On the Lower Arbitrage Bound of American Contingent Claims*

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Abstract

We prove that in a discrete-time market model the lower arbitrage bound of an American contingent claim is itself an arbitrage-free price if and only if it corresponds to the price of the claim optimally exercised under some equivalent martingale measure.

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1 Introduction

An American contingent claim H is a contract which obliges the seller to pay a certain amount $H_\tau \geq 0$ if the buyer of that claim decides to exercise it at a (stopping) time τ. A price π of such an American contingent claim is said to be fair or arbitrage-free if it satisfies the following two conditions. On the one hand, π should not be too expensive from the buyer's point of view, in the sense that there exists an exercise time τ such that π is a fair price for the payoff H_τ. On the other hand, the price π should not be too cheap from the seller's point of view, meaning that there is no exercise time σ such that the fair prices of the payoff H_σ all exceed π. It is well understood that in an arbitrage-free market the arbitrage-free pricing of H is closely related to an optimal stopping problem. Indeed, let $\pi = E^{\mathbb{Q}}[H_\tau]$ where $\mathbb{Q}$ is an equivalent martingale measure and τ is an optimal exercise time for H under $\mathbb{Q}$, i.e., τ solves

$$E^{\mathbb{Q}}[H_\tau] = \sup\{E^{\mathbb{Q}}[H_\sigma] \mid \sigma \text{ is an exercise time }\}. \tag{1.1}$$

It is easily verified that π is an arbitrage-free price for H. But the converse, that is the fact that every arbitrage-free price of an American contingent claim originates from the solution to (1.1) under some equivalent martingale measure, has not been clear so far. To be more precise, the problem here is the lower arbitrage bound $\underline{\pi}(H)$ of H, i.e., the infimum over all arbitrage-free prices of H, which may or may not be itself an arbitrage-free price. In case $\underline{\pi}(H)$ is an arbitrage-free price, it was an open question whether there exists a minimal equivalent martingale measure in the sense that the solution to (1.1) under that measure yields the price $\underline{\pi}(H)$. In this paper we prove that this is indeed the case, and we also give characterizations of this situation in terms of replicability properties of H (Theorem 2.3).

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In his doctoral thesis [9], Trevino Aguilar studies a closely related problem in a continuous-time framework. Indeed, [9] provided some very useful ideas of how to attack the problem.

The remainder of the paper is organized as follows: In Section 2 we introduce the market model, give a short overview over the arbitrage pricing theory as regards American contingent claims and state our main result in Theorem 2.3. The proof of Theorem 2.3 is then carried out through Section 3. Finally, in Section 4 we provide an example illustrating our main results.

We assume that the reader is familiar with standard multi-period discrete-time arbitrage theory such as outlined in Föllmer and Schied [2]. The book [2] is our main reference, and our setup and notation will to a major extent be adopted from there. As regards the arbitrage pricing theory of American contingent claims and the related theory of Snell envelopes, we also refer the reader to [1, 3, 4, 6, 8].

2 The Main Result

We consider a discrete-time market model in which d assets are priced at times $t = 0, \dots, T$ with $T \in \mathbb{N}$. The information available in the market is modeled by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$ with

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{F}_T = \mathcal{F}.$$

Throughout the paper, all equalities and inequalities between random variables are understood in the $\mathbb{P}$-almost sure sense. Following standard arbitrage theory, we assume the existence of a strictly positive asset which is used as numéraire for discounting. We indicate by $S^i = (S^i_t)_{t=0,\dots,T}$, $i = 1, \dots, d$, the discounted price process of the asset i, which is assumed to be non-negative and adapted to the filtration $(\mathcal{F}_t)_{t=0,\dots,T}$. Let $\mathcal{M}$ be the set of equivalent martingale measures, that is, the set of probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $S = (S^1, \dots, S^d)$ is a (d-dimensional) martingale under $\mathbb{Q}$. We assume that the market S is arbitrage-free which is equivalent to $\mathcal{M} \neq \emptyset$; see [2, Theorem 5.17].

For the remainder of the paper we consider a (discounted) American contingent claim, i.e. a non-negative $(\mathcal{F}_t)$-adapted process $H = (H_t)_{t=0,\dots,T}$. We assume that

$$H_t \in L^1(\Omega, \mathcal{F}, \mathbb{Q}) \quad \text{for all } \ t = 0, \dots, T \ \text{ and } \ \mathbb{Q} \in \mathcal{M}.$$

Let $\mathcal{T}$ denote the set of stopping times $\tau : \Omega \to \{0, \dots, T\}$. For each time $\tau \in \mathcal{T}$, the random variable H_τ is the discounted payoff obtained by exercising the American contingent claim H at time τ. Note that H_τ can be considered as the discounted payoff of a European contingent claim, thus the set of arbitrage-free prices of H_τ is given by

$$\Pi(H_\tau) = \{E^{\mathbb{Q}}[H_\tau] \mid \mathbb{Q} \in \mathcal{M} \text{ and } E^{\mathbb{Q}}[H_\tau] < \infty\},$$

see [2, Theorem 5.30]. We define the set of arbitrage-free prices of an American contingent claim as in [2, Definition 6.31], reflecting the asymmetric connotation of such a contract: the seller must hedge against all possible exercise times, while the buyer only needs to find one favorable exercise strategy.

Definition 2.1. *A real number π is an arbitrage-free price of the American contingent claim H if the following two conditions are satisfied:*

(i) *There exists some $\tau \in \mathcal{T}$ and $\pi' \in \Pi(H_\tau)$ such that $\pi \leq \pi'$.*

(ii) *There is no $\tau \in \mathcal{T}$ such that $\pi < \pi'$ for all $\pi' \in \Pi(H_\tau)$.*

The set of arbitrage-free prices of H is denoted by $\Pi(H)$.

Recall that in case of a European contingent claim, the set of arbitrage-free prices is either an open interval or a singleton, the latter case being equivalent to replicability, that is, to the existence of a self-financing strategy whose discounted terminal value equals the value of the claim; see [2, Theorem 5.33]. In case of an American contingent claim it is well understood that $\Pi(H)$ is a real interval with endpoints

$$\underline{\pi}(H) = \inf_{\mathbb{Q}\in\mathcal{M}} \sup_{\tau\in\mathcal{T}} E^{\mathbb{Q}}[H_\tau] \quad \text{and} \quad \overline{\pi}(H) = \sup_{\mathbb{Q}\in\mathcal{M}} \sup_{\tau\in\mathcal{T}} E^{\mathbb{Q}}[H_\tau],$$

and that $\Pi(H)$ either consists of one single point or does not contain its upper endpoint $\overline{\pi}(H)$; see [2, Theorem 6.33]. In the second case, however, in contrast to the pricing of a European contingent claim, both situations

$$\underline{\pi}(H) \in \Pi(H) \quad \text{and} \quad \underline{\pi}(H) \notin \Pi(H)$$

can occur, see Section 4 and [2, Example 6.34]. Let us now establish the relation between the prices in $\Pi(H)$ and the optimal stopping of H under some $\mathbb{Q} \in \mathcal{M}$.

Definition 2.2. *A stopping time $\tau \in \mathcal{T}$ is an optimal stopping time for H under $\mathbb{Q} \in \mathcal{M}$ if*

$$E^{\mathbb{Q}}[H_\tau] = \sup_{\sigma\in\mathcal{T}} E^{\mathbb{Q}}[H_\sigma].$$

We denote by $\mathcal{T}^*$ the set of all optimal stopping times:

$$\mathcal{T}^* := \{\tau \in \mathcal{T} \mid \tau \text{ is an optimal stopping for } H \text{ under some } \mathbb{Q} \in \mathcal{M}\}.$$

It is well-known that the set of optimal stopping times for H under any $\mathbb{Q} \in \mathcal{M}$ is non-empty; see [2, Theorem 6.20]. Note also that the set

$$\mathcal{P} := \{E^{\mathbb{Q}}[H_\tau] \mid \mathbb{Q} \in \mathcal{M} \text{ and } \tau \in \mathcal{T} \text{ is optimal under } \mathbb{Q}\ \}$$

is an interval with bounds $\underline{\pi}(H)$ and $\overline{\pi}(H)$; see [2, proof of Theorem 6.33]. It is easily verified that $\mathcal{P} \subseteq \Pi(H)$. Hence, if $\underline{\pi}(H) \notin \Pi(H)$, then $\mathcal{P} = \Pi(H)$. However, in case $\underline{\pi}(H) \in \Pi(H)$, it has been an open question whether $\mathcal{P} = \Pi(H)$ too, i.e., whether there exists an equivalent martingale measure $\overline{\mathbb{Q}} \in \mathcal{M}$ and an optimal stopping time τ under $\overline{\mathbb{Q}}$ such that $E^{\overline{\mathbb{Q}}}[H_\tau] = \underline{\pi}(H)$. In Theorem 2.3, which is our main result, we show that this is indeed the case. Moreover, we also give a detailed characterization of this situation in terms of replicability of the European contingent claim corresponding to exercising H at a specific stopping time.

Theorem 2.3. *Let $\hat{\tau} := \operatorname{ess\,inf}\{\tau \mid \tau \in \mathcal{T}^*\}$. Then $\hat{\tau} \in \mathcal{T}$, and the following conditions are equivalent:*

(i) $\underline{\pi}(H) \in \Pi(H)$.

(ii) *$H_{\hat{\tau}}$ is replicable (at price $\underline{\pi}(H)$).*

(iii) *There exists $\mathbb{Q} \in \mathcal{M}$ and an optimal stopping time τ for H under $\mathbb{Q}$ such that $E^{\mathbb{Q}}[H_\tau] = \underline{\pi}(H)$.*

(iv) *There exists $\tau \in \mathcal{T}^*$ such that H_τ is replicable.*

The proof of Theorem 2.3 needs some preparation and will be given at the end of Section 3. Notice that Theorem 2.3 extends the case of European contingent claims. Indeed, let H correspond to a European contingent claim, i.e. $H_t = 0$ for all $t = 0, \ldots, T-1$, and $Y := H_T \geq 0$. Then clearly $H_{\hat{\tau}} = Y$, thus $\underline{\pi}(H) = \inf \Pi(Y)$ is arbitrage-free if and only if Y is replicable.

From our previous remarks and Theorem 2.3 we obtain the following:

Corollary 2.4. $\Pi(H) = \{E^{\mathbb{Q}}[H_\tau] \mid \mathbb{Q} \in \mathcal{M}$ *and* τ *is optimal for* H *under* $\mathbb{Q}\}$.

Remark 2.5. The existence of a "worst-case probability measure" $\overline{\mathbb{Q}}$ for the lower Snell envelope of an American contingent claim H with respect to a convex family $\mathcal{N}$ of equivalent probability measures, in the sense that $\overline{\mathbb{Q}} \in \mathcal{N}$ shall satisfy

$$\sup_{\tau \in \mathcal{T}} E^{\overline{\mathbb{Q}}}[H_\tau] = \inf_{\mathbb{Q} \in \mathcal{N}} \sup_{\tau \in \mathcal{T}} E^{\mathbb{Q}}[H_\tau],$$

has been studied in the literature by for instance [9] and [7]; see also the references therein. Existence results are known under the assumption that the set of densities $\left\{\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathbb{Q} \in \mathcal{N}\right\}$ is a subset of $L^p(\Omega, \mathcal{F}, \mathbb{P})$ and compact in the $\sigma(L^p(\Omega, \mathcal{F}, \mathbb{P}), L^q(\Omega, \mathcal{F}, \mathbb{P}))$-topology for some $p \in [1, \infty)$ and $q := p/(p-1)$ where $1/0 := \infty$. However, when studying the lower arbitrage bound $\underline{\pi}(H)$, the set of test measures $\mathcal{N}$ equals the set of equivalent martingale measures $\mathcal{M}$, for which this compactness assumption is satisfied if and only if the market is complete ($\mathcal{M} = \{\mathbb{Q}\}$). Indeed, if $\mathcal{D} := \left\{\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathbb{Q} \in \mathcal{M}\right\}$ is $\sigma(L^p(\Omega, \mathcal{F}, \mathbb{P}), L^q(\Omega, \mathcal{F}, \mathbb{P}))$-compact, then for each $C \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ the continuous function $\mathcal{D} \ni Z \mapsto E[ZC]$ attains its maximum over $\mathcal{D}$ which means that the upper arbitrage bound of the European contingent claim C is itself an arbitrage-free price. Hence, C is replicable ([2, Theorem 5.33]), and thus the market is complete. Therefore, the mentioned results cannot be applied in our setting. Note that Theorem 2.3 does not require any further condition on the set of equivalent martingale measures $\mathcal{M}$. $\diamondsuit$

3 Discussion and Proof of Theorem 2.3

In what follows we introduce the basic tools needed for the proof of Theorem 2.3.

Definition 3.1. *For* $\mathbb{Q} \in \mathcal{M}$*, the Snell envelope* $U^{\mathbb{Q}} = (U_t^{\mathbb{Q}})_{t=0,\dots,T}$ *of the American contingent claim* H *with respect to the measure* $\mathbb{Q}$ *is defined by*

$$U_t^{\mathbb{Q}} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}, \tau \geq t} E^{\mathbb{Q}}[H_\tau \mid \mathcal{F}_t], \quad t = 0, \dots, T.$$

The lower Snell envelope $U^{\downarrow} = (U_t^{\downarrow})_{t=0,\dots,T}$ *of* H *(w.r. to* $\mathcal{M}$*) is defined by*

$$U_t^{\downarrow} = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{M}} U_t^{\mathbb{Q}} = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{M}} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}, \tau \geq t} E^{\mathbb{Q}}[H_\tau \mid \mathcal{F}_t], \quad t = 0, \dots, T.$$

In particular, $U_0^{\downarrow} = \underline{\pi}(H)$.

The process $U^{\mathbb{Q}}$ is the smallest $\mathbb{Q}$-supermartingale dominating H. It is known that $\tau \in \mathcal{T}$ is an optimal stopping time for H under $\mathbb{Q}$ if and only if $H_\tau = U_\tau^{\mathbb{Q}}$ and the stopped process $(U^{\mathbb{Q}})^\tau := (U_{\tau \wedge t}^{\mathbb{Q}})_{t=0,\dots,T}$ is a $\mathbb{Q}$-martingale. Moreover, $\tau^{\mathbb{Q}} := \inf\{t \geq 0 \mid U_t^{\mathbb{Q}} = H_t\}$ is the minimal optimal stopping time for H under $\mathbb{Q}$; see [2, Proposition 6.22]. In particular, the stopping time $\hat{\tau}$ introduced in Theorem 2.3 satisfies $\hat{\tau} = \operatorname{ess\,inf}_{\mathbb{Q} \in \mathcal{M}} \tau^{\mathbb{Q}}$.

Lemma 3.2. *The set* $\{\tau^{\mathbb{Q}} \mid \mathbb{Q} \in \mathcal{M}\}$ *is downward directed, hence* $\hat{\tau}$ *is a stopping time. In particular, there exists a sequence* $(\mathbb{Q}_k)_{k \in \mathbb{N}} \in \mathcal{M}$ *such that* $\{\tau^{\mathbb{Q}_k} = \hat{\tau}\} \nearrow \Omega$ *for* $k \to \infty$.

Proof. The fact that $\{\tau^{\mathbb{Q}} \mid \mathbb{Q} \in \mathcal{M}\}$ is downward directed follows as in the proof of [9, Theorem 5.6]. This implies that there is a sequence $(\mathbb{Q}_k)_{k \in \mathbb{N}} \subset \mathcal{M}$ such that $\tau^{\mathbb{Q}_k} \searrow \hat{\tau}$. From that it follows that $\hat{\tau} = \operatorname{ess\,inf}\{\tau^{\mathbb{Q}_k} \mid k \in \mathbb{N}\}$ is a stopping time. Moreover, as time is discrete and by monotonicity of the sequence $(\tau^{\mathbb{Q}_k})_{k \in \mathbb{N}}$, we deduce that $\{\tau^{\mathbb{Q}_k} = \hat{\tau}\} \nearrow \Omega$ for $k \to \infty$. $\square$

Notice that, according to Lemma 3.2, for almost all $\omega \in \Omega$ we have $\hat{\tau}(\omega) = \tau^{P_\omega}(\omega)$ for some $P_\omega \in \mathcal{M}$. Hence we obtain that for almost all ω

$$H_{\hat{\tau}}(\omega) = H_{\tau^{P_\omega}}(\omega) = U^{P_\omega}_{\tau^{P_\omega}}(\omega) \geq U^{\downarrow}_{\tau^{P_\omega}}(\omega) = U^{\downarrow}_{\hat{\tau}}(\omega) \geq H_{\hat{\tau}}(\omega).$$

Consequently

$$U^{\downarrow}_{\hat{\tau}} = H_{\hat{\tau}}. \tag{3.1}$$

Proposition 3.3. *The lower Snell envelope $U^{\downarrow}$ satisfies the following properties:*

(i) *$(U^{\downarrow})^{\hat{\tau}}$ is a $\mathcal{M}$-submartingale, i.e., a submartingale under each $\mathbb{Q} \in \mathcal{M}$.*

(ii) *If $H_{\hat{\tau}}$ is replicable at price $\underline{\pi}(H)$, then $(U^{\downarrow})^{\hat{\tau}}$ is a $\mathcal{M}$-martingale.*

Proof. Fix $\mathbb{Q} \in \mathcal{M}$. Notice that for every $t \in \{0, \dots, T\}$ there is a sequence $(\mathbb{Q}_k)_{k \in \mathbb{N}} \subset \mathcal{M}$ such that $U^{\mathbb{Q}_k}_t \searrow U^{\downarrow}_t$ and $\mathbb{Q}_k|_{\mathcal{F}_t} = \mathbb{Q}|_{\mathcal{F}_t}$ for all k; see [2, Proposition 6.45 and Lemma 6.50]. Now, for every $t \in \{1, \dots, T\}$,

$$E^{\mathbb{Q}}[U^{\downarrow}_{\hat{\tau}\wedge t} \mid \mathcal{F}_{t-1}] = U^{\downarrow}_{\hat{\tau}} 1_{\{\hat{\tau} \leq t-1\}} + E^{\mathbb{Q}}[U^{\downarrow}_t \mid \mathcal{F}_{t-1}] 1_{\{\hat{\tau} \geq t\}}$$

and

$$\begin{aligned} E^{\mathbb{Q}}\left[U^{\downarrow}_t \mid \mathcal{F}_{t-1}\right] 1_{\{\hat{\tau} \geq t\}} &= \lim_{k \to \infty} E^{\mathbb{Q}}\left[U^{\mathbb{Q}_k}_t \mid \mathcal{F}_{t-1}\right] 1_{\{\hat{\tau} \geq t\}} = \lim_{k \to \infty} E^{\mathbb{Q}_k}\left[U^{\mathbb{Q}_k}_{\tau^{\mathbb{Q}_k}\wedge t} \mid \mathcal{F}_{t-1}\right] 1_{\{\hat{\tau} \geq t\}} \\ &= \lim_{k \to \infty} U^{\mathbb{Q}_k}_{\tau^{\mathbb{Q}_k}\wedge (t-1)} 1_{\{\hat{\tau} \geq t\}} = \lim_{k \to \infty} U^{\mathbb{Q}_k}_{t-1} 1_{\{\hat{\tau} \geq t\}} \geq U^{\downarrow}_{t-1} 1_{\{\hat{\tau} \geq t\}}, \end{aligned}$$

where we use the dominated convergence theorem in the first equality since $0 \leq U^{\mathbb{Q}_k}_t \leq U^{\mathbb{Q}_1}_t \leq E^{\mathbb{Q}_1}[\sum_{s=t}^T H_s \mid \mathcal{F}_t]$, and the facts that $\mathbb{Q}_k|_{\mathcal{F}_t} = \mathbb{Q}|_{\mathcal{F}_t}$, $\hat{\tau} \leq \tau^{\mathbb{Q}}$, and $(U^{\mathbb{Q}_k})^{\tau^{\mathbb{Q}_k}}$ is a $\mathbb{Q}_k$-martingale for the rest. As $\mathbb{Q} \in \mathcal{M}$ was arbitrary, (i) is proved.

In order to prove (ii), let $H_{\hat{\tau}}$ be replicable at price $\underline{\pi}(H)$ and let $\mathbb{Q} \in \mathcal{M}$. Then in combination with (3.1) and (i) we have for all $t = 0, \dots, T$ that

$$\underline{\pi}(H) = E^{\mathbb{Q}}[H_{\hat{\tau}}] = E^{\mathbb{Q}}[U^{\downarrow}_{\hat{\tau}}] \geq E^{\mathbb{Q}}[U^{\downarrow}_{\hat{\tau}\wedge t}] \geq U^{\downarrow}_0 = \underline{\pi}(H),$$

thus $(U^{\downarrow})^{\hat{\tau}}$ is a martingale under $\mathbb{Q}$. □

Lemma 3.4. *Let $\tau \in \mathcal{T}$ be such that H_τ is replicable, then the unique arbitrage-free price p of H_τ satisfies $p \leq \underline{\pi}(H)$. Moreover, if $\tau \in \mathcal{T}^*$, then $p = \underline{\pi}(H)$.*

Proof. For any $\tau \in \mathcal{T}$ and $\mathbb{Q} \in \mathcal{M}$ we have

$$p = E^{\mathbb{Q}}[H_\tau] \leq \sup_{\sigma \in \mathcal{T}} E^{\mathbb{Q}}[H_\sigma] = U^{\mathbb{Q}}_0, \tag{3.2}$$

and taking the infimum on the right-hand side over all $\mathbb{Q} \in \mathcal{M}$ yields $p \leq \underline{\pi}(H)$. Moreover, if $\tau \in \mathcal{T}^*$, then there exists a $\mathbb{Q} \in \mathcal{M}$ such that equality holds in (3.2). □

Proposition 3.5. *Let $H_{\hat{\tau}}$ be replicable at price $\underline{\pi}(H)$. Then*

$$\mathcal{Q} := \left\{\mathbb{Q} \in \mathcal{M} \mid U^{\mathbb{Q}}_{\hat{\tau}} = H_{\hat{\tau}}\right\} = \left\{\mathbb{Q} \in \mathcal{M} \mid U^{\mathbb{Q}}_0 = \underline{\pi}(H)\right\}. \tag{3.3}$$

Proof. Let $\mathbb{Q} \in \mathcal{Q}$. According to Proposition 3.3, $(U^{\downarrow})^{\hat{\tau}}$ is a $\mathbb{Q}$-martingale. We show that the process

$$\widetilde{U}_t := U_t^{\mathbb{Q}} 1_{\{\hat{\tau}<t\}} + U_t^{\downarrow} 1_{\{\hat{\tau}\geq t\}}$$

is a $\mathbb{Q}$-supermartingale dominating H. Indeed, for any $t \in \{1, \dots T\}$ we have that

$$\begin{aligned} E^{\mathbb{Q}}[\widetilde{U}_t \mid \mathcal{F}_{t-1}] &= E^{\mathbb{Q}}[U_t^{\mathbb{Q}} \mid \mathcal{F}_{t-1}] 1_{\{\hat{\tau}<t\}} + E^{\mathbb{Q}}[U^{\downarrow}_{\hat{\tau}\wedge t} \mid \mathcal{F}_{t-1}] 1_{\{\hat{\tau}\geq t\}} \\ &\leq U_{t-1}^{\mathbb{Q}} 1_{\{\hat{\tau}\leq t-1\}} + U^{\downarrow}_{\hat{\tau}\wedge(t-1)} 1_{\{\hat{\tau}>t-1\}} \quad = \quad \widetilde{U}_{t-1}, \end{aligned}$$

where we use the supermartingale property of $U^{\mathbb{Q}}$ and $(U^{\downarrow})^{\hat{\tau}}$ and the fact that $U^{\mathbb{Q}}_{\hat{\tau}} = H_{\hat{\tau}} = U^{\downarrow}_{\hat{\tau}}$ by (3.1). Therefore $\widetilde{U}$ is a $\mathbb{Q}$-supermartingale which obviously dominates H since both $U^{\mathbb{Q}}$ and $U^{\downarrow}$ do. By [2, Proposition 6.11], $U^{\mathbb{Q}}$ is the smallest $\mathbb{Q}$-supermartingale dominating H, which implies that $U_0^{\mathbb{Q}} \leq \widetilde{U}_0 = \underline{\pi}(H)$. Hence $U_0^{\mathbb{Q}} = \underline{\pi}(H)$, and the inclusion '$\subseteq$' in (3.3) is proved.

Now let $\mathbb{Q} \in \mathcal{M}$ be such that $U_0^{\mathbb{Q}} = \underline{\pi}(H)$. Then, as $U^{\mathbb{Q}}$ is a $\mathbb{Q}$-supermartingale dominating H and $H_{\hat{\tau}}$ is replicable at price $\underline{\pi}(H)$, we have

$$\underline{\pi}(H) = U_0^{\mathbb{Q}} \geq E^{\mathbb{Q}}[U^{\mathbb{Q}}_{\hat{\tau}}] \geq E^{\mathbb{Q}}[H_{\hat{\tau}}] = \underline{\pi}(H).$$

This implies $U^{\mathbb{Q}}_{\hat{\tau}} = H_{\hat{\tau}}$ and concludes the proof of the proposition. □

Proof of Theorem 2.3. In Lemma 3.2 it is shown that $\hat{\tau} \in \mathcal{T}$.
(i) ⇒ (ii): Let $\underline{\pi}(H) \in \Pi(H)$. The second property of Definition 2.1 implies the existence of some $\widetilde{\mathbb{P}} \in \mathcal{M}$ such that $\underline{\pi}(H) \geq E^{\widetilde{\mathbb{P}}}[H_{\hat{\tau}}]$. From Proposition 3.3 (i) we know that $(U^{\downarrow})^{\hat{\tau}}$ is a $\mathcal{M}$-submartingale. In conjunction with (3.1) we obtain for all $\mathbb{Q} \in \mathcal{M}$ that

$$E^{\mathbb{Q}}\left[H_{\hat{\tau}}\right] = E^{\mathbb{Q}}\left[U^{\downarrow}_{\hat{\tau}}\right] \geq U_0^{\downarrow} = \underline{\pi}(H).$$

Taking the infimum over all $\mathbb{Q} \in \mathcal{M}$ we arrive at

$$E^{\widetilde{\mathbb{P}}}[H_{\hat{\tau}}] \leq \underline{\pi}(H) \leq \inf_{\mathbb{Q}\in\mathcal{M}} E^{\mathbb{Q}}[H_{\hat{\tau}}] \leq E^{\widetilde{\mathbb{P}}}[H_{\hat{\tau}}],$$

which yields

$$E^{\widetilde{\mathbb{P}}}[H_{\hat{\tau}}] = \underline{\pi}(H) = \inf_{\mathbb{Q}\in\mathcal{M}} E^{\mathbb{Q}}[H_{\hat{\tau}}].$$

Consequently, the set of arbitrage-free prices for the European contingent claim $H_{\hat{\tau}}$ contains its lower bound. Thus $H_{\hat{\tau}}$ is replicable and $\Pi(H_{\hat{\tau}}) = \{\underline{\pi}(H)\}$; see [2, Theorem 5.33].
(ii) ⇒ (iii): Let $H_{\hat{\tau}}$ be replicable. From Lemma 3.4, and since $E^{\mathbb{Q}}[H_{\hat{\tau}}] \geq \underline{\pi}(H)$ for all $\mathbb{Q} \in \mathcal{M}$ as in the proof of Proposition 3.3 (ii), it follows that the unique price of $H_{\hat{\tau}}$ is $\underline{\pi}(H)$. Now fix $\mathbb{P}^* \in \mathcal{M}$. According to Lemma 3.2, there is a sequence $(\mathbb{Q}_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that $A_k := \{\tau^{\mathbb{Q}_k} = \hat{\tau}\} \nearrow \Omega$. Defining

$$B_k := A_k \setminus \bigcup_{m=1}^{k-1} A_m \in \mathcal{F}_{\hat{\tau}},$$

we get

$$\hat{\tau} = \sum_{k=1}^{\infty} \tau^{\mathbb{Q}_k} 1_{B_k}.$$

Now consider the probability measure $\widetilde{\mathbb{P}}$ obtained by pasting the measure $\mathbb{P}^*$ with the measures $\mathbb{Q}_k$ on B_k in $\hat{\tau}$, i.e., $\widetilde{\mathbb{P}}$ defined via

$$\widetilde{\mathbb{P}}(A) = E^{\mathbb{P}^*}\left[\sum_{k=1}^{\infty} E^{\mathbb{Q}_k}[1_{A\cap B_k} \mid \mathcal{F}_{\hat{\tau}}]\right], \quad A \in \mathcal{F},$$

cf. [2, Lemma 6.49]. Clearly $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$. Moreover, $\widetilde{\mathbb{P}} \in \mathcal{M}$ since for $i = 1, \ldots, d$ and $t = 0, \ldots, T-1$ we have

$$E^{\widetilde{\mathbb{P}}}[S^i_{t+1} \mid \mathcal{F}_t] = E^{\mathbb{P}^*}[S^i_{t+1} \mid \mathcal{F}_t] 1_{\{\hat{\tau} \geq t+1\}} + \sum_{k=1}^{\infty} E^{\mathbb{Q}_k}[S^i_{t+1} \mid \mathcal{F}_t] 1_{B_k \cap \{\hat{\tau} \leq t\}} = S^i_t$$

as $B_k \cap \{\hat{\tau} \leq t\} \in \mathcal{F}_t$. Since on B_k we have $U^{\mathbb{Q}_k}_{\hat{\tau}} = H_{\hat{\tau}}$, by monotone convergence

$$\begin{aligned} H_{\hat{\tau}} &= \sum_{k=1}^{\infty} U^{\mathbb{Q}_k}_{\hat{\tau}} 1_{B_k} = \sum_{k=1}^{\infty} \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}, \sigma \geq \hat{\tau}} E^{\mathbb{Q}_k}[H_\sigma 1_{B_k} \mid \mathcal{F}_{\hat{\tau}}] \\ &= \sum_{k=1}^{\infty} \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}, \sigma \geq \hat{\tau}} E^{\widetilde{\mathbb{P}}}[H_\sigma 1_{B_k} \mid \mathcal{F}_{\hat{\tau}}] \geq \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}, \sigma \geq \hat{\tau}} \sum_{k=1}^{\infty} E^{\widetilde{\mathbb{P}}}[H_\sigma 1_{B_k} \mid \mathcal{F}_{\hat{\tau}}] \\ &= \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}, \sigma \geq \hat{\tau}} E^{\widetilde{\mathbb{P}}}[H_\sigma \mid \mathcal{F}_{\hat{\tau}}] = U^{\widetilde{\mathbb{P}}}_{\hat{\tau}} \geq H_{\hat{\tau}}. \end{aligned}$$

This means that $\widetilde{\mathbb{P}} \in \mathcal{M}$ verifies $U^{\widetilde{\mathbb{P}}}_{\hat{\tau}} = H_{\hat{\tau}}$. Proposition 3.5 then yields $U^{\widetilde{\mathbb{P}}}_0 = \underline{\pi}(H)$ and (iii) follows.
(iii) $\Rightarrow$ (i): As already mentioned, $U^{\mathbb{Q}}_0 = E^{\mathbb{Q}}[H_\tau]$ clearly satisfies both conditions in Definition 2.1.
(iii) $\Rightarrow$ (iv): Let $\mathbb{Q} \in \mathcal{M}$ such that $U^{\mathbb{Q}}_0 = \underline{\pi}(H)$, then, according to the equivalences already proved, $H_{\hat{\tau}}$ is replicable at price $\underline{\pi}(H)$. We show that $\tau^{\mathbb{Q}} = \hat{\tau}$. Indeed,

$$\underline{\pi}(H) = U^{\mathbb{Q}}_0 = E^{\mathbb{Q}}[H_{\tau^{\mathbb{Q}}}] \geq E^{\mathbb{Q}}[H_{\hat{\tau}}] = \underline{\pi}(H)$$

implies that $E^{\mathbb{Q}}[H_{\tau^{\mathbb{Q}}}] = E^{\mathbb{Q}}[H_{\hat{\tau}}]$. Hence $\hat{\tau}$ is optimal under $\mathbb{Q}$ and therefore $\tau^{\mathbb{Q}} = \hat{\tau}$.
(iv) $\Rightarrow$ (iii): This implication follows from Lemma 3.4 □

Our main results are expressed in terms of the stopping time $\hat{\tau}$, for which we know that $U^{\downarrow}_{\hat{\tau}} = H_{\hat{\tau}}$; see (3.1). Let us consider the first time when the lower Snell envelope $U^{\downarrow}$ of H equals H, that is,

$$\tau^{\downarrow} := \inf\{t \geq 0 \mid U^{\downarrow}_t = H_t\}.$$

Clearly we have $\tau^{\downarrow} \leq \hat{\tau}$. It might be expected that $\tau^{\downarrow}$ plays a similarly important role in the analysis of $U^{\downarrow}$ as the stopping times $\tau^{\mathbb{Q}}$ do for $U^{\mathbb{Q}}$. Concerning this matter, see for instance the discussion of the lower Snell envelope as outlined in [2]. A natural question is whether $\tau^{\downarrow}$ and $\hat{\tau}$ do coincide, or in case they do not, whether at least the analysis carried out in this section could also be done replacing $\hat{\tau}$ by the earlier stopping time $\tau^{\downarrow}$. However, the answer to both questions is no. In Section 4 we show that $\tau^{\downarrow}$ and $\hat{\tau}$ need not coincide, and that $H_{\tau^{\downarrow}}$ can be replicable without $\underline{\pi}(H)$ being an arbitrage-free price for H. Consequently, $\tau^{\downarrow}$ is not suited for a characterization of the situation $\underline{\pi}(H) \in \Pi(H)$. Nevertheless, we have the following result:

Proposition 3.6. *$\underline{\pi}(H) \in \Pi(H)$ if and only if both $\tau^{\downarrow} \in \mathcal{T}^*$ and $H_{\tau^{\downarrow}}$ is replicable. In either case $\hat{\tau} = \tau^{\downarrow}$.*

Proof. Suppose that $\underline{\pi}(H) \in \Pi(H)$ and let $\mathbb{Q}$ and τ be as in Theorem 2.3 (iii). Since $(U^{\downarrow})^{\hat{\tau}}$ is a $\mathbb{Q}$-martingale by Proposition 3.3, Doob's stopping theorem yields

$$E^{\mathbb{Q}}[H_{\tau^{\downarrow}}] = E^{\mathbb{Q}}[U^{\downarrow}_{\tau^{\downarrow}}] = U^{\downarrow}_0 = \underline{\pi}(H) = E^{\mathbb{Q}}[H_\tau].$$

Hence $\tau^{\downarrow}$ is optimal under $\mathbb{Q}$, so $\hat{\tau} \leq \tau^{\mathbb{Q}} \leq \tau^{\downarrow} \leq \hat{\tau}$. Therefore $\hat{\tau} = \tau^{\mathbb{Q}} = \tau^{\downarrow}$ and $H_{\tau^{\downarrow}} = H_{\hat{\tau}}$ is replicable by Theorem 2.3. The reverse implication follows directly from Theorem 2.3. □

4 An Illustrating Example

Let X_1, X_2 be standard normal distributed random variables on the probability spaces $(\Omega_i, \mathcal{A}_i, \mathbb{P}_i), i = 1, 2$, respectively, and consider the product space $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{A}_1 \otimes \mathcal{A}_2$, and $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$. We define the random variables $\widetilde{X}_i$ on $(\Omega, \mathcal{F}, \mathbb{P})$ by $\widetilde{X}_i(\omega_1, \omega_2) = -1 + \sqrt{2}X_i(\omega_i), i = 1, 2$. Let the discounted stock price of the risky asset on $(\Omega, \mathcal{F}, \mathbb{P})$ be given by

$$S_0 = 1, \quad S_1 = e^{\widetilde{X}_1}, \quad S_2 = e^{\widetilde{X}_1 + \widetilde{X}_2}.$$

The filtration is

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_1 = \sigma(\widetilde{X}_1), \quad \mathcal{F}_2 = \sigma(\widetilde{X}_1, \widetilde{X}_2).$$

Consider the following discounted American contingent claim:

$$H_0 = 0, \quad H_1 = e^{\widetilde{X}_1}, \quad H_2 = e^{\widetilde{X}_1 + \frac{1}{2}\widetilde{X}_2}.$$

Clearly $\tau^{\mathbb{Q}} \geq 1$ for any equivalent martingale measure $\mathbb{Q} \in \mathcal{M}$. Moreover, note that $\mathbb{P} \in \mathcal{M}$ and that, for any $\tau \in \mathcal{T}$ such that $\tau \geq 1$,

$$E^{\mathbb{P}}[H_\tau] = E^{\mathbb{P}}[e^{\widetilde{X}_1} 1_{\{\tau=1\}} + e^{\widetilde{X}_1 + \frac{1}{2}\widetilde{X}_2} 1_{\{\tau=2\}}] = E^{\mathbb{P}}[e^{\widetilde{X}_1} 1_{\{\tau=1\}}] + E^{\mathbb{P}}[e^{\widetilde{X}_1} 1_{\{\tau=2\}}] \cdot E^{\mathbb{P}}[e^{\frac{1}{2}\widetilde{X}_2}] \leq 1,$$

where the last inequality is strict if $\mathbb{P}(\tau = 2) > 0$ since $E^{\mathbb{P}}[e^{\frac{1}{2}\widetilde{X}_2}] < 1$. In particular this gives $\tau^{\mathbb{P}} = 1$, which in turn implies $\hat{\tau} = 1$. Therefore, $H_{\hat{\tau}} = S_1$ is replicable and Theorem 2.3 ensures that $\underline{\pi}(H)$ is an arbitrage-free price for H.

Now consider another discounted American contingent claim, given by

$$H_0 = 0, \quad H_1 = e^{\widetilde{X}_1}, \quad H_2 = e^{\widetilde{X}_1} Z \quad \text{where } Z = e^{\widetilde{X}_2} 1_{\{\widetilde{X}_2 > 1\}} + 1_{\{\widetilde{X}_2 \leq 1\}}.$$

Since $Z \geq 1$ and $\mathbb{P}(Z > 1) > 0$, for each stopping time $\tau \in \mathcal{T}$ we have $H_\tau \leq H_2$, and one verifies that $\tau^{\mathbb{Q}} = 2$ for all $\mathbb{Q} \in \mathcal{M}$, and thus $\hat{\tau} = 2$. However, one can find a sequence of equivalent martingale measures $(\mathbb{Q}_n)_{n \in \mathbb{N}}$ such that $E^{\mathbb{Q}_n}[Z \mid \mathcal{F}_1] \to 1$ as $n \to \infty$. Therefore $U_1^{\downarrow} = H_1$, hence $\tau^{\downarrow} = 1 < 2 = \hat{\tau}$. In addition we have that $H_{\tau^{\downarrow}} = S_1$ is replicable, whereas $H_{\hat{\tau}}$ is not, so $\underline{\pi}(H)$ is not an arbitrage-free price by Theorem 2.3.

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