INVARIANT MANIFOLDS WITH BOUNDARY FOR JUMP-DIFFUSIONS

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ABSTRACT. We provide necessary and sufficient conditions for stochastic invariance of finite dimensional submanifolds with boundary in Hilbert spaces for stochastic partial differential equations driven by Wiener processes and Poisson random measures.

1. INTRODUCTION

Consider a stochastic partial differential equation (SPDE) of the form

(1.1)
$$\begin{cases} dr_t &= (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt) \\ r_0 &= h_0 \end{cases}$$

on a separable Hilbert space H driven by some trace class Wiener process W on a separable Hilbert space $\mathbb{H}$ and a compensated Poisson random measure μ on some mark space E with $dt \otimes F(dx)$ being its compensator. Throughout this paper, we assume that A is the generator of a C_0-semigroup on H and that the mappings α, $\sigma = (\sigma^j)_{j \in \mathbb{N}}$ and γ satisfy appropriate regularity conditions.

Given a finite dimensional C^3-submanifold $\mathcal{M}$ with boundary of H, we study the *stochastic viability* and *invariance problem* related to the SPDE (1.1). In particular, we provide necessary and sufficient conditions such that for each $h_0 \in \mathcal{M}$ there is a (local) mild solution r to (1.1) with $r_0 = h_0$ which stays (locally) on the submanifold $\mathcal{M}$.

Any finite dimensional invariant submanifold $\mathcal{M}$ for the SPDE (1.1) gives rise to a finite dimensional Markovian realization of the respective particular solution processes r with initial values in $\mathcal{M}$, i.e. a deterministic C^3-function G and a finite dimensional Markov process X such that $r_t = G(X_t)$ up to some stopping time. This proves to be useful in applications, since it renders the stochastic evolution model (1.1) analytically and numerically tractable for initial values in $\mathcal{M}$. An important example is the so-called Heath-Jarrow-Morton (HJM) SPDE that describes the evolution of the interest rate curve. Stochastic invariance for the HJM SPDE has been discussed in detail in [2, 3, 4, 8, 9, 15, 16, 20] for the diffusion case. The present paper completes the results from [10, 15, 16] by providing explicit stochastic invariance conditions for the general case of a SPDE with jumps.

Stochastic invariance has been extensively studied also for other sets than manifolds. In finite dimension the general stochastic invariance problem for closed sets has been treated, e.g., in [5] in the diffusion case, and in [22] in the case of jump-diffusions. In infinite dimension we mention, e.g., the works of [19, 20, 23], where stochastic invariance has been established by means of support theorems for diffusion-type SPDEs.

We shall now present and explain the invariance conditions which we derive in this paper. Let us first consider the situation where the jumps in (1.1) are of finite variation. Then the conditions

$$(1.2) \qquad \mathcal{M} \subset \mathcal{D}(A),$$

$$(1.3) \qquad \sigma^j(h) \in \begin{cases} T_h \mathcal{M}, & h \in \mathcal{M} \setminus \partial \mathcal{M}, \\ T_h \partial \mathcal{M}, & h \in \partial \mathcal{M}, \end{cases} \quad \text{for all } j \in \mathbb{N},$$

$$(1.4) \qquad h + \gamma(h,x) \in \overline{\mathcal{M}} \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in \mathcal{M},$$

$$(1.5) \qquad Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)$$

$$- \int_E \gamma(h,x) F(dx) \in \begin{cases} T_h \mathcal{M}, & h \in \mathcal{M} \setminus \partial \mathcal{M}, \\ (T_h \mathcal{M})_+, & h \in \partial \mathcal{M} \end{cases}$$

are necessary and sufficient for stochastic invariance of $\mathcal{M}$ for (1.1).

Condition (1.2) says that the submanifold $\mathcal{M}$ lies in the domain of the infinitesimal generator A. This ensures that the mapping in (1.5) is well-defined. Condition (1.3) means that the volatilities $h \mapsto \sigma^j(h)$ must be tangential to $\mathcal{M}$ in its interior and tangential to the boundary $\partial \mathcal{M}$ at boundary points. Condition (1.4) says that the functions $h \mapsto h + \gamma(h,x)$ map the submanifold $\mathcal{M}$ into its closure $\overline{\mathcal{M}}$. Condition (1.5) means that the adjusted drift must be tangential to $\mathcal{M}$ in its interior and additionally inward pointing at boundary points.

In the general situation, where the jumps in (1.1) may be of infinite variation, condition (1.5) is replaced by the three conditions

$$(1.6) \qquad \int_E |\langle \eta_h, \gamma(h,x) \rangle| F(dx) < \infty, \quad h \in \partial \mathcal{M},$$

$$(1.7) \qquad Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)$$

$$- \int_E \Pi_{(T_h \mathcal{M})^\perp} \gamma(h,x) F(dx) \in T_h \mathcal{M}, \quad h \in \mathcal{M},$$

$$(1.8) \qquad \langle \eta_h, Ah + \alpha(h) \rangle - \frac{1}{2} \sum_{j \in \mathbb{N}} \langle \eta_h, D\sigma^j(h)\sigma^j(h) \rangle$$

$$- \int_E \langle \eta_h, \gamma(h,x) \rangle F(dx) \geq 0, \quad h \in \partial \mathcal{M},$$

where η_h denotes the inward pointing normal vector to $\partial \mathcal{M}$ at boundary points $h \in \partial \mathcal{M}$.

Condition (1.6) concerns the small jumps of r at the boundary of the submanifold and means that the discontinuous part of the solution must be of finite variation, unless it is parallel to the boundary $\partial \mathcal{M}$. Denoting by Π_K the orthogonal projection on a closed subspace $K \subset H$, we decompose

$$\gamma(h,x) = \Pi_{T_h \mathcal{M}} \gamma(h,x) + \Pi_{(T_h \mathcal{M})^\perp} \gamma(h,x).$$

As we will show, condition (1.4) implies

$$(1.9) \qquad \int_E \|\Pi_{(T_h \mathcal{M})^\perp} \gamma(h,x)\| F(dx) < \infty, \quad h \in \mathcal{M}.$$

The essential idea is to perform a second order Taylor expansion for a parametrization around h to obtain

$$\|\Pi_{(T_h \mathcal{M})^\perp} \gamma(h,x)\| = \|\gamma(h,x) - \Pi_{T_h \mathcal{M}} \gamma(h,x)\| \leq C \|\gamma(h,x)\|^2$$

for some constant $C \geq 0$. By virtue of (1.9), the integral in (1.7) exists, and hence, conditions (1.7), (1.8) correspond to (1.5).

As in previous papers on this subject we are dealing with mild solutions of SPDEs, i.e. stochastic processes taking values in a Hilbert space whose drift characteristic is quite irregular (e.g., not continuous with respect to the state variables). Therefore, the arguments to translate stochastic invariance into conditions on the characteristics are not straightforward. The arguments to prove our stochastic invariance results can be structured as follows: First, we show that we can (pre-)localize the problem by separating big and small jumps. Second, prelocal invariance of parametrized submanifolds can be pulled back to $\mathbb{R}^m$ by a linear projection argument tracing back to [11]. Both steps require a careful analysis of jump structures, which leads to the involved invariance conditions.

The remainder of this paper is organized as follows. In Section 2 we state our main results. In Section 3 we provide some notation and auxiliary results about stochastic invariance. In Section 4 we perform local analysis of the invariance problem on half spaces, in Section 5 we perform local analysis of the invariance problem on submanifolds with boundary, and in Section 6 we perform global analysis of the invariance problem on submanifolds with boundary and prove our main results. For convenience of the reader, the proofs of some technical auxiliary results are deferred to the appendix [14].

2. STATEMENT OF THE MAIN RESULTS

In this section we introduce the necessary terminology and state our main results. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions and let H be a separable Hilbert space.

Let W be a Q-Wiener process (see [6, pages 86, 87]) on some separable Hilbert space $\mathbb{H}$, where the covariance operator Q is a trace class operator.

Let $(E, \mathcal{E})$ be a measurable space which we assume to be a *Blackwell space* (see [7, 17]). We remark that every Polish space with its Borel σ-field is a Blackwell space. Furthermore, let μ be a time-homogeneous Poisson random measure on $\mathbb{R}_+ \times E$, see [18, Definition II.1.20]. Then its compensator is of the form $dt \otimes F(dx)$, where F is a σ-finite measure on $(E, \mathcal{E})$.

In [14] we review some basic facts about SPDEs of the type (1.1) and we recall the concepts of (local) strong, weak and mild solutions. In particular, equation (1.1) can be rewritten equivalently

$$(2.1) \quad \begin{cases} dr_t &= (Ar_t + \alpha(r_t))dt + \sum_{j \in \mathbb{N}} \sigma^j(r_t) d\beta_t^j \\ &\quad + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt) \\ r_0 &= h_0, \end{cases}$$

where $(\beta^j)_{j \in \mathbb{N}}$ is a sequence of real-valued independent standard Wiener processes. We next formulate the concept of stochastic invariance.

2.1. **Definition.** *A non-empty Borel set $B \subset H$ is called* prelocally (locally) *invariant for (2.1), if for all $h_0 \in B$ there exists a local mild solution $r = r^{(h_0)}$ to (2.1) with lifetime $\tau > 0$ such that up to an evanescent set[1]*

$$(r^\tau)_- \in B \text{ and } r^\tau \in \overline{B}$$

$$\left(r^\tau \in B \right).$$

The following standing assumptions prevail throughout this paper:
- *A generates a C_0-semigroup $(S_t)_{t \geq 0}$ on H.*

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[1]A random set $A \subset \Omega \times \mathbb{R}_+$ is called *evanescent* if the set $\{\omega \in \Omega : (\omega, t) \in A \text{ for some } t \in \mathbb{R}_+\}$ is a $\mathbb{P}$-nullset, cf. [18, 1.1.10].

- The mapping $\alpha : H \to H$ is locally Lipschitz continuous, that is, for each $n \in \mathbb{N}$ there is a constant $L_n \geq 0$ such that

$$(2.2) \qquad \|\alpha(h_1) - \alpha(h_2)\| \leq L_n \|h_1 - h_2\|, \quad h_1, h_2 \in H \text{ with } \|h_1\|, \|h_2\| \leq n.$$

- For each $n \in \mathbb{N}$ there exists a sequence $(\kappa_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}_+$ with $\sum_{j\in\mathbb{N}}(\kappa_n^j)^2 < \infty$ such that for all $j \in \mathbb{N}$ the mapping $\sigma^j : H \to H$ satisfies

$$(2.3) \qquad \|\sigma^j(h_1) - \sigma^j(h_2)\| \leq \kappa_n^j \|h_1 - h_2\|, \quad h_1, h_2 \in H \text{ with } \|h_1\|, \|h_2\| \leq n,$$

$$(2.4) \qquad \|\sigma^j(h)\| \leq \kappa_n^j, \quad h \in H \text{ with } \|h\| \leq n.$$

Consequently, for each $j \in \mathbb{N}$ the mapping σ^j is locally Lipschitz continuous.
- The mapping $\gamma : H \times E \to H$ is measurable, and for each $n \in \mathbb{N}$ there exists a measurable function $\rho_n : E \to \mathbb{R}_+$ with

$$(2.5) \qquad \int_E \big(\rho_n(x)^2 \vee \rho_n(x)^4\big) F(dx) < \infty$$

such that for all $x \in E$ the mapping $\gamma(\bullet, x) : H \to H$ satisfies

$$(2.6) \quad \|\gamma(h_1, x) - \gamma(h_2, x)\| \leq \rho_n(x) \|h_1 - h_2\|, \quad h_1, h_2 \in H \text{ with } \|h_1\|, \|h_2\| \leq n,$$

$$(2.7) \qquad \|\gamma(h, x)\| \leq \rho_n(x), \quad h \in H \text{ with } \|h\| \leq n.$$

Consequently, for each $x \in E$ the mapping $\gamma(\bullet, x)$ is locally Lipschitz continuous.
- We assume that for each $j \in \mathbb{N}$ the mapping $\sigma^j : H \to H$ is continuously differentiable, that is

$$(2.8) \qquad \sigma^j \in C^1(H) \quad \text{for all } j \in \mathbb{N}.$$

The first four conditions ensure that we may apply the results about SPDEs from [14]. We furthermore assume that:

- $\mathcal{M}$ is a finite-dimensional C^3-submanifold with boundary of H; that is, for all $h \in \mathcal{M}$ there exist an open neighborhood $U \subset H$ of h, an open set $V \subset \mathbb{R}_+^m = \mathbb{R}_+ \times \mathbb{R}^{m-1}$ (where $m \in \mathbb{N}$ is the dimension of $\mathcal{M}$) and a map $\phi \in C^3(V; H)$ (which we will call a *parametrization* of $\mathcal{M}$ around h and also denote as $\phi : V \subset \mathbb{R}_+^m \to U \cap \mathcal{M}$) such that
  (1) $\phi : V \to U \cap \mathcal{M}$ is a homeomorphism;
  (2) $D\phi(y)$ is one to one for all $y \in V$.
  We refer to [14, Section 3] for further details.

2.2. **Remark.** *We impose that $\mathcal{M}$ is of class C^3, because this ensures that the coefficients a, $(b^j)_{j\in\mathbb{N}}$, c and Θ, $(\Sigma^j)_{j\in\mathbb{N}}$, Γ of the SDEs (5.26), (4.1), which we will define in (5.38)–(5.40) and (5.44)–(5.46), satisfy the regularity conditions (2.2)–(2.4) and (2.6)–(2.8) as well; see Lemma 5.6.*

2.3. **Remark.** *Similarly, instead of (2.5) one would expect the weaker condition*

$$(2.9) \qquad \int_E \rho_n(x)^2 F(dx) < \infty.$$

The reason is that (2.5) is required in order to ensure that the above-mentioned coefficients also satisfy the regularity conditions (2.2)–(2.4) and (2.6)–(2.8), but with (2.5) being replaced by (2.9); see Lemma 5.6.

Our first main result now reads as follows.

2.4. **Theorem.** *The following statements are equivalent:*
  (1) *$\mathcal{M}$ is prelocally invariant for (2.1).*
  (2) *We have (1.2)–(1.4) and (1.6)–(1.8).*

In either case, A and the mapping in (1.7) are continuous on $\mathcal{M}$, and for each $h_0 \in \mathcal{M}$ there is a local strong solution $r = r^{(h_0)}$ to (2.1). Moreover, if instead of (1.4) we even have

$$(2.10) \qquad h + \gamma(h, x) \in \mathcal{M} \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in \mathcal{M},$$

then $\mathcal{M}$ is locally invariant for (2.1).

2.5. **Remark.** *It follows from Theorem 2.4 that (pre-)local invariance of $\mathcal{M}$ is a property which only depends on the parameters $\{\alpha, \sigma^j, \gamma, F\}$ – that is, on the law of the solution to (2.1). It does not depend on the actual stochastic basis $\{(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}), W, \mu\}$.*

Note that local invariance of $\mathcal{M}$ does not imply (2.10), as the following example illustrates:

2.6. **Example.** *Let $H = \mathbb{R}$, $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $\mathcal{M} = [0, 1)$ and consider the SDE*

$$(2.11) \qquad \begin{cases} dr_t &= dt + \int_{\mathbb{R}} \gamma(r_{t-}, x) \mu(dt, dx) \\ r_0 &= h_0, \end{cases}$$

where the compensator $dt \otimes F(dx)$ of μ is given by the Dirac measure $F = \delta_1$ concentrated in 1, and

$$\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \gamma(h, x) = 1 - 2h.$$

Then $\mathcal{M}$ is locally invariant for (2.11). Indeed, let $h_0 \in \mathcal{M}$ be arbitrary. There exists $\epsilon > 0$ with $h_0 + \epsilon < 1$. We define the stopping time $\tau > 0$ as

$$\tau := \inf\{t \geq 0 : r_t = h_0 + \epsilon\} \wedge \inf\{t \geq 0 : \mu([0, t] \times \mathbb{R}) = 1\}.$$

Then we have $(r^{(h_0)})^\tau \in \mathcal{M}$ up to an evanescent set, because

$$h + \gamma(h, x) = 1 - h \in \mathcal{M}, \quad h \in (0, 1)$$

showing that $\mathcal{M}$ is locally invariant for (2.11). However, the jump condition (2.10) is not satisfied, because for $h = 0$ we have

$$h + \gamma(h, x) = 1 \notin \mathcal{M}.$$

Nevertheless, we see that condition (1.4) holds true, because $1 \in \overline{\mathcal{M}}$.

If $\mathcal{M}$ is a closed subset of H and global Lipschitz conditions are satisfied, then we obtain global invariance. This is the content of our second main result, for which we recall the following definition:

2.7. **Definition.** *The semigroup $(S_t)_{t \geq 0}$ is called* pseudo-contractive, *if*

$$\|S_t\| \leq e^{\omega t}, \quad t \geq 0$$

for some constant $\omega \in \mathbb{R}$.

Now our second main result reads as follows:

2.8. **Theorem.** *Assume that the semigroup $(S_t)_{t \geq 0}$ is pseudo-contractive and that conditions (2.2)–(2.7) hold globally, i.e. the coefficients L_n, $(\kappa_n^j)_{j \in \mathbb{N}}$, ρ_n do not depend on $n \in \mathbb{N}$, and with the right-hand sides of (2.4), (2.7) multiplied by $(1 + \|h\|)$. If $\mathcal{M}$ is a closed subset of H, then (1.2)–(1.4) and (1.6)–(1.8) imply that for any $h_0 \in \mathcal{M}$ there exists a unique strong solution $r = r^{(h_0)}$ to (2.1) and $r \in \mathcal{M}$ up to an evanescent set.*

2.9. **Remark.** *Let us comment on the pseudo-contractivity of the semigroup, which we have imposed for Theorem 2.8. Together with the global Lipschitz conditions, it ensures existence and uniqueness of mild solutions to the SPDE (2.1) with càdlàg sample paths, which we require for the proof. In the general situation, where the semigroup fulfills the estimate*

$$\|S_t\| \leq Me^{\omega t}, \quad t \geq 0$$

for constants $M \geq 1$ and $\omega \in \mathbb{R}$, the global Lipschitz conditions ensure existence and uniqueness of mild solutions, but it is generally not known whether they have a càdlàg version. However, we remark that in the continuous case $\gamma \equiv 0$ we obtain the existence of continuous mild solutions without the pseudo-contractivity of the semigroup; see, e.g., [6].

2.10. **Remark.** *Note that we have not imposed the pseudo-contractivity of the semigroup for Theorem 2.4. Under the conditions of this result, the existence of locally invariant mild solutions to the SPDE (2.1) follows from the existence of locally invariant strong solutions to the finite dimensional SDEs (4.1), (5.26), and this does not require assumptions on the semigroup.*

The above two theorems simplify in the case of jumps with finite variation:

2.11. **Theorem.** *Assume that*

$$(2.12) \qquad \int_E \|\gamma(h,x)\| F(dx) < \infty \quad \text{for all } h \in \mathcal{M}.$$

Then the following statements are true:

(1) *Theorems 2.4 and 2.8 remain true with (1.6)–(1.8) being replaced by (1.5).*
(2) *Suppose that even the following stronger condition than (2.12) is satisfied: For each $n \in \mathbb{N}$ there exists a measurable function $\theta_n : E \to \mathbb{R}_+$ with $\int_E \theta_n(x) F(dx) < \infty$ such that*

$$(2.13) \qquad \|\gamma(h,x)\| \leq \theta_n(x) \quad \text{for all } h \in \mathcal{M} \text{ with } \|h\| \leq n \text{ and all } x \in E.$$

Then, in addition to statement (1), the mapping in (1.5) is continuous on $\mathcal{M}$.

3. Notation and auxiliary results about stochastic invariance

In this section, we provide some notation and auxiliary results about stochastic invariance which we will use for the proofs our main results. In the sequel, for $h_0 \in H$ and $\epsilon > 0$ we denote by $B_\epsilon(h_0)$ the open ball

$$B_\epsilon(h_0) = \{h \in H : \|h - h_0\| < \epsilon\}.$$

For technical reasons, we will also need the following concept of prelocal invariance:

3.1. **Definition.** *Let $B_1 \subset B_2 \subset H$ be two nonempty Borel sets. B_1 is called prelocally invariant in B_2 for (2.1), if for all $h_0 \in B_1$ there exists a local mild solution $r = r^{(h_0)}$ to (2.1) with lifetime $\tau > 0$ such that $(r^\tau)_- \in B_1$ and $r^\tau \in B_2$ up to an evanescent set.*

3.2. **Remark.** *Note that any non-empty Borel set $B \subset H$ is prelocally invariant for (2.1) in the sense of Definition 2.1 if and only if B is prelocally invariant in $\overline{B}$ for (2.1) in the sense of Definition 3.1.*

We proceed with some auxiliary results about stochastic invariance which we will use later on. For the proofs we refer to [14, Lemmas 2.11–2.16].

3.3. Lemma. *Let $B_1 \subset B_2 \subset H$ be two Borel sets such that B_1 is prelocally invariant in B_2 for (2.1). Then we have*

$$h + \gamma(h, x) \in \overline{B}_2 \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in B_1.$$

3.4. Lemma. *Let $B_1 \subset B_2 \subset H$ be two Borel sets such that*

$$h + \gamma(h, x) \in B_2 \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in B_1.$$

Let $h_0 : \Omega \to H$ be a $\mathcal{F}_0$-measurable random variable and let $r = r^{(h_0)}$ be a local mild solution to (2.1) with lifetime $\tau > 0$ such that $(r^\tau)_- \in B_1$ and $r^\tau \mathbb{1}_{[\![0,\tau[\![} \in B_2$ up to an evanescent set. Then we have $r^\tau \in B_2$ up to an evanescent set.

3.5. Lemma. *Let $B \subset C \subset H$ be two Borel sets such that C is closed in H and*

$$h + \gamma(h, x) \in C \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in B.$$

Let $h_0 : \Omega \to H$ be a $\mathcal{F}_0$-measurable random variable and let $r = r^{(h_0)}$ be a local mild solution to (2.1) with lifetime $\tau > 0$ such that $(r^\tau)_- \in B$ up to an evanescent set. Then we have $r^\tau \in C$ up to an evanescent set.

3.6. Lemma. *Let G_1, G_2 be metric spaces such that G_1 is separable. Let $B \subset G_1$ be a Borel set, let $C \subset G_2$ be a closed set and let $\delta : G_1 \times E \to G_2$ be a measurable mapping such that $\delta(\bullet, x) : G_1 \to G_2$ is continuous for all $x \in E$. Suppose that*

$$\delta(h, x) \in C \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in B.$$

Then we even have

$$\delta(h, x) \in C \quad \text{for all } h \in B, \quad \text{for } F\text{-almost all } x \in E.$$

3.7. Lemma. *Let $(G, \mathcal{G}, \nu)$ be a σ-finite measure space, let $C \subset H$ be a closed, convex cone and let $f \in \mathcal{L}^1(G; H)$ be such that $f(x) \in C$ for ν-almost all $x \in G$. Then we have*

$$\int_G f \, d\nu \in C.$$

3.8. Lemma. *Let $C \subset H$ be a closed, convex cone and let $\delta : \Omega \times \mathbb{R}_+ \times E \to H$ be an optional process satisfying*

$$\mathbb{P}\left(\int_0^t \int_E \|\delta(s, x)\| \mu(ds, dx) < \infty \right) = 1 \quad \text{for all } t \geq 0$$

such that

$$\delta(\bullet, x) \in C \quad \text{up to an evanescent set}, \quad \text{for } F\text{-almost all } x \in E.$$

Then we have $X \in C$ up to an evanescent set, where X denotes the integral process

$$X_t := \int_0^t \int_E \delta(s, x) \mu(ds, dx), \quad t \geq 0.$$

4. Local analysis of the invariance problem on half spaces

As a first building block for the proof of Theorem 2.4, our goal of this section is the proof of Theorem 4.1, which provides a local version of Theorem 2.4 in the particular situation where the manifold is an open subset of a half space. More precisely, fix an arbitrary $m \in \mathbb{N}$ and consider the $\mathbb{R}^m$-valued SDE

(4.1)
$$\begin{cases} dY_t &= \Theta(Y_t)dt + \sum_{j \in \mathbb{N}} \Sigma^j(Y_t)d\beta_t^j + \int_E \Gamma(Y_{t-}, x)(\mu(dt, dx) - F(dx)dt) \\ Y_0 &= y_0. \end{cases}$$

We assume that the mappings $\Theta : \mathbb{R}^m \to \mathbb{R}^m$, $\Sigma^j : \mathbb{R}^m \to \mathbb{R}^m$, $j \in \mathbb{N}$ and $\Gamma : \mathbb{R}^m \times E \to \mathbb{R}^m$ satisfy the regularity conditions (2.2)–(2.4) and (2.6)–(2.8). Instead

of (2.5), we only demand that the mappings $\rho_n : E \to \mathbb{R}_+$, $n \in \mathbb{N}$ appearing in (2.6), (2.7) satisfy (2.9).

Let V be an open subset of the half space $\mathbb{R}_+^m = \mathbb{R}_+ \times \mathbb{R}^{m-1}$, on which we consider the relative topology. Let $\partial V = \{y \in V : y_1 = 0\}$ be the set of all boundary points of V. Let $O_V \subset C_V \subset V$ be subsets such that O_V is open in V and C_V is compact. In the sequel, we equip $\mathbb{R}^m$ with the Euclidean inner product and denote by $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^m$ the first unit vector.

4.1. **Theorem.** *The following statements are equivalent:*

(1) *O_V is prelocally invariant in C_V for (4.1).*
(2) *We have*

$$(4.2) \qquad \Sigma^j(y) \in T_y \partial V, \quad y \in O_V \cap \partial V, \quad \text{for all } j \in \mathbb{N},$$

$$(4.3) \qquad y + \Gamma(y, x) \in C_V \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } y \in O_V,$$

$$(4.4) \qquad \int_E |\langle e_1, \Gamma(y, x) \rangle| F(dx) < \infty, \quad y \in O_V \cap \partial V,$$

$$(4.5) \qquad \langle e_1, \Theta(y) \rangle - \int_E \langle e_1, \Gamma(y, x) \rangle F(dx) \geq 0, \quad y \in O_V \cap \partial V.$$

Proof. For the sake of simplicity, we agree to write $O := O_V$, $\partial O := O \cap \partial V$ and $C := C_V$ during the proof.

(1) $\Rightarrow$ (2): Let $y \in O$ be arbitrary. Since O is prelocally invariant in C for (4.1), there exists a local strong solution $Y = Y^{(y)}$ to (4.1) with lifetime $\tau > 0$ such that $(Y^\tau)_- \in O$ and $Y^\tau \in C$ up to an evanescent set. Thus, Lemma 3.3 yields (4.3), and for every finite stopping time $\varrho \leq \tau$ we have

$$(4.6) \qquad\qquad \mathbb{P}(\langle e_1, Y_\varrho \rangle \geq 0) = 1.$$

From now on, we assume that $y \in \partial O$. Let $(\Phi^j)_{j \in \mathbb{N}} \subset \mathbb{R}$ be a sequence with $\Phi^j \neq 0$ for only finitely many $j \in \mathbb{N}$, and let $\Psi : E \to \mathbb{R}$ be a measurable function of the form $\Psi = c\mathbb{1}_B$ with $c > -1$ and $B \in \mathcal{E}$ satisfying $F(B) < \infty$. Let Z be the Doléans-Dade exponential

$$Z = \mathcal{E}\left(\sum_{j \in \mathbb{N}} \Phi^j \beta^j + \int_0^\bullet \int_E \Psi(x)(\mu(ds, dx) - F(dx)ds) \right).$$

By [18, Theorem I.4.61] the process Z is a solution of

$$Z_t = 1 + \sum_{j \in \mathbb{N}} \Phi^j \int_0^t Z_s d\beta_s^j + \int_0^t \int_E Z_{s-} \Psi(x)(\mu(ds, dx) - F(dx)ds), \quad t \geq 0$$

and, since $\Psi > -1$, the process Z is a strictly positive local martingale. There exists a strictly positive stopping time τ_1 such that Z^{τ_1} is a martingale. Integration by parts (see [18, Theorem I.4.52]) yields

$$(4.7) \qquad \begin{aligned} \langle e_1, Y_t \rangle Z_t &= \int_0^t \langle e_1, Y_{s-} \rangle dZ_s + \int_0^t Z_{s-} d\langle e_1, Y_s \rangle \\ &\quad + \langle\langle e_1, Y^c \rangle, Z^c \rangle_t + \sum_{s \leq t} \langle e_1, \Delta Y_s \rangle \Delta Z_s, \quad t \geq 0. \end{aligned}$$

Taking into account the dynamics (4.1), we have

$$(4.8) \qquad \langle\langle e_1, Y^c \rangle, Z^c \rangle_t = \sum_{j \in \mathbb{N}} \Phi^j \int_0^t Z_s \langle e_1, \Sigma^j(Y_s) \rangle ds, \quad t \geq 0,$$

$$(4.9) \qquad \sum_{s \leq t} \langle e_1, \Delta Y_s \rangle \Delta Z_s = \int_0^t \int_E Z_{s-} \Psi(x) \langle e_1, \Gamma(Y_{s-}, x) \rangle \mu(ds, dx), \quad t \geq 0.$$

Incorporating (4.1), (4.8) and (4.9) into (4.7), we obtain

$$
(4.10) \quad \langle e_1, Y_t \rangle Z_t = M_t + \int_0^t Z_{s-} \Bigg(\langle e_1, \Theta(Y_{s-}) \rangle + \sum_{j \in \mathbb{N}} \Phi^j \langle e_1, \Sigma^j(Y_{s-}) \rangle \\
+ \int_E \Psi(x) \langle e_1, \Gamma(Y_{s-}, x) \rangle F(dx) \Bigg) ds, \quad t \geq 0,
$$

where M is a local martingale with $M_0 = 0$. There exists a strictly positive stopping time τ_2 such that M^{τ_2} is a martingale.

By the continuity of Θ there exist a strictly positive stopping time τ_3 and a constant $\tilde{\Theta} > 0$ such that

$$
|\langle e_1, \Theta(Y_{(t \wedge \tau_3)-}) \rangle| \leq \tilde{\Theta}, \quad t \geq 0.
$$

Suppose that $\Sigma^j(y) \notin T_y \partial V$, i.e. $\langle e_1, \Sigma^j(y) \rangle \neq 0$, for some $j \in \mathbb{N}$. By the continuity of Σ there exist $\eta > 0$ and a strictly positive stopping time $\tau_4 \leq 1$ such that

$$
|\langle e_1, \Sigma^j(Y_{(t \wedge \tau_4)-}) \rangle| \geq \eta, \quad t \geq 0.
$$

Let $(\Phi_k)_{k \in \mathbb{N}} \subset \mathbb{R}$ be the sequence given by

$$
\Phi^k = \begin{cases} -\operatorname{sign}(\langle e_1, \Sigma^k(y) \rangle) \frac{\tilde{\Theta} + 1}{\eta}, & k = j, \\ 0, & k \neq j. \end{cases}
$$

Furthermore, let $\Psi := 0$ and $\varrho := \tau \wedge \tau_1 \wedge \tau_2 \wedge \tau_3 \wedge \tau_4$. Taking expectation in (4.10) yields $\mathbb{E}[\langle e_1, Y_\varrho \rangle Z_\varrho] < 0$, implying $\mathbb{P}(\langle e_1, Y_\varrho \rangle < 0) > 0$, which contradicts (4.6). This proves (4.2).

Now suppose $\int_E |\langle e_1, \Gamma(y, x) \rangle| F(dx) = \infty$. By the Cauchy-Schwarz inequality, for all $B \in \mathcal{E}$ with $F(B) < \infty$ the map $y \mapsto \int_B \Gamma(y, x) F(dx)$ is continuous. Using the σ-finiteness of F, there exist $B \in \mathcal{E}$ with $F(B) < \infty$ and a strictly positive stopping time $\tau_4 \leq 1$ such that

$$
-\frac{1}{2} \int_B |\langle e_1, \Gamma(Y_{(t \wedge \tau_4)-}, x) \rangle| F(dx) \leq -(\tilde{\Theta} + 1), \quad t \geq 0.
$$

Let $\Phi := 0$, $\Psi := -\frac{1}{2} \mathbb{1}_B$ and $\varrho := \tau \wedge \tau_1 \wedge \tau_2 \wedge \tau_3 \wedge \tau_4$. Taking expectation in (4.10) we obtain $\mathbb{E}[\langle e_1, Y_\varrho \rangle Z_\varrho] < 0$, implying $\mathbb{P}(\langle e_1, Y_\varrho \rangle < 0) > 0$, which contradicts (4.6). This yields (4.4).

Since F is σ-finite, there exists a sequence $(B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ with $B_n \uparrow E$ and $F(B_n) < \infty$, $n \in \mathbb{N}$. We shall show for all $n \in \mathbb{N}$ the relation

$$
(4.11) \quad \langle e_1, \Theta(y) \rangle + \int_E \Psi_n(x) \langle e_1, \Gamma(y, x) \rangle F(dx) \geq 0,
$$

where $\Psi_n := -(1 - \frac{1}{n}) \mathbb{1}_{B_n}$. Suppose, on the contrary, that (4.11) is not satisfied for some $n \in \mathbb{N}$. Then there exist $\eta > 0$ and a strictly positive stopping time $\tau_4 \leq 1$ such that

$$
\langle e_1, \Theta(Y_{(t \wedge \tau_4)-}) \rangle + \int_E \Psi_n(x) \langle e_1, \Gamma(Y_{(t \wedge \tau_4)-}, x) \rangle F(dx) \leq -\eta, \quad t \geq 0.
$$

Let $\Phi := 0$ and $\varrho := \tau \wedge \tau_1 \wedge \tau_2 \wedge \tau_3 \wedge \tau_4$. Taking expectation in (4.10) we obtain $\mathbb{E}[\langle e_1, Y_\varrho \rangle Z_\varrho] < 0$, implying $\mathbb{P}(\langle e_1, Y_\varrho \rangle < 0) > 0$, which contradicts (4.6). This yields (4.11). By (4.11), (4.4) and Lebesgue's dominated convergence theorem, we conclude (4.5).

$(2) \Rightarrow (1)$: The metric projection $\Pi = \Pi_{\mathbb{R}_+^m} : \mathbb{R}^m \to \mathbb{R}_+^m$ on the half space $\mathbb{R}_+^m$ is given by

$$
(4.12) \quad \Pi(y^1, y^2, \ldots, y^m) = ((y^1)^+, y^2, \ldots, y^m),
$$

and therefore, it satisfies

$$\|\Pi(y_1) - \Pi(y_2)\| \leq \|y_1 - y_2\| \quad \text{for all } y_1, y_2 \in \mathbb{R}^m.$$

Consequently, the mappings $\Theta_\Pi : \mathbb{R}^m \to \mathbb{R}^m$, $\Sigma_\Pi^j : \mathbb{R}^m \to \mathbb{R}^m$, $j \in \mathbb{N}$ and $\Gamma_\Pi : \mathbb{R}^m \times E \to \mathbb{R}^m$ defined as

$$\Theta_\Pi := \Theta \circ \Pi, \quad \Sigma_\Pi^j := \Sigma^j \circ \Pi \quad \text{and} \quad \Gamma_\Pi(\bullet, x) := \Gamma(\bullet, x) \circ \Pi$$

also satisfy the regularity conditions (2.2)–(2.4) and (2.6)–(2.8), which ensures existence and uniqueness of local strong solutions to the SDE

$$(4.13) \quad \begin{cases} dY_t &= \Theta_\Pi(Y_t)dt + \sum_{j\in\mathbb{N}} \Sigma_\Pi^j(Y_t)d\beta_t^j \\ &\quad + \int_E \Gamma_\Pi(Y_{t-}, x)(\mu(dt, dx) - F(dx)dt) \\ Y_0 &= y_0. \end{cases}$$

Now, let $y_0 \in O$ be arbitrary. Then there exists a local strong solution Y to (4.13) with $Y_0 = y_0$ and some lifetime $\tau > 0$. First, suppose that $y_0 \notin \partial O$. Then there exists $\epsilon > 0$ such that $\overline{B_\epsilon(y_0)} \subset O$. We define the strictly positive stopping time

$$\varrho := \inf\{t \geq 0 : Y_t \notin B_\epsilon(y_0)\} \wedge \tau.$$

Then we have

$$(Y^\varrho)_- \in \overline{B_\epsilon(y_0)} \subset O.$$

Using (4.3) and Lemma 3.4 we obtain $Y^\varrho \in C$ up to an evanescent set.

From now on, we suppose that $y_0 \in \partial O$. Then there exists $\epsilon > 0$ such that $\overline{B_\epsilon(y_0)} \cap \mathbb{R}_+^m \subset O$. We define the strictly positive stopping time

$$\varrho := \inf\{t \geq 0 : Y_t \notin B_\epsilon(y_0)\} \wedge \tau.$$

Setting

$$P := \overline{B_\epsilon(y_0)} \quad \text{and} \quad \mathbb{R}_-^m := \{y \in \mathbb{R}^m : y_1 \leq 0\},$$

by taking into account that the metric projection Π on $\mathbb{R}_+^m$ is given by (4.12), we have

$$(4.14) \quad \Pi(y) \in \partial O, \quad y \in P \cap \mathbb{R}_-^m.$$

By (4.12) and (4.3), for all $y \in P \cap \mathbb{R}_+^m$ we have

$$\langle e_1, y + \xi\Gamma_\Pi(y, x)\rangle = (1 - \xi)\langle e_1, y\rangle + \xi(\langle e_1, y\rangle + \langle e_1, \Gamma_\Pi(y, x)\rangle)$$
$$(4.15) \quad = (1 - \xi)\langle e_1, y\rangle + \xi\langle e_1, y + \Gamma(y, x)\rangle \geq 0 \quad \text{for all } \xi \in [0, 1],$$
$$\text{for } F\text{-almost all } x \in E.$$

Furthermore, by (4.2)–(4.5) and (4.14), for all $y \in P \cap \mathbb{R}_-^m$ we have

$$(4.16) \quad \langle e_1, \Sigma_\Pi^j(y)\rangle = \langle e_1, \Sigma^j(\Pi(y))\rangle = 0, \quad \text{for all } j \in \mathbb{N},$$

$$(4.17) \quad \langle e_1, \Gamma_\Pi(y, x)\rangle = \langle e_1, \Pi(y)\rangle + \langle e_1, \Gamma(\Pi(y), x)\rangle$$
$$= \langle e_1, \Pi(y) + \Gamma(\Pi(y), x)\rangle \geq 0, \quad \text{for } F\text{-almost all } x \in E,$$

$$(4.18) \quad \int_E |\langle e_1, \Gamma_\Pi(y)\rangle|F(dx) = \int_E |\langle e_1, \Gamma(\Pi(y))\rangle|F(dx) < \infty,$$

$$(4.19) \quad \langle e_1, \Theta_\Pi(y)\rangle - \int_E \langle e_1, \Gamma_\Pi(y, x)\rangle F(dx)$$

$$= \langle e_1, \Theta(\Pi(y))\rangle - \int_E \langle e_1, \Gamma(\Pi(y), x)\rangle F(dx) \geq 0.$$

The function $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(y) := (-y^3)^+$ is of class $C^2(\mathbb{R})$ and we have $\phi'(y) < 0$ for $y < 0$ and $\phi'(y) = \phi''(y) = 0$ for $y \geq 0$. By (4.15)–(4.19) and Lemma 3.6, we obtain

$$(4.20) \qquad \phi'(\langle e_1, y \rangle)\left(\langle e_1, \Theta_\Pi(y) \rangle - \int_E \langle e_1, \Gamma_\Pi(y, x) \rangle F(dx)\right) \leq 0, \quad y \in P$$

$$(4.21) \qquad \phi''(\langle e_1, y \rangle)|\langle e_1, \Sigma_\Pi^j(y) \rangle|^2 = 0, \quad y \in P, \quad \text{for all } j \in \mathbb{N}$$

$$(4.22) \qquad \phi'(\langle e_1, y \rangle)\langle e_1, \Sigma_\Pi^j(y) \rangle = 0, \qquad y \in P, \quad \text{for all } j \in \mathbb{N}$$

$$(4.23) \qquad \left(\int_0^1 \phi'(\langle e_1, y + \xi\Gamma_\Pi(y, x) \rangle)d\xi\right)\langle e_1, \Gamma_\Pi(y, x) \rangle \leq 0 \quad \text{for all } y \in P,$$
$$\text{for } F\text{-almost all } x \in E.$$

Applying Itô's formula (see [18, Theorem I.4.57]) yields $\mathbb{P}$-almost surely

$$\phi(\langle e_1, Y_{t\wedge\varrho} \rangle) = \phi(\langle e_1, y_0 \rangle)$$
$$+ \int_0^{t\wedge\varrho}\left(\phi'(\langle e_1, Y_s \rangle)\langle e_1, \Theta_\Pi(Y_s) \rangle + \frac{1}{2}\sum_{j\in\mathbb{N}}\phi''(\langle e_1, Y_s \rangle)|\langle e_1, \Sigma_\Pi^j(Y_s) \rangle|^2\right.$$
$$+ \int_E\left(\phi(\langle e_1, Y_s + \Gamma_\Pi(Y_s, x) \rangle) - \phi(\langle e_1, Y_s \rangle)\right.$$
$$\left.\left. - \phi'(\langle e_1, Y_s \rangle)\langle e_1, \Gamma_\Pi(Y_s, x) \rangle\right)F(dx)\right)ds$$
$$+ \sum_{j\in\mathbb{N}}\int_0^{t\wedge\varrho}\phi'(\langle e_1, Y_s \rangle)\langle e_1, \Sigma_\Pi^j(Y_s) \rangle d\beta_s^j$$
$$+ \int_0^{t\wedge\varrho}\int_E\left(\langle e_1, \phi(\langle e_1, Y_{s-} + \Gamma_\Pi(Y_{s-}, x) \rangle) - \phi(\langle e_1, Y_{s-} \rangle)\right)$$
$$(\mu(ds, dx) - F(dx)ds), \quad t \geq 0.$$

By (4.18) and Taylor's theorem we obtain $\mathbb{P}$-almost surely

$$\phi(\langle e_1, Y_{t\wedge\varrho} \rangle)$$
$$= \int_0^{t\wedge\varrho}\left[\phi'(\langle e_1, Y_s \rangle)\left(\langle e_1, \Theta_\Pi(Y_s) \rangle - \int_E\langle e_1, \Gamma_\Pi(Y_s, x) \rangle F(dx)\right)\right.$$
$$\left. + \frac{1}{2}\sum_{j\in\mathbb{N}}\phi''(\langle e_1, Y_s \rangle)|\langle e_1, \Sigma_\Pi^j(Y_s) \rangle|^2\right]ds$$
$$+ \sum_{j\in\mathbb{N}}\int_0^{t\wedge\varrho}\phi'(\langle e_1, Y_s \rangle)\langle e_1, \Sigma_\Pi^j(Y_s) \rangle d\beta_s^j$$
$$+ \int_0^{t\wedge\varrho}\int_E\left(\int_0^1\phi'(\langle e_1, Y_{s-} + \xi\Gamma_\Pi(Y_{s-}, x) \rangle)d\xi\right)\langle e_1, \Gamma_\Pi(Y_{s-}, x) \rangle$$
$$\mu(ds, dx), \quad t \geq 0.$$

By (4.20)–(4.23) and Lemmas 3.7 and 3.8, we deduce that $\phi(\langle e_1, Y^\varrho \rangle) \leq 0$ up to an evanescent set. Therefore, we obtain on up to an evanescent set

$$(Y^\tau)_- \in \overline{B_\epsilon(y_0)} \cap \mathbb{R}_+^m \subset O.$$

Using (4.3) and Lemma 3.5 we obtain $Y^\tau \in C$ up to an evanescent set. Since $\Theta|_C = \Theta_\Pi|_C$, $\Sigma^j|_C = \Sigma_\Pi^j|_C$ for all $j \in \mathbb{N}$ and $\Gamma(\bullet, x)|_C = \Gamma_\Pi(\bullet, x)|_C$ for all $x \in E$, the process Y is also a local strong solution to (4.1) with lifetime ϱ, proving that O is prelocally invariant in C for (4.1). $\qquad\square$

Note that V is a m-dimensional C^3-submanifold with boundary of $\mathbb{R}^m$, and that for $y \in \partial V$ the inward pointing normal vector to ∂V at y is given by the first unit vector $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^m$. In order to see that for the submanifold V conditions (4.2)–(4.5) resemble conditions (1.2)–(1.4) and (1.6)–(1.8), we require the following auxiliary result.

4.2. Lemma. *Suppose that (4.2) is satisfied. Then for all $j \in \mathbb{N}$ we have*

$$\langle e_1, D\Sigma^j(y)\Sigma^j(y)\rangle = 0, \quad y \in O_V \cap \partial V.$$

Proof. The statement is a consequence of [14, Lemma 3.13]. $\qquad\square$

5. Local analysis of the invariance problem on submanifolds with boundary

As next building block for the proof of Theorem 2.4, our goal of this section is the proof of Theorem 5.3, which provides a local version of Theorem 2.4. We assume that for the m-dimensional C^3-submanifold $\mathcal{M}$ with boundary of H there exist

- a m-dimensional C^3-submanifold $\mathcal{N}$ with boundary of $\mathbb{R}^m$,
- parametrizations $\phi : V \subset \mathbb{R}^m_+ \to \mathcal{M}$ and $\psi : V \subset \mathbb{R}^m_+ \to \mathcal{N}$,
- and elements $\zeta_1, \ldots, \zeta_m \in \mathcal{D}(A^*)$ such that the mapping $f := \phi \circ \psi^{-1} : \mathcal{N} \to \mathcal{M}$ has the inverse

(5.1) $\qquad f^{-1} : \mathcal{M} \to \mathcal{N}, \quad f^{-1}(h) = \langle \zeta, h\rangle := (\langle\zeta_1, h\rangle, \ldots, \langle\zeta_m, h\rangle).$

In other words, the diagram

(5.2)
$$
\begin{array}{ccc}
\mathcal{N} \subset \mathbb{R}^m & \xrightleftharpoons[\langle\zeta,\bullet\rangle]{f} & \mathcal{M} \subset H \\
& {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle\phi} & \\
& V \subset \mathbb{R}^m_+ &
\end{array}
$$

commutes.

5.1. Remark. *According to [14, Proposition 3.11], for an arbitrary C^3-submanifold $\mathcal{M}$ with boundary of H and an arbitrary point $h_0 \in \mathcal{M}$ there always exists a neighborhood of h_0 such that a diagram of form (5.2) exists and commutes. We will use this result for the global analysis of the invariance problem in Section 6.*

5.2. Remark. *For a C^3-submanifold $\mathcal{M}$ without boundary there even exist local parametrizations $\phi : V \subset \mathbb{R}^m \to U \cap \mathcal{M}$ with inverses being of the form $\langle\zeta, \bullet\rangle$ for some $\zeta_1, \ldots, \zeta_m \in \mathcal{D}(A^*)$, see [11]. In the present situation, where $\mathcal{M}$ is a submanifold with boundary, this is generally not possible, and thus, we consider the situation where the diagram (5.2) commutes.*

Let $O_\mathcal{M} \subset C_\mathcal{M} \subset \mathcal{M}$ be subsets. We assume that $O_\mathcal{M}$ is open in $\mathcal{M}$ and $C_\mathcal{M}$ is compact. Our announced main result of this section reads as follows.

5.3. Theorem. *The following statements are equivalent:*

(1) $O_\mathcal{M}$ *is prelocally invariant in* $C_\mathcal{M}$ *for (2.1).*

(2) *The following conditions are satisfied:*

(5.3) $O_{\mathcal{M}} \subset \mathcal{D}(A),$

(5.4) $\sigma^j(h) \in T_h\mathcal{M}, \quad h \in O_{\mathcal{M}}, \quad j \in \mathbb{N},$

(5.5) $\sigma^j(h) \in T_h\partial\mathcal{M}, \quad h \in O_{\mathcal{M}} \cap \partial\mathcal{M}, \quad j \in \mathbb{N},$

(5.6) $h + \gamma(h,x) \in C_{\mathcal{M}} \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in O_{\mathcal{M}},$

(5.7) $\displaystyle\int_E |\langle \eta_h, \gamma(h,x)\rangle| F(dx) < \infty, \quad h \in O_{\mathcal{M}} \cap \partial\mathcal{M},$

(5.8) $\displaystyle Ah + \alpha(h) - \frac{1}{2}\sum_{j\in\mathbb{N}} D\sigma^j(h)\sigma^j(h)$

$\displaystyle \qquad\qquad - \int_E \Pi_{(T_h\mathcal{M})^\perp}\gamma(h,x)F(dx) \in T_h\mathcal{M}, \quad h \in O_{\mathcal{M}},$

(5.9) $\displaystyle \langle \eta_h, Ah + \alpha(h)\rangle - \frac{1}{2}\sum_{j\in\mathbb{N}}\langle \eta_h, D\sigma^j(h)\sigma^j(h)\rangle$

$\displaystyle \qquad\qquad - \int_E \langle \eta_h, \gamma(h,x)\rangle F(dx) \geq 0, \quad h \in O_{\mathcal{M}} \cap \partial\mathcal{M}.$

In either case, A and the mapping in (5.8) are continuous on $O_{\mathcal{M}}$.

Our strategy for proving Theorem 5.3 can be divided into the following steps:

- Define the $\mathbb{R}^m$-valued SDE (5.26), whose coefficients a, b^j, c are given by pull-backs in terms of α, σ^j, γ.
- Define the $\mathbb{R}^m$-valued SDE (4.1), whose coefficients Θ, Σ^j, Γ are given by pull-backs in terms of a, b^j, c.
- Provide conditions (4.2)–(4.5) for invariance of V for the SDE (4.1); this has already been established in Theorem 4.1.
- Translate these conditions into conditions (5.17)–(5.22) regarding invariance of $\mathcal{N}$ for the SDE (5.26).
- Translate these conditions into conditions (5.3)–(5.9) regarding invariance of $\mathcal{M}$ for the original SPDE (2.1).

Now, we start with the formal proofs. First, we prepare an auxiliary result.

5.4. Lemma. *The following statements are true:*

(1) *For each $h \in H$ we have*

$$\sum_{j\in\mathbb{N}} \|D\sigma^j(h)\sigma^j(h)\| < \infty,$$

and the mapping

$$H \to H, \quad h \mapsto \sum_{j\in\mathbb{N}} D\sigma^j(h)\sigma^j(h)$$

is continuous.

(2) *If (5.6) is satisfied, then for each $h \in O_{\mathcal{M}}$ we have*

$$\int_E \|\Pi_{(T_h\mathcal{M})^\perp}\gamma(h,x)\|F(dx) < \infty,$$

and the mapping

$$O_{\mathcal{M}} \to H, \quad h \mapsto \int_E \Pi_{(T_h\mathcal{M})^\perp}\gamma(h,x)F(dx)$$

is continuous.

Proof. This follows from [14, Lemma 2.17 and Corollary 3.28]. $\qquad\square$

Let G be another separable Hilbert space. For any $k \in \mathbb{N}$ we denote by $C_b^k(G; H)$ the linear space consisting of all $f \in C^k(G; H)$ such that $D^i f$ is bounded for all $i = 1, \ldots, k$. In particular, for each $f \in C_b^k(G; H)$ the mappings $D^i f$, $i = 0, \ldots, k-1$ are Lipschitz continuous. We do not demand that f itself is bounded, as this would exclude continuous linear operators $f \in L(G; H)$.

5.5. Definition. *Let $\alpha : H \to H$, $\sigma^j : H \to H$, $j \in \mathbb{N}$ and $\gamma : H \times E \to H$ be mappings satisfying*

$$(5.10) \qquad \sum_{j \in \mathbb{N}} \|\sigma^j(h)\|^2 < \infty \quad and \quad \int_E \|\gamma(h, x)\|^2 F(dx) < \infty$$

for all $h \in H$, and let $f : G \to H$ and $g \in C_b^2(H; G)$ be mappings. We define the mappings $(f, g)_\lambda^\star \alpha : G \to G$, $(f, g)_W^\star \sigma^j : G \to G$, $j \in \mathbb{N}$ and $(f, g)_\mu^\star \gamma : G \times E \to G$ as

$$(5.11) \qquad ((f, g)_\lambda^\star \alpha)(z) := Dg(h)\alpha(h) + \frac{1}{2} \sum_{j \in \mathbb{N}} D^2 g(h)(\sigma^j(h), \sigma^j(h))$$

$$+ \int_E \big(g(h + \gamma(h, x)) - g(h) - Dg(h)\gamma(h, x)\big) F(dx),$$

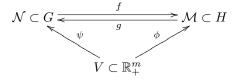$$(5.12) \qquad ((f, g)_W^\star \sigma^j)(z) := Dg(h)\sigma^j(h),$$

$$(5.13) \qquad ((f, g)_\mu^\star \gamma)(z, x) := g(h + \gamma(h, x)) - g(h),$$

where $h = f(z)$.

The following results show that the mappings from Definition 5.5 may be regarded as pull-backs for jump-diffusions. First, we provide sufficient conditions which ensure that the regularity conditions (2.2)–(2.4) and (2.6)–(2.8) are preserved.

5.6. Lemma. *Let $\alpha : H \to H$, $\sigma^j : H \to H$, $j \in \mathbb{N}$ and $\gamma : H \times E \to H$ be mappings satisfying the regularity conditions (2.2)–(2.4) and (2.6)–(2.8). Furthermore, let $f \in C_b^1(G; H)$ and $g \in C_b^3(H; G)$ be arbitrary. Then the following statements are true:*

(1) *The mappings $(f, g)_\lambda^\star \alpha$, $((f, g)_W^\star \sigma^j)_{j \in \mathbb{N}}$ and $(f, g)_\mu^\star \gamma$ also fulfill the regularity conditions (2.2)–(2.4) and (2.6)–(2.8), but with the mappings $\rho_n : E \to \mathbb{R}_+$, $n \in \mathbb{N}$ appearing in (2.6), (2.7) only satisfying (2.9) instead of (2.5).*

(2) *If $g \in L(H; G)$, then the mappings $\rho_n : E \to \mathbb{R}_+$, $n \in \mathbb{N}$ appearing in (2.6), (2.7) even satisfy (2.5).*

Proof. See [14, Lemma 2.24]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Recall that $\mathcal{M}$ denotes a C^3-submanifold with boundary of the separable Hilbert space H. Let $\mathcal{N}$ be a C^3-submanifold with boundary of G. We assume there exist parametrizations $\phi : V \to \mathcal{M}$ and $\psi : V \to \mathcal{N}$. Let $f := \phi \circ \psi^{-1} : \mathcal{N} \to \mathcal{M}$ and $g := f^{-1} : \mathcal{M} \to \mathcal{N}$. Then the diagram

$$\mathcal{N} \subset G \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{M} \subset H$$
$$\psi \searrow \qquad \swarrow \phi$$
$$V \subset \mathbb{R}_+^m$$

commutes. We assume that ϕ, ψ, $\Phi := \phi^{-1}$, $\Psi := \psi^{-1}$ have extensions $\phi \in C_b^3(\mathbb{R}^m; H)$, $\psi \in C_b^3(\mathbb{R}^m; G)$, $\Phi \in C_b^3(H; \mathbb{R}^m)$, $\Psi \in C_b^3(G; \mathbb{R}^m)$. Consequently, the mappings f, g have extensions $f \in C_b^3(G; H)$, $g \in C_b^3(H; G)$.

We define the subsets $O_\mathcal{N} \subset C_\mathcal{N} \subset \mathcal{N}$ by $O_\mathcal{N} := g(O_\mathcal{M})$ and $C_\mathcal{N} := g(C_\mathcal{M})$.

5.7. **Definition.** *Let $\beta : O_{\mathcal{M}} \to H$, $\sigma^j : O_{\mathcal{M}} \to H$, $j \in \mathbb{N}$ and $\gamma : O_{\mathcal{M}} \times E \to H$ be mappings satisfying (5.10) for all $h \in O_{\mathcal{M}}$. We define the mappings $f_\lambda^\star \beta : O_{\mathcal{N}} \to G$, $f_W^\star \sigma^j : O_{\mathcal{N}} \to G$, $j \in \mathbb{N}$ and $f_\mu^\star \gamma : O_{\mathcal{N}} \times E \to G$ as*

$$(f_\lambda^\star \beta)(z) := ((f,g)_\lambda^\star \beta)(z),$$
$$(f_W^\star \sigma^j)(z) := ((f,g)_W^\star \sigma^j)(z),$$
$$(f_\mu^\star \gamma)(z,x) := ((f,g)_\mu^\star \gamma)(z,x)$$

according to (5.11)–(5.13).

Let $a : G \to G$, $b^j : G \to G$, $j \in \mathbb{N}$ and $c : G \times E \to G$ be mappings satisfying the regularity conditions (2.2)–(2.4) and (2.6)–(2.8). In the sequel, for $z \in \partial \mathcal{N}$ the vector ξ_z denotes the inward pointing normal vector to $\partial \mathcal{N}$ at z.

The following result shows how the invariance conditions of Theorem 5.3 translate when we change to another manifold, and how this is related to the just defined pull-backs.

5.8. **Proposition.** *Suppose we have (5.3) and define $\beta : O_{\mathcal{M}} \to H$ as*

$$\beta(h) := Ah + \alpha(h), \quad h \in O_{\mathcal{M}}.$$

Moreover, we suppose that

$$(5.14) \qquad a(z) = (f_\lambda^\star \beta)(z), \quad z \in O_{\mathcal{N}},$$

$$(5.15) \qquad b^j(z) = (f_W^\star \sigma^j)(z), \quad j \in \mathbb{N} \text{ and } z \in O_{\mathcal{N}},$$

$$(5.16) \qquad c(z,x) = (f_\mu^\star \gamma)(z,x) \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } z \in O_{\mathcal{N}}.$$

Then the following statements are true:

(1) *If conditions (5.4)–(5.9) are satisfied, then we also have*

$$(5.17) \qquad b^j(z) \in T_z \mathcal{N}, \quad z \in O_{\mathcal{N}}, \quad j \in \mathbb{N},$$

$$(5.18) \qquad b^j(z) \in T_z \partial \mathcal{N}, \quad z \in O_{\mathcal{N}} \cap \partial \mathcal{N}, \quad j \in \mathbb{N},$$

$$(5.19) \qquad z + c(z,x) \in C_{\mathcal{N}} \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } z \in O_{\mathcal{N}},$$

$$(5.20) \qquad \int_E |\langle \xi_z, c(z,x) \rangle| F(dx) < \infty, \quad z \in O_{\mathcal{N}} \cap \partial \mathcal{N},$$

$$(5.21) \qquad a(z) - \frac{1}{2} \sum_{j \in \mathbb{N}} Db^j(z) b^j(z)$$
$$- \int_E \Pi_{(T_z \mathcal{N})^\perp} c(z,x) F(dx) \in T_z \mathcal{N}, \quad z \in O_{\mathcal{N}},$$

$$(5.22) \qquad \langle \xi_z, a(z) \rangle - \frac{1}{2} \sum_{j \in \mathbb{N}} \langle \xi_z, Db^j(z) b^j(z) \rangle$$
$$- \int_E \langle \xi_z, c(z,x) \rangle F(dx) \geq 0, \quad z \in O_{\mathcal{N}} \cap \partial \mathcal{N}.$$

(2) *If we have (5.4), (5.6) and (5.8), then we also have*

$$(5.23) \qquad \beta(h) = (g_\lambda^\star a)(h), \quad h \in O_{\mathcal{M}},$$

$$(5.24) \qquad \sigma^j(h) = (g_W^\star b^j)(h), \quad j \in \mathbb{N} \text{ and } h \in O_{\mathcal{M}},$$

$$(5.25) \qquad \gamma(h,x) = (g_\mu^\star c)(h,x) \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in O_{\mathcal{M}}.$$

Proof. This follows from [14, Propositions 3.23 and 3.33]. $\qquad \square$

Now, we consider the G-valued SDE

(5.26)
$$\begin{cases} dZ_t &=& a(Z_t)dt + \sum_{j\in\mathbb{N}} b^j(Z_t)d\beta_t^j + \int_E c(Z_{t-},x)(\mu(dt,dx) - F(dx)dt) \\ Z_0 &=& z_0. \end{cases}$$

For our subsequent analysis, the following technical definition will be useful.

5.9. **Definition.** *The set $O_\mathcal{M}$ is called* prelocally invariant *in $C_\mathcal{M}$ for (2.1) with solutions given by (5.26) and f, if for all $h_0 \in O_\mathcal{M}$ there exists a local strong solution $Z = Z^{(g(h_0))}$ to (5.26) with lifetime $\tau > 0$ such that $(Z^\tau)_- \in O_\mathcal{N}$ and $Z^\tau \in C_\mathcal{N}$ up to an evanescent set and $f(Z)$ is a local mild solution to (2.1) with initial condition h_0 and lifetime τ.*

5.10. **Lemma.** *Suppose $O_\mathcal{M}$ is prelocally invariant in $C_\mathcal{M}$ for (2.1) with solutions given by (5.26) and f. Then the following statements are true:*

(1) *$O_\mathcal{M}$ is prelocally invariant in $C_\mathcal{M}$ for (2.1).*
(2) *$O_\mathcal{N}$ is prelocally invariant in $C_\mathcal{N}$ for (5.26).*

Proof. This is an immediate consequence of Definitions 3.1 and 5.9. $\square$

The following result shows how the coefficients of locally invariant jump-diffusions translate when we change to another manifold; they are given by the respective pull-backs.

5.11. **Proposition.** *Let Z be a local strong solution to (5.26) for some initial condition $z_0 \in O_\mathcal{N}$ with lifetime $\tau > 0$ such that $(Z^\tau)_- \in O_\mathcal{N}$ and $Z^\tau \in C_\mathcal{N}$ up to an evanescent set. Then $r := f(Z)$ is a local strong solution to the SDE*

(5.27)
$$\begin{cases} dr_t &=& (g_\lambda^* a)(r_t)dt + \sum_{j\in\mathbb{N}} (g_W^* b^j)(r_t)d\beta_t^j \\ && + \int_E (g_\mu^* c)(r_{t-},x)(\mu(dt,dx) - F(dx)dt) \\ r_0 &=& h_0 \end{cases}$$

with initial condition $h_0 = f(z_0)$ and lifetime τ.

Proof. This follows from Itô's formula for jump-diffusions in infinite dimension; see [14, Proposition 2.25]. $\square$

If the generator A is continuous, then the just introduced invariance concept transfers to the sets $O_\mathcal{N}$ and $C_\mathcal{N}$.

5.12. **Lemma.** *Suppose $A \in L(H)$. Then the following statements are equivalent:*

(1) *$O_\mathcal{M}$ is prelocally invariant in $C_\mathcal{M}$ for (2.1) with solutions given by (5.26) and f.*
(2) *$O_\mathcal{N}$ is prelocally invariant in $C_\mathcal{N}$ for (5.26) with solutions given by (2.1) and g.*

Proof. (1) $\Rightarrow$ (2): Let $z_0 \in O_\mathcal{N}$ be arbitrary and set $h_0 := f(z_0) \in O_\mathcal{M}$. There exists a local strong solution $Z = Z^{(g(h_0))} = Z^{(z_0)}$ to (5.26) with lifetime $\tau > 0$ such that $(Z^\tau)_- \in O_\mathcal{N}$ and $Z^\tau \in C_\mathcal{N}$ up to an evanescent set, and, since $A \in L(H)$, the process $r = f(Z)$ is a local strong solution to (2.1) with initial condition $h_0 = f(z_0)$. Therefore, we have $(r^\tau)_- \in O_\mathcal{M}$ and $r^\tau \in C_\mathcal{M}$ up to an evanescent set, and $g(r)$ is a local strong solution to (5.26) with initial condition z_0 and lifetime τ, because $Z^\tau = g(r^\tau)$.
(2) $\Rightarrow$ (1): This implication is proven analogously. $\square$

5.13. **Proposition.** *The following statements are equivalent:*

(1) *$O_\mathcal{M}$ is prelocally invariant in $C_\mathcal{M}$ for (2.1) with solutions given by (5.26) and f.*

(2) $O_{\mathcal{N}}$ is prelocally invariant in $C_{\mathcal{N}}$ for (5.26) and we have

$$(5.28) \qquad O_{\mathcal{M}} \subset \mathcal{D}(A),$$

$$(5.29) \qquad (A + \alpha)(h) = (g_{\lambda}^{\star} a)(h) \quad \text{for all } h \in O_{\mathcal{M}},$$

$$(5.30) \qquad \sigma^j(h) = (g_W^{\star} b^j)(h) \quad \text{for all } j \in \mathbb{N}, \quad \text{for all } h \in O_{\mathcal{M}},$$

$$(5.31) \qquad \gamma(h, x) = (g_{\mu}^{\star} c)(h, x) \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in O_{\mathcal{M}}.$$

In either case, A is continuous on $O_{\mathcal{M}}$.

Proof. $(1) \Rightarrow (2)$: By Lemma 5.10 the set $O_{\mathcal{N}}$ is prelocally invariant in $C_{\mathcal{N}}$ for (5.26). Let $h \in O_{\mathcal{M}}$ be arbitrary. Since $O_{\mathcal{M}}$ is prelocally invariant in $C_{\mathcal{M}}$ for (2.1) with solutions given by (5.26) and f, there exists a local strong solution $Z = Z^{(g(h))}$ to (5.26) with lifetime $\tau > 0$ such that $(Z^{\tau})_- \in O_{\mathcal{N}}$ and $Z^{\tau} \in C_{\mathcal{N}}$ up to an evanescent set and $r := f(Z)$ is a local mild solution to (2.1) with initial condition h and lifetime τ. By Proposition 5.11 the process r is a local strong solution to (5.27) with initial condition $h = f(z)$ and lifetime τ.

Let $\zeta \in \mathcal{D}(A^*)$ be arbitrary. Since r is also a local weak solution to (2.1) with lifetime τ, we have $\mathbb{P}$-almost surely

$$
\begin{aligned}
\langle \zeta, r_{t \wedge \tau} \rangle = \langle \zeta, h \rangle &+ \int_0^{t \wedge \tau} (\langle A^* \zeta, r_s \rangle + \langle \zeta, \alpha(r_s) \rangle) ds \\
&+ \sum_{j \in \mathbb{N}} \int_0^{t \wedge \tau} \langle \zeta, \sigma^j(r_s) \rangle d\beta_s^j \\
&+ \int_0^{t \wedge \tau} \int_E \langle \zeta, \gamma(r_{s-}, x) \rangle (\mu(ds, dx) - F(dx)ds), \quad t \geq 0.
\end{aligned}
$$

Therefore, we get up to an evanescent set

$$B + M^c + M^d = 0,$$

where the processes B, M^c, M^d are given by

$$
\begin{aligned}
B_t &:= \int_0^{t \wedge \tau} \left(\langle A^* \zeta, r_s \rangle + \langle \zeta, \alpha(r_s) - (g_{\lambda}^{\star} a)(r_s) \rangle \right) ds, \\
M_t^c &:= \sum_{j \in \mathbb{N}} \int_0^{t \wedge \tau} \langle \zeta, \sigma^j(r_s) - (g_W^{\star} b^j)(r_s) \rangle d\beta_s^j, \\
M_t^d &:= \int_0^{t \wedge \tau} \int_E \langle \zeta, \gamma(r_{s-}, x) - (g_{\mu}^{\star} c)(r_{s-}, x) \rangle (\mu(ds, dx) - F(dx)ds).
\end{aligned}
$$

The process B is a finite variation process which is continuous, and hence predictable, M^c is a continuous square-integrable martingale and M^d is a purely discontinuous square-integrable martingale. Therefore $B + M^c + M^d$ is a special semimartingale. Since the decomposition $B + M$ of a special semimartingale into a finite variation process B and a local martingale M is unique (see [18, Corollary I.3.16]) and the decomposition of a local martingale $M = M^c + M^d$ into a continuous local martingale M^c and a purely discontinuous local martingale M^d is unique (see [18, Theorem I.4.18]), we deduce that $B = M^c = M^d = 0$ up to an evanescent set. By

the Itô isometry, we obtain $\mathbb{P}$-almost surely

$$(5.32) \qquad \int_0^{t\wedge\tau} \big(\langle A^*\zeta, r_s\rangle + \langle\zeta, \alpha(r_s) - (g_\lambda^\star a)(r_s)\rangle\big)ds = 0, \quad t \geq 0,$$

$$(5.33) \qquad \int_0^{t\wedge\tau} \bigg(\sum_{j\in\mathbb{N}} |\langle\zeta, \sigma^j(r_s) - (g_W^\star b^j)(r_s)\rangle|^2\bigg)ds = 0, \quad t \geq 0,$$

$$(5.34) \qquad \int_0^{t\wedge\tau} \bigg(\int_E |\langle\zeta, \gamma(r_{s-}, x) - (g_\mu^\star c)(r_{s-}, x)\rangle|^2 F(dx)\bigg)ds = 0, \quad t \geq 0.$$

Since the process r is càdlàg, by Lemma 5.6 and Lebesgue's dominated convergence theorem (applied to the sum $\sum_{j\in\mathbb{N}}$ and to the integral $\int_E$) the integrands appearing in (5.32)–(5.34) are continuous in $s = 0$, and hence, we get

$$(5.35) \qquad \langle A^*\zeta, h\rangle + \langle\zeta, \alpha(h) - (g_\lambda^\star a)(h)\rangle = 0,$$

$$(5.36) \qquad \sum_{j\in\mathbb{N}} |\langle\zeta, \sigma^j(h) - (g_W^\star b^j)(h)\rangle|^2 = 0,$$

$$(5.37) \qquad \int_E |\langle\zeta, \gamma(h, x) - (g_\mu^\star c)(h, x)\rangle|^2 F(dx) = 0.$$

Identity (5.35) shows that $\zeta \mapsto \langle A^*\zeta, h\rangle$ is continuous on $\mathcal{D}(A^*)$, proving $h \in \mathcal{D}(A^{**})$. Since $A = A^{**}$, see [21, Theorem 13.12], we obtain $h \in \mathcal{D}(A)$, which yields (5.28). Using the identity $\langle A^*\zeta, h\rangle = \langle\zeta, Ah\rangle$, we obtain

$$\langle\zeta, Ah + \alpha(h) - (g_\lambda^\star a)(h)\rangle = 0 \quad \text{for all } \zeta \in \mathcal{D}(A^*),$$

and hence (5.29). For an arbitrary $j \in \mathbb{N}$ we obtain, by using (5.36),

$$\langle\zeta, \sigma^j(h) - (g_W^\star b^j)(h)\rangle = 0 \quad \text{for all } \zeta \in \mathcal{D}(A^*),$$

showing (5.30). By (5.37), for all $\zeta \in \mathcal{D}(A^*)$ we have

$$\langle\zeta, \gamma(h, x) - (g_\mu^\star c)(h, x)\rangle = 0 \quad \text{for } F\text{-almost all } x \in E.$$

Using Lemma 3.6, for F-almost all $x \in E$ we obtain

$$\langle\zeta, \gamma(h, x) - (g_\mu^\star c)(h, x)\rangle = 0 \quad \text{for all } \zeta \in \mathcal{D}(A^*),$$

which proves (5.31).

(2) $\Rightarrow$ (1): Let $h_0 \in O_\mathcal{M}$ be arbitrary. Since $O_\mathcal{N}$ is prelocally invariant in $C_\mathcal{N}$ for (5.26), there exists a local strong solution $Z = Z^{(g(h_0))}$ to (5.26) with lifetime $\tau > 0$ such that $(Z^\tau)_- \in O_\mathcal{N}$ and $Z^\tau \in C_\mathcal{N}$ up to an evanescent set. By Proposition 5.11 and conditions (5.28)–(5.31), the process $r := f(Z)$ is a local strong solution to (2.1) with initial condition h_0 and lifetime τ, showing that $O_\mathcal{M}$ is prelocally invariant in $C_\mathcal{M}$ for (2.1) with solutions given by (5.26) and f.

Additional Statement: If conditions (5.28), (5.29) are satisfied, then we have

$$Ah = (g_\lambda^\star a)(h) - \alpha(h), \quad h \in O_\mathcal{M},$$

and hence, the continuity of A on $O_\mathcal{M}$ follows from Lemma 5.6. $\qquad\square$

For the rest of this section, let $G = \mathbb{R}^m$, where $m \in \mathbb{N}$ denotes the dimension of the submanifold $\mathcal{M}$. We assume there exist elements $\zeta_1, \ldots, \zeta_m \in \mathcal{D}(A^*)$ such that the mapping $f : \mathcal{N} \to \mathcal{M}$ has the inverse (5.1), that is, diagram (5.2) commutes.

We define the subsets $O_V \subset C_V \subset V$ by $O_V := \psi^{-1}(O_\mathcal{N})$ and $C_V := \psi^{-1}(C_\mathcal{N})$. Recall that $O_\mathcal{M}$ is open in $\mathcal{M}$ and $C_\mathcal{M}$ is compact. Since $f : \mathcal{N} \to \mathcal{M}$ is a homeomorphism, $O_\mathcal{N}$ is open in $\mathcal{N}$ and $C_\mathcal{N}$ is compact. Furthermore, since $\psi : V \to \mathcal{N}$ is

a homeomorphism, O_V is open in V and C_V is compact. We define the mappings for the $\mathbb{R}^m$-valued SDE (5.26) as

$$a := \langle A^*\zeta, f \rangle + (f, \langle \zeta, \bullet \rangle)^\star_\lambda \alpha : \mathbb{R}^m \to \mathbb{R}^m, \tag{5.38}$$

$$b^j := (f, \langle \zeta, \bullet \rangle)^\star_W \sigma^j : \mathbb{R}^m \to \mathbb{R}^m \quad \text{for } j \in \mathbb{N}, \tag{5.39}$$

$$c := (f, \langle \zeta, \bullet \rangle)^\star_\mu \gamma : \mathbb{R}^m \times E \to \mathbb{R}^m, \tag{5.40}$$

where $\langle A^*\zeta, f \rangle := (\langle A^*\zeta_1, f \rangle, \ldots, \langle A^*\zeta_m, f \rangle)$. Then for each $h \in O_\mathcal{M}$ we have

$$a(z) = \langle A^*\zeta, h \rangle + \langle \zeta, \alpha(h) \rangle, \tag{5.41}$$

$$b^j(z) = \langle \zeta, \sigma^j(h) \rangle, \quad j \in \mathbb{N} \tag{5.42}$$

$$c(z, x) = \langle \zeta, \gamma(h, x) \rangle, \quad x \in E \tag{5.43}$$

where $z = \langle \zeta, h \rangle \in O_\mathcal{N}$. Furthermore, we define the mappings

$$\Theta := (\psi, \Psi)^\star_\lambda a : \mathbb{R}^m \to \mathbb{R}^m, \tag{5.44}$$

$$\Sigma^j := (\psi, \Psi)^\star_W b^j : \mathbb{R}^m \to \mathbb{R}^m, \quad \text{for } j \in \mathbb{N}, \tag{5.45}$$

$$\Gamma := (\psi, \Psi)^\star_\mu c : \mathbb{R}^m \times E \to \mathbb{R}^m \tag{5.46}$$

and consider the $\mathbb{R}^m$-valued SDE (4.1). According to Lemma 5.6, the mappings a, $(b^j)_{j \in \mathbb{N}}$, c as well as Θ, $(\Sigma^j)_{j \in \mathbb{N}}$, Γ satisfy the regularity conditions (2.2)–(2.4) and (2.6)–(2.8). Note that

$$\Theta(y) = (\psi^\star_\lambda a)(y), \quad y \in O_V \tag{5.47}$$

$$\Sigma^j(y) = (\psi^\star_W b^j)(y), \quad j \in \mathbb{N} \text{ and } y \in O_V \tag{5.48}$$

$$\Gamma(y, x) = (\psi^\star_\mu c)(y, x), \quad x \in E \text{ and } y \in O_V. \tag{5.49}$$

Note that V is a m-dimensional C^3-submanifold with boundary of $\mathbb{R}^m$, and that for $y \in \partial V$ the inward pointing normal vector to ∂V at y is given by the first unit vector $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^m$. Therefore, Theorem 4.1 together with Lemma 4.2 provides the statement of Theorem 5.3 for the particular case, where the submanifold is an open subset in the half space $\mathbb{R}^m_+$.

5.14. **Lemma.** *Suppose that $O_\mathcal{M}$ is prelocally invariant in $C_\mathcal{M}$ for (2.1). Then the set $O_\mathcal{M}$ is prelocally invariant in $C_\mathcal{M}$ for (2.1) with solutions given by (5.26) and f.*

Proof. Let $h_0 \in O_\mathcal{M}$ be arbitrary. Since $O_\mathcal{M}$ is prelocally invariant in $C_\mathcal{M}$ for (5.26), there exists a local mild solution $r = r^{(h_0)}$ to (2.1) with lifetime $\tau > 0$ such that $(r^\tau)_- \in O_\mathcal{M}$ and $r^\tau \in C_\mathcal{M}$ up to an evanescent set. Since $\zeta_1, \ldots, \zeta_m \in \mathcal{D}(A^*)$ and r is also a local weak solution to (2.1), setting $Z := \langle \zeta, r \rangle$ we have, by taking into account (5.41)–(5.43), $\mathbb{P}$-almost surely

$$Z_{t \wedge \tau} = \langle \zeta, r_{t \wedge \tau} \rangle = \langle \zeta, h_0 \rangle + \int_0^{t \wedge \tau} (\langle A^*\zeta, r_s \rangle + \langle \zeta, \alpha(r_s) \rangle) ds$$

$$+ \sum_{j \in \mathbb{N}} \int_0^{t \wedge \tau} \langle \zeta, \sigma^j(r_s) \rangle d\beta_s^j + \int_0^{t \wedge \tau} \int_E \langle \zeta, \gamma(r_{s-}, x) \rangle (\mu(ds, dx) - F(dx) ds)$$

$$= \langle \zeta, h_0 \rangle + \int_0^{t \wedge \tau} a(Z_s) ds + \sum_{j \in \mathbb{N}} \int_0^{t \wedge \tau} b^j(Z_s) d\beta_s^j$$

$$+ \int_0^{t \wedge \tau} \int_E c(Z_{s-}, x)(\mu(ds, dx) - F(dx) ds), \quad t \geq 0.$$

Therefore, the process Z is a local strong solution to (5.26) with initial condition $\langle \zeta, h_0 \rangle$ and lifetime τ such that $(Z^\tau)_- \in O_\mathcal{N}$ and $Z^\tau \in C_\mathcal{N}$ up to an evanescent set. By (5.1) we have $f(Z^\tau) = r^\tau$, and hence, the process $f(Z)$ is a local mild solution to (2.1) with initial condition h_0 and lifetime τ. $\square$

Now, we are ready to provide the proof of Theorem 5.3.

Proof of Theorem 5.3. $(1) \Rightarrow (2)$: By Lemma 5.14, the set $O_{\mathcal{M}}$ is prelocally invariant in $C_{\mathcal{M}}$ for (2.1) with solutions given by (5.26) and f. Therefore, we have two implications:

- Proposition 5.13 yields (5.3) and

$$(5.50) \quad (A + \alpha)(h) = (\langle \zeta, \bullet \rangle_\lambda^\star a)(h), \quad h \in O_{\mathcal{M}},$$

$$(5.51) \quad \sigma^j(h) = (\langle \zeta, \bullet \rangle_W^\star b^j)(h) \quad \text{for all } j \in \mathbb{N}, \quad \text{for all } h \in O_{\mathcal{M}},$$

$$(5.52) \quad \gamma(h, x) = (\langle \zeta, \bullet \rangle_\mu^\star c)(h, x) \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in O_{\mathcal{M}}.$$

- By Lemma 5.10, the set $O_{\mathcal{N}}$ is prelocally invariant in $C_{\mathcal{N}}$ for (5.26). Hence, by (5.47)–(5.49) and Proposition 5.13, the set O_V is prelocally invariant in C_V for (4.1) with solutions given by (5.26) and Ψ.

The latter statement has two further consequences:

- By Lemma 5.12, the set $O_{\mathcal{N}}$ is prelocally invariant in $C_{\mathcal{N}}$ for (5.26) with solutions given by (4.1) and ψ. Thus, Proposition 5.13 yields

$$(5.53) \quad a(z) = (\Psi_\lambda^\star \Theta)(z), \quad z \in O_{\mathcal{N}},$$

$$(5.54) \quad b^j(z) = (\Psi_W^\star \Sigma^j)(z), \quad j \in \mathbb{N} \text{ and } z \in O_{\mathcal{N}},$$

$$(5.55) \quad c(z, x) = (\Psi_\mu^\star \Gamma)(z, x) \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } z \in O_{\mathcal{N}}.$$

- By Lemma 5.10, the set O_V is prelocally invariant in C_V for (4.1). Theorem 4.1 implies that conditions (4.2)–(4.5) are satisfied.

In view of (4.2)–(4.5), Lemma 4.2, identities (5.53)–(5.55) and Proposition 5.8 we obtain (5.17)–(5.22), where ξ_z denotes the inward pointing normal vector to $\partial \mathcal{N}$ at z. Taking into account (5.50)–(5.52), applying Proposition 5.8 we arrive at (5.4)–(5.9).

$(2) \Rightarrow (1)$: Suppose that conditions (5.3)–(5.9) are satisfied. By (5.3) and (5.41), for all $z \in O_{\mathcal{N}}$ we obtain

$$a(z) = \langle A^* \zeta, h \rangle + \langle \zeta, \alpha(h) \rangle = \langle \zeta, Ah + \alpha(h) \rangle = (f_\lambda^\star(A + \alpha))(z),$$

where $h = f(z) \in O_{\mathcal{M}}$. Thus, we have

$$(5.56) \quad a(z) = (f_\lambda^\star(A + \alpha))(z), \quad z \in O_{\mathcal{N}},$$

$$(5.57) \quad b^j(z) = (f_W^\star \sigma^j)(z), \quad j \in \mathbb{N} \text{ and } z \in O_{\mathcal{N}},$$

$$(5.58) \quad c(z, x) = (f_\mu^\star \gamma)(z, x), \quad x \in E \text{ and } z \in O_{\mathcal{N}},$$

which has two implications:

- By (5.4), (5.6), (5.8) and Proposition 5.8 we obtain (5.50)–(5.52).
- By (5.4)–(5.9) and Proposition 5.8 we have (5.17)–(5.22).

In view of (5.47)–(5.49), we obtain the following consequences:

- By (5.17), (5.19), (5.21) and Proposition 5.8 we obtain (5.53)–(5.55).
- By (5.17)–(5.22), Proposition 5.8 and Lemma 4.2 we have (4.2)–(4.5).

Therefore, by Theorem 4.1, the set O_V is prelocally invariant in C_V for (4.1). By (5.53)–(5.55) and Proposition 5.13, the set $O_{\mathcal{N}}$ is prelocally invariant in $C_{\mathcal{N}}$ for (5.26) with with solutions given by (4.1) and ψ. According to Lemma 5.10, the set $O_{\mathcal{N}}$ is prelocally invariant in $C_{\mathcal{N}}$ for (5.26). By (5.3), (5.50)–(5.52) and Proposition 5.13, the set $O_{\mathcal{M}}$ is prelocally invariant in $C_{\mathcal{M}}$ for (2.1) with solutions given by (5.26) and f.

Additional Statement: If $O_{\mathcal{M}}$ is prelocally invariant in $C_{\mathcal{M}}$ for (2.1) with solutions given by (5.26) and f, then Proposition 5.13 implies that A is continuous on $O_{\mathcal{M}}$. Using Lemma 5.4, we obtain that the mapping in (5.8) is continuous on $O_{\mathcal{M}}$.   □

6. Global analysis of the invariance problem on submanifolds with boundary and proofs of the main results

In this section, we perform global analysis of the invariance problem and prove our main results. The idea is to localize the invariance problem and to apply Theorem 5.3 from the previous section. In order to realize this idea, we will switch between the original SPDE (2.1) and the SPDE (6.3), which only makes sufficiently small jumps.

Before we start with the proofs of our main results, we prepare some auxiliary results. Let $B \in \mathcal{E}$ be a set with $F(B^c) < \infty$.

6.1. Lemma. *The mappings $\alpha^B : H \to H$ and $\gamma^B : H \times E \to H$ defined as*

$$(6.1) \qquad \alpha^B(h) := \alpha(h) - \int_{B^c} \gamma(h, x) F(dx),$$

$$(6.2) \qquad \gamma^B(h, x) := \gamma(h, x) \mathbb{1}_B(x)$$

also satisfy the regularity conditions (2.2), (2.6), (2.7).

Proof. See [14, Lemma 2.18]. □

Now, we consider the SPDE

$$(6.3) \qquad \begin{cases} dr_t^B &= (Ar_t^B + \alpha^B(r_t^B))dt + \sum_{j \in \mathbb{N}} \sigma^j(r_t^B) d\beta_t^j \\ &\quad + \int_E \gamma^B(r_{t-}^B, x)(\mu(dt, dx) - F(dx)dt) \\ r_0^B &= h_0. \end{cases}$$

We define ϱ^B as the first time where the Poisson random measure makes a jump outside B; that is

$$\varrho^B = \inf\{t \geq 0 : \mu([0, t] \times B^c) = 1\}$$

6.2. Lemma. *The mapping ϱ^B is a strictly positive stopping time.*

Proof. See [14, Lemma 2.20]. □

The following result shows that the SPDEs (2.1) and (6.3) locally have the same mild solutions.

6.3. Proposition. *Let $h_0 : \Omega \to H$ be a $\mathcal{F}_0$-measurable random variable, let $B \in \mathcal{E}$ be a set with $F(B^c) < \infty$, and let $0 < \tau \leq \varrho^B$ be a stopping time. Then the following statements are true:*

(1) *If there exists a local mild solution r to (2.1) with lifetime τ, then there also exists a local mild solution r^B to (6.3) with lifetime τ such that*

$$(6.4) \qquad r^\tau \mathbb{1}_{[\![0,\tau[\![} = (r^B)^\tau \mathbb{1}_{[\![0,\tau[\![}.$$

(2) *If there exists a local mild solution r^B to (6.3) with lifetime τ, then there also exists a local mild solution r to (2.1) with lifetime τ such that (6.4) is satisfied.*

In particular, in either case we have $(r^\tau)_- = ((r^B)^\tau)_-$.

Proof. See [14, Proposition 2.21]. □

Recall that $\mathcal{M}$ denotes a C^3-submanifold with boundary of H. The following result shows how the invariance conditions regarding α, σ^j, γ and $\alpha^B, \sigma^j, \gamma^B$ are related.

6.4. **Proposition.** *Let $O_\mathcal{M} \subset \mathcal{M}$ be a subset which is open in $\mathcal{M}$, and suppose that*

$$O_\mathcal{M} \subset \mathcal{D}(A),$$

$$h + \gamma(h,x) \in \overline{\mathcal{M}} \quad \text{for F-almost all } x \in E, \quad \text{for all } h \in O_\mathcal{M}.$$

Then the following statements are true:

(1) *We have (5.7)–(5.9) if and only if*

$$\text{(6.5)} \qquad \int_E |\langle \eta_h, \gamma^B(h,x)\rangle| F(dx) < \infty, \quad h \in O_\mathcal{M} \cap \partial\mathcal{M},$$

$$\text{(6.6)} \qquad Ah + \alpha^B(h) - \frac{1}{2}\sum_{j\in\mathbb{N}} D\sigma^j(h)\sigma^j(h)$$

$$- \int_E \Pi_{(T_h\mathcal{M})^\perp}\gamma^B(h,x)F(dx) \in T_h\mathcal{M}, \quad h \in O_\mathcal{M},$$

$$\text{(6.7)} \qquad \langle \eta_h, Ah + \alpha^B(h)\rangle - \frac{1}{2}\sum_{j\in\mathbb{N}}\langle \eta_h, D\sigma^j(h)\sigma^j(h)\rangle$$

$$- \int_E \langle \eta_h, \gamma^B(h,x)\rangle F(dx) \geq 0, \quad h \in O_\mathcal{M} \cap \partial\mathcal{M}.$$

(2) *The mapping in (5.8) is continuous on $O_\mathcal{M}$ if and only if the mapping in (6.6) is continuous on $O_\mathcal{M}$.*

Proof. This follows from [14, Lemma 3.27 and Proposition 3.19]. $\qquad\square$

The following auxiliary result shows that for each $h_0 \in \mathcal{M}$ there exists a neighborhood of h_0 such that the assumptions from Section 5 are fulfilled, and that the global jump condition (1.4) can be localized by choosing the set $B \in \mathcal{E}$ for γ^B appropriately.

6.5. **Proposition.** *Suppose that condition (1.4) is satisfied. Then, for all $h_0 \in \mathcal{M}$ there exist*

(i) *a constant $\epsilon > 0$ such that $B_\epsilon(h_0) \cap \mathcal{M}$ is a submanifold as in Section 5, i.e., diagram (5.2) commutes,*

(ii) *subsets $O_\mathcal{M} \subset C_\mathcal{M} \subset B_\epsilon(h_0) \cap \mathcal{M}$ with $h_0 \in O_\mathcal{M}$ as in Section 5, i.e., $O_\mathcal{M}$ is open in $B_\epsilon(h_0) \cap \mathcal{M}$ and $C_\mathcal{M}$ is compact,*

(iii) *and a set $B \in \mathcal{E}$ with $F(B^c) < \infty$*

such that we have

$$\text{(6.8)} \qquad h + \gamma^B(h,x) \in C_\mathcal{M} \quad \text{for F-almost all } x \in E, \quad \text{for all } h \in O_\mathcal{M}.$$

Proof. This follows from [14, Proposition 3.11 and Lemma 3.15]. $\qquad\square$

Finally, we require the following result about the existence of strong solutions to (2.1) under stochastic invariance.

6.6. **Lemma.** *Suppose that $\mathcal{M} \subset \mathcal{D}(A)$ and that A is continuous on $\mathcal{M}$. Let $h_0 \in \mathcal{M}$ be arbitrary, and let $r = r^{(h_0)}$ be a local weak solution to (2.1) with initial condition h_0 lifetime $\tau > 0$ such that $(r^\tau)_- \in \mathcal{M}$ up to an evanescent set. Then r is also a local strong solution to (2.1) with lifetime τ.*

Proof. See [14, Lemma 2.7]. $\qquad\square$

Now, we are ready to provide the proofs of our main results.

Proof of Theorem 2.4. (1) $\Rightarrow$ (2): We will prove that prelocal invariance of $\mathcal{M}$ for (2.1) implies conditions (1.2)–(1.4), (1.6)–(1.8), the continuity of A and the mapping in (1.7) on $\mathcal{M}$, and and that for each $h_0 \in \mathcal{M}$ there is a local strong solution $r = r^{(h_0)}$ to (2.1).

According to Lemma 3.3 we have (1.4). Let $h_0 \in \mathcal{M}$ be arbitrary. By Proposition 6.5 there exist quantities as in (i)–(iii) such that condition (6.8) is satisfied.

We will show that $O_{\mathcal{M}}$ is prelocally invariant in $C_{\mathcal{M}}$ for (6.3). Indeed, let $g_0 \in O_{\mathcal{M}}$ be arbitrary. Since $O_{\mathcal{M}}$ is open in $B_\epsilon(h_0) \cap \mathcal{M}$, there exists $\delta > 0$ such that $\overline{B_\delta(g_0)} \cap \mathcal{M} \subset O_{\mathcal{M}}$. Since $\mathcal{M}$ is prelocally invariant for (2.1), there exist a local mild solution $r = r^{(g_0)}$ to (2.1) with lifetime $0 < \tau \leq \varrho^B$ such that $(r^\tau)_- \in \mathcal{M}$ up to an evanescent set. According to Proposition 6.3, there exists a local mild solution $r^B = r^{B,(g_0)}$ to (6.3) with lifetime τ such that $(r^\tau)_- = ((r^B)^\tau)_-$. The mapping

$$\varrho := \inf\{t \geq 0 : r_t \notin B_\delta(g_0)\} \wedge \tau$$

is a strictly positive stopping time, and we obtain up to an evanescent set

$$((r^B)^\varrho)_- = (r^\varrho)_- \in \overline{B_\delta(g_0)} \cap \mathcal{M} \subset O_{\mathcal{M}}.$$

Furthermore, using (6.8) and Lemma 3.5 we obtain $(r^B)^\varrho \in C_{\mathcal{M}}$ up to an evanescent set. Hence, the set $O_{\mathcal{M}}$ is prelocally invariant in $C_{\mathcal{M}}$ for (6.3).

Theorem 5.3, applied to the SPDE (6.3), yields (5.3)–(5.5), (6.5)–(6.7) and that A and the mapping in (6.6) are continuous on $O_{\mathcal{M}}$. Since (5.3) and (1.4) are satisfied, by Proposition 6.4 we also have (5.7)–(5.9) and the mapping in (5.8) is continuous on $O_{\mathcal{M}}$. Since $h_0 \in \mathcal{M}$ was arbitrary, we deduce (1.2), (1.3), (1.6)–(1.8) and that A and the mapping in (1.7) are continuous on $\mathcal{M}$. By Lemma 6.6, for each $h_0 \in \mathcal{M}$ there is a local strong solution $r = r^{(h_0)}$ to (2.1).

(2) $\Rightarrow$ (1): Now, we will prove that conditions (1.2)–(1.4) and (1.6)–(1.8) imply prelocal invariance of $\mathcal{M}$ for (2.1) and the statement regarding local invariance.

Let $h_0 \in \mathcal{M}$ be arbitrary. By Proposition 6.5 there exist quantities as in (i)–(iii) such that condition (6.8) is satisfied.

We will show that $C_{\mathcal{M}}$ is prelocally invariant in $O_{\mathcal{M}}$ for (6.3). By (1.2), (1.3) and (1.6)–(1.8) we have (5.3)–(5.5) and (5.7)–(5.9). Since (5.3) and (1.4) are satisfied, by Proposition 6.4 we also have (6.5)–(6.7). Consequently, by (5.3)–(5.5), (6.8), (6.5)–(6.7) and Theorem 5.3, the set $C_{\mathcal{M}}$ is prelocally invariant in $O_{\mathcal{M}}$ for (6.3).

Now, we will show that $\mathcal{M}$ is prelocally invariant for (2.1). Since $C_{\mathcal{M}}$ is prelocally invariant in $O_{\mathcal{M}}$ for (6.3), there exists a local mild solution r^B to (6.3) with lifetime $0 < \tau \leq \varrho^B$ such that up to an evanescent set

$$((r^B)^\tau)_- \in O_{\mathcal{M}} \quad \text{and} \quad (r^B)^\tau \in C_{\mathcal{M}}.$$

According to Proposition 6.3, there exists a local mild solution r to (2.1) with lifetime τ such that $(r)^\tau \mathbb{1}_{[\![0,\tau[\![}} = (r^B)^\tau \mathbb{1}_{[\![0,\tau[\![}}$. We obtain up to an evanescent set

$$(r^\tau)_- = ((r^B)^\tau)_- \in O_{\mathcal{M}} \subset \mathcal{M}$$

as well as

$$r^\tau \mathbb{1}_{[\![0,\tau[\![}} = (r^B)^\tau \mathbb{1}_{[\![0,\tau[\![}} \in C_{\mathcal{M}} \subset \mathcal{M}.$$

Using Lemma 3.4, by (1.4) we obtain $r^\tau \in \overline{\mathcal{M}}$ up to an evanescent set, proving that $\mathcal{M}$ is prelocally invariant for (2.1).

If even condition (2.10) is satisfied, then by Lemma 3.4 we obtain $r^\tau \in \mathcal{M}$ up to an evanescent set, and hence, $\mathcal{M}$ is locally invariant for (2.1). $\square$

Proof of Theorem 2.8. Let $h_0 \in \mathcal{M}$ be arbitrary. Then there exists a unique mild and weak solution $r = r^{(h_0)}$ to (2.1); see, e.g., [12, Corollary 10.9]. Defining the stopping time

(6.9) $$\tau := \inf\{t \geq 0 : r_t \notin \mathcal{M}\},$$

we claim that

(6.10) $$\mathbb{P}(\tau = \infty) = 1.$$

Suppose, on the contrary, that (6.10) is not satisfied. Then there exists $N \in \mathbb{N}$ such that $\mathbb{P}(\tau \leq N) > 0$. We define the bounded stopping time $\tau_0 := \tau \wedge N$. By the closedness of $\mathcal{M}$ in H, we have $(r^{\tau_0})_- \in \mathcal{M}$ up to an evanescent set. Therefore, by relation (1.4) and Lemma 3.5 we obtain $r^{\tau_0} \in \mathcal{M}$ up to an evanescent set. We define the filtration $\mathbb{F}^{(\tau_0)} := (\mathcal{F}_{\tau_0+t})_{t \geq 0}$, the sequence $(\beta^{(\tau_0),j})_{j \in \mathbb{N}}$ of real-valued processes by

$$(6.11) \qquad \beta_t^{(\tau_0),j} := \beta_{\tau_0+t}^j - \beta_{\tau_0}^j, \quad t \geq 0,$$

and the random measure $\mu^{(\tau_0)}$ on $\mathbb{R}_+ \times E$ by

$$(6.12) \qquad \mu^{(\tau_0)}(\omega; B) := \mu(\omega; B_{\tau_0(\omega)}), \quad \omega \in \Omega \text{ and } B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E},$$

where we use the notation

$$B_{\tau_0} := \{(t + \tau_0, x) \in \mathbb{R}_+ \times E : (t, x) \in B\}.$$

According to [13, Lemma 4.6], the sequence $(\beta^{(\tau_0),j})_{j \in \mathbb{N}}$ is a sequence of real-valued independent standard Wiener processes, adapted to $\mathbb{F}^{(\tau_0)}$, and $\mu^{(\tau_0)}$ is a time-homogeneous Poisson random measure relative to the filtration $\mathbb{F}^{(\tau_0)}$ with compensator $dt \otimes F(dx)$. The process $r_{\tau_0+\bullet}$ is a weak solution to the time-shifted SPDE

$$(6.13) \qquad \begin{cases} dr_t = (Ar_t + \alpha(r_t))dt + \sum_{j \in \mathbb{N}} \sigma^j(r_t)d\beta_t^{(\tau_0),j} \\ \qquad\quad + \int_E \gamma(r_{t-}, x)(\mu^{(\tau_0)}(dt, dx) - F(dx)dt) \\ r_0 = h_0 \end{cases}$$

with initial condition r_{τ_0}, because for each $\zeta \in \mathcal{D}(A^*)$ we have $\mathbb{P}$-almost surely

$$\langle \zeta, r_{\tau_0+t} \rangle = \langle \zeta, r_{\tau_0} \rangle + \langle \zeta, r_{\tau_0+t} - r_{\tau_0} \rangle$$

$$= \langle \zeta, r_{\tau_0} \rangle + \int_{\tau_0}^{\tau_0+t} \left(\langle A^*\zeta, r_s \rangle + \langle \zeta, \alpha(r_s) \rangle \right) ds + \sum_{j \in \mathbb{N}} \int_{\tau_0}^{\tau_0+t} \langle \zeta, \sigma^j(r_s) \rangle d\beta_s^j$$

$$+ \int_{\tau_0}^{\tau_0+t} \int_E \langle \zeta, \gamma(r_{s-}, x) \rangle (\mu(ds, dx) - F(dx)ds)$$

$$= \langle \zeta, r_{\tau_0} \rangle + \int_0^t \left(\langle A^*\zeta, r_{\tau_0+s} \rangle + \langle \zeta, \alpha(r_{\tau_0+s}) \rangle \right) ds + \sum_{j \in \mathbb{N}} \int_0^t \langle \zeta, \sigma(r_{\tau_0+s}) \rangle d\beta_s^{(\tau_0),j}$$

$$+ \int_0^t \int_E \langle \zeta, \gamma(r_{(\tau_0+s)-}, x) \rangle (\mu^{(\tau_0)}(ds, dx) - F(dx)ds), \quad t \geq 0.$$

There exists $K \in \mathbb{N}$ such that $\mathbb{P}(\Gamma) > 0$, where

$$\Gamma := \{\tau \leq N\} \cap \{\|r_{\tau_0}\| \leq K\}.$$

By choosing a suitable covering $\mathcal{M} = \bigcup_{k \in \mathbb{N}} \mathcal{M}_k$ according to Lindelöf's Lemma [1, Lemma 1.1.6] and arguing as in the second part of the proof of Theorem 2.4, there exists a local weak solution r^K to the time-shifted SPDE (6.13) with the $\mathcal{F}_{\tau_0}$-measurable initial condition $r_{\tau_0} \mathbb{1}_{\{\|r_{\tau_0}\| \leq K\}}$ and lifetime $\varrho > 0$ such that $(r^K)^\varrho \in \mathcal{M}$ up to an evanescent set. Noting that $\{\tau \leq N\} = \{\tau = \tau_0\}$, by the uniqueness of weak solutions to (6.13) we obtain up to an evanescent set

$$(r_{\tau+\bullet})^\varrho \mathbb{1}_\Gamma = (r_{\tau_0+\bullet})^\varrho \mathbb{1}_\Gamma = (r^K)^\varrho \mathbb{1}_\Gamma \in \mathcal{M},$$

which contradicts the definition (6.9) of τ. Therefore, relation (6.10) is satisfied and we obtain $r \in \mathcal{M}$ up to an evanescent set. Hence, Lemma 6.6 implies that r is a strong solution to (2.1). $\qquad \square$

For the proof of Theorem 2.11 we prepare an auxiliary result.

6.7. Lemma. *For all $h \in \partial\mathcal{M}$ we have $(T_h\mathcal{M})_+ = T_h\mathcal{M} \cap \{\eta_h\}^+$, where*

$$\{\eta_h\}^+ = \{g \in H : \langle\eta_h, g\rangle \geq 0\}$$

Proof. See [14, Lemma 3.7]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 2.11. Relation (2.12) implies (1.6). Furthermore, presuming (1.2), we have (1.5) if and only if (1.7), (1.8) are satisfied. Indeed, noting that

$$Ah + \alpha(h) - \frac{1}{2}\sum_{j\in\mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_E \gamma(h,x)F(dx)$$

(6.14)
$$= Ah + \alpha(h) - \frac{1}{2}\sum_{j\in\mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_E \Pi_{(T_h\mathcal{M})^\perp}\gamma(h,x)F(dx)$$

$$- \Pi_{T_h\mathcal{M}} \int_E \gamma(h,x)F(dx), \quad h \in \mathcal{M},$$

we have (1.7) if and only if

$$Ah + \alpha(h) - \frac{1}{2}\sum_{j\in\mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_E \gamma(h,x)F(dx) \in T_h\mathcal{M}, \quad h \in \mathcal{M},$$

and, by Lemma 6.7, we have (1.8) if and only if

$$Ah + \alpha(h) - \frac{1}{2}\sum_{j\in\mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_E \gamma(h,x)F(dx) \in (T_h\mathcal{M})_+, \quad h \in \partial\mathcal{M},$$

showing that condition (1.5) is equivalent to (1.7), (1.8).

Now, suppose that even condition (2.13) is satisfied. Since, by Theorem 2.4, the mapping in (1.7) is continuous on $\mathcal{M}$, identity (6.14) together with relations (2.6), (2.13) and Lebesgue's dominated convergence theorem shows that the mapping in (1.5) is continuous $\mathcal{M}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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