Comonotone Pareto Optimal Allocations for Law Invariant Robust Utilities on L^1 (extended working paper version)

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Abstract

We prove the existence of comonotone Pareto optimal allocations satisfying utility constraints when decision makers have probabilistic sophisticated variational preferences and thus representing criteria in the class of law invariant robust utilities. The total endowment is only required to be integrable.

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1 Introduction

In this paper we prove the existence of Pareto optimal allocations of integrable random endowments when decision makers have probabilistic sophisticated variational preferences. Variational preferences were introduced and axiomatically characterized by Maccheroni, Marinacci, and Rustichini (2006). This broad class of preferences allows to model ambiguity aversion and includes several subclasses of preferences that have been extensively studied in the economic literature. In mathematical finance, variational preferences are known as robust utilities; see Föllmer, Schied,

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and Weber (2009) and the references therein. In particular, Föllmer et al. (2009) establish the connection between variational preferences and robust utilities as representing choice criteria of variational preferences on random endowments corresponding to Savage acts; see also Remark 2.2 for a brief summary of these results. Our study focuses on variational preferences which are in addition assumed to be probabilistic sophisticated[1] and thus can be represented by robust utilities which are law invariant. Probabilistic sophistication or law invariance means that the decision maker sees any two random endowments that have the same distribution under a reference probability measure as equivalent. To be consistent both with the literature on decision making and (mathematical) finance, we use the term *probabilistic sophistication* whenever referring to preferences, and *law invariance* whenever referring to the representing robust utility. We pursue this approach – in the spirit of Föllmer et al. (2009) – in an attempt to unify the current knowledge on Pareto optimal allocations, and to emphasize the range of the results presented in this paper.

The class of law invariant robust utilities which represent probabilistic sophisticated variational preferences are of the following type:

$$\mathcal{U}(X) = \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q})\right), \quad X \in L^1, \tag{1.1}$$

where $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is a (not necessarily strictly) concave (not necessarily strictly) increasing utility function, $\mathcal{Q}$ is a set of probability measures which is closed under densities with the same distribution, and $\alpha(\mathbb{Q})$ is a suited law invariant penalization on $\mathbb{Q} \in \mathcal{Q}$; see Definition 2.1 for the details. This broad class nests many well-known choice criteria studied in the economic and finance literature, in particular, the von Neumann and Morgenstern (1947) *expected utility*, the probabilistic sophisticated *maxmin expected utility preferences* introduced by Gilboa and Schmeidler (1989), the *multiplier preferences* introduced by Hansen and Sargent (2000, 2001), and (apart from the sign) the law invariant *cash (sub)-additive convex risk measures* introduced by Artzner, Delbaen, Eber, and Heath (1999), Föllmer and Schied (2002), and Frittelli and Rosazza Gianin (2005) with the property of cash-invariance, and by El Karoui and Ravanelli (2010) with the generalized property of cash (sub)-additivity.

[1]Probabilistic sophisticated preferences were introduced by Machina and Schmeidler (1992), further studied by Marinacci (2002) and by Maccheroni et al. (2006) and Strzalecki (2011) for the case of variational preferences.

We assume that the decision makers have preferences on a space of future random payoff profiles which we identify with $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$, i.e. the space of $\mathbb{P}$-integrable random variables on a fixed non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ modulo $\mathbb{P}$-almost sure equality, where $\mathbb{P}$ is the reference probability measure. So far, most of the existing literature makes the assumption that payoff profiles are bounded, i.e. in L^∞, or even that the state space is finite. These assumptions are justified in many settings. But for applications in finance where nearly all models involve unbounded distributions, the boundedness assumption is not appropriate. This suggests the model space L^1 and the probabilistic sophistication of the preferences indeed allows for that; see Remark 2.2 or Filipović and Svindland (2012).

In this paper we consider $n \geq 2$ decision makers with probabilistic sophisticated variational preferences on L^1 represented by law invariant robust utilities as in (1.1). All decision makers share the same reference probability. Given the initial endowments $W_i \in L^1$, $i = 1, \ldots, n$, of the decision makers, we prove the existence of comonotone Pareto optimal allocations of the aggregate endowment $W = W_1 + \ldots + W_n$ which satisfy individual rationality constraints. Indeed we show that under some mild conditions comonotone Pareto optima exist for basically any constraints on the utilities of the allocations as long as there is at least one allocation satisfying these constraints. Note that comonotonicity means that the endowments in the allocation are continuous increasing functions of the aggregate endowment W. Comonotonicity of the Pareto optima is a consequence of the law invariant robust utilities preserving second order stochastic dominance; see Section 3.2. The possibility to restrain the set of allocations to the comonotone ones in the optimization problem corresponding to Pareto optima is also a major ingredient in our existence proof.

The existence of Pareto optimal allocations has so far only been established for a few subclasses of law invariant robust utilities. In case that all decision makers have von Neumann–Morgenstern expected utilities, existence results were already proved in the sixties by Borch (1962), Arrow (1963) and Wilson (1968). More recent is the proof of the existence of Pareto optimal allocations when all decision makers apply law invariant convex risk measures on L^∞; see Jouini, Schachermayer, and Touzi (2008) (and also Acciaio (2007) and Barrieu and El Karoui (2005)) and references therein. Filipović and Svindland (2008) extend this result to integrable (not necessarily bounded) aggregate endowments. Even more recently, Dana (2011) proves the existence of Pareto optimal allocations when the decision makers have choice criteria within a class of law invariant, finitely valued, continuous, concave utility functions on L^∞ not necessarily representing variational preferences. In Dana (2011) at least one utility function is required to be cash additive and the others are assumed to be strictly concave. Note that the cash additivity assumption is very useful when proving the existence of optimal allocations because, in conjunction with a comonotone improvement result, it immediately allows us to restrain the optimal allocation problem to an essentially compact set. Dropping the cash additivity assumption generates some difficulties which we are able to solve: Indeed, we also reduce the problem to an optimization over an essentially compact set of comonotone allocations. However, in contrast to the cash additive case, in which this reduction can be simply imposed due to the invariance towards constant re-sharing of sure payoffs, in the concave (non cash additive) case it follows as a necessity from the concavity of the utility functions u in (1.1). Other results on the existence of Pareto optimal allocations are found in Kiesel and Rüschendorf (2008). Here the existence of optimal allocations is proved for convex risk functionals on L^∞ which are not necessarily law invariant, however, under the assumption that the aggregate endowment W is in the interior of the domain of the (infimal-)convolution of the convex risk functionals, and that there exists an interior point in the intersection of the domains of the dual functions of these risk functionals. Such interior point conditions are standard when arguing by means of convex duality theory. Unfortunately, such results are not applicable in our case. The reason is that on large model spaces like L^1 the interior of the domains of the robust utilities (1.1), as well as of their (sup-)convolutions, are in general empty. Reducing the model space to e.g. L^∞ could solve that problem, but then monotonicity implies that the interior of the domain of the dual function is always empty because it is concentrated on the positive cone of the dual space. Another result in the economic literature is provided by Rigotti, Shannon, and Strzalecki (2008). Here the authors prove the existence of Pareto optimal allocations for variational preferences on the positive cone of L^∞ under the strong assumption of mutual absolute continuity, which is in general not satisfied in our case.

The paper is organized as follows. In Section 2 we introduce probabilistic sophisticated variational preferences and law invariant robust utilities on L^1 and we recall some useful properties. The Pareto optimal allocations problem is studied throughout Section 3 in which we state our main results. Section 4 briefly summarizes our main result on the existence of comonotone Pareto optimal allocations whereas in Section 5 we provide examples illustrating our findings.

2 Setup

Throughout this paper $(\Omega, \mathcal{F}, \mathbb{P})$ is an atom-less probability space, i.e. a probability space supporting a random variable with continuous distribution. All equalities and inequalities between random variables are understood in the $\mathbb{P}$-a.s. sense. Given two random variables X and Y we write $X \overset{d}{=} Y$ to indicate that both random variables have the same distribution under the reference probability measure $\mathbb{P}$. The expectation (if well-defined) of a random variable X under a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ will be denoted by $\mathbb{E}_{\mathbb{Q}}[X]$. In case $\mathbb{Q} = \mathbb{P}$ we also write $\mathbb{E}[X] := \mathbb{E}_{\mathbb{P}}[X]$. We denote by $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$ the space of $\mathbb{P}$-integrable random variables modulo $\mathbb{P}$-almost sure equality.

2.1 Probabilistic Sophisticated Variational Preferences on L^1

Variational preferences were introduced by Maccheroni et al. (2006). In the same paper the authors also study the subclass of probabilistic sophisticated variational preferences. Probabilistic sophistication means that $X \overset{d}{=} Y$ implies $X \sim Y$ in the preference order.

Definition 2.1. (i) *A function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is a* utility function *(on $\mathbb{R}$) if it is concave, right-continuous, increasing,* $\operatorname{dom} u := \{x \in \mathbb{R} \mid u(x) > -\infty\} \neq \emptyset$, *and not constant in the sense that there exist $x, y \in \operatorname{dom} u$ such that $u(x) \neq u(y)$.*

(ii) *Let Δ denote the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which are absolutely continuous and have bounded densities with respect to $\mathbb{P}$, i.e. $\forall A \in \mathcal{F}$, $\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0$, and there exists $K > 0$ such that $\mathbb{P}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} < K\right) = 1$. A set of probability measures $\mathcal{Q} \subset \Delta$ is* closed under densities with the same distribution *if $\mathbb{Q} \in \mathcal{Q}$ and $\widehat{\mathbb{Q}} \in \Delta$ with $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \overset{d}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}$ implies that $\widehat{\mathbb{Q}} \in \mathcal{Q}$.*

(iii) *A decision maker has* probabilistic sophisticated variational preferences $\succeq$ *if for all $X, Y \in L^1$:*

$$X \succeq Y \quad \Leftrightarrow \quad \inf_{\mathbb{Q} \in \mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q})\right) \geq \inf_{\mathbb{Q} \in \mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(Y)] + \alpha(\mathbb{Q})\right)$$

where u is a utility function, $\emptyset \neq \mathcal{Q} \subset \Delta$ is convex and closed under densities with the same distribution, and $\alpha : \mathcal{Q} \to \mathbb{R}$ is a convex and law invariant function on $\mathcal{Q}$ in the sense that $\mathbb{Q}, \widehat{\mathbb{Q}} \in \mathcal{Q}$ with $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \stackrel{d}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}$ implies $\alpha(\widehat{\mathbb{Q}}) = \alpha(\mathbb{Q})$. In addition α satisfies $\inf_{\mathbb{Q}\in\mathcal{Q}} \alpha(\mathbb{Q}) > -\infty$. The numerical representation

$$\mathcal{U} : L^1 \to \mathbb{R} \cup \{-\infty\},\ X \mapsto \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q})\right), \tag{2.1}$$

is the law invariant robust utility *used by the decision maker to quantify the utility of a payoff profile $X \in L^1$.*

The law invariant robust utility $\mathcal{U}$ in (2.1) is, clearly, law invariant[2] ($X \stackrel{d}{=} Y$ implies $\mathcal{U}(X) = \mathcal{U}(Y)$) and posseses some other useful properties which are collected in Lemma 2.3 below. Due to Jensen's inequality for concave functions, the expectations in (2.1) are all well-defined, possibly taking the value $-\infty$. Note that $\mathcal{U}(X) = -\infty$ is possible for some $X \in L^1$. The interpretation is that the payoff profiles with utility $-\infty$ are totally unacceptable.

We remark that what we call a utility function in Definition 2.1 (i) satisfies relatively weak requirements and nests the vast majority of utilities proposed in the economic, finance, and insurance literature (like CARA and CRRA), also including extreme cases such as increasing linear or affine functions (Convex Risk Measures). Notice that by allowing u to take the value $-\infty$ we incorporate the cases when the domain of the utility function u is bounded from below, as e.g. for the power utilities or the logarithmic utilities.

Remark 2.2. A standard approach to modeling preferences in presence of model ambiguity (Knightian uncertainty) is to consider preference orders on the set $\mathcal{M}$ of all Markov kernels $\mathcal{X}(\omega, dy)$ from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (where $\mathcal{B}(\mathbb{R})$ denotes the Borel-σ-algebra) for which there exists a $k > 0$ such that $\mathcal{X}(\omega, [-k, k]) = 1$ for all $\omega \in \Omega$. It can then be shown under some mild additional assumptions that a preference order on $\mathcal{M}$ is in the class of variational preferences if and only if it admits a numerical representation of the form

$$\mathcal{U}(\mathcal{X}) = \inf_{Q\in C} \left(\int \int u(y)\mathcal{X}(\omega, dy)\, dQ(\omega) + \alpha(Q) \right), \quad \mathcal{X} \in \mathcal{M}. \tag{2.2}$$

[2]Since $\mathcal{U}$ is defined via the law invariant penalization α, it is law invariant. This follows as in Föllmer and Schied (2004) Theorem 4.54. Conversely, the dual function in the convex duality sense of a law invariant concave function $\mathcal{U} : L^1 \to \mathbb{R} \cup \{-\infty\}$, which in particular can serve as a penalization α in the sense of (2.1), is always law invariant; again see Föllmer and Schied (2004) Theorem 4.54.

Here u is a utility function on $\mathbb{R}$, and without loss of generality we may assume that C is the set of all *finitely additive* normalized measures, and $\alpha : C \to \mathbb{R} \cup \{\infty\}$ is a convex law invariant function with the additional property of being the minimal function for which $\mathcal{U}$ can be represented as in (2.2). For an axiomatic definition of variational preferences on Markov kernels and the details on their numerical representation (2.2) we refer to Föllmer et al. (2009). Notice that the space of all bounded payoff profiles L^∞ is naturally embedded into the space $\mathcal{M}$ by identifying each $X \in L^\infty$ with the associated kernel $\mathcal{X}(\omega, dy) = \delta_{X(\omega)}(dy)$ where δ_x denotes the Dirac measure given $x \in \mathbb{R}$. The restriction of $\mathcal{U}$ to L^∞ then takes the form

$$\mathcal{U}(X) = \inf_{Q \in C} \left(\int u(X)\, dQ + \alpha(Q) \right), \quad X \in L^\infty, \tag{2.3}$$

which is a robust utility. In case of probabilistic sophistication/law invarinace, using results in Svindland (2010a), it follows that $\mathcal{U}$ is $\sigma(L^\infty, L^\infty)$-upper semi continuous and thus we obtain a representation of $\mathcal{U}$ as an infimum over *σ-additive* probability measures in Δ:

$$\mathcal{U}(X) = \inf_{\mathbb{Q} \in \mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q}) \right), \quad X \in L^\infty, \tag{2.4}$$

where $\mathcal{Q} :=$ dom $\alpha \cap \Delta$. Hence, these preferences on L^∞ are indeed consistent with our definition of probabilistic sophisticated variational preferences on L^1; see Definition 2.1. Moreover, the representation (2.4) shows that the robust utility $\mathcal{U}$ and thus the corresponding preference order is canonically extended from L^∞ to L^1. $\diamondsuit$

2.2 Properties of the Law Invariant Robust Utilities

In the following Lemma 2.3 we collect some well-known properties of law invariant robust utilities on L^1 which we will make frequently use of. The proofs can be found or easily derived from results in for instance Dana (2005), Föllmer and Schied (2004) and Maccheroni et al. (2006). For the sake of completeness we provide a proof in Section A.

Lemma 2.3. *Consider a law invariant robust utility $\mathcal{U}$ as in* (2.1). *Then $\mathcal{U}$ has the following properties:*

(i) *properness: $\mathcal{U} < \infty$ and the domain* $\operatorname{dom} \mathcal{U} := \{X \in L^1 \mid \mathcal{U}(X) > -\infty\}$ *is not empty.*

(ii) *concavity: $\mathcal{U}(\lambda X + (1-\lambda)Y) \geq \lambda \mathcal{U}(X) + (1-\lambda)\mathcal{U}(Y)$ for all $\lambda \in [0, 1]$.*

(iii) *monotonicity:* $X \geq Y$ *implies* $\mathcal{U}(X) \geq \mathcal{U}(Y)$.

(iv) $\succeq_{ssd}$*-monotonicity:* $X \succeq_{ssd} Y$ *implies* $\mathcal{U}(X) \geq \mathcal{U}(Y)$*, where*

$$X \succeq_{ssd} Y \quad \Leftrightarrow \quad \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)], \text{ for all utility functions } u : \mathbb{R} \to \mathbb{R},$$

is the second order stochastic dominance order.

(v) *upper semi-continuity: If* $(X_n) \subset L^1$ *converges to* $X \in L^1$ *(with respect to* $\|\cdot\|_1 := \mathbb{E}[|\cdot|]$*), then* $\mathcal{U}(X) \geq \limsup_{n\to\infty} \mathcal{U}(X_n)$.

Probabilistic sophisticated variational preferences do not only preserve second order stochastic dominance (Lemma 2.3 (iv)) but consequently also the concave order, i.e. $X \succeq_{co} Y$ implies $\mathcal{U}(X) \geq \mathcal{U}(Y)$, where $\succeq_{co}$ denotes the concave order, that is

$$X \succeq_{co} Y \quad \Leftrightarrow \quad \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)] \text{ for all concave functions } u : \mathbb{R} \to \mathbb{R}. \tag{2.5}$$

Clearly, $X \succeq_{co} Y$ implies $X \succeq_{ssd} Y$. The property of preserving $\succeq_{co}$ is often referred to as Schur concavity of $\mathcal{U}$. Indeed, in case of monotone concave upper semi-continuous functions $\succeq_{ssd}$-monotonicity is equivalent to Schur concavity.

3 Comonotone Pareto Optimal Allocations for Probabilistic Sophisticated Variational Preferences

Consider $n \geq 2$ decision makers with initial endowments $W_i \in L^1$. All decision makers are assumed to have probabilistic sophisticated variational preferences on L^1 and corresponding law invariant robust utilities

$$\mathcal{U}_i(X) = \inf_{\mathbb{Q}\in\mathcal{Q}_i} \left(\mathbb{E}_{\mathbb{Q}}[u_i(X)] + \alpha_i(\mathbb{Q})\right), \quad X \in L^1, \ i = 1, \dots, n, \tag{3.1}$$

as defined in (2.1). We assume that $\mathcal{U}_i(W_i) > -\infty$ for all $i = 1, \dots, n$, and let $W := W_1 + \ldots + W_n$ be the aggregate endowment.

Denote by $\mathbb{A}(W)$ the set of all allocations of W, i.e.

$$\mathbb{A}(W) = \{(X_1, \dots, X_n) \in (L^1)^n \mid \sum_{i=1}^n X_i = W\}.$$

Recall that an allocation $(X_1, \ldots, X_n) \in \mathbb{A}(W)$ is *Pareto optimal* if $(Y_1, \ldots, Y_n) \in \mathbb{A}(W)$ and $\mathcal{U}_i(Y_i) \geq \mathcal{U}_i(X_i)$ for $i = 1, \ldots, n$ implies that $\mathcal{U}_i(Y_i) = \mathcal{U}_i(X_i)$ for all $i = 1, \ldots, n$. We are interested in those Pareto optimal allocations which are in addition acceptable in the following sense: Define the set $\mathbb{A}_c(W)$ of all *acceptable* allocations of W as those $(X_1, \ldots, X_n) \in \mathbb{A}(W)$ such that $\mathcal{U}_i(X_i) > -\infty$ and

$$\mathcal{U}_i(X_i) \geq \mathcal{U}_i(W_i) - c_i \tag{3.2}$$

for all $i = 1, \ldots, n$, where $c_i \in \mathbb{R} \cup \{\infty\}$. The condition (3.2) expresses *the individual (rationality) constraint* of decision maker i, specifying which payoff profiles X_i in a new re-allocation of W she is willing to accept. Clearly, $c_i = 0$ represents the (classical) case when the decision maker will not accept any allocation which allots her an endowment which is not at least as good as her initial one. An extreme is $c_i = \infty$ which means that the decision maker is willing to accept any allocation with finite utility.[3] We also allow for situations in which the decision maker is to some bounded extent willing to accept a worsening as compared to her initial endowment ($c_i > 0$), or requires an improvement ($c_i < 0$). Note that if $c_i \geq 0$ for all $i \in \{1, \ldots, n\}$, then the initial allocation $(W_1, \ldots, W_n)$ is acceptable, so in particular $\mathbb{A}_c(W) \neq \emptyset$. However, if some agents demand a strict improvement, it is in general not clear whether the set of acceptable allocations is non-empty. Hence, we make the following assumption.

Assumption 3.1. $\mathbb{A}_c(W) \neq \emptyset$.

3.1 Characterization of Pareto Optimal Allocations

It is well-known that Pareto optima can be characterized as solutions to a weighted sup-convolution optimization problem

$$\text{Maximize } \sum_{i=1}^{n} \lambda_i \mathcal{U}_i(X_i) \quad \text{subject to } (X_1, \ldots, X_n) \in \mathbb{A}(W), \tag{3.3}$$

where $\lambda_i \geq 0$, $i = 1, \ldots, n$, are Negishi weights associated to the Pareto optimal allocation. Notice that we do not require acceptability in (3.3). However, replacing $\mathbb{A}(W)$ by $\mathbb{A}_c(W)$ in (3.3) characterizes acceptable Pareto optima as solutions to

$$\text{Maximize } \sum_{i=1}^{n} \lambda_i \mathcal{U}_i(X_i) \quad \text{subject to } (X_1, \ldots, X_n) \in \mathbb{A}_c(W) \tag{3.4}$$

[3] $c_i = \infty$ is understood as the restriction $\mathcal{U}_i(X_i) \geq \mathcal{U}_i(W_i) - \infty := -\infty$ being redundant.

where again $\lambda_i \geq 0$, $i = 1, \ldots, n$. The relation of (3.3) and (3.4) to (acceptable) Pareto optima is given in the following proposition.

Proposition 3.2. *If* $(X_1, \ldots, X_n) \in \mathbb{A}(W)\,(\in \mathbb{A}_c(W))$ *is Pareto optimal (and acceptable), then there exist weights* $\lambda_i \geq 0$, $i = 1, \ldots, n$, *not all equal to zero, such that the allocation* $(X_1, \ldots, X_n)$ *solves* (3.3) *(or* (3.4)*) with these weights. Conversely, if* $(X_1, \ldots, X_n)$ *solves* (3.3) *(or* (3.4)*) for some strictly positive weights* $\lambda_i > 0$, $i = 1, \ldots, n$, *then* $(X_1, \ldots, X_n)$ *is Pareto optimal (and acceptable).*

Let us briefly comment on the differences between the problems (3.3) and (3.4). It is easily verified that any solution $(X_1, \ldots, X_n)$ to either (3.3) or (3.4) for some strictly positive weights $\lambda_i > 0$ is Pareto optimal. In case of (3.4) this follows from the fact that any other allocation $(Y_1, \ldots, Y_n)$ with $\mathcal{U}_i(Y_i) \geq \mathcal{U}_i(X_i)$ for all i would have to satisfy $(Y_1, \ldots, Y_n) \in \mathbb{A}_c(W)$ too. However, the solutions to (3.4) are *acceptable* Pareto optima. On the other hand, any acceptable Pareto optimum $(X_1, \ldots, X_n)$ can be characterized as a solution to both (3.3) and (3.4). Clearly, for any Negishi weights $(\lambda_1, \ldots, \lambda_n)$ associated to $(X_1, \ldots, X_n)$ via (3.3), $(X_1, \ldots, X_n)$ also solves (3.4) with the same weights because $\mathbb{A}_c(W) \subset \mathbb{A}(W)$. But in general the converse is not true. The set of weights for which $(X_1, \ldots, X_n)$ solves (3.4) does in general depend on the individual constraints c_i. This set increases as the set $\mathbb{A}_c(W)$ decreases[4], and is thus always a superset of the set of Negishi weights associated to $(X_1, \ldots, X_n)$ via (3.3). This latter set corresponds to the limiting case where $c_i = \infty$ for all i and is valid for and therefore independent of any possible individual constraints. Also notice that for given weights $(\lambda_1, \ldots, \lambda_n)$ the solution to problem (3.4) may depend on the individual constraints in the sense that changing the constraints gives a different or no solution.[5] When proving the existence of acceptable Pareto optima later on, we will need both (3.3) and (3.4). More precisely, in Theorems 3.8 and 3.9 we will show that there is an invariant (under all possible individual constraints) set of Negishi weights for which (3.4) (and thus also the limiting case (3.3)) always admits a comonotone solution. This set of Negishi weights however is given by the limiting case (3.3).

[4]See Example 5.4.

[5]The individual constraints may for instance imply that the closed set $\mathbb{A}_c(W)$ is essentially bounded. Let us think of it as compact. Then (3.4) will allow for a solution for any positive weights whereas the optimization (3.3) over all allocations only works for certain weights.

In the characterization of Pareto optima given in Proposition 3.2 the associated Negishi weights in (3.3) or (3.4) can in general only be shown to be non-negative. So we may have $\lambda_j = 0$ for some j.[6] As we will see in Section 3.3, our techniques allow us to prove the existence of solutions to (3.4) only in case all Negishi weights are strictly positive, i.e. $\lambda_i > 0$ for all $i \in \{1, \ldots, n\}$ (see Theorems 3.8 and 3.9). So the question may arise what kind of Pareto optimal allocations we are neglecting. The following Lemma 3.3 shows that very often there are strictly positive Negishi weights for any Pareto optimum.

Lemma 3.3. *Let $(X_1, \ldots, X_n) \in \mathbb{A}(W)$ be Pareto optimal and suppose that there is some $\epsilon > 0$ such that for any $i = 1, \ldots, n$ it holds that $\mathcal{U}_i(X_i - \epsilon) \in \operatorname{dom} \mathcal{U}_i$ and that*

$$(0, \epsilon) \ni m \mapsto \mathcal{U}_i(X_i + m) \quad \textit{is strictly increasing.}$$

Then all Negishi weights associated to $(X_1, \ldots, X_n)$ via (3.3) are strictly positive.

Proof. Let $(X_1, \ldots, X_n) \in \mathbb{A}(W)$ be Pareto optimal and consider any set of Negishi weights $(\lambda_1, \ldots, \lambda_n)$ associated to $(X_1, \ldots, X_n)$ via (3.3). Assume that for some $j \in \{1, \ldots, n\}$ we have $\lambda_j = 0$ and pick some $k \in \{1, \ldots, n\}$ such that $\lambda_k > 0$ (Proposition 3.2). Consider the allocation $(\widetilde{X}_1, \ldots, \widetilde{X}_n) \in \mathbb{A}(W)$ where $\widetilde{X}_i = X_i$ for all $i \neq j, k$ and $\widetilde{X}_j = X_j - \delta$ and $\widetilde{X}_k = X_k + \delta$ for some $\delta \in (0, \epsilon)$. As $\lambda_j = 0$, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i \mathcal{U}_i(\widetilde{X}_i) &= \sum_{i=1; i \neq k,j}^{n} \lambda_i \mathcal{U}_i(X_i) + \lambda_k \mathcal{U}_k(X_k + \delta) \\ &> \sum_{i=1; i \neq k,j}^{n} \lambda_i \mathcal{U}_i(X_i) + \lambda_k \mathcal{U}_k(X_k) \quad = \quad \sum_{k=1}^{n} \lambda_i \mathcal{U}_i(X_i). \end{aligned} \tag{3.5}$$

which is a contradiction. □

An immediate consequence of Lemma 3.3 and Proposition 3.2 is the following:

Corollary 3.4. *Suppose that the $\mathcal{U}_i$ satisfy the following conditions for all $i = 1, \ldots, n$:*

- $\operatorname{dom} \mathcal{U}_i = \operatorname{dom} \mathcal{U}_i + \mathbb{R}$,
- $\mathbb{R} \ni m \mapsto \mathcal{U}_i(X + m)$ *is strictly increasing for all* $X \in \operatorname{dom} \mathcal{U}_i$.

[6] Note that $\lambda_j = 0$ implies that the decision maker j is not considered in the social welfare maximization problem (3.4).

Then $(X_1,\dots,X_n)\in\mathbb{A}(W)$ *is Pareto optimal if and only if it solves* (3.3) *for some strictly positive weights* $\lambda_i>0$, $i=1\dots,n$.

3.2 Comonotone Pareto Optimal Allocations

When proving the existence of solutions to (3.4), and thus of acceptable Pareto optimal allocations, we will profit from the fact that due to the $\succeq_{ssd}$-monotonicity of the $\mathcal{U}_i$ we may restrict our attention to the set of comonotone acceptable allocations defined next; see proofs of Theorems 3.8 and 3.9 in Appendix C.

Definition 3.5. *We denote by* CF *the set of all n-tuples* $(f_1,\dots,f_n)$ *of increasing functions* $f_i:\mathbb{R}\to\mathbb{R}$, $i=1,\dots,n$, *such that* $\sum_{i=1}^n f_i=\mathrm{Id}_{\mathbb{R}}$. *These functions* f_i *are necessarily 1-Lipschitz-continuous. An allocation* $(Y_1,\dots,Y_n)\in\mathbb{A}(W)$, *is* comonotone *if there exists* $(f_i)_{i=1}^n\in\mathrm{CF}$ *such that* $Y_i=f_i(W)$ *for all* $i=1,\dots,n$.

In particular, it is known that if there exists a Pareto optimal allocation in our setting, then there is also a comonotone one. This follows from the following Proposition.

Proposition 3.6. *For any* $(X_1,\dots,X_n)\in\mathbb{A}_c(W)$ *there exists a comonotone acceptable allocation* $(Y_1,\dots,Y_n)\in\mathbb{A}_c(W)$ *such that* $Y_i\succeq_{co}X_i$ *(and thus* $Y_i\succeq_{ssd}X_i$*) for all* $i=1,\dots,n$.

Proof. First of all we recall a result which is often referred to as comonotone improvement: for any allocation $(X_1,\dots,X_n)\in\mathbb{A}(W)$ there exists a comonotone allocation $(Y_1,\dots,Y_n)\in\mathbb{A}(W)$ such that $Y_i\succeq_{co}X_i$ for all $i=1,\dots,n$. The proof for the case when W is supported by a finite set goes back to Landsberger and Meilijson (1994). This has been further extended to aggregate endowments $W\in L^1$ by Filipović and Svindland (2008), Dana and Meilijson (2011), and Ludkovski and Rüschendorf (2008). Finally, by $\succeq_{ssd}$-monotonicity of the $\mathcal{U}_i$ (see Lemma 2.3 (iv)) it follows that if $(X_1,\dots,X_n)\in\mathbb{A}_c(W)$, then any comonotone improvement $(Y_1,\dots,Y_n)$ of $(X_1,\dots,X_n)$ is acceptable as well, i.e. $(Y_1,\dots,Y_n)\in\mathbb{A}_c(W)$. □

Remark 3.7. The comonotone allocations have another desirable property. Suppose that $W\in L^p\subset L^1$ for some $p\in[1,\infty]$, and let $(f_i(W))_{i=1}^n$ be a comonotone allocation of W, i.e. $(f_i)_{i=1}^n\in\mathrm{CF}$. Then, by the 1-Lipschitz continuity of the f_i, it is easily verified that $(f_i(W))_{i=1}^n\in(L^p)^n$. Hence, any comonotone Pareto optimal allocation will posses the same integrability/boundedness properties as the aggregate endowment W. In that sense, further restricting the set of acceptable allocations by imposing additional integrability or even boundedness constraints in the formulation of problem (3.4) (or (3.3)) will yield the same comonotone solutions as solving the unrestricted problem. $\Diamond$

3.3 Main Results

Let $s_i := \inf \operatorname{dom} u_i \in \mathbb{R} \cup \{-\infty\}$, $i = 1, \dots, n$, and $d_H^i := \lim_{x \to s_i} u_i'(x)$ (which may be ∞) and $d_L^i := \lim_{x \to \infty} u_i'(x) (\geq 0)$ where u' denotes the right-hand-derivative of u. Finally, let $N \subset \{1, \dots, n\}$ be the set of all indices such that $d_H^i = d_L^i$[7], and $M := \{1, \dots, n\} \setminus N$ the set of all indices such that $d_L^i < d_H^i$. Note that $N = \emptyset$ or $M = \emptyset$ is possible.

In the following theorems we specify a non-empty set of Negishi weights $(\lambda_1, \dots, \lambda_n)$ for which the associated optimization problem (3.4) admits a solution.

Theorem 3.8. *Suppose that $s_i > -\infty$, $i = 1, \dots, n$. Then for every set of strictly positive weights $\lambda_i > 0$, $i = 1, \dots, n$,* (3.4) *admits a comonotone solution.*

Theorem 3.9. *Consider the following bounds on the weights λ_i, $i = 1, \dots, n$, and some $\delta > 0$:*

$$\begin{aligned} \lambda_i &= \frac{\delta}{d_H^i} && \text{for all } i \in N, \\ \lambda_i d_L^i < \delta &< \lambda_i d_H^i && \text{for all } i \in M. \end{aligned} \tag{3.6}$$

We consider two cases:

(i) *Suppose that $N = \emptyset$ or $|N| = 1$. Then* (3.4) *admits a comonotone solution for every set of weights $\lambda_i > 0$, $i = 1, \dots, n$, satisfying the constraints* (3.6).

(ii) *Suppose that $|N| \geq 2$. If $s_i = -\infty$ for all $i \in N$, and $\mathcal{U}_j(-W^-) > -\infty$ for all $j \in N$ such that $c_j \in \mathbb{R}$, then* (3.4) *admits a comonotone solution for every set of weights $\lambda_i > 0$, $i = 1, \dots, n$, satisfying the constraints* (3.6). *In particular, if $|N| = n$, the solutions are given up to a reallocation of cash. That is, if $(X_1, \dots, X_n)$ is a solution to* (3.4) *for some given weights, then also $(X_1 + m_1, \dots, X_n + m_n)$ is a solution to* (3.4) *with that weights whenever the numbers $m_i \in \mathbb{R}$ satisfy $\sum_{i=1}^n m_i = 0$ and $(X_1 + m_1, \dots, X_n + m_n) \in \mathbb{A}_c(W)$.*

[7] When $d_L^i = d_H^i = d^i$ and $s_i = -\infty$ (i.e. $\operatorname{dom} u_i = \mathbb{R}$), then the corresponding robust utility $\mathcal{U}_i$ is cash additive in the sense that $\mathcal{U}_i(X + m) = \mathcal{U}_i(X) + d^i m$ for all $m \in \mathbb{R}$ and $X \in L^1$ and thus corresponds to a convex risk measure (if $d^i = 1$ and multiplied by -1).

We remark that when $N = \emptyset$, the two conditions in (3.6) reduce to the last one. Note that Dana (2011) derives similar bounds on the weights λ_i given in (3.6) in her setting. The proofs of Theorems 3.8 and 3.9 are provided in Appendix C. Theorem 3.9 is discussed in several examples in Section 5.2 in which we illustrate that if we drop one of the conditions on the weights stated in (3.6), we cannot in general expect the existence of solutions to (3.3) any longer.

If $d^i_H = \infty$ and $d^i_L = 0$ for all $i = 1, \ldots, n$, then the bounds in (3.6) are void. Hence, Theorem 3.9 (i) implies the following Corollary.

Corollary 3.10. *If $d^i_H = \infty$ and $d^i_L = 0$ for all $i = 1, \ldots, n$ (as for instance when the decision makers' utilities u_i are chosen amongst the exponential, logarithmic or power utilities), then there exists a comonotone solution to* (3.4) *for any set of strictly positive weights $\lambda_i > 0$, $i = 1, \ldots, n$.*

As regards the uniqueness of Pareto optimal allocations, we have the following result. To this end we recall that a function $\mathcal{U} : L^1 \to \mathbb{R} \cup \{-\infty\}$ is strictly concave if $\mathcal{U}(\lambda X + (1-\lambda)Y) > \lambda\mathcal{U}(X) + (1-\lambda)\mathcal{U}(Y)$ whenever $\lambda \in (0, 1)$ and $X \neq Y$.

Corollary 3.11. *Suppose that under the conditions stated in Theorem 3.8 (and Theorem 3.9, respectively) $(n-1)$ among the n law invariant robust utilities $\mathcal{U}_i$ are strictly concave. Then, for any given set of weights $\lambda_i > 0$, $i = 1, \ldots, n$, (and satisfying the bounds (3.6), respectively), the Pareto optimal allocation which solves the optimization problem* (3.4) *associated to* $(\lambda_1, \ldots, \lambda_n)$ *is unique and comonotone.*

Proof. For any given vector of positive weights $(\lambda_1, \ldots, \lambda_n)$ the set of solutions to the associated optimization problem (3.4) is convex because the $\mathcal{U}_i$ are concave. This together with the strict concavity of $(n-1)$ robust utilities implies that the solution to (3.4), if it exists, is unique. $\square$

4 Conclusion: The Existence Theorem

Proposition 3.2 and Theorems 3.8 and 3.9 immediately imply the existence of acceptable comonotone Pareto optimal allocations. This is summarized in the following theorem.

Theorem 4.1. (i) *If $s_i > -\infty$ for all $i = 1, \ldots, n$, then there exists an acceptable comonotone Pareto optimal allocation.*

(ii) *If* $|N| \leq 1$, *then there exists an acceptable comonotone Pareto optimal allocation.*

(iii) *Suppose that* $|N| \geq 2$. *If* $s_i = -\infty$ *for all* $i \in N$, *and* $\mathcal{U}_j(-W^-) > -\infty$ *for all* $j \in N$ *such that* $c_j \in \mathbb{R}$, *then there exists an acceptable comonotone Pareto optimal allocation.*

Moreover, in the situation of (iii), *if* $|N| = n$, *then the Pareto optimal allocations have the property of being up to a reallocation of cash. That is, if* $(X_1, \ldots, X_n)$ *is a Pareto optimal allocation of* W, *then also* $(X_1 + m_1, \ldots, X_n + m_n)$ *is a Pareto optimal allocation whenever the numbers* $m_i \in \mathbb{R}$ *satisfy* $\sum_{i=1}^n m_i = 0$ *and* $(X_1 + m_1, \ldots, X_n + m_n) \in \mathbb{A}_c(W)$.

5 Examples

5.1 Yaari Preferences Versus Multiplier Preferences

Suppose that decision maker 1 has Yaari (1987) type preferences represented by the robust utility

$$\mathcal{U}_1(X) = \frac{1}{\alpha} \int_0^\alpha q_X(s)\, ds, \quad X \in L^1, \tag{5.1}$$

where $\alpha \in (0,1)$ and $q_X(s) := \inf\{x : \mathbb{P}(X \leq x) \geq s\}$ with $s \in (0,1)$ being the quantile function of X. Note that $-\mathcal{U}_1$ is the well-known Average Value at Risk (AVaR), that is

$$\mathcal{U}_1(X) = -\mathrm{AVaR}_\alpha(X) = \min_{\mathbb{Q} \in \mathcal{Q}_1} \mathbb{E}_\mathbb{Q}[X] = \min_{\mathbb{Q} \in \mathcal{Q}_1} \int_0^1 q_X(s) q_{\frac{d\mathbb{Q}}{d\mathbb{P}}}(1-s)\, ds$$

where $\mathcal{Q}_1 := \{\mathbb{Q} \ll \mathbb{P} \mid \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\alpha}\}$; see e.g. Föllmer and Schied (2004) Theorems 4.47 and 4.54. As for decision maker 2, her probabilistic sophisticated variational preferences are represented by a law invariant robust utility $\mathcal{U}_2$ as in (2.1) satisfying some additional properties which are listed in Proposition 5.2. Examples of such robust utilities are multiplier preferences or semi-deviation utilities with any strictly increasing utility $u_2 : \mathbb{R} \to \mathbb{R}$.

Our case study is inspired by and extends an example in Jouini et al. (2008), Proposition 3.2. In Jouini et al. (2008) $\mathcal{U}_2$ is required to be a monetary utility, i.e. cash additive ($u_2 \equiv \mathrm{Id}_\mathbb{R}$). Proposition 5.2 below shows that the functional form of the Pareto optimal allocations obtained in Jouini et al. (2008) stays the same also when we allow for a larger class of preferences for the second agent. The proof of Proposition 5.2 is essentially the same as in Jouini et al. (2008).

For the sake of completeness we provide it in Section D. Before giving the result, we recall the definition of strict risk aversion conditional on lower-tail events.

Definition 5.1. (i) *Let* $X \in L^1$ *and* $A \in \mathcal{F}$ *with* $\mathbb{P}(A) > 0$*. The set* A *is a* lower tail-event for X if $\operatorname{ess\,inf}_A X < \operatorname{ess\,sup}_A X \le \operatorname{ess\,inf}_{A^c} X$ where $\operatorname{ess\,inf}_A X := \sup\{m \in \mathbb{R} \mid \mathbb{P}(X > m \mid A) = 1\}$ ($\sup \emptyset := -\infty$) and $\operatorname{ess\,sup}_A X := \inf\{m \in \mathbb{R} \mid \mathbb{P}(X \le m \mid A) = 1\}$ ($\inf \emptyset := \infty$).

(ii) A function $\mathcal{U} : L^1 \to \mathbb{R} \cup \{-\infty\}$ is *strictly risk averse conditional on lower tail-events if* $\mathcal{U}(X) < \mathcal{U}(X 1_{A^c} + \mathbb{E}[X \mid A] 1_A)$ *for every* $X \in \operatorname{dom} \mathcal{U}$ *and any set* A *which is a lower tail-event for* X*.*

Proposition 5.2. *Let decision maker 1 be represented by* (5.1) *and decision maker 2 by a law invariant robust utility*

$$\mathcal{U}_2(X) = \inf_{\mathbb{Q} \in \mathcal{Q}} (\mathbb{E}_{\mathbb{Q}}[u_2(X)] + \alpha_2(\mathbb{Q}))$$

as in (2.1) *with the following additional properties*

- $\operatorname{dom} \mathcal{U}_2 = \operatorname{dom} \mathcal{U}_2 + \mathbb{R}$*,*
- $\mathcal{U}_2$ *is strictly monotone, i.e.* $X \ge Y$ *and* $\mathbb{P}(X > Y) > 0$ *implies* $\mathcal{U}_2(X) > \mathcal{U}_2(Y)$*,*
- $\mathcal{U}_2$ *is strictly risk averse conditional on lower-tail events.*

Given any initial endowments $W_i \in \operatorname{dom} \mathcal{U}_i$*,* $i = 1, 2$*, and* $c_1, c_2 \in \mathbb{R} \cup \{\infty\}$ *such that* $\mathbb{A}_c(W) \neq \emptyset$*, the comonotone Pareto optimal allocations of the aggregate endowment* $W = W_1 + W_2$ *are of the following form*

$$(X_1, X_2) = (-(W - l)_- + k, W \vee l - k) \text{ where } l \in \mathbb{R} \cup \{-\infty\} \text{ and } k \in \mathbb{R} \text{ with } k \ge \mathcal{U}_1(W_1) - \mathcal{U}_1(-(W - l)_-) - c_1. \tag{5.2}$$

If $\mathcal{U}_2$ is in addition strictly concave, then according to Corollary 3.11 all Pareto optimal allocations are comonotone and of shape (5.2).

5.2 Examples Illustrating the Bounds (3.6)

Since our examples will only involve two decision makers and as the bounds (3.6) are determined by the limiting case with trivial individual constraints, in the following we give a version of Theorem 3.9 for two decision makers with $c_1 = c_2 = \infty$.

Theorem 5.3. *Suppose $n = 2$. We consider two cases:*

(i) *Suppose that $d_L^i < d_H^i$ for at least one $i \in \{1, 2\}$. If the weights $\lambda_1 > 0$ and $\lambda_2 > 0$ satisfy*

$$\frac{\lambda_2}{\lambda_1} \in \left(\frac{d_L^1}{d_H^2}, \frac{d_H^1}{d_L^2} \right), \tag{5.3}$$

then there exists a comonotone solution to (3.3).[8]

(ii) *Suppose that $d_H^1 = d_L^1$, $d_H^2 = d_L^2$, $s_1 = s_2 = -\infty$. If $\lambda_i = \frac{\delta}{d_H^i}$, $i = 1, 2$, for some $\delta > 0$, then there exists a comonotone solution to* (3.3).

Example 5.4. Illustration of Theorem 5.3 (i) and of the differences between (3.3) and (3.4): Let $\mathcal{U}_1(X) = \mathbb{E}[d_L X^+ - d_H X^-]$ and $\mathcal{U}_2(X) := \mathbb{E}[X]$, $X \in L^1$, where $0 < d_L < 1 < d_H$. Suppose that $W \geq 0$. If $\frac{\lambda_2}{\lambda_1} < d_L$, consider the allocations $(W + k, -k) \in \mathbb{A}(W)$, $k \in \mathbb{R}_+$. Then

$$\begin{aligned} \lambda_1 \mathcal{U}_1(W + k) + \lambda_2 \mathcal{U}_2(-k) &= \lambda_1 \mathbb{E}[d_L(W + k)] - \lambda_2 k \\ &= \lambda_1 \mathbb{E}[d_L W] + (\lambda_1 d_L - \lambda_2)k \to \infty \quad \text{for } k \to \infty \end{aligned}$$

because $\lambda_1 d_L - \lambda_2 > 0$. Hence, (3.3) admits no solution. Analogously, (3.3) admits no solution in case $\frac{\lambda_2}{\lambda_1} > d_H$. However for $\frac{\lambda_2}{\lambda_1} \in [d_L, d_H]$ the comonotone allocation $(0, W)$ solves (3.3). Now suppose for simplicity that $W = 0$ and that $c_i \in \mathbb{R}$ for $i = 1, 2$. By Proposition 3.6 we only need to consider the comonotone allocations $(-k, k)$, $k \in \mathbb{R}$, when solving (3.4). Since acceptability implies that k must be bounded (for instance $k \leq c_1/d_H$ if $k \geq 0$), (3.4) admits a solution for any $\lambda_i > 0$, $i = 1, 2$. This solution obviously depends on how far we can push k in an optimal direction, hence on the constraints c_i. Apparently, all allocations $(-k, k)$, $k \in \mathbb{R}$, are Pareto optimal and also acceptable for some c_i. Also note that if $c_1 = \infty$ and $c_2 \in \mathbb{R}$, then (3.4) admits a solution if and only if $\frac{\lambda_2}{\lambda_1} \leq d_H$ while if $c_1 \in \mathbb{R}$ and $c_2 = \infty$ there is a solution to (3.4) if and only if $\frac{\lambda_2}{\lambda_1} \geq d_L$. ◊

Notice that in Example 5.4 there exists a solution to (3.3) even if $\frac{\lambda_2}{\lambda_1}$ equals one of the bounds given in (5.3). However, if the robust utility of one of the decision makers is strictly concave, then often (3.3) admits no solution at the interval bounds of (5.3) either:

[8] Here $\frac{0}{\infty} := 0$ and $\frac{\infty}{0} := \infty$.

Example 5.5. Illustration of Theorem 5.3 (i): Let $W = 0$ and consider two utility functions u_1 and u_2 with $d_H^i < \infty$ and $d_L^i > 0$, $i = 1, 2$. Let $\mathcal{U}_i(X) := \mathbb{E}[u_i(X)]$, $X \in L^1$. Suppose that $u_2'(a) = d_L^2 + \frac{1}{2\sqrt{a}}$ for large $a > 0$ (in particular u_2' does not attain d_L^2) and that $u_1'(-a) = d_H^1$ for $a > 0$. Then for $\frac{\lambda_2}{\lambda_1} = \frac{d_H^1}{d_L^2}$ and some constant $k > 0$ we have

$$\sup_{a\in\mathbb{R}} \lambda_1 \mathcal{U}_1(-a) + \lambda_2 \mathcal{U}_2(a) = \sup_{a\in\mathbb{R}} \lambda_1 u_1(-a) + \lambda_2 u_2(a) \geq \sup_{a\geq 0} \lambda_2 \sqrt{a} + k = \infty.$$

Similar arguments show that in general we cannot expect the existence of solutions to (3.3) in case $\frac{\lambda_2}{\lambda_1}$ equals the lower bound $\frac{d_L^1}{d_H^2}$ either. ◊

Example 5.6. Illustration of Theorem 5.3 (ii): Suppose that $u_1(x) = d^1 x$ and $u_2(x) = d^2 x$ for some $d^1, d^2 > 0$. If $\frac{\lambda_2}{\lambda_1} \neq \frac{d^1}{d^2}$, then $(\lambda_1 d^1 - \lambda_2 d^2) \neq 0$ and considering the allocations of type $(W + k, -k) \in \mathbb{A}(W)$ for some constant k yields

$$\sup_{k\in\mathbb{R}} \lambda_1 \mathcal{U}_1(W + k) + \lambda_2 \mathcal{U}_2(-k) = \lambda_1 \mathcal{U}_1(W) + \lambda_2 \mathcal{U}_2(0) + \sup_{k\in\mathbb{R}} (\lambda_1 d^1 - \lambda_2 d^2) k = \infty.$$

Hence, (3.3) admits no solution. ◊

5.3 (Non-)Existence of Pareto Optima in case the Decision Makers are not Probabilistic Sophisticated with respect to the same Reference Measure

Consider two decision makers with expected utility choice criteria $\mathcal{U}_1(X) = \mathbb{E}_{\mathbb{P}}[u_1(X)]$ and $\mathcal{U}_2(X) = \mathbb{E}_{\widetilde{\mathbb{P}}}[u_2(X)]$, $X \in L^\infty$, where the probability measures $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent but not equal, and $u_i : \mathbb{R} \to \mathbb{R}$ are utility functions with $u_i(0) = 0$, $i = 1, 2$. Notice that the decision makers are probabilistic sophisticated in different worlds, i.e. with respect to different reference probabilities. Hence there is $\epsilon > 0$ such that the sets $A := \{\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \geq 1 + \epsilon\}$ and $B := \{\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \leq 1 - \epsilon\}$ have positive probability (under $\mathbb{P}$). Suppose that $c_1 = c_2 = \infty$ and that $(Y_1, Y_2) \in \mathcal{A}(0)$ is a Pareto optimal allocation. According to Lemma 3.3 - which does not rely on probabilistic sophistication - if the u_i are 'nice', then (Y_1, Y_2) is the solution to

$$\lambda_1 \mathcal{U}_1(Y_1) + \lambda_2 \mathcal{U}_2(Y_2) = \sup_{Y\in L^1} \lambda_1 \mathcal{U}_1(-Y) + \lambda_2 \mathcal{U}_2(Y)$$

for some weights $\lambda_i > 0$, $i = 1, 2$. However,

$$\begin{aligned}
\lambda_1 \mathcal{U}_1(Y_1) + \lambda_2 \mathcal{U}_2(Y_2) &\geq \sup_{t>0} \lambda_1 \mathbb{E}_{\mathbb{P}}[u_1(-t1_A)] + \lambda_2 \mathbb{E}_{\mathbb{P}}\left[u_2(t1_A)\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right] \\
&\geq \sup_{t>0} (\lambda_1 u_1(-t) + \lambda_2 (1+\epsilon) u_2(t)) \mathbb{P}(A) \qquad (5.4)
\end{aligned}$$

and similarly

$$
\begin{aligned}
\lambda_1\mathcal{U}_1(Y_1)+\lambda_2\mathcal{U}_2(Y_2) &\geq \sup_{t>0}\lambda_1\mathbb{E}_{\mathbb{P}}[u_1(t1_B)]+\lambda_2\mathbb{E}_{\mathbb{P}}\left[u_2(-t1_B)\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right]\\
&\geq \sup_{t>0}(\lambda_1 u_1(t)+\lambda_2(1-\epsilon)u_2(-t))\mathbb{P}(B). \qquad (5.5)
\end{aligned}
$$

Now it is easy to construct situations in which (5.4) or (5.5) explode and thus contradict the Pareto optimality of (Y_1, Y_2). If for instance $d^i_L > 0$ and $d^i_H < \infty$ for $i = 1, 2$, then

$$(5.4) \geq \sup_{t>0}(\lambda_2(1+\epsilon)d^2_L - \lambda_1 d^1_H)\mathbb{P}(A)t \qquad (5.6)$$

and

$$(5.5) \geq \sup_{t>0}(\lambda_1 d^1_L - \lambda_2(1-\epsilon)d^2_H)\mathbb{P}(B)t. \qquad (5.7)$$

(5.6) or (5.7) explode apart from the case[9]

$$\frac{(1+\epsilon)d^2_L}{d^1_H} \leq \frac{\lambda_1}{\lambda_2} \leq \frac{(1-\epsilon)d^2_H}{d^1_L}. \qquad (5.8)$$

So in particular we must have that

$$\frac{(1+\epsilon)d^2_L}{d^1_H} \leq \frac{(1-\epsilon)d^2_H}{d^1_L}. \qquad (5.9)$$

However, if e.g. $1-\epsilon/2 < d^i_L \leq d^i_H < 1+\epsilon/2$, $i = 1, 2$, then (5.9) is not satisfied which in the end contradicts the Pareto optimality of (Y_1, Y_2).

But there are also cases in which Pareto optimal allocations exists. Suppose that there are constants $K > 1 > k > 0$ such that $k \leq d\widetilde{\mathbb{P}}/d\mathbb{P} \leq K$ and suppose that u_2 is such that $\frac{d^2_H}{d^2_L} \geq \frac{K}{k}$. The latter condition implies that u_2 is concave enough in the sense that there is a utility function $\widetilde{u}_2$ which dominates $v(x) := ku_2(x)1_{\{x<0\}} + Ku_2(x)1_{\{x\geq 0\}}$, $x \in \mathbb{R}$. Indeed, as v is concave on the half axises $x < 0$ and $x \geq 0$ respectively, and by the requirement on the concavity of u_2, there are $x_0 < 0$ and $x_1 > 0$ and a joint constant $L > 0$ such that $ku_2'(x_0) > Ku_2'(x_1)$ and $x \mapsto ku_2'(x_0)x + L$ dominates v on $x < 0$ and $x \mapsto Ku_2'(x_1)x + L$ dominates v on $x \geq 0$, so $\widetilde{u}_2(x) := Ku_2'(x_1)x^+ - ku_2'(x_0)x^- + L$ does the job. Consequently for all $W \in L^1$ and for any

[9]Note the similarity between the bounds in (5.8) and the bounds in (5.3).

$\lambda_1, \lambda_2 > 0$ we have

$$\sup_{(X_1,X_2)\in\mathcal{A}(W)} \lambda_1\mathcal{U}_1(X_1) + \lambda_2\mathcal{U}_2(X_2) = \sup_{(X_1,X_2)\in\mathcal{A}(W)} \lambda_1\mathcal{U}_1(X_1) + \lambda_2\mathbb{E}_{\mathbb{P}}\left[u_2(X_2)\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right] \tag{5.10}$$

$$\leq \sup_{(X_1,X_2)\in\mathcal{A}(W)} \lambda_1\mathcal{U}_1(X_1) + \lambda_2\mathbb{E}_{\mathbb{P}}\left[v(X_2)\right]$$

$$\leq \sup_{(X_1,X_2)\in\mathcal{A}(W)} \lambda_1\mathcal{U}_1(X_1) + \lambda_2\mathbb{E}_{\mathbb{P}}\left[\widetilde{u}_2(X_2)\right]. \tag{5.11}$$

Since $\widetilde{\mathcal{U}}_2(\cdot) := \mathbb{E}_{\mathbb{P}}\left[\widetilde{u}_2(\cdot)\right]$ is of type (2.1) we know that (5.11) is bounded and admits a comonotone solution if λ_1, λ_2 satisfy the conditions stated in Theorem 5.3. Now it is easy to construct situations in which the above inequalities are indeed equalities, and solutions to (5.11) thus coincide with solutions to the left hand side of (5.10). Suppose for instance that $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = k1_B + K1_{B^c}$ for some set $B \in \mathcal{F}$ with $\mathbb{P}(B) = \frac{K-1}{K-k}$, and that $u_2(x) = d_L^2 x^+ - d_H^2 x^-$. Then we may choose $\widetilde{u}_2(x) = v(x) = Kd_L^2 x^+ - kd_H^2 x^-$. Depending on u_1, any situation in which the extreme allocation $(W, 0)$ is a solution to (5.11) (like in Example 5.4), this allocation obviously also solves the left hand side of (5.10). Furthermore, whenever there is a solution (Y_1, Y_2) to (5.11) such that $B = \{Y_2 \leq 0\}$, the allocation (Y_1, Y_2) solves (5.10) too, because then $u_2(Y_2)\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \widetilde{u}_2(Y_2)$. Apparently, if $\widetilde{\mathbb{P}} \neq \mathbb{P}$, the existence of Pareto optima depends on parameters such as the deviation of the measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ from each other relative to the concavity of the utilities u_i. Hence, if the decision makers are not probabilistic sophisticated with respect to the same reference probability measure, then existence results like Theorem 4.1 do not hold in general any longer.

A Proof of Lemma 2.3

In the proof of Lemma 2.3 and Theorems 3.8 and 3.9 we will apply the following facts about law invariant monetary utilities which apart from the sign are convex risk measures.

Lemma A.1. *Let*

$$\mathcal{U}(X) = \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q})\right), \quad X \in L^1,$$

be a law invariant robust utility as (2.1)*. Define*

$$U(X) := \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}\left[X\right] + \alpha(\mathbb{Q})\right), \quad X \in L^1, \tag{A.1}$$

so that $\mathcal{U}(\cdot) = U(u(\cdot))$. Then, U is a proper, law invariant, $\succeq_{ssd}$-monotone, upper semi-continuous, monotone, cash additive ($U(X+m) = U(X) + m$ for all $m \in \mathbb{R}$), and concave function. Moreover, we have that

$$U(X) \leq \mathbb{E}[X] + U(0) \quad \textit{for all } X \in L^1. \tag{A.2}$$

Proof. We give a brief version of the proof since many of the presented arguments are standard and can for instance be found in Föllmer and Schied (2004). Cash additivity and monotonicity are obvious by definition of U and properness follows from $\inf_{\mathbb{Q}\in\mathcal{Q}} \alpha(\mathbb{Q}) > -\infty$. Concavity and upper semi-continuity follow from the fact that U is a point-wise infimum over continuous affine functions. To see that U is law invariant we note that for $X \in L^1$ and $Z \in L^\infty$ we have

$$\sup_{\widetilde{Z} \overset{d}{=} Z} \mathbb{E}[X\widetilde{Z}] = \int_0^1 q_X(s) q_Z(s)\, ds \tag{A.3}$$

where $q_Y(s) := \inf\{x \mid \mathbb{P}(Y \leq x) \geq s\}$ denotes the (left-continuous) quantile function of a random variable Y. The relation (A.3) is a consequence of the (upper) Hardy-Littlewood inequality and some analysis. A proof can be found in Föllmer and Schied (2004), Lemma 4.55, or in a slightly more general version in Svindland (2010b), Lemma C.2. As e.g. in the proof of Föllmer and Schied (2004), Theorem 4.54, by (A.3) and law invariance of $\mathcal{Q}$ and α we obtain that

$$\begin{aligned}
U(X) &= \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} X\right] + \alpha(\mathbb{Q}) \right) = \inf_{\mathbb{Q}\in\mathcal{Q}} \inf_{\widetilde{\mathbb{Q}} \overset{d}{=} \mathbb{Q}} \left(\mathbb{E}\left[\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} X\right] + \alpha(\widetilde{\mathbb{Q}}) \right) \\
&= \inf_{\mathbb{Q}\in\mathcal{Q}} \left(- \left(\sup_{\widetilde{\mathbb{Q}} \overset{d}{=} \mathbb{Q}} \mathbb{E}\left[-\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} X \right] \right) + \alpha(\mathbb{Q}) \right) \\
&= \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\int_0^1 -q_{-X}(s) q_{\frac{d\mathbb{Q}}{d\mathbb{P}}}(s)\, ds + \alpha(\mathbb{Q}) \right) \\
&= \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\int_0^1 q_X(1-s) q_{\frac{d\mathbb{Q}}{d\mathbb{P}}}(s)\, ds + \alpha(\mathbb{Q}) \right)
\end{aligned}$$

in which, with some abuse of notation, we write $\widetilde{\mathbb{Q}} \overset{d}{=} \mathbb{Q}$ instead of $\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} \overset{d}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}$ and use the fact that $q_X(1-s) = -q_{-X}(s)$ for almost all $s \in (0,1)$. Clearly, the last term in the equations only depends on the distribution of X under $\mathbb{P}$. Hence, the law invariance follows. According to Dana (2005), Theorem 4.1, an upper semi-continuous monotone concave function is law invariant if and only if it is $\succeq_{ssd}$-monotone as defined in Lemma 2.3 (iv). The final statement follows from the fact that $\mathbb{E}[X] \succeq_{ssd} X$ by Jensen's inequality for concave functions. Thus $\succeq_{ssd}$-monotonicity and cash additivity imply that $U(X) \leq U(\mathbb{E}[X]) = \mathbb{E}[X] + U(0)$. □

Proof of Lemma 2.3. Let

$$\mathcal{U}(X) := \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q})\right), \quad X \in L^1,$$

as in (2.1) and let U be defined as in Lemma A.1 such that $\mathcal{U}(\cdot) = U(u(\cdot))$.

(i): This follows from Jensen's inequality for concave functions and the fact that by definition dom $u \neq \emptyset$ and $\inf_{\mathbb{Q}\in\mathcal{Q}} \alpha(\mathbb{Q}) > -\infty$.

(ii): is obvious.

(iii): This follows from the concavity of u and the monotonicity and concavity of U.

(iv): According Dana (2005), Theorem 4.1, an upper semi-continuous monotone concave function is law invariant if and only if it is $\succeq_{ssd}$-monotone. The upper semi-continuity of $\mathcal{U}$ is proved in the next item.

(v): Let $k \in \mathbb{R}$ and $(X_n)_{n\in\mathbb{N}} \subset E_k := \{X \in L^1 \mid \mathcal{U}(X) \geq k\}$ be a sequence converging in $(L^1, \|\cdot\|_1)$ to some X. Then we may choose a subsequence which we also denote by $(X_n)_{n\in\mathbb{N}}$ which converges $\mathbb{P}$-a.s. to X too. We consider the following two cases: either the right-hand derivative u' of u is bounded on the domain of u or it is unbounded. In the first case, if the right-hand derivative u' of u is bounded on the domain of u, let $C > 0$ such that $u'(x) \leq C$ for all $x \in$ dom u. The right continuity of u implies that in this case dom u is closed in $\mathbb{R}$. Since $X_n \in$ dom U for all $n \in \mathbb{N}$, we must have that $X_n \in$ dom u $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$ (see (A.2)) and therefore $X \in$ dom u $\mathbb{P}$-a.s. Monotonicity and concavity of u imply that

$$|u(X_n) - u(X)| \leq (u'(X) \vee u'(X_n))|X_n - X| \leq C|X_n - X|.$$

Hence, we conclude that the sequence $u(X_n)$ converges to $u(X)$ in L^1, and by upper semi-continuity of U we infer that

$$\mathcal{U}(X) = U(u(X)) \geq \limsup_{n\to\infty} U(u(X_n)) \geq k,$$

so E_k is closed. Now suppose that u' is unbounded on the domain of u. Then there exists a strictly decreasing sequence $(a_r)_{r\in\mathbb{N}} \subset$ dom u such that $u'(a_1) > 0$, $u'(a_r) < \infty$, and

$\lim_{r\to\infty} u'(a_r) = \infty$. By the same arguments as presented in the first case the sequence $u(X_n \vee a_r)$ converges in L^1 to $u(X \vee a_r)$, because u' is bounded on $[a_r, \infty]$ and $(X_n \vee a_r)_{n\in\mathbb{N}}$ converges $\mathbb{P}$-a.s. and in $(L^1, \|\cdot\|_1)$ to $X \vee a_r$. Hence, by upper semi-continuity and monotonicity of U (Lemma A.1) as well as monotonicity of u we obtain

$$\mathcal{U}(X \vee a_r) = U(u(X \vee a_r)) \geq \limsup_{n\to\infty} U(u(X_n \vee a_r)) \geq \limsup_{n\to\infty} U(u(X_n)) \geq k.$$

Now let $a := \lim_{r\to\infty} a_r \geq -\infty$. Then $\mathrm{dom}\, u \subset [a, \infty)$, and $X_n \geq a$ $\mathbb{P}$-a.s., because $X_n \in \mathrm{dom}\, \mathcal{U}$. Hence, $X = \lim_{n\to\infty} X_n \geq a$ $\mathbb{P}$-a.s. too and thus $\lim_{r\to\infty} X \vee a_r = X$. Moreover, by right-continuity and monotonicity of u we have $\lim_{r\to\infty} u(X \vee a_r) = u(X)$ monotonously. Therefore, we infer from applying the monotone convergence theorem that

$$\begin{aligned}
\mathcal{U}(X) &= \inf_{\mathbb{Q}\in\mathcal{Q}} (\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q})) = \inf_{\mathbb{Q}\in\mathcal{Q}} \lim_{r\to\infty} (\mathbb{E}_{\mathbb{Q}}[u(X \vee a_r)] + \alpha(\mathbb{Q})) \\
&\geq \limsup_{r\to\infty} \inf_{\mathbb{Q}\in\mathcal{Q}} (\mathbb{E}_{\mathbb{Q}}[u(X \vee a_r)] + \alpha(\mathbb{Q})) = \limsup_{r\to\infty} \mathcal{U}(X \vee a_r) \\
&\geq k.
\end{aligned}$$

Hence, also in this case E_k is closed, so $\mathcal{U}$ is upper semi-continuous. □

B Proof of Proposition 3.2

We prove the case of Pareto optima. For acceptable Pareto optima in what follows simply replace $\mathbb{A}(W)$ by $\mathbb{A}_c(W)$.

Now let $(X_1, \ldots, X_n) \in \mathbb{A}(W)$ be Pareto Optimal. Then the non-empty convex sets

$$C := \{(\mathcal{U}_1(X_1), \ldots, \mathcal{U}_n(X_n))\}$$

and

$$V = \{(\mathcal{U}_1(Y_1), \ldots, \mathcal{U}_n(Y_n)) \mid (Y_1, \ldots, Y_n) \in \mathbb{A}(W)\} - \mathbb{R}^n_{++}$$

in $\mathbb{R}^n$, where $\mathbb{R}^n_{++} := \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_i > 0, i = 1, \ldots, n\}$, have empty intersection due to the Pareto optimality of $(X_1, \ldots, X_n)$. Hence, there exists a non-trivial linear functional $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that

(B.1) $$\sum_{i=1}^n \lambda_i \mathcal{U}_i(X_i) \geq \sum_{i=1}^n \lambda_i (\mathcal{U}_i(Y_1) - y_i)$$

for all $(Y_1, \ldots, Y_n) \in \mathbb{A}(W)$ and $(y_1, \ldots, y_n) \in \mathbb{R}^n_{++}$; see Rockafellar (1974), Theorem 11.2. We infer that $\lambda_i \geq 0$ for all i because otherwise choosing $y_i >> 0$ would yield a contradiction. The last assertion of Proposition 3.2 is obvious.

C Proofs of Theorems 3.8 and 3.9

The next Lemma C.1 is an Arzela-Ascoli type argument which will also play a crucial role in the proof of Theorems 3.8 and 3.9. For a proof see e.g. Filipović and Svindland (2008).

Lemma C.1. *Let $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of increasing 1-Lipschitz-continuous functions such that $f_n(0) \in [-K, K]$ for all $n \in \mathbb{N}$ where $K \geq 0$ is a constant. Then there is a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ and an increasing 1-Lipschitz-continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{k \to \infty} f_{n_k}(x) = f(x)$ for all $x \in \mathbb{R}$.*

Let $\mathrm{CFN} := \{(f_i)_{i=1}^n \in \mathrm{CF} \mid f_1(0) = \ldots = f_n(0) = 0\}$. Note that

$$\mathrm{CF} = \{(f_i + a_i)_{i=1}^n \mid (f_i)_{i=1}^n \in \mathrm{CFN}, a_i \in \mathbb{R}, \sum_{i=1}^n a_i = 0\}.$$

According to Proposition 3.6 there exists a solution to (3.4) for some given weights $(\lambda_1, \ldots, \lambda_n)$ if and only if there is a solution to

$$\text{(C.1)} \qquad \text{Maximize} \sum_{i=1}^n \lambda_i \mathcal{U}_i(f_i(W) + a_i) \qquad \text{subject to } (f_i)_{i=1}^n \in \mathrm{CFN},\ a_i \in \mathbb{R},$$
$$\sum_{i=1}^n a_i = 0, (f_i(W) + a_i)_{i=1}^n \in \mathbb{A}_c(W).$$

Proof of Theorem 3.9. We will prove the existence of a solution to (C.1). Fix a set of weights $(\lambda_1, \ldots, \lambda_n)$ satisfying the conditions (3.6). First of all, we observe that if for some $j \in M$ the right-hand-derivative u_j' does not attain the values d_H^j and/or d_L^j, we can always find non-negative numbers $\widetilde{d}_H^j$ and/or $\widetilde{d}_L^j$ in the image of u_j' such that, for the already given set of weights $(\lambda_1, \ldots, \lambda_n)$, the conditions (3.6) still hold true if we replace the d_H^j and/or d_L^j by $\widetilde{d}_H^j$ and/or $\widetilde{d}_H^j$. We assume that all d_H^j and/or d_L^j which are not attained by the corresponding u_j' are replaced as in the described manner, and for the sake of simplicity we keep the notation d_H^j and d_L^j. By concavity of the u_i there is a constant k such that for all $i = 1, \ldots, n$ the affine functions

$\mathbb{R} \ni x \mapsto d_L^i x + k$ and $\mathbb{R} \ni x \mapsto d_H^i x + k$ both dominate u_i[10]. Using this, we will show that

$$\text{(C.2)} \qquad P := \sup\Big\{ \sum_{i=1}^n \lambda_i \mathcal{U}_i(f_i(W) + a_i) \mid (f_i)_{i=1}^n \in \text{CFN},\ a_i \in \mathbb{R}, \sum_{i=1}^n a_i = 0, (f_i(W) + a_i)_{i=1}^n \in \mathbb{A}_c(W) \Big\} < \infty,$$

and that this supremum is realized over a bounded set of comonotone allocations where the bound is given by W. More precisely, we will prove that there exists some constant $K > 0$ depending on W such that

$$\text{(C.3)} \qquad P = \sup\Big\{ \sum_{i=1}^n \lambda_i \mathcal{U}_i(f_i(W) + a_i) \mid (f_i)_{i=1}^n \in \text{CFN},\ a_i \in [-K, K], \sum_{i=1}^n a_i = 0, (f_i(W) + a_i)_{i=1}^n \in \mathbb{A}_c(W) \Big\}.$$

To this end, we define the functions

$$U_i(X) := \inf_{\mathbb{Q} \in \mathcal{Q}_i} \left(\mathbb{E}_{\mathbb{Q}}[X] + \alpha_i(\mathbb{Q}) \right), \quad X \in L^1,\ i = 1, \ldots, n,$$

as in Lemma A.1. Consider $(f_i)_{i=1}^n \in \text{CFN}$ and $a_i \in \mathbb{R}$ such that $\sum_{i=1}^n a_i = 0$ and $(f_i(W) + a_i)_{i=1}^n \in \mathbb{A}_c(W)$. Let $I := \{i \in \{1, \ldots, n\} \mid a_i < 0\}$ and $J := \{1, \ldots, n\} \setminus I$. By applying

[10] Note that if u is a concave function, then it is always dominated by $x \mapsto u'(y)(x - y) + u(y)$.

monotonicity, cash additivity and finally property (A.2) of the U_i (see Lemma A.1) we obtain:

$$\sum_{i=1}^n \lambda_i \mathcal{U}_i(f_i(W)+a_i) = \sum_{i=1}^n \lambda_i U_i(u_i(f_i(W)+a_i))$$
$$\leq \sum_{i\in I\cap M} \lambda_i U_i(d_H^i(f_i(W)+a_i)+k) + \sum_{j\in J\cap M} \lambda_j U_j(d_L^j(f_j(W)+a_j)+k)+$$
$$\sum_{l\in N} \lambda_l U_l(d_H^l(f_l(W)+a_l)+k)$$
$$\leq k\sum_{i=1}^n \lambda_i + \sum_{i\in I\cap M} \lambda_i U_i(d_H^i f_i(W))+$$
$$\sum_{j\in J\cap M} \lambda_j U_j(d_L^j f_j(W)) + \sum_{l\in N} \lambda_l U_l(d_H^l f_l(W))+$$
$$\left(\min_{i\in I\cap M} \lambda_i d_H^i\right) \sum_{i\in I\cap M} a_i + \left(\max_{j\in J\cap M} \lambda_j d_L^j\right) \sum_{j\in J\cap M} a_j + \delta \sum_{l\in N} a_l$$
$$\leq k\sum_{i=1}^n \lambda_i + \mathbb{E}[W^+]\sum_{i=1}^n \lambda_i d_H^i + \sum_{i=1}^n \lambda_i U_i(0)+$$
$$\left(\min_{i\in I\cap M} \lambda_i d_H^i\right) \sum_{i\in I\cap M} a_i + \left(\max_{j\in J\cap M} \lambda_j d_L^j\right) \sum_{j\in J\cap M} a_j + \delta \sum_{l\in N} a_l. \tag{C.4}$$

Suppose that $N=\emptyset$, then we further estimate

$$(C.4) \leq \mathbb{E}[W^+]\sum_{i=1}^n \lambda_i d_H^i + k\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \lambda_i U_i(0) - \left(\min_{i\in I} \lambda_i d_H^i - \max_{j\in J} \lambda_j d_L^j\right) a \tag{C.5}$$

where $a := \sum_{i\in J} a_i\ (\geq 0)$. If $N\neq\emptyset$ and $\sum_{l\in N} a_l < 0$, then we estimate

$$(C.4) \leq \mathbb{E}[W^+]\sum_{i=1}^n \lambda_i d_H^i + k\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \lambda_i U_i(0) - \left(\delta - \max_{j\in J\cap M} \lambda_j d_L^j\right) \tilde{a} \tag{C.6}$$

for $\tilde{a} := \sum_{i\in J\cap M} a_i\ (\geq 0)$ using (3.6). And similarly, if $N\neq\emptyset$ and $\sum_{l\in N} a_l \geq 0$, then we estimate

$$(C.4) \leq \mathbb{E}[W^+]\sum_{i=1}^n \lambda_i d_H^i + k\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \lambda_i U_i(0) - \left(\min_{i\in I\cap M} \lambda_i d_H^i - \delta\right) \hat{a} \tag{C.7}$$

for $\hat{a} := \sum_{i\in J\cup N} a_i\ (\geq 0)$. Consequently we infer that

$$P \leq \mathbb{E}[W^+]\sum_{i=1}^n \lambda_i d_H^i + k\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \lambda_i U_i(0) < \infty.$$

Choose any allocation $(X_1,\ldots,X_n)\in\mathbb{A}_c(W)$ (Assumption 3.1). Then we have that $P \geq \sum_{i=1}^n \lambda_i \mathcal{U}_i(X_i) =: \widetilde{k}$. Letting

$$A := \min_{i=1,\ldots,n} \lambda_i d_H^i - \max_{j=1,\ldots,n} \lambda_j d_L^j$$

if $N = \emptyset$, or

$$A := \min\left\{\left(\delta - \max_{j\in M}\lambda_j d_L^j\right), \left(\min_{i\in M}\lambda_i d_H^i - \delta\right)\right\}$$

if $N \neq \emptyset$, we infer from (C.5), (C.6), and (C.7) that the supremum in (C.2) is realized over allocations such that

$$|a_i| \leq \frac{|\widetilde{k}| + \mathbb{E}[W^+]\sum_{i=1}^n \lambda_i d_H^i + k\sum_{i=1}^n \lambda_i + |\sum_{i=1}^n \lambda_i U_i(0)|}{A} =: \overline{K}$$

for all $i \in M$, and $|\sum_{i\in N} a_i| \leq \overline{K}$ too. Note that $A > 0$ due to the conditions (3.6) on the weights λ_i. In the following we argue that in case $|N| > 1$ we may also assume that the a_i belonging to $i \in N$ are bounded due to the insensitivity of the cash additive $\mathcal{U}_i$, $i \in N$, to constant re-sharings of 0 amongst themselves. To this end note that the choice of the λ_i and the requirement $s_i = -\infty$ for $i \in N$ implies

$$\text{(C.8)} \quad \sum_{i\in N}\lambda_i\mathcal{U}_i(f_i(W) + a_i + m_i) = \sum_{i\in N}\lambda_i\mathcal{U}_i(f_i(W) + a_i) + \delta\sum_{i\in N} m_i = \sum_{i\in N}\lambda_i\mathcal{U}_i(f_i(W) + a_i),$$

whenever $m_i \in \mathbb{R}$ such that $\sum_{i\in N} m_i = 0$. Hence, adding constants m_i such that $\sum_{i\in N} m_i = 0$ to the endowments of the decision makers in N does not affect the contribution of the allocation to P. This immediately implies that we may assume $|a_i| \leq \overline{K}$ for all $i \in N$ if $c_i = \infty$ for all $i \in N$, because in that case we may choose $m_1 = \sum_{i\in N} a_i - a_1$ and $m_i = -a_i$ for all $i \neq 1$ in (C.8), and the altered allocation satisfies the bound ($|m_1 + a_1| = |\sum_{i\in N} a_i| \leq \overline{K}$, see above). If the set $N_b \subset N$ of indices $i \in N$ such that $c_i \in \mathbb{R}$ is not empty, we also need to consider cash amounts that might be needed to make the endowment $f_i(W) + a_i$ acceptable. This is the point where the assumption $\mathcal{U}_i(-W^-) > -\infty$ for all $i \in N_b$ enters (whenever $|N| > 1$). Using the cash additivity ($\mathcal{U}_i(Y + z) = \mathcal{U}_i(Y) + d_H^i z$, $z \in \mathbb{R}$) and monotonicity of $\mathcal{U}_i$ we obtain that $f_i(W) + z$ is acceptable for decision maker $i \in N_b$ whenever

$$z \geq \frac{\mathcal{U}_i(W_i) - \mathcal{U}_i(-W^-) - c_i}{d_H^i}.$$

Moreover, acceptability of $f_i(W) + a_i$ implies (again using cash additivity and monotonicity of $\mathcal{U}_i$) for all $i \in N_b$:

$$a_i \geq \frac{\mathcal{U}_i(W_i) - c_i - \mathcal{U}_i(f_i(W_i))}{d_H^i} \geq \frac{\mathcal{U}_i(W_i) - c_i - \mathcal{U}_i(0)}{d_H^i} - \mathbb{E}[W^+]$$

where we have used that $\mathcal{U}_i(f_i(W_i)) \leq \mathcal{U}_i(W^+) \leq \mathbb{E}[d_H^i W^+] + \mathcal{U}_i(0)$ according to (A.2). Therefore we know that there exists a joint constant $\widehat{K} > 0$ such that the a_i, $i \in N_b$, are bounded from below by $-\widehat{K}$ and such that the endowments $f(W_i)+a_i+m_i$ stay acceptable for $m_i = -[(a_i-\widehat{K})\vee 0]$, $i\in N_b$. If $N_b = N$, then choosing some $j\in N_b$ and letting $m_i = -[(a_i-\widehat{K})\vee 0]$ for $i\in N_b\setminus\{j\}$ and $m_j = -\sum_{i\in N\setminus\{j\}} m_i$, we obtain that $\sum_{i\in N} m_i = 0$, $|a_i+m_i|\le \widehat{K}$ for all $i\in N_b\setminus\{j\}$, and

$$|a_j+m_j| \le |\sum_{i\in N} a_i| + |\sum_{i\in N_b\setminus\{j\}} a_i+m_i| \le \overline{K} + |N_b|\widehat{K} =: K.$$

Moreover, since $a_j \le a_j+m_j$, also $f(W_j)+a_j+m_j$ is acceptable, i.e. $\mathcal{U}_j(f(W_j)+a_j+m_j) \ge \mathcal{U}_j(W_j)-c_j$. If $|N_u|>1$ where $N_u := N\setminus N_b$, then we choose some $j\in N_u$ and let $m_i = -[(a_i-\widehat{K})\vee 0]$ for all $i\in N_b$ and $m_i=-a_i$ for all $i\in N_u\setminus\{j\}$, whereas $m_j = -\sum_{i\in N\setminus\{j\}} m_i$. Again, $\sum_{i\in N} m_i = 0$, and $a_i+m_i = 0$ for all $i\in N_u\setminus\{j\}$, $|a_i+m_i|\le\widehat{K}$ for all $i\in N_b$, and

$$|a_j+m_j| \le |\sum_{i\in N} a_i| + |\sum_{i\in N_b} a_i+m_i| \le \overline{K} + |N_b|\widehat{K} =: K.$$

Hence, (C.2) and (C.3) are proved. By virtue of (C.3) we may choose a sequence $((f_i^p)_{i=1}^n)_{p\in\mathbb{N}} \subset$ CF with $f_i^p(0)\in[-K,K]$ for all $i=1,\dots,n$ and $p\in\mathbb{N}$ such that $(f_i^p(W))_{i=1}^n \in \mathbb{A}_c(W)$ for all $p\in\mathbb{N}$ and

$$P = \lim_{p\to\infty}\sum_{i=1}^n \lambda_i\mathcal{U}_i(f_i^p(W)).$$

According to Lemma C.1 there exists a subsequence, which we for the sake of simplicity also denote by $(f_i^p)_{i=1}^n$, which converges pointwise to some $(f_i)_{i=1}^n \in$ CF. As $|f_i^p(W)| \le |W|+K$ for all $i=1,\dots,n$ and $p\in\mathbb{N}$, we may apply the dominated convergence theorem which yields $f_i(W)\in L^1$, and $\lim_{p\to\infty}\mathbb{E}[|f_i(W)-f_i^p(W)|]=0$ for all $i=1,\dots,n$. By upper semi-continuity of the $\mathcal{U}_i$ (Lemma 2.3) we have

$$\mathcal{U}_i(W_i)-c_i \le \limsup_{p\to\infty}\mathcal{U}_i(f_i^p(W)) \le \mathcal{U}_i(f_i(W)),$$

and

$$\begin{aligned} P &= \lim_{p\to\infty}\sum_{i=1}^n \lambda_i\mathcal{U}_i(f_i^p(W)) \quad\le\quad \sum_{i=1}^n \lambda_i \limsup_{p\to\infty}\mathcal{U}_i(f_i^p(W)) \\ &\le \sum_{i=1}^n \lambda_i\mathcal{U}_i(f_i(W)). \end{aligned}$$

Hence, we infer that $(f_i(W))_{i=1}^n\in\mathbb{A}_c(W)$ (since $P>-\infty$) and

$$P = \sum_{i=1}^n \lambda_i\mathcal{U}_i(f_i(W)).$$

For the last part of (ii) suppose that $|N| = n$, and let $(X_1, \dots, X_n)$ be a solution to (3.4) for the given weights. If $m_i \in \mathbb{R}$ such that $\sum_{i=1}^n m_i = 0$, then the same computation as in (C.8) yields $\sum_{i=1}^n \lambda_i \mathcal{U}_i(X_i + m_i) = \sum_{i=1}^n \lambda_i \mathcal{U}_i(X_i)$. □

Proof of Theorem 3.8. Recall (C.1) and let $(f_i)_{i=1}^n \in \mathrm{CFN}$, $a_i \in \mathbb{R}$ with $\sum_{i=1}^n a_i = 0$ such that $(f_i(W) + a_i)_{i=1}^n \in \mathbb{A}_c(W)$. Since in particular $\mathcal{U}_i(f_i(W) + a_i) > -\infty$, we must have that $f_i(W) + a_i \geq s_i$ for all $i = 1, \dots, n$. Let $\widetilde{K} := \sum_{i=1}^n |s_i|$. Then

$$-(|W| + \widetilde{K}) \leq f_i(W) + a_i = W - (\sum_{j \neq i} f_j(W) + a_j) \leq |W| + \widetilde{K}.$$

Hence, we deduce that (C.3) holds with $K := 2 \operatorname{essinf} |W| + \widetilde{K}$. The rest of the proof now follows the lines of the proof of Theorem 3.9. □

D Proof of Proposition 5.2

For the proof of Proposition 5.2 we will need some (additional) tools from convex duality theory which we briefly introduce in the following. The details and proofs of the statements can e.g. be found in Ekeland and Téman (1999). Let $\mathcal{U}$ be a law invariant robust utility as in (2.1). The dual function of $\mathcal{U}$ is

$$\mathcal{U}^*(Z) := \sup_{Y \in L^1} \mathcal{U}(Y) - E[YZ], \quad Z \in L^\infty,$$

which is convex and $\sigma(L^\infty, L^1)$-lower semi-continuous, i.e. the level sets $E_k := \{Z \in L^\infty \mid \mathcal{U}^*(Z) \leq k\}$ are closed in the $\sigma(L^\infty, L^1)$-topology for all $k \in \mathbb{R}$. Moreover, $\mathcal{U}^*$ is law invariant by the same arguments as applied in the proof of Lemma A.1 ($-\mathcal{U}^*$ is concave and upper semi-continuous) and therefore $\mathcal{U}^*$ is $\succeq_c$-antitone according to Dana (2005), Theorem 4.1. Since $\mathcal{U}$ is concave and upper semi-continuous (Lemma 2.3), it follows from the Fenchel-Moreau theorem that

$$\mathcal{U}(X) = \mathcal{U}^{**}(X) := \inf_{Z \in L^\infty} E[ZX] + \mathcal{U}^*(Z), \quad X \in L^1. \tag{D.1}$$

Again the very same techniques as in the proof of Lemma A.1 show that $\mathcal{U}$ is law invariant if and only if $\mathcal{U}^*$ is law invariant. The superdifferential of $\mathcal{U}$ at some $X \in L^1$ is

$$\partial \mathcal{U}(X) := \{Z \in L^\infty \mid \mathcal{U}(Y) \leq \mathcal{U}(X) + \mathbb{E}[Z(Y - X)] \, \forall Y \in L^1\}.$$

Notice that

$$Z \in \partial\mathcal{U}(X) \quad \Leftrightarrow \quad \mathcal{U}(X) = E[ZX] + \mathcal{U}^*(Z) \tag{D.2}$$

and that monotonicity of $\mathcal{U}$ implies $\partial\mathcal{U}(X) \subset \operatorname{dom} \mathcal{U}^* \subset L^\infty_+$.

Lemma D.1. *Let $\mathcal{U}$ be a law invariant robust utility as in* (2.1) *and let $X \in L^1$ such that $\partial\mathcal{U}(X) \neq \emptyset$. Then there exists a decreasing function $h : \mathbb{R} \to [0,\infty)$ such that $h(X) \in \partial\mathcal{U}(X)$.*

Proof. Let $Z \in \partial\mathcal{U}(X)$ and $h : \mathbb{R} \to \mathbb{R}_+$ be a measurable function such that $h(X) = E[Z \mid X]$. By (D.2), $E[Z \mid X] \succeq_c Z$ (Jensen's inequality), $\succeq_c$-antitonicity of $\mathcal{U}^*$ and finally (D.1) it follows that

$$\mathcal{U}(X) = E[ZX] + \mathcal{U}^*(Z) \geq E[E[Z \mid X]X] + \mathcal{U}^*(E[Z \mid X]) \geq \mathcal{U}(X).$$

Thus $h(X) \in \partial\mathcal{U}(X)$ too. Note that (A.3) and law invariance of $\mathcal{U}^*$ imply

$$\mathcal{U}(X) \quad \leq \quad \int_0^1 q_{h(X)}(1-t)q_X(t)\,dt + \mathcal{U}^*(h(X))$$

in the same way as the similar argument presented in the proof of Lemma A.1. Hence we obtain that

$$\begin{aligned} \mathcal{U}(X) \quad &\leq \quad \int_0^1 q_{h(X)}(1-t)q_X(t)\,dt + \mathcal{U}^*(h(X)) \\ &\leq \quad E[h(X)X] + \mathcal{U}^*(h(X)) \quad = \quad \mathcal{U}(X) \end{aligned}$$

where we applied the Hardy-Littlewood inequalities in the second step; see Föllmer and Schied (2004) Theorem A.24. Consequently $E[h(X)X] = \int_0^1 q_{h(X)}(1-t)q_X(t)\,dt$ which guarantees that h might be chosen as to be decreasing; see again Föllmer and Schied (2004) Theorem A.24. □

Lemma D.2. *Let $\mathcal{U}$ be a law invariant robust utility as in* (2.1) *which is strictly risk averse conditional on lower-tail events. Let $(X,Z) \in L^1 \times L^\infty$ be such that $Z \in \partial\mathcal{U}(X)$ and $X = f(W)$, $Z = h(W)$ for some $W \in L^1$ and an increasing function $f : \mathbb{R} \to \mathbb{R}$ and a decreasing function $h : \mathbb{R} \to \mathbb{R}_+$. Consider the set $A := \{Z = \operatorname{ess\,sup} Z\}$. If $\mathbb{P}(A) > 0$, then X is constant on the set A.*

Proof. Assume that $\mathbb{P}(A) > 0$ and, by contradiction, that X is not constant on A. Since f is increasing and h is decreasing A is a lower tail-event of X. As $\mathcal{U}$ is strictly risk averse conditional on lower-tail events it follows that

$$\mathcal{U}(X) < \mathcal{U}(\overline{X}) \tag{D.3}$$

where $\overline{X} = X 1_{A^c} + \mathbb{E}[X \mid A] 1_A$. But $\mathbb{E}[ZX] = \mathbb{E}[Z\overline{X}]$ and $Z \in \partial\mathcal{U}(X)$ imply $\mathcal{U}(\overline{X}) \leq \mathcal{U}(X) + \mathbb{E}[Z(\overline{X} - X)] = \mathcal{U}(X)$ which contradicts (D.3). □

Lemma D.3. *Let $\mathcal{U}$ be a law invariant robust utility as in* (2.1) *which is in addition strictly monotone on* dom $\mathcal{U}$ *and let $(X, Z) \in L^1 \times L^\infty$ such that $Z \in \partial\mathcal{U}(X)$. Then $Z > 0$ a.s.*

Proof. Let $A := \{Z = 0\}$. As $\mathcal{U}(X + 1_A) \leq \mathcal{U}(X) + \mathbb{E}[Z 1_A] = \mathcal{U}(X)$, strict monotonicity of $\mathcal{U}$ implies $\mathbb{P}(A) = 0$. □

Proof of Proposition 5.2. In the following we may and will assume that $c_1 = c_2 = \infty$, because any Pareto optimum remains optimal if we replace the rationality constraints by the trivial ones. Given that we are only concerned about the shape of the allocation, the constraints c_i do not play any role apart from giving the bound on the constant k in (5.2). This bound in turn follows from the translation invariance of $\mathcal{U}_1$ implying that

$$\mathcal{U}_1(X_1) = \mathcal{U}_1(-(W - l)_-) + k \geq \mathcal{U}_1(W_1) - c_1.$$

Hence, let $(X_1, X_2) \in \mathbb{A}(W)$ (where $c_1 = c_2 = \infty$) be a comonotone Pareto optimal allocation of W, and let $f, g : \mathbb{R} \to \mathbb{R}$ be increasing functions such that $f + g = \mathrm{Id}_{\mathbb{R}}$ and $(X_1, X_2) = (f(W), g(W))$. According to Lemma 3.4 there exists $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\lambda_1 \mathcal{U}_1(X_1) + \lambda_2 \mathcal{U}_2(X_2) = \max_{(Y_1, Y_2) \in \mathbb{A}(W)} \lambda_1 \mathcal{U}_1(Y_1) + \lambda_2 \mathcal{U}_2(Y_2). \tag{D.4}$$

Note that the function

$$\mathcal{U}_{\lambda_1, \lambda_2}(Y) := \sup_{(Y_1, Y_2) \in \mathbb{A}(Y)} \lambda_1 \mathcal{U}_1(Y_1) + \lambda_2 \mathcal{U}_2(Y_2), \quad Y \in L^1,$$

is concave, increasing and

$$\mathrm{dom}\, \mathcal{U}_{\lambda_1, \lambda_2} = \mathrm{dom}\, \mathcal{U}_1 + \mathrm{dom}\, \mathcal{U}_2 = L^1 + \mathrm{dom}\, \mathcal{U}_2 = L^1.$$

Moreover $\mathcal{U}_{\lambda_1, \lambda_2}(\cdot) \geq \lambda_1 \mathcal{U}_1(\cdot) + \lambda_2 \mathcal{U}_2(0)$ implies that there exists an open set in L^1 on which $\mathcal{U}_{\lambda_1, \lambda_2}$ is bounded from below, because $\lambda_1 \mathcal{U}_1$ as a continuous concave function has this property;

see Ekeland and Téman (1999) Proposition 2.5. Hence $\mathcal{U}_{\lambda_1,\lambda_2}$ is continuous on L^1 and therefore everywhere superdifferentiable; see e.g. Ekeland and Téman (1999) Proposition 2.5 and Proposition 5.2. $\mathcal{U}_{\lambda_1,\lambda_2}$ is also law invariant. This can be deduced by verifying that the dual

$$\mathcal{U}^*_{\lambda_1,\lambda_2} = (\lambda_1\mathcal{U}_1)^* + (\lambda_2\mathcal{U}_2)^*$$

is law invariant as a sum of law invariant functions; see also the introductory remarks of this section. According to Lemma D.1 there exists a decreasing function $h:\mathbb{R}\to\mathbb{R}_+$ such that

$$Z := h(W) \in \partial\mathcal{U}_{\lambda_1,\lambda_2}(W) = \partial\lambda_1\mathcal{U}_1(X_1)\cap\partial\lambda_2\mathcal{U}_2(X_2). \tag{D.5}$$

The inclusion $\subset$ in (D.5) is due to the fact that for all $Z\in\partial\mathcal{U}_{\lambda_1,\lambda_2}(W)$ we have that

$$\mathcal{U}_{\lambda_1,\lambda_2}(Y) \le \mathcal{U}_{\lambda_1,\lambda_2}(W) + \mathbb{E}[Z(Y-W)] \quad \text{for all } Y\in L^1$$

and thus by (D.4) and definition of $\mathcal{U}_{\lambda_1,\lambda_2}$ that

$$\lambda_1\mathcal{U}_1(Y_1)+\lambda_2\mathcal{U}_2(Y_2) \le \lambda_1\mathcal{U}_1(X_1)+\lambda_2\mathcal{U}_2(X_2)+\mathbb{E}[Z(Y_1+Y_2-(X_1+X_2))]$$

for all $Y_1, Y_2\in L^1$. Now $Z\in\partial\lambda_i\mathcal{U}_i(X_i)$, $i=1,2$, follows. The converse inclusion in (D.5) follows similarly. By definition of the supergradient, this implies that $\frac{Z}{\lambda_1}\in\partial\mathcal{U}_1(X_1)$ and $\frac{Z}{\lambda_2}\in\partial\mathcal{U}_2(X_2)$. From $\frac{Z}{\lambda_2}\in\partial\mathcal{U}_2(X_2)$ and the strict monotonicity of $\mathcal{U}_2$ it follows that $\mathbb{P}(Z=0)=0$; see Lemma D.3. Note that

$$\mathcal{U}_1(Y) = \int_0^1 q_Y(t)\,d\varphi(t), \quad Y\in L^1,$$

where $\varphi(t) := \frac{t}{\alpha}\wedge 1$ for $t\in[0,1]$ is an increasing continuous (on $(0,1)$) function. Since $\frac{Z}{\lambda_1}\in\partial\mathcal{U}_1(X_1)$ we have also that

$$\mathcal{U}_1(X_1) = \int_0^1 q_{X_1}(t)q_{\frac{Z}{\lambda_1}}(1-t)\,dt = \int_0^1 q_{X_1}(t)\,d\psi(t)$$

where $\psi(t) := \frac{1}{\lambda_1}\int_0^t q_Z(1-s)\,ds$, $t\in[0,1]$, is another increasing continuous function. Hence, we obtain that

$$\int_0^1 q_{X_1}(t)\,d\varphi(t) - \int_0^1 q_{X_1}(t)\,d\psi(t) = \mathcal{U}_1(X_1)-\mathcal{U}_1(X_1) = 0$$

and integration by parts Dunford and Schwartz (1976, III.6.21, Theorem 22) of each integral in combination with a limiting argument by means of $X_1^n := -n\vee X_1\wedge n$, $n\in\mathbb{N}$, yields

$$\int_0^1 (\psi(t)-\varphi(t))\,dq_{X_1}(t) = 0. \tag{D.6}$$

As $\frac{Z}{\lambda_1} \in \mathcal{Q}_1$, we observe that $\psi \le \varphi$. Hence (D.6) can only be satisfied if q_{X_1} is constant on $\{\psi < \varphi\}$.

Since $\mathbb{P}(Z = 0) = 0$, we have $q_Z(1-s) > 0$ for any $s \in (0,1)$, so in particular $\psi(t) < 1$ for all $t < 1$. Moreover, $\psi(0) = 0$ and the slope of ψ is at most $\frac{1}{\alpha}$. Therefore we have $\beta := \inf\{t \mid \psi(t) < \varphi(t)\} \in [0,1)$ and

$$q_{X_1} \text{ is constant on } (\beta, 1) \subset \{\psi < \varphi\}. \tag{D.7}$$

If $\beta > 0$, it follows for any $t \in [0,\beta]$ that $\frac{t}{\alpha} = \psi(t) = \int_{1-t}^{1} \frac{1}{\lambda_1} q_Z(s)ds$. As $q_{\frac{Z}{\lambda_1}} \le \frac{1}{\alpha}$, we deduce that

$$q_Z(s) = \frac{\lambda_1}{\alpha} = \operatorname{ess\,sup} Z \text{ for all } s \in (1-\beta, 1]. \tag{D.8}$$

Since $\frac{Z}{\lambda_2} \in \partial\mathcal{U}_2(X_2)$ and as $\mathcal{U}_2$ is strictly risk averse conditionally on lower tail-events, X_2 is constant on $\{Z = \operatorname{ess\,sup} Z\}$; see Lemma D.2. Recall that $X_1 = f(W)$, $X_2 = g(W)$ and $Z = h(W)$ for increasing functions g and f and a decreasing function h. Consequently (D.7) implies that $f(W)$ is constant on $W^{-1}(l,\infty)$ whereas (D.8) implies that $h(W)$ and thus $g(W)$ are constant on $W^{-1}(-\infty, l)$ for $l := q_W(\beta)$ ($:= -\infty$ if $\beta = 0$). In conjunction with the fact that f, g are continuous we deduce that X_1 and X_2 ought to be of the following form

$$X_1 = (W - l)1_{\{W \le l\}} + k, \quad X_2 = l1_{\{W \le l\}} + W1_{\{l \le W\}} - k, \quad \text{where } k \in \mathbb{R}.$$

□

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