

is Pareto-optimal. Last, we note that an optimal payoff for  $X$  is given by  $\rho^A(X)\mathbf{1}_A + (\Lambda(X) - \rho^A(X) - \rho^C(X))\mathbf{1}_B + \rho^C(X)\mathbf{1}_C \in \mathcal{M}$ .

## 5. LAW-INVARIANT ACCEPTANCE SETS

In this section we discuss the risk sharing problem for agent systems with *law-invariant acceptance sets*, the second case study exemplifying the results in Sects. 2 and 3.

**5.1. The setting.** Throughout we fix an *atomless* probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , i.e., there is a random variable  $U : \Omega \rightarrow \mathbb{R}$  such that the cumulative distribution function  $\mathbb{R} \ni x \mapsto \mathbb{P}(U \leq x)$  of  $U$  under  $\mathbb{P}$  is continuous. By  $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$  and  $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$  we denote the spaces of (equivalence classes of)  $\mathbb{P}$ -integrable and bounded random variables, respectively. They are Banach lattices when equipped with the usual  $\mathbb{P}$ -almost sure (a.s.) order and the topologies arising from their natural norms  $\|\cdot\|_1 : X \mapsto \mathbb{E}[|X|]$  and

$$\|\cdot\|_\infty : X \mapsto \inf\{m > 0 : \mathbb{P}(|X| \leq m) = 1\}.$$

All appearing (in)equalities between random variables are understood in the a.s. sense.

**Definition 5.1.** A subset  $\mathcal{C} \subseteq L^1$  is  *$\mathbb{P}$ -law-invariant* if  $X \in \mathcal{C}$  whenever there is  $Y \in \mathcal{C}$  which is equal to  $X$  in law under  $\mathbb{P}$ , i.e. the two Borel probability measures  $\mathbb{P} \circ X^{-1}$  and  $\mathbb{P} \circ Y^{-1}$  on  $(\mathbb{R}, \mathbb{B}(\mathbb{R}))$  agree. Given a  $\mathbb{P}$ -law-invariant set  $\emptyset \neq \mathcal{C} \subseteq L^1$  and some other set  $S \neq \emptyset$ , a function  $f : \mathcal{C} \rightarrow S$  is called  *$\mathbb{P}$ -law-invariant* if  $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ Y^{-1}$  implies  $f(X) = f(Y)$ .

Let us first specify the setting in the case of the ambient market space  $\mathcal{X}$  agreeing with the space  $L^1$  of all integrable random variables. This allows a better grasp of its respective aspects. We will consider a more general setting later.

*Model space assumptions.* Throughout this section, all agents  $i \in [n]$  operate on the same model space  $\mathcal{X}_i = \mathcal{X} = L^1$  consisting of equivalence classes of integrable random variables. In Sect. 5.2, the results will be generalised to a wide class of model spaces  $L^\infty \subseteq \mathcal{X} \subseteq L^1$ , always under the assumption that the model spaces coincide, i.e.,  $\mathcal{X}_i = \mathcal{X}$ ,  $i \in [n]$ . The reason for this is that, in principle, the individual model spaces  $\mathcal{X}_i$  should be law-invariant and closed ideals in  $\mathcal{X}$ . This has strong implications however. In fact, if  $\mathcal{X}$  is a law-invariant Banach lattice of random variables which carries a law-invariant lattice norm like we assume below, and  $\mathcal{X}_i$  is supposed to be a non-trivial closed and law-invariant ideal in  $\mathcal{X}$ , then  $\mathcal{X}_i = \mathcal{X}$ .

*Acceptance sets.* Each agent  $i \in [n]$  deems a loss profile adequately capitalised if it belongs to a closed  $\mathbb{P}$ -law-invariant acceptance set  $\mathcal{A}_i \subseteq L^1$  which contains a riskless payoff, i.e.,

$$\mathbb{R} \cap \mathcal{A}_i \neq \emptyset. \tag{5.1}$$

As the dual space of  $L^1$  may be identified with  $L^\infty$ , we may see the respective *support functions*  $\sigma_{\mathcal{A}_i}$ ,  $i \in [n]$ , as law-invariant mappings

$$\sigma_{\mathcal{A}_i} : L^\infty \rightarrow (-\infty, \infty], \quad Q \mapsto \sup_{Y \in \mathcal{A}_i} \mathbb{E}[QY];$$

cf. Appendix A.1. Due to monotonicity of the sets  $\mathcal{A}_i$ ,  $\text{dom}(\sigma_{\mathcal{A}_i}) \subseteq L_+^\infty$  holds. The reader may think of acceptance sets arising, for instance, from the Average Value at Risk (Expected Shortfall) or distortion risk measures.

*Security markets.* Regarding the security markets, we require there is a linear functional  $\pi : \mathcal{M} \rightarrow \mathbb{R}$  on the global security space  $\mathcal{M}$  such that the individual pricing functionals are given by  $\mathbf{p}_i = \pi|_{\mathcal{S}_i}$ ,  $i \in [n]$ ; the agents operate on different sub-markets  $(\mathcal{S}_i, \pi|_{\mathcal{S}_i})$  of  $(\mathcal{M}, \pi)$ . In particular, conditions  $(\star)$  and (A1) are satisfied. Moreover, we assume

**Assumption 5.2.**  $\pi$  is of the shape  $\pi(Z) = \mathbb{E}[(Q + \delta)Z]$ ,  $Z \in \mathcal{M}$ , where  $\delta > 0$  is a constant and  $Q \in \bigcap_{i \in [n]} \text{dom}(\sigma_{\mathcal{A}_i}) \subseteq L_+^\infty$ .

Our assumption on the pricing functionals is very flexible as illustrated by Example 5.14 below; the pricing functional given in Example 2.2 also conforms to Assumption 5.2. Note that the constant function  $1 = \mathbf{1}_\Omega$  is an element of  $\text{dom}(\sigma_{\mathcal{A}_i})$ ,  $i \in [n]$ ; cf. (5.4) below. As the intersection  $\bigcap_{i \in [n]} \text{dom}(\sigma_{\mathcal{A}_i})$  is a cone,  $Q + \delta \in \bigcap_{i \in [n]} \text{dom}(\sigma_{\mathcal{A}_i})$  for every  $Q \in \bigcap_{i \in [n]} \text{dom}(\sigma_{\mathcal{A}_i})$  and every  $\delta > 0$ . In particular, any jointly relevant density with arbitrarily small constant perturbation can be used for pricing.

Recall from the introduction that assuming the individual acceptance sets  $\mathcal{A}_i$  to be law-invariant means that being acceptable or not is merely a statistical property of the loss profile. Mathematically, this intuition necessitates introducing the hypothetical physical measure  $\mathbb{P}$ . Prices in the security market can, e.g., be determined by a suitable equivalent martingale measure  $\mathbb{Q}$  though. For the remainder of this section we assume that Assumption 5.2 is satisfied.

Let us at last introduce the notion of *comonotone partitions of the identity*, or *comonotone functions*, i.e., functions in the set

$$\mathfrak{C} := \{\mathbf{f} = (f_1, \dots, f_n) : \mathbb{R} \rightarrow \mathbb{R}^n : f_i \text{ nondecreasing, } \sum_{i=1}^n f_i = id_{\mathbb{R}}\}.$$

**5.2. Main results.** We will first formulate the main results concerning the existence of Pareto-optimal allocations in the case of  $\mathcal{X} = \mathcal{X}_i = L^1$ ,  $i \in [n]$ .

**Theorem 5.3.** *Suppose the assumptions of this section are met.*

- (1) *The set  $\mathcal{A}_+ + \ker(\pi)$  is a closed and proper subset of  $L^1$ , and  $\Lambda$  is proper and l.s.c.*
- (2) *All  $X \in \text{dom}(\Lambda)$  admit an optimal payoff  $Z^X \in \mathcal{M}$ . In particular, for any  $X \in \text{dom}(\Lambda)$ , there exists a Pareto-optimal allocation  $\mathbf{X}$  of the shape*

$$X_i = A_i - N_i + \Lambda(X)U_i, \quad A_i := f_i(X - \Lambda(X)U + N) \in \mathcal{A}_i, \quad i \in [n], \quad (5.2)$$

*where  $N \in \ker(\pi)$  is an  $X$ -dependent zero cost global security and  $\mathbf{f} \in \mathfrak{C}$  is  $X$ -dependent, whereas  $\mathbf{N} \in \mathbb{A}_N^s$  is arbitrary and  $U_i \in \mathcal{S}_i \cap L_{++}^1$ ,  $i \in [n]$ , are chosen such that  $U := \sum_{i=1}^n U_i$  satisfies  $\pi(U) = 1$ .*

**Remark 5.4.** If  $n = 1$ ,  $\Lambda = \rho_{\mathcal{R}}$  and Theorem 5.3 in fact solves the *optimal payoff problem* studied in [7]. The proof of Theorem 5.3(1) shows that for every single agent  $i \in [n]$ , the Minkowski sum  $\mathcal{A}_i + \ker(\mathbf{p}_i)$  is a closed and proper subset of  $L^1$ , and  $\rho_i$  is l.s.c.

**Corollary 5.5.** *In the situation of Theorem 5.3, suppose that the agent system checks (A2). Then for every  $\mathbf{W} \in (L^1)^n$  such that  $W = \sum_{i=1}^n W_i \in \text{int dom}(\Lambda)$  there is an equilibrium  $(\mathbf{X}, \phi)$ .*

**Remark 5.6.** Finding elements in the interior of  $\text{dom}(\Lambda)$  usually requires stronger continuity properties of the involved risk measures and is an important motivation for studying the risk sharing problem on general model spaces endowed with a stronger topology than  $\|\cdot\|_1$ . We will do this shortly in Theorem 5.8 and Lemma 5.12. Given a loss  $W \in L^1$ , the trick is to find a suitable model space  $(\mathcal{X}, \|\cdot\|)$  such that  $W \in \text{int}_{\|\cdot\|} \text{dom}(\Lambda|_{\mathcal{X}})$ ; see, e.g., [17, 33, 36, 41].

By Lemma A.5 there is a continuous selection  $\Psi : \mathcal{M} \rightarrow \prod_{i=1}^n \mathcal{S}_i$  of  $\mathcal{M} \ni Z \mapsto \mathbb{A}_Z^s$ . Hence, the correspondence  $\widehat{\mathcal{P}} : L^1 \rightarrow (L^1)^n$  mapping  $X$  to Pareto-optimal allocations of shape (5.2) such that, additionally, the security allocation of  $N \in \ker(\pi)$  is given by  $\mathbf{N} = \Psi(N)$ <sup>8</sup> has nonempty values on  $\text{dom}(\Lambda)$  by Theorem 5.3. Although it might be the case that not all Pareto-optimal allocations of  $X \in \text{dom}(\Lambda)$  are elements of  $\widehat{\mathcal{P}}(X)$ ,  $\widehat{\mathcal{P}}$  has the advantage of being upper hemicontinuous on the interior of the domain of  $\Lambda$ .

**Theorem 5.7.** *In the situation of Theorem 5.3 suppose  $\mathcal{A}_+$  does not agree with one of the level sets  $\{X \in L^1 : \mathbb{E}[X] \leq c\}$ ,  $c \in \mathbb{R}$ . Then  $\widehat{\mathcal{P}}$  is upper hemicontinuous at every continuity point  $X \in \text{dom}(\Lambda)$  of  $\Lambda$  and, a fortiori, on  $\text{int} \text{dom}(\Lambda)$ .*

We already advertised that the assumption that all agents operate on the space  $\mathcal{X} = L^1$  does not restrict the generality of Theorems 5.3 and 5.7 and Corollary 5.5. Indeed, the market space  $\mathcal{X}$  may be chosen to be any law-invariant ideal within  $L^1$  with respect to the  $\mathbb{P}$ -a.s. order falling in one of the following two categories:

- (BC) *Bounded case:*  $\mathcal{X} = L^\infty$  equipped with the supremum norm  $\|\cdot\|_\infty$ .
- (UC) *Unbounded case:*  $L^\infty \subseteq \mathcal{X} \subseteq L^1$  is a  $\mathbb{P}$ -law invariant Banach lattice endowed with an order continuous law-invariant lattice norm  $\|\cdot\|$ .<sup>9</sup>

In the unbounded case, one can show that the identity embeddings

$$L^\infty \hookrightarrow \mathcal{X} \hookrightarrow L^1$$

are continuous, i.e. there are constants  $\kappa, K > 0$  such that

$$\forall X \in L^\infty \forall Y \in \mathcal{X} : \|X\| \leq \kappa \|X\|_\infty \text{ and } \|Y\|_1 \leq K \|Y\|. \quad (5.3)$$

Moreover, for all  $\phi \in \mathcal{X}^*$  there is a unique  $Q \in L^1$  such that  $QX \in L^1$  and  $\phi(X) = \mathbb{E}[QX]$  hold for all  $X \in \mathcal{X}$ . The reader may think here of  $L^p$ -spaces with  $1 < p < \infty$ , or more generally Orlicz hearts equipped with a Luxemburg norm as for instance in [14, 17, 28].

In view of Lemma 5.12 we will assume that

- each individual acceptance set  $\mathcal{A}_i \subseteq \mathcal{X}$  is closed, law-invariant and satisfies  $\mathcal{A}_i \cap \mathbb{R} \neq \emptyset$ ;
- the security markets  $(\mathcal{S}_i, \mathfrak{p}_i)$  agree with Assumption 5.2.

Our main result is

**Theorem 5.8.** *Let  $\mathcal{X}$  be a Banach lattice satisfying (BC) or (UC). Assume the agent system  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$  is such that each individual acceptance set  $\mathcal{A}_i \subseteq \mathcal{X}$  is closed, law-invariant and*

<sup>8</sup>Recall that  $\mathbf{N}$  in (5.2) can be chosen arbitrarily.

<sup>9</sup>As  $\mathcal{X}$  will be a super Dedekind complete Riesz space, this translates as the fact that whenever  $X_n \downarrow 0$  in order,  $\|X_n\| \downarrow 0$  holds as well; cf. [4, Definition 1.43] and [27, Theorem A.33].

satisfies  $\mathcal{A}_i \cap \mathbb{R} \neq \emptyset$ , and the security markets  $(\mathcal{S}_i, \mathbf{p}_i)$  agree with Assumption 5.2. Then Theorems 5.3 and 5.7 and Corollary 5.5 hold verbatim when  $\mathcal{X}$  replaces  $L^1$  and  $\|\cdot\|$  replaces  $\|\cdot\|_1$ .

For the final result on upper hemicontinuity of the equilibrium correspondence, recall that the finite risk measure  $\rho_{\mathcal{R}} : L^\infty \rightarrow \mathbb{R}$  resulting from a risk measurement regime  $\mathcal{R}$  on  $L^\infty$  is continuous from above if  $\rho_{\mathcal{R}}(X_n) \downarrow \rho_{\mathcal{R}}(X)$  whenever  $(X_n)_{n \in \mathbb{N}} \subseteq L^\infty$  and  $X \in L^\infty$  are such that  $X_n \downarrow X$  a.s.

**Theorem 5.9.** *Assume that (A2) is satisfied and that in case (BC)  $\rho_1$  is continuous from above, whereas in case (UC)  $\mathcal{X}$  is reflexive. Suppose furthermore that  $\mathcal{A}_+$  does not agree with a level set  $\mathcal{L}_c(\mathbb{E}[\cdot])$  and consider the correspondence  $\mathcal{E} : \mathcal{X}^n \rightarrow \mathcal{X}^n \times \mathcal{X}^*$  mapping  $\mathbf{W}$  to equilibrium allocations  $(\mathbf{X}, \phi)$  of shape*

$$X_i = Y_i + \frac{\phi(W_i - Y_i)}{\phi(\tilde{Z})} \tilde{Z}, \quad i \in [n],$$

where  $\mathbf{Y} \in \widehat{\mathcal{P}}(W_1 + \dots + W_n)$ ,  $\tilde{Z} \in \tilde{\mathcal{S}}$  with  $\pi(\tilde{Z}) \neq 0$ , and  $\phi$  is a subgradient of  $\Lambda$  at  $W_1 + \dots + W_n$ . Then  $\mathcal{E}$  is upper hemicontinuous in the following sense: whenever  $(\mathbf{W}^k)_{k \in \mathbb{N}} \subseteq \prod_{i=1}^n \text{int dom}(\rho_i)$  and  $\mathbf{W}^k \rightarrow \mathbf{W} \in \prod_{i=1}^n \text{int dom}(\rho_i)$  as  $k \rightarrow \infty$  and  $(\mathbf{X}^k, \phi_k) \in \mathcal{E}(\mathbf{W}^k)$  is chosen arbitrarily,  $k \in \mathbb{N}$ , there is a subsequence  $(k_\lambda)_{\lambda \in \mathbb{N}}$  such that  $(\mathbf{X}, \phi) := \lim_{\lambda \rightarrow \infty} (\mathbf{X}^{k_\lambda}, \phi_{k_\lambda}) \in \mathcal{E}(\mathbf{W})$ .

**5.3. Ancillary results and proofs.** We begin with the existence of optimal payoffs and Pareto-optimal allocations. This is a direct consequence of Proposition 3.4 in the case  $\mathcal{X} = L^1$ , provided we can prove the properness of  $\Lambda$  and closedness of  $\mathcal{A}_+ + \ker(\pi)$ . First, we characterise the recession cone  $0^+ \mathcal{A}$  of a convex law-invariant acceptance set  $\mathcal{A}$ , a result of independent interest. For the definition of a recession cone and of the support function  $\sigma_{\mathcal{A}}$ , we refer to Appendix A.1. With slight modifications, Proposition 5.10 also holds true for general closed, convex and law-invariant sets  $\mathcal{C}$  which do not agree with one of the sets  $\{X \in L^1 : c_- \leq \mathbb{E}[X] \leq c_+\}$ , where  $-\infty \leq c_- \leq c_+ \leq \infty$ .

**Proposition 5.10.** *Suppose  $\emptyset \neq \mathcal{A} \subsetneq L^1$  is a law-invariant and closed acceptance set.*

- (1)  $0^+ \mathcal{A}$  is law-invariant.
- (2) *Suppose furthermore that  $\mathcal{A}$  does not agree with one of the level sets  $\mathcal{L}_c(\mathbb{E}[\cdot])$  for some  $c \in \mathbb{R}$ . Let  $Q \in \text{dom}(\sigma_{\mathcal{A}})$  and  $\delta > 0$  be arbitrary. If  $U \in 0^+ \mathcal{A}$  satisfies  $\mathbb{E}[(Q + \delta)U] = 0$ , then  $U = 0$ .*

*Proof.* (1) As  $\mathcal{A}$  is norm closed and convex, we may apply the Hahn-Banach separation theorem to obtain the representation

$$\mathcal{A} = \{X \in L^1 : \mathbb{E}[QX] \leq \sigma_{\mathcal{A}}(Q), \forall Q \in \text{dom}(\sigma_{\mathcal{A}})\},$$

where  $\sigma_{\mathcal{A}}$  is the support function of  $\mathcal{A}$ . It is well-known that  $\text{dom}(\sigma_{\mathcal{A}})$  is a law-invariant and convex cone in  $L_+^\infty$ . The law-invariance of  $\text{dom}(\sigma_{\mathcal{A}})$  combined with Lemma A.1 shows that the recession cone  $0^+ \mathcal{A}$  is law-invariant and closed as well.

(2) By (1) and [40, Lemma 1.3], for any  $U \in 0^+\mathcal{A}$ ,  $Q \in \text{dom}(\sigma_{\mathcal{A}})$ , and sub- $\sigma$ -algebra  $\mathcal{H} \subseteq \mathcal{F}$ , we have

$$\mathbb{E}[U|\mathcal{H}] \in 0^+\mathcal{A} \quad \text{and} \quad \mathbb{E}[Q|\mathcal{H}] \in \text{dom}(\sigma_{\mathcal{A}}). \quad (5.4)$$

Choosing  $\mathcal{H} = \{\emptyset, \Omega\}$ , we obtain that  $E[Q] \in \text{dom}(\sigma_{\mathcal{A}})$  for all  $Q \in \text{dom}(\sigma_{\mathcal{A}})$ . Moreover, by choosing  $Q \in \text{dom}(\sigma_{\mathcal{A}}) \subseteq L_+^\infty$  appropriately, we obtain that  $1 \in \text{dom}(\sigma_{\mathcal{A}})$ .

Now suppose there is no  $c \in \mathbb{R}$  such that  $\mathcal{A} = \mathcal{L}_c(\mathbb{E}[\cdot])$ , and assume a direction  $U \in 0^+\mathcal{A}$  is not constant. In a first step, we will exclude the possibility that  $\mathbb{E}[U] = 0$ . To this end let  $Q \in \text{dom}(\sigma_{\mathcal{A}})$  be non-constant. Such a  $Q$  exists because  $\mathcal{A}$  does not agree with one of the lower level sets of  $\mathbb{E}[\cdot]$ . As  $(\Omega, \mathcal{F}, \mathbb{P})$  is non-atomic, for  $k \geq 2$  large enough there is a finite measurable partition<sup>10</sup>  $\Pi := (A_1, \dots, A_k)$  of  $\Omega$  such that  $\mathbb{P}(A_j) = \frac{1}{k}$ ,  $j \in [k]$ , and

$$\widehat{U} = \mathbb{E}[U|\sigma(\Pi)] = \sum_{i=1}^k u_i \mathbf{1}_{A_i} \quad \text{and} \quad \widehat{Q} = \mathbb{E}[Q|\sigma(\Pi)] = \sum_{i=1}^k q_i \mathbf{1}_{A_i}$$

are both non-constant. For any permutation  $\tau : [k] \rightarrow [k]$  the random variable given by  $\widehat{U}_\tau := \sum_{i=1}^k u_{\tau(i)} \mathbf{1}_{A_i}$  has the same distribution under  $\mathbb{P}$  as  $\widehat{U}$ . Hence, by (5.4) and (1),  $\widehat{U}_\tau \in 0^+\mathcal{A}$  follows. Similarly,  $\widehat{Q}_\tau := \sum_{i=1}^k q_{\tau(i)} \mathbf{1}_{A_i} \in \text{dom}(\sigma_{\mathcal{A}})$ . For our argument we will hence assume without loss of generality that the vectors  $u$  and  $q$  satisfy  $u_1 \leq \dots \leq u_k$  and  $q_1 \leq \dots \leq q_k$ . In both chains of inequalities, at least one inequality has to be strict. We estimate

$$\mathbb{E}[\widehat{Q}]\mathbb{E}[\widehat{U}] = \left( \frac{1}{k} \sum_{i=1}^k q_i \right) \cdot \left( \frac{1}{k} \sum_{i=1}^k u_i \right) < \frac{1}{k} \sum_{i=1}^k q_i u_i = \mathbb{E}[\widehat{Q} \cdot \widehat{U}] \leq 0.$$

Here, the first strict inequality is due to Chebyshev's sum inequality [29, Theorem 43] and  $u$  and  $q$  being non-constant. The last inequality is due to  $\widehat{U} \in 0^+\mathcal{A}$ ,  $\widehat{Q} \in \text{dom}(\sigma_{\mathcal{A}})$ , and Lemma A.1.  $\mathbb{E}[\widehat{U}] = \mathbb{E}[U] = 0$  is hence impossible.

In a second step, let  $Q \in \text{dom}(\sigma_{\mathcal{A}})$  and  $\delta > 0$  be arbitrary. Suppose  $U \in 0^+\mathcal{A}$  satisfies  $\mathbb{E}[(Q + \delta)U] = \mathbb{E}[QU] + \delta\mathbb{E}[U] = 0$ . As  $Q, \delta \in \text{dom}(\sigma_{\mathcal{A}})$ , Lemma A.1 yields  $\mathbb{E}[QU] \leq 0$  and  $\delta\mathbb{E}[U] \leq 0$ . Combining these facts leads to the identities  $\mathbb{E}[U] = \mathbb{E}[QU] = 0$ , whence  $U = 0$  follows with the first step.  $\square$

We continue with a result for comonotone functions. One easily verifies that for  $\mathbf{f} \in \mathfrak{C}$  and  $i \in [n]$  the coordinate function  $f_i$  is Lipschitz continuous with Lipschitz constant 1. Moreover, for  $\gamma > 0$ , we set

$$\mathfrak{C}_\gamma := \{\mathbf{f} \in \mathfrak{C} : \mathbf{f}(0) \in [-\gamma, \gamma]^n\}.$$

From [26, Lemma B.1] we recall the following compactness result:

**Lemma 5.11.** *For every  $\gamma > 0$ ,  $\mathfrak{C}_\gamma \subseteq (\mathbb{R}^n)^\mathbb{R}$  is sequentially compact in the topology of pointwise convergence: for any sequence  $(\mathbf{f}^k)_{k \in \mathbb{N}} \subseteq \mathfrak{C}_\gamma$  there is a subsequence  $(k_\lambda)_{\lambda \in \mathbb{N}}$  and  $\mathbf{f} \in \mathfrak{C}_\gamma$  such that*

$$\forall x \in \mathbb{R} : \mathbf{f}^{k_\lambda}(x) \rightarrow \mathbf{f}(x), \quad \lambda \rightarrow \infty.$$

We are now ready to prove Theorem 5.3.

<sup>10</sup>That is the sets are pairwise disjoint, measurable, and their union is  $\Omega$ .

*Proof of Theorem 5.3.* (1) The individual acceptance sets  $\mathcal{A}_i$  may be used to define  $\mathbb{P}$ -law-invariant l.s.c. monetary base risk measures  $\xi_i$  by

$$\xi_i(X) := \inf\{m \in \mathbb{R} : X - m \in \mathcal{A}_i\} \in (-\infty, \infty], \quad X \in L^1.$$

By (5.1),  $\xi_i(Y) \in \mathbb{R}$  holds for all bounded random variables  $Y \in L^\infty$ . The lower level sets  $\mathcal{L}_c(\xi_i)$ ,  $c \in \mathbb{R}$ , may be written as  $\mathcal{L}_c(\xi_i) = c + \mathcal{A}_i$ . The risk measures  $\xi_i$  admit a dual representation

$$\xi_i(X) = \sup_{Q \in \text{dom}(\xi_i^*)} \mathbb{E}[QX] - \xi_i^*(Q), \quad X \in L^1, \quad (5.5)$$

where cash-additivity implies that

$$\text{dom}(\xi_i^*) \subseteq \{Q \in L_+^\infty : \mathbb{E}[Q] = 1\} \text{ and } \xi_i^*(Q) = \sigma_{\mathcal{A}_i}(Q), \quad Q \in \text{dom}(\xi_i^*). \quad (5.6)$$

Moreover, the infimal convolution  $\xi := \square_{i=1}^n \xi_i > -\infty$  is a  $\mathbb{P}$ -law-invariant monetary risk measure on  $L^1$  as well and  $\xi^* = \sum_{i=1}^n \xi_i^*$  by Lemma A.4. Now, by [26, Corollary 2.7],  $\xi$  is l.s.c., and for each  $X \in \text{dom}(\xi)$  there is  $\mathbf{f} \in \mathfrak{C}$  such that

$$\xi(X) = \sum_{i=1}^n \xi_i(f_i(X)). \quad (5.7)$$

Suppose now  $X \in L^1$  satisfies  $\xi(X) \leq 0$  and let  $\mathbf{f}$  as in (5.7). For all  $i \in [n]$  we may choose  $c_i \in \mathbb{R}$  such that  $\xi_i(f_i(X) - c_i) = \xi_i(f_i(X)) - c_i \leq 0$  and  $\sum_{i=1}^n c_i = 0$ . If  $g_i := f_i - c_i$ ,  $g_i(X) \in \mathcal{L}_0(\xi_i) = \mathcal{A}_i$ ,  $i \in [n]$ . Hence,

$$X = \sum_{i=1}^n g_i(X) \in \sum_{i=1}^n \mathcal{A}_i = \mathcal{A}_+.$$

We have thus shown that

$$\mathcal{L}_0(\xi) = \mathcal{A}_+.$$

As  $\xi$  is l.s.c. the left-hand set (and thus also the right-hand set) is norm closed.

Let  $\pi(\cdot) = \mathbb{E}[(Q+\delta)\cdot]$  as in Assumption 5.2. Suppose first that, for some  $c \in \mathbb{R}$ ,  $\mathcal{A}_+ = \mathcal{L}_c(\mathbb{E}[\cdot])$ . Then  $Q \in \mathbb{R}_+$  holds and  $\pi = p\mathbb{E}[\cdot]$  for a suitable  $p > 0$ . We obtain that  $0^+ \mathcal{A}_+ \cap \ker(\pi) = \ker(\pi)$  is a subspace. By Dieudonné's theorem [45, Theorem 1.1.8],  $\mathcal{A}_+ + \ker(\pi)$  is closed.

If  $\mathcal{A}_+$  does not agree with one of the lower level sets of  $\mathbb{E}[\cdot]$ , Proposition 5.10(2) allows us to infer that  $0^+ \mathcal{A}_+ \cap \ker(\pi) = \{0\}$ , a subspace. Again, Dieudonné's theorem yields the closedness of  $\mathcal{A}_+ + \ker(\pi)$ .

For properness of  $\Lambda$ , let  $X \in L^1$  be arbitrary. Suppose  $Z \in \mathcal{M}$  is such that  $X - Z \in \mathcal{A}_+$ , i.e.  $\xi(X - Z) \leq 0$ . Let  $Q \in L_+^\infty$  and  $\delta > 0$  be chosen as in Assumption 5.2. By (5.6),

$$Q^* := \frac{Q + \delta}{\mathbb{E}[Q] + \delta} \in \text{dom}(\xi^*).$$

Moreover,

$$0 \geq \mathbb{E}[Q^*(X - Z)] - \xi^*(Q^*) = \mathbb{E}[Q^*X] - \xi^*(Q^*) - (\mathbb{E}[Q] + \delta)\pi(Z),$$

which implies

$$\pi(Z) \geq \frac{\mathbb{E}[Q^*X] - \xi^*(Q^*)}{\mathbb{E}[Q] + \delta} > -\infty.$$

The properness follows with the representation of  $\Lambda$  given in Theorem 3.1(2). The lower semicontinuity of  $\Lambda$  is due to Proposition 3.4.

(2) By (1),  $\Lambda$  is proper and  $\mathcal{A}_+ + \ker(\pi)$  is closed. By Proposition 3.4, every  $X \in \text{dom}(\Lambda)$  admits an optimal payoff  $Z^X$  and thus a Pareto-optimal allocation by Theorem 3.3. For the concrete shape of  $Z^X$  and the Pareto-optimal allocation, let  $\mathbf{U} \in \prod_{i=1}^n \mathcal{S}_i$  be as in the assertion. As in the proof of (1), we may find  $\mathbf{f} \in \mathfrak{C}$  such that  $f_i(X - Z^X) \in \mathcal{A}_i$ ,  $i \in [n]$ . As  $\pi(Z^X) = \Lambda(X)$ , we have  $N := \Lambda(X)U - Z^X \in \ker(\pi)$ . For any  $\mathbf{N} \in \mathbb{A}_N^s$ , the security allocation  $\mathbf{Z}^X := \Lambda(X)\mathbf{U} - \mathbf{N}$  lies in  $\mathbb{A}_{Z^X}^s$ . According to Theorem 3.3,

$$\mathbf{f}(X - Z^X) + \mathbf{Z}^X = \mathbf{f}(X - \Lambda(X)U + N) + \Lambda(X)\mathbf{U} - \mathbf{N}$$

is a Pareto-optimal allocation of  $X$  with  $\mathbf{f}(X - \Lambda(X)U + N) \in \prod_{i=1}^n \mathcal{A}_i$ . This proves (5.2).  $\square$

*Proof of Corollary 5.5.* Combine Theorem 5.3 and Theorem 3.5.  $\square$

*Proof of Theorem 5.7.* We start with any sequence  $(X^k)_{k \in \mathbb{N}} \subseteq \text{int dom}(\Lambda)$  that converges to  $X \in \text{int dom}(\Lambda)$ . For all  $k \in \mathbb{N}$  let  $\mathbf{X}^k = (X_i^k)_{i \in [n]} \in \widehat{\mathcal{P}}(X^k)$ . By Appendix A.3, it is enough to show that there is a subsequence  $(k_\lambda)_{\lambda \in \mathbb{N}}$  and an allocation  $\mathbf{X} \in \widehat{\mathcal{P}}(X)$  such that  $\mathbf{X}^{k_\lambda} \rightarrow \mathbf{X}$  coordinatewise for  $\lambda \rightarrow \infty$ . To this end, we first recall the construction of  $\mathbf{X}^k$ ,  $k \in \mathbb{N}$ , from Theorem 5.3: There are sequences  $(N^k)_{k \in \mathbb{N}} \subseteq \ker(\pi)$  and  $(\mathbf{f}^k)_{k \in \mathbb{N}} \subseteq \mathfrak{C}$  such that

- $A_i^k := f_i^k(X^k - \Lambda(X^k)U + N^k) \in \mathcal{A}_i$ ,  $i \in [n]$ ;
- $\mathbf{X}^k = \mathbf{A}^k + \Lambda(X^k)\mathbf{U} - \mathbf{N}^k$ , where  $\mathbf{N}^k = \Psi(N^k)$ .

We will establish in three steps that  $(N^k)_{k \in \mathbb{N}}$  and  $(\mathbf{f}^k)_{k \in \mathbb{N}}$  lie in suitable relatively sequentially compact sets, which will allow us to choose a convergent subsequence.

First, as  $\Lambda$  is continuous on  $\text{int dom}(\Lambda)$  by [20, Corollary 2.5], the sequence  $(X^k - \Lambda(X^k)U)_{k \in \mathbb{N}}$  is bounded.

The second step is to prove that  $(N^k)_{k \in \mathbb{N}}$  is a norm bounded sequence as well. We assume for contradiction we can select a subsequence  $(k_\lambda)_{\lambda \in \mathbb{N}}$  such that  $1 \leq \|N^{k_\lambda}\|_1 \uparrow \infty$ . Using compactness of the unit sphere in the finite-dimensional space  $\ker(\pi)$  and potentially passing to another subsequence, we may furthermore assume

$$\frac{1}{\|N^{k_\lambda}\|_1} N^{k_\lambda} \rightarrow N^* \in \ker(\pi) \setminus \{0\}, \quad \lambda \rightarrow \infty,$$

Let  $Y \in \mathcal{A}_+$  be arbitrary and note that

$$Y + N^* = \lim_{\lambda \rightarrow \infty} (1 - \|N^{k_\lambda}\|_1^{-1})Y + \|N^{k_\lambda}\|_1^{-1} (X^{k_\lambda} - \Lambda(X^{k_\lambda})U + N^{k_\lambda}) \in \mathcal{A}_+,$$

as the latter set is closed and convex and the sequence  $(X^{k_\lambda} - \Lambda(X^{k_\lambda})U)_{\lambda \in \mathbb{N}}$  is norm bounded. Hence,  $N^* \in 0^+ \mathcal{A}_+ \cap \ker(\pi)$  which is trivial by Assumption 5.2 and Proposition 5.10(2), leading to the desired *contradiction*.  $(N^k)_{k \in \mathbb{N}}$  has to be bounded and  $\{N^k : k \in \mathbb{N}\} \subseteq \ker(\pi)$  is relatively (sequentially) compact by the finite dimension of the latter space.

In a third step, we establish relative sequential compactness for the set  $\{\mathbf{f}^k : k \in \mathbb{N}\}$ . To this end, recall the definition of the monetary risk measures  $\xi_i$  in the proof of Theorem 5.3. As  $1 \in \text{dom}(\sigma_{\mathcal{A}_i})$  by the proof of Proposition 5.10 and  $\mathbb{E}[1] = 1$ , (5.6) implies  $\xi_i^*(1) < \infty$  for all  $i \in [n]$ . Now fix  $k \in \mathbb{N}$  and let  $I := \{i \in [n] : f_i^k(0) > 0\}$  and  $J := [n] \setminus I$ . If  $I$  is empty,  $f_i^k(0) = 0$  has to hold for all  $i \in [n]$ . Now suppose we can choose  $i \in I$ . We abbreviate  $W^k := X^k - \Lambda(X^k)U + N^k$  and estimate

$$\begin{aligned} -\mathbb{E}[|W^k|] &\leq -\mathbb{E}[|f_i^k(W^k) - f_i^k(0)|] \leq \mathbb{E}[f_i^k(W^k) - f_i^k(0)] \\ &\leq \xi_i(f_i^k(W^k)) + \xi_i^*(1) - f_i^k(0) \leq \xi_i^*(1) - f_i^k(0), \end{aligned}$$

where we used that  $A_i^k = f_i^k(W^k) \in \mathcal{A}_i$ . Hence,

$$\forall i \in I : |f_i^k(0)| \leq \xi_i^*(1) + \|W^k\|_1. \quad (5.8)$$

If  $j \in J$ , we obtain from the requirement  $f_1^k + \dots + f_n^k = id_{\mathbb{R}}$

$$|f_j^k(0)| = -f_j^k(0) \leq -\sum_{i \in J} f_i^k(0) = \sum_{i \in I} f_i^k(0) \leq \sum_{i \in [n]} \xi_i^*(1) + n\|W^k\|_1 =: \gamma_k.$$

Thus,  $\mathbf{f}^k \in \mathfrak{C}_{\gamma_k}$ . As the bound  $\gamma_k$  depends on  $k$  only in terms of  $\|W^k\|_1$  which is uniformly bounded in  $k$  by the first and the second step,  $\gamma := \sup_{k \in \mathbb{N}} \gamma_k < \infty$  and  $(\mathbf{f}^k)_{k \in \mathbb{N}} \subseteq \mathfrak{C}_\gamma$ . After passing to subsequences two times, we can find a subsequence  $(k_\lambda)_{\lambda \in \mathbb{N}}$  such that

- $\ker(\pi) \ni N := \lim_{\lambda \rightarrow \infty} N^{k_\lambda}$  exists and thus  $\Psi(N^{k_\lambda}) \rightarrow \Psi(N)$  for  $\lambda \rightarrow \infty$ .
- for a suitable  $\mathbf{f} \in \mathfrak{C}_\gamma$  it holds that  $\max_{i \in [n]} |f_i^{k_\lambda} - f_i| \rightarrow 0$  pointwise for  $\lambda \rightarrow \infty$ , cf. Lemma 5.11.

It remains to show that  $(f_i(X - \Lambda(X)U + N) + \Lambda(X)U_i + \Psi(N)_i)_{i \in [n]} \in \widehat{\mathcal{P}}(X)$  and that it is the limit of the subsequence of the Pareto-optimal allocations chosen initially. To this end, we set  $\mathbf{A} := \mathbf{f}(X - \Lambda(X)U + N)$  and  $g_i^{(k_\lambda)} := f_i^{(k_\lambda)} - f_i^{(k_\lambda)}(0)$ .  $\mathbb{P}$ -a.s., the estimate

$$\begin{aligned} \left| A_i - A_i^{k_\lambda} \right| &\leq \left| (g_i - g_i^{k_\lambda})(X - \Lambda(X)U + N) \right| \\ &\quad + \left| f_i^{k_\lambda}(X - \Lambda(X)U + N) - f_i^{k_\lambda}(X^{k_\lambda} - \Lambda(X^{k_\lambda})U + N^{k_\lambda}) \right| \\ &\quad + \left| f_i(0) - f_i^{k_\lambda}(0) \right| \end{aligned} \quad (5.9)$$

holds. The third term vanishes for  $\lambda \rightarrow \infty$ . The first term vanishes in norm due to dominated convergence. From the estimate

$$\begin{aligned} &\left\| f_i^{k_\lambda}(X - \Lambda(X)U + N) - f_i^{k_\lambda}(X^{k_\lambda} - \Lambda(X^{k_\lambda})U + N^{k_\lambda}) \right\|_1 \\ &\leq \left\| X - X^{k_\lambda} - (\Lambda(X) - \Lambda(X^{k_\lambda}))U + N - N^{k_\lambda} \right\|_1, \end{aligned}$$

we infer the second term vanishes in norm, as well. Set  $\mathbf{N} := \Psi(N)$ . Lower semicontinuity of  $\rho_i$  — which follows from Theorem 5.3 applied in the case  $n = 1$ , see Remark 5.4 — yields

$$\begin{aligned} \sum_{i=1}^n \rho_i(A_i + \Lambda(X)U_i - N_i) &\leq \liminf_{\lambda \rightarrow \infty} \sum_{i=1}^n \rho_i(A_i^{k_\lambda} + \Lambda(X^{k_\lambda})U_i - N_i^{k_\lambda}) \\ &= \liminf_{\lambda \rightarrow \infty} \Lambda(X^{k_\lambda}) = \Lambda(X). \end{aligned}$$

The definition of  $\Lambda$  eventually yields that the inequality is actually an equality, i.e.

$$\sum_{i=1}^n \rho_i(A_i + \Lambda(X)U_i - N_i) = \Lambda(X).$$

We have proved that  $(A_i + \Lambda(X)U_i - N_i)_{i \in [n]} \in \widehat{\mathcal{P}}(X)$  and thus upper hemicontinuity, cf. Appendix A.3.

The same proof applies if  $X \in \text{dom}(\Lambda)$  is such that  $\Lambda$  is continuous at  $X$ .  $\square$

We now turn our attention to localising the results to the case when for all  $i \in [n]$ ,  $\mathcal{X} = \mathcal{X}_i$  and the space  $\mathcal{X}$  conforms with one of the cases **(BC)** or **(UC)**. As a first step, we need the following crucial extension result:

**Lemma 5.12.** *Let  $\mathcal{R} := (\mathcal{A}, \mathcal{S}, \mathfrak{p})$  be a risk measurement regime on a Banach lattice  $\mathcal{X}$  satisfying **(BC)** or **(UC)**. Suppose that  $\mathcal{A}$  is  $\|\cdot\|$ -closed, law-invariant and satisfies  $\mathcal{A} \cap \mathbb{R} \neq \emptyset$ , and  $\mathfrak{p}(Z) = \mathbb{E}[QZ]$  for some  $Q \in \text{dom}(\sigma_{\mathcal{A}}) \cap L^\infty$ . If we set  $\mathcal{B} := \text{cl}_{\|\cdot\|_1}(\mathcal{A})$ ,  $\mathfrak{R} := (\mathcal{B}, \mathcal{S}, \mathfrak{p})$  is a risk measurement regime on  $L^1$  and  $\rho_{\mathfrak{R}}|_{\mathcal{X}} = \rho_{\mathcal{R}}$ .*

*Proof.* We first prove that

$$\mathcal{A} = \{Y \in \mathcal{X} : \forall Q \in \text{dom}(\sigma_{\mathcal{A}}) \cap L^\infty (\mathbb{E}[QY] \leq \sigma_{\mathcal{A}}(Q))\}. \quad (5.10)$$

In case **(BC)** this follows from  $\mathcal{A}$  being closed in the  $\sigma(L^\infty, L^\infty)$ -topology, the weak topology associated to the dual pairing  $\langle L^\infty, L^\infty \rangle$ ; cf. [40, Proposition 1.2]. Now consider the case **(UC)** and define the *convex indicator* of  $\mathcal{A}$  to be

$$\delta_{\mathcal{A}} : \mathcal{X} \ni X \mapsto \begin{cases} \infty, & X \notin \mathcal{A}, \\ 0, & X \in \mathcal{A}. \end{cases}$$

This is a convex and law-invariant function. Without loss of generality we may assume  $\mathcal{X} \neq L^1$ . As  $\delta_{\mathcal{A}}$  has the Fatou property in the sense of [12],  $\delta_{\mathcal{A}}$  is l.s.c. in the  $\sigma(\mathcal{X}, L^\infty)$ -topology by [12, Proposition 2.11]. This directly implies (5.10).

Furthermore, the identities  $\text{dom}(\sigma_{\mathcal{B}}) = \text{dom}(\sigma_{\mathcal{A}}) \cap L^\infty$  and  $\sigma_{\mathcal{B}} = \sigma_{\mathcal{A}}|_{L^\infty}$  are easily verified. By Lemma A.1,

$$\begin{aligned} \mathcal{B} &= \{Y \in L^1 : \mathbb{E}[QY] \leq \sigma_{\mathcal{B}}(Q), \forall Q \in \text{dom}(\sigma_{\mathcal{B}})\} \\ &= \{Y \in L^1 : \mathbb{E}[QY] \leq \sigma_{\mathcal{A}}(Q), \forall Q \in \text{dom}(\sigma_{\mathcal{A}}) \cap L^\infty\}. \end{aligned}$$

This shows that  $\mathcal{B}$  is an acceptance set and that  $\mathcal{A} = \mathcal{B} \cap \mathcal{X}$ .

In order to verify (2.1), suppose  $X \in L^1$  and  $Z \in \mathcal{S}$  satisfy  $X + Z \in \mathcal{B}$ . Then

$$\mathfrak{p}(Z) = \mathbb{E}[QZ] = \mathbb{E}[Q(X + Z)] - \mathbb{E}[QX] \leq \sigma_{\mathcal{B}}(Q) - \mathbb{E}[QX] < \infty.$$

Hence,  $\mathfrak{R}$  is a risk measurement regime on  $L^1$ . For the identity  $\rho_{\mathfrak{R}}|_{\mathcal{X}} = \rho_{\mathcal{R}}$ , note that for  $X \in \mathcal{X}$  and for  $Z \in \mathcal{S}$ ,  $X - Z \in \mathcal{B}$  if and only if  $X - Z \in \mathcal{B} \cap \mathcal{X} = \mathcal{A}$ . We infer  $\rho_{\mathfrak{R}}(X) = \rho_{\mathcal{R}}(X)$ ,  $X \in \mathcal{X}$ .  $\square$

For  $\mathbf{f} \in \mathfrak{C}$ ,  $i \in [n]$  and  $X \in \mathcal{X}$ , 1-Lipschitz continuity of the function  $f_i$  yields  $|f_i(X)| \leq |X| + |f_i(0)| \in \mathcal{X}$   $\mathbb{P}$ -a.s. As  $\mathcal{X}$  is an ideal,  $f_i(X) \in \mathcal{X}$  holds as well; hence,  $\mathbf{f}(\mathcal{X}) \subseteq \mathcal{X}^n$ , and if we plug in  $X \in \mathcal{X}$  in (5.2), the resulting Pareto-optimal allocation lies in  $\mathcal{X}^n$  because  $\mathbf{U}, \mathbf{N} \in \mathcal{X}^n$  as  $\mathcal{S}_i \subseteq \mathcal{X}$  for all  $i \in [n]$ . We can now give the proof of Theorem 5.8.

*Proof of Theorem 5.8.* Let  $\mathfrak{R}_i$  denote the extension of the risk measurement regime  $\mathcal{R}_i$  to  $L^1$  as in Lemma 5.12. Apply Theorem 5.3 to  $\rho_{\mathfrak{R}_1}, \dots, \rho_{\mathfrak{R}_n}$  and  $X \in \mathcal{X}$  to obtain generalised versions of Theorem 5.3(2) and Corollary 5.5. This in conjunction with Proposition 3.4 generalises Theorem 5.3(1). The proof of Theorem 5.7 only needs to be altered at (5.8) and (5.9). We may replace  $\|W^k\|_1$  by  $K\|W^k\|$  in the first and use the order continuity of  $\|\cdot\|$  in the second instance, where the constant  $K$  is chosen as in (5.3).  $\square$

Finally we turn to the upper hemicontinuity of the equilibrium correspondence  $\mathcal{E}$  as formulated in Theorem 5.9, and we prove this theorem.

*Proof of Theorem 5.9.* Let  $\mathbf{W}$  be such that  $W := \sum_{i=1}^n W_i \in \text{int dom}(\Lambda)$ . From the proof of Theorem 3.5 we infer that, indeed, every  $(\mathbf{X}, \phi) \in \mathcal{E}(\mathbf{W})$  is an equilibrium of  $\mathbf{W}$ . For upper hemicontinuity, we shall first establish that the equilibrium prices of an approximating sequence lie in a sequentially relatively compact set in the dual  $\mathcal{X}^*$ . We shall hence prove that there is  $\varepsilon > 0$  and constants  $c_1$  and  $c_2$  only depending on  $W$  such that, given any  $X \in \mathcal{X}$  with  $\|X - W\| \leq \varepsilon$  and any subgradient  $\phi$  of  $\Lambda$  at  $X$ , it holds that

$$\|\phi\|_* \leq c_1 \text{ and } \Lambda^*(\phi) = \sum_{i=1}^n \rho_i^*(\phi) \leq c_2.$$

As we shall elaborate later, these bound imply that all subgradients of  $\Lambda$  at vectors in a closed ball around  $W$  lie in a  $\sigma(\mathcal{X}^*, \mathcal{X})$ -sequentially compact set.

In order to prove the assertion, continuity of  $\Lambda$  on  $\text{int dom}(\Lambda)$  (see [20, Corollary 2.5]) allows us to choose  $\varepsilon > 0$  such that  $|\Lambda(W + Y) - \Lambda(W)| \leq 1$  whenever  $\|Y\| \leq 2\varepsilon$ . Let now  $\delta > 0$  be such that  $\delta\varepsilon + \delta\|W\| \leq \varepsilon$  and fix  $X$  such that  $\|X - W\| \leq \varepsilon$  and a subgradient  $\phi$  of  $\Lambda$  at  $X$ . Moreover, suppose  $Y \in \mathcal{X}$  is such that  $\|Y\| \leq 1$ . We obtain from the subgradient inequality

$$\Lambda(X) + \varepsilon\phi(Y) \leq \Lambda(X + \varepsilon Y) \leq \Lambda(W) + 1.$$

Rearranging this inequality yields

$$\|\phi\|_* = \sup_{\|Y\| \leq 1} \phi(Y) \leq \frac{\Lambda(W) + 1 - \Lambda(X)}{\varepsilon} \leq \frac{2}{\varepsilon} =: c_2.$$

Moreover,

$$\begin{aligned}\Lambda(X) &= \phi(X) - \Lambda^*(\phi) = \frac{1}{1+\delta} (\phi((1+\delta)X) - \Lambda^*(\phi)) - \frac{\delta}{1+\delta} \Lambda^*(\phi) \\ &\leq \frac{1}{1+\delta} \Lambda((1+\delta)X) - \frac{\delta}{1+\delta} \Lambda^*(\phi).\end{aligned}$$

By rearranging this inequality we obtain

$$\sum_{i=1}^n \rho_i^*(\phi) = \Lambda^*(\phi) \leq \frac{1}{\delta} \Lambda((1+\delta)X) - \frac{1+\delta}{\delta} \Lambda(X) \leq \frac{2+\delta}{\delta} - \Lambda(W) =: c_1,$$

where we have used  $\|(1+\delta)X - W\| \leq 2\varepsilon$  following from the choice of  $\delta$ .

Now consider a sequence  $(\mathbf{W}^k)_{k \in \mathbb{N}} \subseteq \prod_{i=1}^n \text{int dom}(\rho_i)$  such that, for all  $i \in [n]$ ,  $W_i^k \rightarrow W_i$ ,  $k \rightarrow \infty$ , holds. Without loss of generality, we may assume that  $W^k := W_1^k + \dots + W_n^k$  lies in the ball around  $W$  with radius  $\varepsilon$ . For each  $k \in \mathbb{N}$  assume that  $(\mathbf{X}^k, \phi_k) \in \mathcal{E}(\mathbf{W}^k)$ ,  $k \in \mathbb{N}$ . We set

$$X_i^k = Y_i^k + \frac{\phi_k(W_i^k - Y_i^k)}{\phi(\tilde{Z})} \tilde{Z}, \quad i \in [n].$$

As  $\mathbf{Y}^k \in \widehat{\mathcal{P}}(W^k)$  and  $W^k \rightarrow W$ ,  $k \rightarrow \infty$ , we may assume, after passing to a subsequence, that  $\mathbf{Y}^k \rightarrow \mathbf{Y} \in \widehat{\mathcal{P}}(W)$  by Theorem 5.7.

We shall now select a convergent subsequence  $(\phi^k)_{k \in \mathbb{N}}$ . In case **(BC)**, we conclude from [33, Proposition 3.1(iii)] and Lemma A.4 that

$$\text{dom}(\Lambda^*) \subseteq \text{dom}(\rho_1^*) \subseteq L^1,$$

which implies that all subgradients  $\psi$  of  $\Lambda$  have the shape  $\psi = \mathbb{E}[\bar{Q} \cdot]$  for a unique  $\bar{Q} \in L_+^1$ . Hence, the equilibrium prices are given by  $\phi_k = \mathbb{E}[Q_k \cdot]$  for a unique  $Q_k \in L_+^1$ . Moreover, all subgradients  $Q_k$  lie in the  $\sigma(L^1, L^\infty)$ -compact set  $\mathcal{L}_{c_1}(\rho_1^*)$ . We may invoke the Eberlein-Šmulian theorem [3, Theorem 6.34] to find a subsequence  $(k_\lambda)_{\lambda \in \mathbb{N}}$  such that  $Q_{k_\lambda} \rightarrow Q \in L^1$  weakly, or equivalently  $\phi_{k_\lambda} \rightarrow \phi = \mathbb{E}[Q \cdot]$  in  $\sigma(\mathcal{X}^*, \mathcal{X})$ . In case **(UC)**, reflexivity of  $\mathcal{X}$ , the Banach-Alaoglu theorem and the bounds above imply the existence of a sequentially relatively compact set  $\Gamma$  such that  $\phi \in \Gamma$  whenever  $\|X - W\| \leq \varepsilon$  and  $\phi$  is a subgradient of  $\Lambda$  at  $X$ . Hence there is a  $\sigma(\mathcal{X}^*, \mathcal{X})$ -convergent subsequence  $(\phi_{k_\lambda})_{\lambda \in \mathbb{N}}$ .

Consequently, in both cases,

$$\phi_{k_\lambda}(W_i^{k_\lambda} - Y_i^{k_\lambda}) \rightarrow \phi(W_i - Y_i), \quad \lambda \rightarrow \infty.$$

It remains to prove that  $\phi$  is a subgradient of  $\Lambda$  at  $W$ . But as  $\Lambda^*$  is l.s.c. in the  $\sigma(\mathcal{X}^*, \mathcal{X})$ -topology and  $\phi_{k_\lambda}(W^{k_\lambda}) \rightarrow \phi(W)$ , we obtain

$$\begin{aligned}\Lambda(W) &= \limsup_{\lambda \rightarrow \infty} \phi_{k_\lambda}(W^{k_\lambda}) - \Lambda^*(\phi_{k_\lambda}) = \phi(W) - \liminf_{\lambda \rightarrow \infty} \Lambda^*(\phi_{k_\lambda}) \\ &\leq \phi(W) - \Lambda^*(\phi),\end{aligned}$$

which implies that, necessarily,  $\Lambda(W) = \phi(W) - \Lambda^*(\phi)$  and  $\phi$  is a subgradient of  $\Lambda$  at  $W$ .  $\square$

5.4. **Examples.** We conclude with two examples.

**Example 5.13.** We consider the model space  $\mathcal{X} := L^1$  on which two agents operate with acceptability criteria given by the *entropic risk measure*. More precisely, we choose  $0 < \beta \leq \gamma$  arbitrary and define

$$\mathcal{A}_1 := \{X \in L^1 : \xi_\beta(X) \leq 0\}, \quad \mathcal{A}_2 := \{X \in L^1 : \xi_\gamma(X) \leq 0\},$$

where, for  $\alpha > 0$ ,  $\xi_\alpha(X) := \frac{1}{\alpha} \log(\mathbb{E}[e^{\alpha X}])$ ,  $X \in L^1$ . It is well-known, cf. [26, Example 2.9], that

$$\xi := \xi_\beta \square \xi_\gamma = \xi_{\frac{\beta\gamma}{\beta+\gamma}}.$$

The convex conjugate  $\xi_\alpha^*$  of  $\xi_\alpha$  is given in terms of the relative entropy: for all  $Q \in L^1_+$  such that  $\mathbb{E}[Q] = 1$ , we have

$$\xi_\alpha^*(Q) = \frac{1}{\alpha} \mathbb{E}[Q \log(Q)] < \infty.$$

In order to satisfy Assumption 5.2, we may hence choose any pricing density  $Q^* \in L^1_+$  such that  $Q^* \geq \delta > 0$  for some  $\delta > 0$ . The pricing functionals are given by  $\mathbf{p}_i = \mathbb{E}[Q^* \cdot]$ . Moreover, we choose  $A \in \mathcal{F}$  such that  $\mathbb{E}[Q^* \mathbf{1}_A] = \mathbb{E}[Q^* \mathbf{1}_{A^c}]$ ,  $\mathcal{S}_1 = \mathcal{M} = \text{span}\{\mathbf{1}_A, \mathbf{1}_{A^c}\}$ , and  $\mathcal{S}_2 = \text{span}\{\mathbf{1}_A\}$ . Given these specifications,  $(\mathcal{R}_1, \mathcal{R}_2)$  is an agent system.

Note that  $\ker(\pi) = \{N_r := r\mathbf{1}_A - r\mathbf{1}_{A^c} : r \in \mathbb{R}\}$ . We will now characterise  $\mathcal{A}_+ + \ker(\pi)$  and set, for the sake of brevity,

$$\alpha := \frac{\beta\gamma}{\beta + \gamma}.$$

Given the characterisation of  $\mathcal{A}_+$ ,  $X - N_r \in \mathcal{A}_+$  for some  $r \in \mathbb{R}$  if and only if  $\mathbb{E}[e^{\alpha X} \mathbf{1}_A] \cdot \mathbb{E}[e^{\alpha X} \mathbf{1}_{A^c}] \leq \frac{1}{4}$ , as there is then a solution  $r \in \mathbb{R}$  to

$$0 \geq \frac{1}{\alpha} \log\left(\mathbb{E}[e^{\alpha(X-N_r)}]\right) = \frac{1}{\alpha} \log\left(e^{-\alpha r} \mathbb{E}[e^{\alpha X} \mathbf{1}_A] + e^{\alpha r} \mathbb{E}[e^{\alpha X} \mathbf{1}_{A^c}]\right).$$

Now, for arbitrary  $X \in \text{dom}(\Lambda) = \text{dom}(\xi_\alpha)$ , we note that

$$\begin{aligned} \Lambda(X) &= \inf\{\pi(r\mathbf{1}) : r \in \mathbb{R}, X - r\mathbf{1} \in \mathcal{A}_+ + \ker(\pi)\} \\ &= \inf\left\{r\mathbb{E}[Q^*] : r \in \mathbb{R}, e^{-\alpha r} \mathbb{E}[e^{\alpha X} \mathbf{1}_A] \cdot \mathbb{E}[e^{\alpha X} \mathbf{1}_{A^c}] \leq \frac{1}{4}\right\} \\ &= \frac{\mathbb{E}[Q^*]}{\alpha} (\log \mathbb{E}[e^{\alpha X} \mathbf{1}_A] + \log \mathbb{E}[e^{\alpha X} \mathbf{1}_{A^c}] + 2 \log(2)). \end{aligned}$$

Hereafter, we choose a solution  $r^*$  of

$$e^{-\alpha r^*} \mathbb{E}[e^{\alpha(X-\Lambda(X))} \mathbf{1}_A] + e^{\alpha r^*} \mathbb{E}[e^{\alpha(X-\Lambda(X))} \mathbf{1}_{A^c}] = 1,$$

e.g.

$$r^* := \log\left(\frac{2\mathbb{E}[e^{\alpha(X-\Lambda(X))} \mathbf{1}_A]}{\sqrt{1 - 4\mathbb{E}[e^{\alpha(X-\Lambda(X))} \mathbf{1}_A] \cdot \mathbb{E}[e^{\alpha(X-\Lambda(X))} \mathbf{1}_{A^c}] + 1}}\right).$$

Using the results from [26, Example 2.9],

$$\left(\frac{\gamma}{\beta + \gamma}(X - \Lambda(X)\mathbf{1} - N_{r^*}), \frac{\beta}{\beta + \gamma}(X - \Lambda(X)\mathbf{1} - N_{r^*})\right) \in \mathcal{A}_1 \times \mathcal{A}_2.$$

Consequently, the following is a Pareto-optimal allocation of  $X$ :

$$\left( \frac{\gamma}{\beta + \gamma} (X - \Lambda(X)\mathbf{1} - N_{r^*}) + \Lambda(X)\mathbf{1} + N_{r^*}, \frac{\beta}{\beta + \gamma} (X - \Lambda(X)\mathbf{1} - N_{r^*}) \right)$$

**Example 5.14.** Here, we choose the model space  $\mathcal{X} = L^\infty$  and illustrate the existence of Pareto-optimal allocations for two agents with acceptance sets less similar than in Example 5.13. To this end, we fix two parameters  $\beta \in (0, 1)$  and  $\gamma > 0$  and suppose that acceptability for agent 1 is based on the *Average Value at Risk*, i.e.

$$\begin{aligned} \mathcal{A}_1 &= \{X \in L^\infty : \xi_1(X) := \text{AVaR}_\beta(X) \leq 0\} \\ &= \{X \in L^\infty : \forall Q \in \mathcal{Q} (\mathbb{E}[QX] \leq 0)\}, \end{aligned}$$

where  $\mathcal{Q} = \{Q \in L_+^\infty : 0 \leq Q \leq \frac{1}{1-\beta} \text{ P-a.s.}, \mathbb{E}[Q] = 1\}$ . The acceptance set of agent 2 is, as in Example 5.13, given by an entropic risk measure, i.e.

$$\mathcal{A}_2 := \{X \in L^\infty : \xi_2(X) := \frac{1}{\gamma} \log(\mathbb{E}[e^{\gamma X}]) \leq 0\}.$$

By [27, Example 4.34 & Theorem 4.52], the support function of  $\mathcal{A}_+ = \mathcal{A}_1 + \mathcal{A}_2$  is given for  $Q \in L_+^\infty$  by

$$\sigma_{\mathcal{A}_+}(Q) = (\sigma_{\mathcal{A}_1} + \sigma_{\mathcal{A}_2})(Q) = \begin{cases} 0, & Q = 0 \\ \frac{1}{\gamma} \mathbb{E} \left[ Q \log \left( \frac{Q}{\mathbb{E}[Q]} \right) \right], & \text{if } Q \neq 0 \text{ and } \frac{Q}{\mathbb{E}[Q]} \in \mathcal{Q}, \\ \infty & \text{otherwise.} \end{cases}$$

As in Example 2.2, we choose a pricing density  $Q^* \in L_+^\infty$  such that, for some  $\delta \in (0, 1)$ ,  $\delta \leq Q^* \leq \frac{1-\delta}{1-\beta} + \delta$  holds and such that  $\mathbb{E}[Q^*] = 1$ . In this case,  $Q^* = \delta + (1-\delta)Q$ , where  $Q = \frac{Q^* - \delta}{1-\delta} \in \mathcal{Q}$ , hence  $Q^*$  satisfies Assumption 5.2.

Suppose the security spaces  $\mathcal{S}_i$ ,  $i = 1, 2$ , are given as in Example 5.13 for some nonempty  $A \in \mathcal{F}$ . As pricing rules we set  $\mathbf{p}_i := \mathbb{E}[Q^* \cdot]$ ,  $i = 1, 2$ , which results in

$$\ker(\pi) = \text{span}\{N := \mathbf{1}_A - r^* \mathbf{1}_{A^c}\}, \quad r^* = \frac{\mathbb{E}[Q^* \mathbf{1}_A]}{1 - \mathbb{E}[Q^* \mathbf{1}_A]}.$$

Let  $X \in L^\infty$  be any aggregated loss. Using [24, Theorem 3], we obtain the dual representation

$$\Lambda(X) = \max_{Q \in \tilde{\mathcal{Q}}} \mathbb{E}[QX] - \frac{1}{\gamma} \mathbb{E}[Q \log(Q)],$$

where  $\tilde{\mathcal{Q}} = \{Q \in \mathcal{Q} : \mathbb{E}[Q \mathbf{1}_A] = \mathbb{E}[Q^* \mathbf{1}_A]\}$ . We will now compute the right scaling factor  $s \in \mathbb{R}$  such that  $X - \Lambda(X) - sN \in \mathcal{A}_+$ . This is the case if and only if we have for all  $Q \in \mathcal{Q} \setminus \tilde{\mathcal{Q}}$

$$\mathbb{E}[QX] - \frac{1}{\gamma} \mathbb{E}[Q \log(Q)] - \Lambda(X) \leq s \mathbb{E}[QN].$$

We obtain

$$s \geq \sup_{Q \in \mathcal{Q} \setminus \tilde{\mathcal{Q}}: \mathbb{E}[Q \mathbf{1}_A] > \mathbb{E}[Q^* \mathbf{1}_A]} \frac{\mathbb{E}[QX] - \frac{1}{\gamma} \mathbb{E}[Q \log(Q)] + \Lambda(X)}{\mathbb{E}[QN]}$$

and

$$s \leq \inf_{Q \in \mathcal{Q} \setminus \tilde{\mathcal{Q}}: \mathbb{E}[Q\mathbf{1}_A] < \mathbb{E}[Q^*\mathbf{1}_A]} \frac{\frac{1}{\gamma} \mathbb{E}[Q \log(Q)] + \Lambda(X) - \mathbb{E}[QX]}{|\mathbb{E}[QN]|},$$

and the bounds describe an *a priori* nonempty interval. Choose any  $s^*$  in this interval. Combining [31, Proposition 3.2 & Sect. 3.5], we obtain that

$$((X - \Lambda(X) - s^*N - \zeta)^+, (X - \Lambda(X) - s^*N) \wedge \zeta) \in \mathcal{A}_1 \times \mathcal{A}_2.$$

for a suitable  $\zeta \in \mathbb{R}$ . Thus  $(X_1, X_2)$  given by

$$\begin{aligned} X_1(\zeta) &= (X - \Lambda(X) - s^*N - \zeta)^+ - s^*r^*\mathbf{1}_{A^c} + \Lambda(X), \\ X_2(\zeta) &= (X - \Lambda(X) - s^*N) \wedge \zeta + s^*\mathbf{1}_A, \end{aligned}$$

is a Pareto-optimal allocation of  $X$ .

## 6. OPTIMAL PORTFOLIO SPLITS

In this section we study the existence of optimal portfolio splits. For a thorough discussion of this problem, we refer to Tsanakas [42], although the problem we consider is rather akin to Wang [43]. A financial institution holds a portfolio which yields the future loss  $W$ . In order to diversify the risk posed by  $W$ , it may consider dividing the portfolio into  $n$  sub-portfolios  $X_1, \dots, X_n \in \mathcal{X}$ ,  $X_1 + \dots + X_n = W$ , and transfer these sub-portfolios to, e.g., distinct legal entities such as subsidiaries which operate under potentially varying regulatory regimes. As observed by Tsanakas, for convex, but not positively homogeneous risk measures, *without* market frictions like transaction costs risk can usually be reduced arbitrarily by introducing more subsidiaries, and hence, there is no incentive to stop this splitting procedure. However, since  $n$  can be arbitrarily large, transaction costs should not be neglected in this setting, and we will study the problem of finding *cost-optimal portfolio splits* under market frictions.

To be more precise, we model the subsidiaries as a family  $(\rho_i)_{i \in \mathbb{N}}$  of *normalised* risk measures on one and the same Fréchet lattice  $(\mathcal{X}, \preceq, \tau)$  – which entails  $\rho_i^* \geq 0$  for all  $i \in \mathbb{N}$  – such that the associated risk measurement regimes  $(\mathcal{R}_i)_{i \in \mathbb{N}}$  check infinite supportability (A4): as one and the same parent company splits the losses into  $n$  sub-portfolios, assuming that, for each  $n \in \mathbb{N}$ , the set of subsidiaries  $(\rho_i)_{i \in [n]}$  forms an agent system satisfying (A3) seems natural. Let further  $\mathfrak{c} : \mathbb{N} \rightarrow [0, \infty)$  be a nondecreasing cost function. The transaction costs of introducing subsidiaries  $i \in [n]$  and splitting a portfolio among them are given by  $\mathfrak{c}(n)$ . The condition  $\lim_{n \rightarrow \infty} \mathfrak{c}(n) = \infty$  prevents infinite splitting. At last we introduce  $\Lambda_n(X) := \inf_{\mathbf{X} \in \mathbb{A}_X} \sum_{i=1}^n \rho_i(X_i)$ ,  $X \in \mathcal{X}$ , the usual risk sharing functional associated to  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ . Note that for all  $X \in \mathcal{X}$ ,  $n \in \mathbb{N}$ , and every  $\mathbf{X} \in \mathcal{X}^n$  with  $\sum_{i=1}^n X_i = X$ , the estimate  $\sum_{i=1}^n \rho_i(X_i) = \sum_{i=1}^n \rho_i(X_i) + \rho_{n+1}(0) \geq \Lambda_{n+1}(X)$  holds, which entails  $\Lambda_n(X) \geq \Lambda_{n+1}(X)$ ,  $n \in \mathbb{N}$ . In this setting, optimal portfolio splits exist if each  $\Lambda_n$  is exact on  $\text{dom}(\Lambda_n)$ :

**Theorem 6.1.** *Suppose  $(\mathcal{R}_i)_{i \in \mathbb{N}}$  is a sequence of risk measurement regimes on a Fréchet lattice  $\mathcal{X}$  which checks (A4) and results in all  $\rho_i$  being normalised. Moreover, assume that*

the cost function satisfies

$$\lim_{n \rightarrow \infty} \mathbf{c}(n) = \infty$$

and that  $\Lambda_n$  is exact on  $\text{dom}(\Lambda_n)$  for all  $n \in \mathbb{N}$  and let  $W \in \sum_{i=1}^m \text{dom}(\rho_i)$  for some  $m \in \mathbb{N}$ . Then there is  $(n_*, X_1, \dots, X_{n_*})$ , where  $n_* \in \mathbb{N}$  and  $(X_1, \dots, X_{n_*})$  is an attainable allocation of  $W$ , which is a solution of

$$\sum_{i=1}^n \rho_i(X_i) + \mathbf{c}(n) \rightarrow \min \quad \text{subject to } n \in \mathbb{N} \text{ and } \mathbf{X} \in \mathcal{X}^n \text{ with } \sum_{i=1}^n X_i = W. \quad (6.1)$$

*Proof.* Note that (A4) can be rewritten as

$$\exists \phi_0 \in \bigcap_{i=1}^{\infty} \text{dom}(\rho_i^*) : \sum_{i=1}^{\infty} \rho_i^*(\phi_0) < \infty. \quad (6.2)$$

Let

$$m_* := \min\{m \in \mathbb{N} : \Lambda_m(W) < \infty\} = \min\{m \in \mathbb{N} : W \in \sum_{i=1}^m \text{dom}(\rho_i)\} < \infty.$$

By (6.2), we have  $\Lambda_n(W) \geq \phi_0(W) - \sum_{i=1}^{\infty} \rho_i^*(\phi_0) > -\infty$  for all  $n \geq m_*$ . Thus,  $\Lambda_n(W) + \mathbf{c}(n) = \infty$  whenever  $n < m_*$  and

$$\liminf_{n \rightarrow \infty} \Lambda_n(W) + \mathbf{c}(n) \geq \phi_0(W) - \sum_{i=1}^{\infty} \rho_i^*(\phi_0) + \lim_{n \rightarrow \infty} \mathbf{c}(n) = \infty.$$

Therefore, we can find  $n_* \in \mathbb{N}$  such that

$$\Lambda_{n_*}(W) + \mathbf{c}(n_*) = \inf_{n \in \mathbb{N}} \Lambda_n(W) + \mathbf{c}(n) \in \mathbb{R}.$$

In order to obtain a solution to (6.1), choose an attainable allocation  $\mathbf{X} \in \mathcal{X}^{n_*}$  of  $X$  such that  $\Lambda_{n_*}(X) = \sum_{i=1}^{n_*} \rho_i(X_i)$ .  $\square$

**Corollary 6.2.** *Suppose  $(\mathcal{R}_i)_{i \in \mathbb{N}}$  is a sequence of risk measurement regimes on a Fréchet lattice  $\mathcal{X}$  such that all  $\rho_i$  are normalised. Then the assertion of Theorem 6.1 holds under each of the following conditions:*

- (1) *The risk measures  $(\rho_1, \dots, \rho_n)$  comply with Theorem 5.8 for each  $n \in \mathbb{N}$  and the pricing functionals are given by  $\mathbf{p}_i = \mathbb{E}[(Q + \delta) \cdot] |_{\mathcal{S}_i}$  for a fixed  $\delta > 0$  and  $Q \in L_+^\infty$  with  $\sup_{Y \in \mathcal{A}_i} \mathbb{E}[(Q + \delta)Y] \leq 0$ ,  $i \in \mathbb{N}$ .*
- (2) *(A4) is satisfied, and for each  $n \in \mathbb{N}$ ,  $(\mathcal{R}_1, \dots, \mathcal{R}_n)$  is a polyhedral agent system.*

*Proof.* (1) Let  $Q \in L_+^\infty$  and  $\delta > 0$  be as described in the assertion and set  $Q^* := Q + \delta$ . Assumption 5.2 is satisfied. Let  $i \in \mathbb{N}$  be arbitrary and recall the definition of the cash-additive risk measures  $\xi_i$  in the proof of Theorem 5.3. By (5.6),  $\xi_i^*(\frac{Q^*}{\mathbb{E}[Q^*]}) \leq 0$ . Theorem 5.8 in the case  $n = 1$  (see Remark 5.4) yields that each  $X \in \text{dom}(\rho_i)$  admits an optimal payoff

$Z^X \in \mathcal{S}_i$ , i.e.  $X - Z^X \in \mathcal{A}_i$  and  $\mathbb{E}[Q^* Z^X] = \mathbf{p}_i(Z^X) = \rho_i(X)$ . Hence,

$$\begin{aligned} \rho_i^*(Q^*) &= \sup_{X \in \text{dom}(\rho_i)} \mathbb{E}[Q^* X] - \rho_i(X) = \sup_{X \in \text{dom}(\rho_i)} \mathbb{E}[Q^*(X - Z^X)] \\ &\leq \mathbb{E}[Q^*] \xi_i^*\left(\frac{Q^*}{\mathbb{E}[Q^*]}\right) \leq 0. \end{aligned}$$

Conversely, as  $\rho_i$  is normalised, we have  $\rho_i^*(Q^*) \geq 0$ . Hence, (A4) holds and  $\phi_0$  in (6.2) may be chosen as  $\phi_0 = \mathbb{E}[Q^* \cdot]$ . The solvability of (6.1) under (1) follows from Theorems 5.8 and 6.1.

(2) By Theorem 4.3  $\Lambda_n$  is exact on  $\text{dom}(\Lambda_n)$  for every  $n \in \mathbb{N}$ . Apply Theorem 6.1.  $\square$

**Remark 6.3.** Suppose that in the situation of Corollary 6.2(1) each of the monetary base risk measures  $\xi_i(X) := \inf\{m \in \mathbb{R} : X - m \in \mathcal{A}_i\}$ ,  $X \in \mathcal{X}$ , is normalised. Then each  $\delta > 0$  and each  $Q \in \mathbb{R}_+$  satisfy the assumptions of part (1). This follows from the fact that  $\xi_i^*(1) = 0$  holds for every  $i \in \mathbb{N}$ . Indeed, by arguments similar to the proof of Proposition 5.10,  $\xi_i^*(\mathbb{E}[Q|\mathcal{H}]) \leq \xi_i^*(Q)$  holds for all  $Q \in \text{dom}(\xi_i^*)$  and all sub- $\sigma$ -algebras  $\mathcal{H} \subseteq \mathcal{F}$ . Hence,

$$\xi_i^*(1) = \inf_{Q \in \text{dom}(\xi_i^*)} \xi_i^*(Q) = - \sup_{Q \in \text{dom}(\xi_i^*)} -\xi_i^*(Q) = -\xi_i(0) = 0, \quad i \in \mathbb{N}.$$

## APPENDIX A. TECHNICAL SUPPLEMENTS

**A.1. The geometry of convex sets.** Fix a nonempty convex subset  $\mathcal{C}$  of a locally convex Hausdorff topological Riesz space  $(\mathcal{X}, \preceq, \tau)$  with dual space  $\mathcal{X}^*$ . The *support function* of  $\mathcal{C}$  is the functional

$$\sigma_{\mathcal{C}} : \mathcal{X}^* \rightarrow (-\infty, \infty], \quad \phi \mapsto \sup_{Y \in \mathcal{C}} \phi(Y).$$

The *recession cone* of  $\mathcal{C}$  is the set

$$0^+\mathcal{C} := \{U \in \mathcal{X} : Y + kU \in \mathcal{C}, \forall Y \in \mathcal{C}, \forall k \geq 0\}.$$

A vector  $U$  lies in  $0^+\mathcal{C}$  if and only if  $Y + U \in \mathcal{C}$  holds for all  $Y \in \mathcal{C}$ .  $U$  is then called a *direction* of  $\mathcal{C}$ . The *lineality space* of  $\mathcal{C}$  is the vector space  $\text{lin}(\mathcal{C}) := 0^+\mathcal{C} \cap (-0^+\mathcal{C})$ . In the case of an acceptance set  $\mathcal{A}$ , monotonicity implies  $\text{dom}(\sigma_{\mathcal{A}}) \subseteq \mathcal{X}_+^*$ . If  $\mathcal{C}$  is closed, the Hahn-Banach separation theorem shows that

$$\mathcal{C} = \{Y \in \mathcal{X} : \phi(Y) \leq \sigma_{\mathcal{C}}(\phi), \forall \phi \in \text{dom}(\sigma_{\mathcal{C}})\}.$$

Combining this identity with the definition of the recession cone and the lineality space yields

**Lemma A.1.** *If  $\mathcal{C} \subseteq \mathcal{X}$  is closed and convex and  $\mathcal{J} \subseteq \text{dom}(\sigma_{\mathcal{C}})$  is such that*

$$\mathcal{C} = \{X \in \mathcal{X} : \phi(X) \leq \sigma_{\mathcal{C}}(\phi), \forall \phi \in \mathcal{J}\},$$

*then*

$$0^+\mathcal{C} = \bigcap_{\phi \in \mathcal{J}} \mathcal{L}_0(\phi) = \{U \in \mathcal{X} : \phi(U) \leq 0, \forall \phi \in \mathcal{J}\} \quad \text{and} \quad \text{lin}(\mathcal{A}) = \bigcap_{\phi \in \mathcal{J}} \ker(\phi).$$

Last we state a decomposition result for closed convex sets specific to finite-dimensional spaces. It follows from arguments in the proofs of [9, Lemmas II.16.2 and II.16.3].

**Lemma A.2.** Let  $\mathcal{C} \subseteq \mathbb{R}^d$  be convex and closed and  $\mathcal{V} := \text{lin}(\mathcal{C})^\perp$ . If  $\text{ext}(\mathcal{C} \cap \mathcal{V})$  denotes the set of extreme points of  $\mathcal{C} \cap \mathcal{V}$  and  $\text{co}(\cdot)$  is the convex hull operator,  $\mathcal{C}$  can be written as

$$\mathcal{C} = \text{co}(\text{ext}(\mathcal{C} \cap \mathcal{V})) + 0^+\mathcal{C}.$$

**A.2. Infimal convolution.** Let  $(\mathcal{X}, \preceq)$  be a Riesz space and suppose that functions  $g_i : \mathcal{X} \rightarrow (-\infty, \infty]$ ,  $i \in [n]$ , are given. The *infimal convolution* or *epi-sum* of  $g_1, \dots, g_n$  is the function  $\square_{i=1}^n g_i : \mathcal{X} \rightarrow [-\infty, \infty]$  defined by

$$\square_{i=1}^n g_i(X) := \inf \left\{ \sum_{i=1}^n g_i(X_i) : X_1, \dots, X_n \in \mathcal{X}, \sum_{i=1}^n X_i = X \right\}, \quad X \in \mathcal{X}.$$

The convolution is said to be *exact* at  $X \in \mathcal{X}$  if  $(\square_{i=1}^n g_i)(X) \in \mathbb{R}$  and there is  $X_1, \dots, X_n \in \mathcal{X}$  with  $\sum_{i=1}^n X_i = X$  such that

$$\sum_{i=1}^n g_i(X_i) = (\square_{i=1}^n g_i)(X).$$

**Lemma A.3.** Suppose  $\mathcal{X}_i \subseteq \mathcal{X}$ ,  $i \in [n]$ , are ideals in a Riesz space  $(\mathcal{X}, \preceq)$  such that  $\mathcal{X} = \sum_{i=1}^n \mathcal{X}_i$ .

- (1) If all  $g_i : \mathcal{X} \rightarrow (-\infty, \infty]$  are convex, then  $\square_{i=1}^n g_i$  is convex.
- (2) If  $g_i$  is monotone on  $\mathcal{X}_i$  with respect to  $\preceq$  for all  $i \in [n]$ , i.e.,  $X, Y \in \mathcal{X}_i$ ,  $X \preceq Y$ , implies  $g_i(X) \leq g_i(Y)$ , and  $g_i|_{\mathcal{X} \setminus \mathcal{X}_i} \equiv \infty$ , then  $\square_{i=1}^n g_i$  is monotone on  $\mathcal{X}$ .

*Proof.* We only prove (2). Let  $X, Y \in \mathcal{X}$ ,  $X \preceq Y$ , and let  $\mathbf{X}, \mathbf{Y} \in \prod_{i=1}^n \mathcal{X}_i$  with  $\sum_{i=1}^n X_i = X$  and  $\sum_{i=1}^n Y_i = Y$ . We thus have

$$0 \preceq Y - X = |Y - X| \preceq \sum_{i=1}^n |Y_i - X_i|.$$

By the Riesz space property of  $\mathcal{X}$  and the Riesz Decomposition Property (cf. [3, Sect. 8.5]), there is a vector  $\mathbf{Z} \in (\mathcal{X}_+)^n$  such that  $Y - X = \sum_{i=1}^n Z_i$  and such that  $Z_i = |Z_i| \preceq |Y_i - X_i|$ ,  $i \in [n]$ .  $\mathcal{X}_i$  being an ideal yields that in fact  $\mathbf{Z} \in \prod_{i=1}^n \mathcal{X}_i$ . By monotonicity of  $g_i$  on  $\mathcal{X}_i$ ,  $i \in [n]$ , we obtain

$$(\square_{i=1}^n g_i)(X) \leq \sum_{i=1}^n g_i(Y_i - Z_i) \leq \sum_{i=1}^n g_i(Y_i).$$

As  $(\square_{i=1}^n g_i)(Y) = \inf \{ \sum_{i=1}^n g_i(Y_i) : \mathbf{Y} \in \prod_{i=1}^n \mathcal{X}_i \}$  by the assumption that  $g_i|_{\mathcal{X} \setminus \mathcal{X}_i} \equiv \infty$ , taking the infimum over suitable  $\mathbf{Y}$  on the right-hand side proves the assertion.  $\square$

Note that the risk sharing functional satisfies  $\Lambda = \square_{i=1}^n g_i$ , where the functions  $g_i$  are defined by  $g_i(X) = \rho_i(X)$  if  $X \in \mathcal{X}_i$  and  $g_i(X) = \infty$  otherwise,  $X \in \mathcal{X}$ ,  $i \in [n]$ . These functions  $g_i$  inherit convexity on  $\mathcal{X}$  and monotonicity on  $\mathcal{X}_i$  from  $\rho_i$ .

**Lemma A.4.** Given a locally convex Hausdorff topological Riesz space  $(\mathcal{X}, \preceq, \tau)$  and proper functions  $g_i : \mathcal{X} \rightarrow (-\infty, \infty]$ ,  $i \in [n]$ , the following identities hold:

$$(\square_{i=1}^n g_i)^* = \sum_{i=1}^n g_i^* \quad \text{and} \quad \text{dom}((\square_{i=1}^n g_i)^*) = \bigcap_{i=1}^n \text{dom}(g_i^*).$$

**A.3. Correspondences.** Given two nonempty sets  $A$  and  $B$ , a map  $\Gamma : A \rightarrow 2^B$  mapping elements of  $A$  to subsets of  $B$  is called a *correspondence* and will be denoted by  $\Gamma : A \rightarrow B$ . Assume now that  $(\mathcal{X}, \tau)$  and  $(\mathcal{Y}, \sigma)$  are topological spaces, and let  $\Gamma : \mathcal{X} \rightarrow \mathcal{Y}$  be a correspondence.

A continuous function  $\Psi : \mathcal{X} \rightarrow \mathcal{Y}$  is a *continuous selection* for the correspondence  $\Gamma$  if  $\Psi(x) \in \Gamma(x)$  holds for all  $x \in \mathcal{X}$ .

If  $(\mathcal{X}, \sigma)$  is first countable,  $\Gamma$  is *upper hemicontinuous* at  $x \in \mathcal{X}$  if, whenever  $(x_k)_{k \in \mathbb{N}}$  is a sequence  $\sigma$ -convergent to  $x$  and  $(y_k)_{k \in \mathbb{N}} \subseteq \mathcal{Y}$  is such that, for each  $k \in \mathbb{N}$ ,  $y_k \in \Gamma(x_k)$ , there is a limit point  $y \in \Gamma(x)$  of  $(y_k)_{k \in \mathbb{N}}$ . If both topological spaces are first countable,  $\Gamma$  is *lower hemicontinuous* at  $x \in \mathcal{X}$  if, whenever  $(x_k)_{k \in \mathbb{N}}$  is a sequence  $\sigma$ -convergent to  $x$  and  $y \in \Gamma(x)$ , there is a subsequence  $(k_\lambda)_{\lambda \in \mathbb{N}}$  and  $y_\lambda \in \Gamma(x_{k_\lambda})$ ,  $\lambda \in \mathbb{N}$ , such that  $y_\lambda \rightarrow y$  with respect to  $\tau$  as  $\lambda \rightarrow \infty$ .<sup>11</sup> An example of a lower hemicontinuous correspondence relevant for our investigations is the security allocation map

$$\mathbb{A}^s : \mathcal{M} \ni Z \mapsto \mathbb{A}_Z \cap \prod_{i=1}^n \mathcal{S}_i.$$

**Lemma A.5.** *The correspondence  $\mathbb{A}^s$  is lower hemicontinuous on the global security market  $\mathcal{M}$  and admits a continuous selection  $\Psi : \mathcal{M} \rightarrow \prod_{i=1}^n \mathcal{S}_i$  with respect to any norm on  $\mathcal{M}$ .*

*Proof.* Let  $\langle \cdot, \cdot \rangle$  be an inner product on  $\mathcal{M}$ . Set  $\mathcal{S}_0 := \{0\}$ . We claim that there are natural numbers  $0 = m_0 < m_1 \leq \dots \leq m_n$  and  $Z_1, \dots, Z_{m_n} \in \bigcup_{i=1}^n \mathcal{S}_i$  such that for all  $i \in [n]$ , it holds that  $\{Z_{m_{i-1}+1}, \dots, Z_{m_i}\}$  is an orthonormal basis of  $\{X \in \mathcal{S}_i : X \perp \text{span}\{Z_1, \dots, Z_{m_{i-1}}\}\}$ . Note that every  $Z \in \mathcal{M}$  can be expressed as  $Z = \sum_{i=1}^{m_n} \langle Z_i, Z \rangle Z_i$ , hence the mapping  $\Psi : Z \mapsto \mathbb{A}_Z^s$  defined by

$$\Psi(Z)_i := \sum_{i=m_{i-1}+1}^{m_i} \langle Z_i, Z \rangle Z_i, \quad i \in [n],$$

is a selection of  $\mathbb{A}^s$  and continuous with respect to the unique locally convex Hausdorff topology on  $\mathcal{M}$ . Lower hemicontinuity follows immediately.  $\square$

**Acknowledgements.** We should like to thank the associate editor and an anonymous referee for helpful suggestions which aided in improving the original draft of this manuscript.

## REFERENCES

- [1] Acciaio, B. (2009), Short Note on Inf-Convolution Preserving the Fatou Property. *Annals of Finance*, 5:281-287.
- [2] Acciaio, B., and G. Svindland (2009), Optimal Risk Sharing with Different Reference Probabilities. *Insurance: Mathematics and Economics*, 44(3):426-433.
- [3] Aliprantis, C. and K. Border (2006), *Infinite Dimensional Analysis – A Hitchhiker’s Guide*. 3rd edition, Springer.
- [4] Aliprantis, C., and O. Burkinshaw (2003), *Locally Solid Riesz Spaces with Applications to Economics*. American Mathematical Society.
- [5] Arrow, K. J. (1963), Uncertainty and the Welfare Economics of Medical Care. *American Economic Review*, 5:941-973.

<sup>11</sup>We define lower and upper hemicontinuity using sequences rather than nets and tacitly use that this is sufficient under the given assumptions, cf. [3, Theorems 17.20 & 17.21].

- [6] Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath (1999), Coherent Measures of Risk. *Mathematical Finance*, 9(3):203-228.
- [7] Baes, M., P. Koch-Medina, and C. Munari (2019), Existence, Uniqueness, and Stability of Optimal Portfolios of Eligible Assets. Forthcoming in *Mathematical Finance*. Available online at <https://doi.org/10.1111/mafi.12205>.
- [8] Barrieu, P., and N. El Karoui (2005), Inf-Convolution of Risk Measures and Optimal Risk Transfer. *Finance and Stochastics*, 9(2):269-298.
- [9] Barvinok, A. (2012), *A Course in Convexity*. Springer.
- [10] Borch, K. (1962), Equilibrium in a Reinsurance Market. *Econometrica*, 30:424-444.
- [11] Carr, P., H. Geman, and D. Madan (2001), Pricing and Hedging in Incomplete Markets. *Journal of Financial Economics*, 62:131-167.
- [12] Chen, S., N. Gao, and F. Xanthos (2018), The Strong Fatou Property of Risk Measures. *Dependence Modeling*, 6(1):183-196.
- [13] Cheridito, P., M. Kupper, and L. Tangpi (2017), Duality Formulas for Robust Pricing and Hedging in Discrete Time. *SIAM Journal on Financial Mathematics*, 8(1):738-765.
- [14] Cheridito, P., and T. Li (2009), Risk Measures on Orlicz Hearts. *Mathematical Finance*, 19(2):189-214.
- [15] Dana, R.-A., and C. Le Van (2000), Arbitrage, Duality and Asset Equilibria. *Journal of Mathematical Economics* 34(3):397-413.
- [16] Dana, R.-A., C. Le Van, and F. Magnien (1999), On the Different Notions of Arbitrage and Existence of Equilibrium. *Journal of Economic Theory*, 87(1):169-193.
- [17] Delbaen, F. (2002), Coherent Risk Measures on General Probability Spaces. In: *Advances in Finance and Stochastics: Essays in Honor of Dieter Sondermann*, pp. 1-37. Springer.
- [18] Delbaen, F. (2012), *Monetary Utility Functions*. Osaka University CSFI Lecture Note Series, Vol. 3.
- [19] Embrechts, P., H. Liu, and R. Wang (2017), Quantile-Based Risk Sharing. *Operations Research*, 66(4):936-949.
- [20] Ekeland, I. and Témam, R. (1999), *Convex Analysis and Variational Problems*. SIAM.
- [21] El Karoui, N., and C. Ravanelli (2009), Cash Subadditive Risk Measures under Interest Rate Ambiguity. *Mathematical Finance*, 19:561-590.
- [22] Farkas, E. W., P. Koch Medina, and C. Munari (2013), Beyond Cash-Additive Risk Measures: When Changing the Numéraire Fails. *Finance and Stochastics*, 18(1):145-173.
- [23] Farkas, E. W., P. Koch Medina, and C. Munari (2014), Capital Requirements with Defaultable Securities. *Insurance: Mathematics and Economics*, 55:58-67.
- [24] Farkas, E. W., P. Koch Medina, and C. Munari (2015), Measuring Risk with Multiple Eligible Assets. *Mathematics and Financial Economics*, 9(1):3-27.
- [25] Filipović, D., and M. Kupper (2008), Equilibrium Prices for Monetary Utility Functions. *International Journal of Theoretical and Applied Finance*, 11:325-343.
- [26] Filipović, D., and G. Svindland (2008), Optimal Capital and Risk Allocations for Law- and Cash-Invariant Convex Functions. *Finance and Stochastics*, 12(3):423-439.
- [27] Föllmer, H., and A. Schied (2011), *Stochastic Finance: An Introduction in Discrete Time*. 3rd edition, De Gruyter.
- [28] Gao, N., D. Leung, C. Munari, and F. Xanthos (2018), Fatou Property, Representations, and Extensions of Law-Invariant Risk Measures on General Orlicz Spaces. *Finance and Stochastics*, 22(2):395-415.
- [29] Hardy, G. H., J. E. Littlewood, and G. Pólya (1952), *Inequalities*. 2nd edition, Cambridge University Press.
- [30] Husain, T. (1965), *The Open Mapping and Closed Graph Theorems in Topological Vector Spaces*. Vieweg.
- [31] Jouini, E., W. Schachermayer, and N. Touzi (2008), Optimal Risk Sharing for Law Invariant Monetary Utility Functions. *Mathematical Finance*, 18:269-292.
- [32] Klöppel, S., and M. Schweizer (2007), Dynamic Indifference Valuation via Convex Risk Measures. *Mathematical Finance*, 17(4):599-627.
- [33] Liebrich, F.-B., and G. Svindland (2017), Model Spaces for Risk Measures. *Insurance: Mathematics and Economics*, 77:150-165.
- [34] Mas-Colell, A., and W. Zame (1991), Equilibrium Theory in Infinite Dimensional Spaces. In: Hildenbrand, W., and H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, Vol. 4, pp. 1835-1898, Elsevier.
- [35] Munari, C. (2015), *Measuring Risk Beyond the Cash-Additive Paradigm*. PhD thesis, ETH Zürich. Available online at <https://doi.org/10.3929/ethz-a-010475509>.

- [36] Pichler, A. (2013), The Natural Banach Space for Version Independent Risk Measures. *Insurance: Mathematics and Economics*, 53:405-415.
- [37] Ravanelli, C., and G. Svindland (2014), Comonotone Pareto Optimal Allocations for Law Invariant Robust Utilities on  $L^1$ . *Finance and Stochastics*, 18:249-269.
- [38] Rockafellar, R.T. (1970), *Convex Analysis*. Princeton University Press.
- [39] Rudin, M. E. (1969), A New Proof that Metric Spaces are Paracompact. *Proceedings of the American Mathematical Society*, 20:603.
- [40] Svindland, G. (2010), Continuity Properties of Law-Invariant (Quasi-)Convex Risk Functions on  $L^\infty$ . *Mathematics and Financial Economics*, 3(1):39-43.
- [41] Svindland, G. (2010), Subgradients of law-invariant convex risk measures on  $L^1$ . *Statistics & Decisions*, 27(2):169-199.
- [42] Tsanakas, A. (2009), To Split or Not to Split: Capital Allocation with Convex Risk Measures. *Insurance: Mathematics and Economics*, 44(2):268-277.
- [43] Wang, R. (2016), Regulatory Arbitrage of Risk Measures. *Quantitative Finance*, 16(3):337-347.
- [44] Wilson, R. (1968), The Theory of Syndicates. *Econometrica*, 36:119-132.
- [45] Zălinescu, C. (2002), *Convex Analysis in General Vector Spaces*. World Scientific.
- [46] Zheng, X. Y. (2009), Pareto Solutions of Polyhedral-valued Vector Optimization Problems in Banach Spaces. *Set-Valued Analysis*, 17:389-408.