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Sensitivity of life insurance reserves via Markov semigroups

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Fahrenwaldt M. Sensitivity of life insurance reserves via Markov semigroups. Scandinavian Actuarial Journal. We consider Thiele's differential equation for the reserve of a multi-state insurance contract with functional dependence on the surplus. In an analytic approach based on semigroups, we obtain existence and uniqueness results and prove growth and regularity properties. Moreover, we investigate the sensitivity of the reserves with respect to the surplus, payment rate, and transition assumptions in terms of uniform and pointwise estimates. The approach can easily be generalized.

Keywords: market reserve; Thiele's differential equation; non-autonomous parabolic partial differential equations; evolution systems; sensitivity of reserves

1. Introduction

Modern insurance contracts combine insurance and financial risks in sophisticated ways. Their pricing and reserving depends on a large number of parameters many of which cannot be accurately determined. Prudent risk management therefore needs to understand how each parameter affects the insurance reserves:

(i) constant premiums are usually determined at the inception of the contract and must be sufficient over its lifetime;
(ii) the Solvency II standard model measures capital charges as changes in the net asset value under defined uniform parameter shifts;
(iii) on top of reserves risk margins may be required which are also determined in certain parameter scenarios.

Mathematically, reserves for insurance contracts are typically modeled by the conditional expectation of benefit payments. This formulation is natural for modeling purposes but does not lend itself easily to a sensitivity analysis.

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Instead we propose to investigate Thiele's differential equation, the system of partial differential equations (PDEs) satisfied by the reserve, using semigroup techniques. This is a well-established method for obtaining qualitative information about PDE solutions by studying properties of the differential operators. We illustrate this in the case of a surplus-linked life insurance contract as in Steffensen (2006) where the surplus may be invested in a risky asset modeled by an Itô process. Main results are as follows:

(i) existence, uniqueness, and regularity of a solution of the PDE system which gives the reserve as a function of time and the surplus;
(ii) estimates in terms of Hölder norms of the sensitivity of the market reserve with respect to payment rate and transition assumptions (mortality and disability);
(iii) pointwise bounds on the first derivative of the reserve with respect to the surplus, the bounds are given in terms of a solution of another PDE system;
(iv) a factorization of the reserve function into risk types (financial, mortality/disability, payment) which exhibits the impact of these factors.

Key novel contributions of this paper to the actuarial literature include the use of evolution systems generated by differential operators with unbounded coefficients on non-compact manifolds. This approach is dictated by the surplus-dependence and can easily be applied to more general situations and automatically yields qualitative information on the solutions. Our approach is a natural stepping stone for considering market reserves driven by Lévy or affine processes.

Compared to the literature, our method stands out for its basis in operator theory and the easy generalizations. We divide the relevant literature in several strands (same typology as in the introduction of Christiansen (2008), see that paper for more exhaustive references). The first is of a qualitative nature where the magnitude of reserve changes cannot be determined as represented by Lidstone (1905) and Norberg (1985) for simple standard contracts, Hoem (1988), Ramlau-Hansen (1988) and Linnemann (1993). The second strand concerns the comparison of several well-defined scenarios as done in Olivieri (2001) and Khalaf-Allah *et al.* (2006). The third strand uses classical derivatives with respect to finitely many parameters such as in Bowers *et al.* (1997) and Kalashnikov and Norberg (2003). Here, the authors assume that the valuation basis depends smoothly on a set of parameters. Generalizing this, Christiansen (2008) employed functional derivatives to analyze sensitivities: the idea is to view insurance reserves as functionals defined on the valuation basis consisting of the interest rate and transition intensities which are viewed as members of function spaces. The derivatives can be made explicit in concrete applications.

More recent attention in the context of sensitivities has focused on defining worst-case scenarios. Here, the idea of first-order approximations using functional derivatives is employed in Christiansen (in press) to investigate implicit safety margins of life and health insurance contracts and in Christiansen (2011) to assess netting effects in life insurance contracts (caused by the difference between payments during sojourns and upon transition between states). Also, Christiansen and Steffensen (2011) assume dependent interest and transition intensities and use dynamic programming and a recursive scheme to obtain a unique worst-case scenario within given constraints. This is in succession to Christiansen (2010) where the worst-case scenario in a consideration without interest rate dependence is given as the solution of a Thiele-type integral equation.

This paper is organized as follows. The next section introduces the notation for function spaces and collects all necessary assumptions and results on evolution systems generated by certain differential operators. This is followed by a summary and discussion of the key results whose proofs are sketched in the subsequent section. The final part of this paper is concerned with a discussion of potential generalizations. An appendix contains technical details of proofs.

2. Preliminaries

We introduce the following Banach spaces of real-valued functions. References for these spaces are Evans (2010) and Lorenzi and Bertoldi (2007).

Let $D \subseteq \mathbb{R}^p$ be a domain. We denote by $C_b(D)$ the linear space of bounded real-valued continuous functions on D, a metric space under the sup norm $||f||_\infty = \sup_{x \in D} |f(x)|$. Let $C_b([0,T] \times D)$ be the Banach space of bounded continuous functions in two variables with norm $||f||_\infty = \sup_{(t,x)\in[0,T]\times D} |f(t,x)|$.

The Hölder continuous functions with parameter $0 < \alpha < 1$ are given by

$$C^\alpha(D) = \left\{ f \in C_b(D) \,\middle|\, [f]_\alpha = \sup_{\substack{x,y \in D \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty \right\}$$

with norm $||f||_{C^\alpha(D)} = ||f||_\infty + [f]_\alpha$. Similarly for $C^{\alpha,\beta}([0,T] \times D)$ with norm

$$||f||_{C^{\alpha,\beta}([0,T]\times D)} = \sup_{x \in D} ||f(\cdot, x)||_{C^\alpha([0,T])} + \sup_{t \in [0,T]} ||f(t, \cdot)||_{C^\beta(D)}.$$

As a special case, we define the parabolic Hölder spaces as

$$C^{\alpha/2,\alpha}([0,T] \times D) = C^{\alpha/2,0}([0,T] \times D) \cap C^{0,\alpha}([0,T] \times D)$$

with norm $||f||_{C^{\alpha/2,\alpha}} = ||f||_{C^{\alpha/2,0}} + ||f||_{C^{0,\alpha}}$ for $x \geq 0$.

A subscript *loc* to any such space means that we regard it as a locally convex space with a collection of seminorms given by the supremum on compact sets.

Let $L^\infty(\mathbb{R})$ be the space of measurable real-valued functions on the real line under the essential supremum norm also denoted by $|| \cdot ||_\infty$.

For X a Banach space we let $X \otimes \mathbb{R}^n$ be the direct sum of n copies of X, it is a Banach space under the norm $||(x_1, \ldots, x_n)||^2 = \sum_{j=1}^n ||x_i||_X^2$.

3. Statement of the key results

The following subsection gives a version of Thiele's PDE, transforms it and sets out the motivation for our approach. Subsequently, the key results for uniform and pointwise sensitivities are stated.

3.1. Thiele's PDE, assumptions, and solution strategy

We consider the reserve of a life insurance contract with dividend payments linked to a surplus. The equations are slightly simplified versions of the Thiele differential equations for market reserves developed in Section 4 of Steffensen (2006) and Steffensen (2007), Section 10.3.2; the simplification consists ignoring jumps in the contribution and dividend functions. The insurance contract has n states with the reserve given as a vector $(V^1, \dots, V^n)^\top$ in a suitable function space. (Here, $^\top$ denotes the transpose of a vector.) The system of PDEs governing the reserve reads for $j = 1, \dots, n$ as follows

$$\left.\begin{aligned} 0 &= \partial_t V^j(t,x) + \mathcal{A}(t)V^j(t,x) + \beta^j(t,x) - rV^j(t,x) \\ 0 &= V^j(T,x) \end{aligned}\right\} \tag{1}$$

on $[0, T] \times \mathbb{R}$ where

$$\begin{aligned} \mathcal{A}(t) =& \tfrac{1}{2}\pi(t,x)^2\sigma^2x^2\partial_x^2 + \left(rx + c^j(t) - \delta^j(t,x)\right)\partial_x \\ &+ \sum_{k\neq j} \mu^{jk}(t)\left(V^k(t, x + c^{jk}(t) - \delta^{jk}(t,x)) - V^j(t,x)\right), \\ \beta^j(t,x) =& b^j(t) + \delta^j(t,x) + \sum_{k\neq j}\mu^{jk}(t)\left(b^{jk}(t) + \delta^{jk}(t,x)\right). \end{aligned} \tag{2}$$

The variables and functions have the following meaning (here $j \to k$ denotes the transition from state j to state k)

t = time,

x = value of the surplus,

T = time of expiry of the contract,

$V^j(t,x)$ = reserve of the contract in state j,

r = constant risk-free interest rate,

$\pi(t,x)$ = share of the surplus invested in a risky asset,

σ = constant diffusion coefficient in the surplus dynamics,

$b^{jk}(t)$ = benefit payments due upon transitions $j \to k$,

$b^j(t)$ = benefit payments due during sojourns in state j,

$\mu^{jk}(t)$ = intensities for transitions $j \to k$,

$\delta^j(t, x)$ = dividends paid from surplus during sojourns in state j,

$\delta^{jk}(t, x)$ = dividends paid from surplus upon transitions $j \to k$,

$c^j(t)$ = contribution to the surplus in sojourns in state j,

$c^{jk}(t)$ = contribution to the surplus upon transitions $j \to k$.

The aim now is to analyze the well-posedness of the PDE (1), i.e. the existence and uniqueness of solutions and their continuous dependence on the data. We cannot expect to find closed-form solutions except in very simple cases. Thus, we apply semigroup methods that allow an algebraic formulation of the solution so that further analysis is possible by analytic methods, cf. Pazy (1983), Amann (1995) and Lunardi (1995) for standard references.

The idea is as follows: we regard the function $V(t, x)$ as a map $t \mapsto V(t, \cdot)$ from $[0, T]$ to a function space X, e.g. $C(\mathbb{R})$. This allows to interpret the PDE system (1) as a first-order evolution equation of the form $\partial_t V(t) = \mathcal{L}V(t) + f(t)$ describing the motion of a point $V(t)$ in the Banach space X driven by a linear operator $\mathcal{L}$. Here, f is an element of $C([0, T]; X)$, i.e. a continuous function $[0, T] \to X$. We can then formally write the solution of this equation as $V(t) = e^{t\mathcal{L}}V(0) + \int_0^t e^{(t-s)\mathcal{L}} f(s)ds$. Although this is no closed-form solution, it yields qualitative information in terms of regularity, growth, asymptotics, etc. through a study of the operators $e^{t\mathcal{L}}$.

The key difficulty lies in showing that $e^{t\mathcal{L}}$ exists and that it has certain mapping properties between function spaces. This object has of course the semigroup property $e^{(t+s)\mathcal{L}} = e^{t\mathcal{L}}e^{s\mathcal{L}}$ and moreover $\partial_t e^{t\mathcal{L}} = \mathcal{L}e^{t\mathcal{L}}$ so that it allows to construct a solution of the PDE.

In our case the operator $\mathcal{L}$ will be time-dependent $\mathcal{L} = \mathcal{L}(t)$ and the semigroup $e^{t\mathcal{L}}$ will be replaced by an *evolution family* of linear operators $G(t, s)$ indexed by two time variables. It has the properties $G(t, r)G(r, s) = G(t, s)$, $G(s, s) = 1$ and $\partial_t G(t, s) = \mathcal{L}(t)G(t, s)$. In probabilistic terms this can be interpreted as the transition semigroup corresponding to the Markov process for the behavior of the risky asset, cf. Rogers and Williams (2000).

Alternative solution methods are based on fundamental solutions, i.e. the integral kernels of the operators $G(t, s)$. This line of study was taken by the Russian school, we refer to Ladyzhenskaya *et al.* (1968) for a classic reference and to Eidelman and Ivasyshen (2004) for a discussion of the present dissipative case. However, these techniques do not easily generalize to more complex situations such as risky assets driven by Lévy processes.

We state and motivate the hypotheses on the data and indicate how they can be relaxed.

HYPOTHESIS 1 *Core assumptions on the coefficients are as follows.*

(i) *Existence of the evolution family*

(a) *there is a* $\pi_0 > 0$ *such that* $\pi(x) \geq \pi_0$ *for all* $x \in \mathbb{R}_+$,
(b) π, c^j *and* δ^j *belong to* $C^{\alpha/2,\alpha}_{loc}([0,T] \times \mathbb{R}_+)$ *for some* $\alpha \in (0,1)$,
(c) *the function* π *is bounded above and the* c^j *are nonnegative.*

(ii) *Regularity of payments, dividends upon transition, transition intensities*

(a) b^j *and* μ^{jk} *belong to* $C([0,T])$,
(b) δ^{jk} *belong to* $C^{0,\alpha}([0,T] \times \mathbb{R})$ *for all* j, k.

(iii) *Boundedness of dividend payments*

(a) *there is a constant* $k > 0$ *such that* $0 \leq \delta^j(t,x) \leq kx$,
(b) *the expression* $\frac{\log x}{x(1+\log^2 x)}\delta^j(t,x)$ *remains bounded for* $x \to 0$,
(c) *for all* x, t *we have* $x + c^{jk}(t) - \delta^{jk}(t,x) \geq 0$.

The assumptions in (i) are standard and guarantee the existence of the evolution family $G(t,s)$. They consist of the uniform ellipticity assumption and put lax constraints on the regularity of the coefficients of the partial differential operators in (2). Economically, uniform ellipticity means that there is always a minimum share of the surplus invested in the risk asset. The regularity assumption poses no serious limitation on modeling as they are phrased in local Hölder spaces, i.e. they are valid on compact sets. The boundedness of π and the non-negativity of c^j are clear. It is possible to relax assumptions (i)(a) and (b), i.e. to allow $\pi(x) \geq 0$ (zero investment in the risky asset is possible) and have c^j, δ^j only measurable. This could be done in the framework of so-called weak solutions, cf. Evans (2010). The degenerate case $\pi(x) \geq 0$ would then be treated by a regularizing argument where one would consider $\mathcal{A}(t) + \epsilon\Delta$ for $\epsilon > 0$ to obtain a uniformly elliptic equation and consider the limit as $\epsilon \to 0$, cf. Lions (1965). Here, Δ denotes the Laplacian.

The assumptions in (ii) just require that the functions modeling benefit payments, transition intensities and dividend payments upon state changes be (Hölder) continuous. In practical modeling exercises this is no significant constraint. A relaxation to purely measurable functions is possible at the expense of all our results being phrased in L^∞-spaces.

The final group of assumptions (iii) expresses the protection of the insurance company and the policyholder. The linear bound on the dividend payments reflects the idea that not more than the surplus should be paid out in dividends. The second condition limits the insurer from paying large dividends if the surplus is too small; this technically looking condition is satisfied for example if δ^j is zero for x below a certain threshold. The final assumption simply says that upon jumps between states of the contract, the surplus before the transition is not turned negative by net contributions and dividends. These conditions cannot be relaxed in our framework. However, any dividend scheme that is linear for large x and approaches 0 fast enough for $x \to 0$ is admissible, including the important linear dividend rule $\delta^j(t,x) = q(t)x$.

Additional regularity assumptions on the coefficients lead to sharper results.

Hypothesis 2 (i) *π and its first spatial derivative belong to $C^{\alpha/2,\alpha}_{loc}([0,T]\times\mathbb{R})$ for some $\alpha\in(0,1)$,*

(ii) *$|x\partial_x\pi|$ is bounded on $[0,T]\times\mathbb{R}_+$,*

(iii) *$c^j, c^{jk}, \delta^j, \delta^{jk}$ also satisfy (i) for all $j,k\in\{1,\dots,n\}$,*

(iv) *$|\partial_x\delta^j|$ are bounded on $[0,T]\times\mathbb{R}_+$ for $j=1,\dots,n$.*

These assumptions allow for higher regularity of the insurance reserves as a function of the value of the risky asset as is pointed out in the key theorems.

These assumptions are driven by and particular to the semigroup approach. The other approaches to sensitivities work with differentiable functions (Kalashnikov and Norberg (2003)) or in spaces of functions of bounded variation, cf. Christiansen (2008) and subsequent papers using that approach.

Our initial step is to rewrite the the PDE system (1) in matrix form. Define a time-dependent linear operator $\mathbf{T}=\mathbf{T}(t)$ acting on $C_b([0,T]\times\mathbb{R}_+)\otimes\mathbb{R}^n$ by

$$\mathbf{T}=\begin{pmatrix} -\sum_{k\neq 1}\mu^{1k}1 & \mu^{12}T^{12} & \mu^{13}T^{13} & \dots & \mu^{1n}T^{1n} \\ \mu^{21}T^{21} & -\sum_{k\neq 2}\mu^{2k}1 & \mu^{23}T^{23} & \dots & \mu^{2n}T^{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu^{n1}T^{n2} & \mu^{n2}T^{n2} & \mu^{n3}T^{n3} & \dots & -\sum_{k\neq n}\mu^{nk}1 \end{pmatrix}$$

where 1 denotes the identity operator. Here, $\mu^{jk}=\mu^{jk}(t)$ and the $T^{jk}=T^{jk}(t)$ are linear operators acting on functions on $[0,T]\times\mathbb{R}_+$ as

$$\left(T^{ik}f\right)(t,x)=f\left(r,x+c^{jk}(t)-\delta^{jk}(t,x)\right),$$

which makes sense as we required $x+c^{jk}-\delta^{jk}\geq 0$ for all $(t,x)\in[0,T]\times\mathbb{R}_+$. Now make a change of variables converting the PDE into an initial-value problem on the new manifold $\mathbb{R}$. We revert time as $\tau=T-t$, the space transformation is given by $y=\log x$. *By abuse of notation, we denote the reserve in the transformed (τ,y)-frame also by* $\mathbf{V}$. The market reserve then satisfies the PDE system (the 'Thiele PDE')

$$\left.\begin{array}{l} \partial_\tau\mathbf{V}=(\mathcal{A}(\tau)+\mathbf{T}-r)\,\mathbf{V}+e^{r\tau}\beta \\ \mathbf{V}(0)=0 \end{array}\right\} \tag{3}$$

with the diagonal operator

$$\mathcal{A}=\begin{pmatrix}\mathcal{A}^1 & & \\ & \ddots & \\ & & \mathcal{A}^n\end{pmatrix}$$

where (omitting explicit space and time dependence)

$$\mathcal{A}^j=\tfrac{1}{2}\pi^2\sigma^2\partial_y^2+\left(r+\left(c^j-\delta^j\right)e^{-y}-\tfrac{1}{2}\pi\sigma^2\right)\partial_y$$

are linear second-order partial differential operators.

3.2. Existence and uniqueness of a solution

Now let $\mathbf{G}$ be the evolution system generated by $\mathcal{A}$, i.e. a family of operators $\mathbf{G}(\tau, s)$ acting between Banach spaces of Hölder continuous functions such that

$$\mathbf{G}(\tau, s)\mathbf{G}(s, r) = \mathbf{G}(\tau, r)$$

for $r \leq s \leq \tau$ and $\mathbf{G}(\tau, \tau) = id$. We will prove later that such a family exists. We say that $\mathbf{V}$ is a *mild solution* of (3) if it satisfies the Duhamel formula

$$\mathbf{V}(\tau) = \int_0^\tau \mathbf{G}(\tau, s) \left[\mathbf{TV}(s) + e^{-r(\tau-s)}\beta(s) \right] ds. \tag{4}$$

Equation (4) is an integral version of the Thiele equation. The obvious advantage of the concept of integral equations and mild solutions is that it does not entail derivatives of $\mathbf{V}$ so that one can go beyond classically differentiable functions. In Milbrodt and Stracke (1997), the authors introduce a similar integral equation to unify continuous and discrete approaches to insurance reserves. Their equation also appears as the natural basis for recent sensitivity analyses, foremost Christiansen (2011), Equation (2.4) of Christiansen (2010) and Equation (3.1) of Christiansen and Steffensen (2011). However, these approaches only have reserves depending on the time variable t whereas the surplus-dependent insurance requires an additional variable x for the value of the surplus. Thus, our integral Equation (4) is a true generalization of the Thiele equations used in the cited papers and it can only be obtained via semigroups.

The existence results now read as follows.

THEOREM 3 *Under Hypothesis 1 the system of PDEs (3) has a unique mild solution* $\mathbf{V}$ *in* $C_b([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n$ *satisfying (4). If we also assume Hypothesis 2, then the solution belongs to* $C^{0,\alpha}([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n$*, where* α *denotes the Hölder class of the coefficients of* $\mathcal{A}$.

We can interpret the Duhamel decomposition (4) economically. Indeed it shows that $\mathbf{V}$ factorizes into terms corresponding to

(i) financial risk from the investment in the risky asset as represented by $\mathbf{G}$,
(ii) the effect of payments and dividends as represented by β, and
(iii) insurance risk as represented by $\mathbf{T}$.

In Steffensen (2006), the author defines the market reserve as the conditional expectation (which of course also solves (1)) on the (t, x)-spacetime manifold as

$$\mathbf{V}^{Z(t)}(t, X(t)) = \mathbb{E}^{\mathbb{Q}} \left[\int_t^T e^{-r(s-t)} d(B + D)(s) \,|Z(t), X(t) \right] \tag{5}$$

where X is the process of the surplus, Z is the process giving the state of the insurance contract, and B, D correspond to b and δ, respectively. Here, $\mathbb{Q}$ is a risk-neutral product measure expressing both insurance and financial risks.

To compare this with our approach, we consider the economically significant case of an insurance contract where the surplus does not change upon transition from one state to another. This translates to $c^{jk}(t) - \delta^{jk}(t, x) \equiv 0$ and the solution of (4) can be expressed as

$$\mathbf{V}(\tau, y) = \int_0^\tau e^{-r(\tau-s)} \left[\mathbf{G}(\tau, s) \exp \mathbf{M}(s)\beta(s)\right](y)\, ds. \tag{6}$$

Here, we set $\mathbf{M}(s) = \int_0^s \mathbf{T}(s')\, ds'$ by componentwise integration of the matrices. The matrix exponentials $\exp \mathbf{M}$ appear since $\mathbf{T}$ has no space-dependence so that $\mathcal{A}$ and $\mathbf{T}$ commute. Disregarding the terms $-r\mathbf{V} + e^{r\tau}$ in (3) which can be removed by a coordinate transformation, the solution operator becomes $\mathbf{G} \exp \mathbf{M}$. Note that (6) is precisely of the form of (5) where the product of commuting evolution operators $\mathbf{G} \exp \mathbf{M}$ corresponds to the product measure $\mathbb{Q}$.

3.3. *Uniform and pointwise sensitivity estimates*

We first formulate uniform estimates for the sensitivities of the market reserve in terms of its key parameters and also give pointwise estimates for its dependence upon the surplus. In the following, any constants can be computed in principle so that the theorems provide simple tools to assess uniform shifts in valuation bases as requested by Solvency II for example.

THEOREM 4 *Assume Hypothesis 1 and set $\varphi(\tau) = \int_0^\tau e^{-r(\tau-s)} ds$. Define two Banach spaces $Y_1 = C_b(\mathbb{R}) \otimes \mathbb{R}^n$ and $Y_2 = C_b([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n$.*

(i) *Growth of the market reserve: let $\hat{T} = \sup_\tau ||\mathbf{T}(\tau)||$ where $||\cdot||$ denotes the operator norm on Y_2. Then*

$$||\mathbf{V}(\tau)||_{Y_1} \le ||\beta||_{Y_2} \left(\hat{T} e^{c\hat{T}\tau} \int_0^\tau \varphi(s) ds + \varphi(\tau) \right)$$

for $\tau \in [0, T]$.

(ii) *Dependence on the payment rate:*

$$||\mathbf{V}_1(\tau) - \mathbf{V}_2(\tau)||_{Y_1} \le ||\beta_1 - \beta_2||_{Y_2} \left(\hat{T} e^{\hat{T}\tau} \int_0^\tau \varphi(s) ds + \varphi(\tau) \right)$$

for $\tau \in [0, T]$ where $\mathbf{V}_i$ denotes the reserve corresponding to β_i.

(iii) *Dependence on insurance risk: suppose there are two sets of transition intensities μ_i^j and μ_i^{jk} for $i = 1, 2$ with corresponding operators $\mathbf{T}_i$. Then*

$$||\mathbf{V}_1(\tau) - \mathbf{V}_2(\tau)||_{Y_1} \le ||\beta||_{Y_2} \sup_s ||\mathbf{T}_1(s) - \mathbf{T}_2(s)|| C(\mathbf{T}_1, \mathbf{T}_2; \tau),$$

for $\tau \in [0, T]$ where

$$C(\mathbf{T}_1, \mathbf{T}_2; \tau) = \hat{T}_1 e^{\hat{T}_1 \tau} \int_0^\tau \int_0^s \varphi(u) du ds + e^{\hat{T}_2 \tau} \int_0^\tau \varphi(s) ds.$$

Here, $\mathbf{V}_i$ is the reserve corresponding to μ_i^j, μ_i^{jk}, and $\hat{T}_i = \sup_\tau ||\mathbf{T}_i(\tau)||$.

Simultaneous changes in several assumptions can be treated similarly.

It is instructive to compare this result with the literature. To ease the presentation, let X denote the space representing the valuation bases in each approach. In Kalashnikov and Norberg (2003), the authors view the reserves as differentiable functions $V^j : [0, T] \times X \to \mathbb{R}$ where X is a finite-dimensional vector space. Sensitivities are given as derivatives of the reserve with respect to X viewed as finitely many parameters. Christiansen (2008) and related papers treat the reserve as maps $V^j : [0, T] \to X'$ where X' is the dual space of X and X is a product of function spaces, each factor representing a valuation basis. Sensitivities for fixed t are then expressed as functional derivatives (measured in the topology of X') leading to first-order approximations in an L^1-norm. Now, Theorem 4 due to the surplus-dependence views reserves as Hölder functions $V^j : [0, T] \times X \to C_b(\mathbb{R})$ with X a product of function spaces with each factor representing an assumption (e.g. μ^{ij} or c^j.) Recall that the surplus-dependence triggers the use of semigroups acting on function spaces. We thus express continuity of the reserves in the uniform topology of $C_b(\mathbb{R})$ so that from a structural point of view our results are similar to Christiansen (2008). However, our Lipschitz continuity results are phrased in uniform norms (in t, x) which are practically more relevant than L^1-norms (in t).

As regards to the cited recent worst-case analyses, these build on Christiansen (2008) and exploit the first-order approximation. The above theorem provides a crude notion of worst-case deviations within the geometry of Banach spaces. A more detailed analysis in the direction of the cited works can be based on the linear Equation (7) below and a deeper analysis of the operator difference $\mathbf{T}_1 - \mathbf{T}_2$ which corresponds to the Gâteaux differential in the other approach. As this is not the main point of the present article we have, however, not expanded on this.

Assuming also Hypothesis 2, we have sharper results in that the theorem holds in $Y_1 = C_b^{\alpha}(\mathbb{R}) \otimes \mathbb{R}^n$ and $Y_2 = C_b^{0,\alpha}([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n$ with constants c depending on α being introduced in the right-hand sides in front of the β- and $\hat{T}$-terms. This has implications for numerical schemes as they converge faster if the functions involved are more regular.

Generally, pointwise estimates in x are elusive so that we resort to pointwise upper bounds of the gradient. We can obtain these in the simplification $c^{jk}(t) - \delta^{jk}(t, x) \equiv 0$ so that the solution of (3) is given in the form (6). Note that by standard results, $\mathbf{V}$ is a classical solution in this case, i.e. it belongs to $C^{1,2}([0, T] \times \mathbb{R}^n) \otimes \mathbb{R}^n$. For the pointwise estimates we assume in addition to Hypothesis 1 and 2.

HYPOTHESIS 5 *The coefficient functions π, c^j, and δ^j and their first order spatial derivatives belong to the space $C_{loc}^{\alpha/2,\alpha}([0, T] \times \mathbb{R})$ for some $\alpha \in (0, 1)$.*

We also define

$$\beta'(\tau, y) = \exp \mathbf{M}(\tau) \beta(\tau) e^{r\tau}$$

to represent both the insurance risk and payment rate. We then have

THEOREM 6 *For $p > 1$ there is a $\sigma(p)$ such that if $W^j(\tau, y)$ is the solution of the PDE*

$$\left.\begin{array}{l} \partial_\tau W^j = \mathcal{A}(\tau)W^j - (rp - \sigma)W^j + \left|T^{1-\frac{1}{p}}\partial_y \beta'^j(\tau, \cdot)\right|^p \\ W^j(0) = 0, \end{array}\right\}$$

then the derivative of the actuarial reserve V^j with respect to y is pointwise dominated by W^j via

$$|\partial_y V^j(\tau, y)|^p \leq W^j(\tau, y)$$

for $(\tau, y) \in [0, T] \times \mathbb{R}$.

This result has no direct correspondences in the literature as it addresses a unique feature of surplus-linked life insurance, namely the dependence on the value of the surplus. This is somewhat similar to Kalashnikov and Norberg (2003) as we derive a new PDE from the Thiele PDE to express sensitivities.

4. Proof of the key results

The idea of the proofs is to view the reserves as members of a suitable state space (spaces of Hölder continuous functions) and represent time evolution by a family of linear operators with the semigroup property. Further properties of the time evolution operators yield the sensitivity results in the spirit of operator algebra. We only present the main roads of the proofs in this section with details in the appendix.

We record the fact that the second-order partial differential operators $\mathcal{A}^j$ generate an evolution family on uniform spaces. These operators are bounded and have smoothing properties.

PROPOSITION 7 *Under Hypothesis 1, each operator $\mathcal{A}^j$ gives rise to an evolution system $G^j(\tau, s)$ of bounded linear operators with $||G^j(\tau, s)f||_{C_b(\mathbb{R})} \leq ||f||_{C_b(\mathbb{R})}$.*

Under the additional Hypothesis 2 for any $0 \leq \alpha \leq \gamma < 1$ there is a constant $c = c(\alpha, \gamma)$ such that $G^j(\tau, s) : C_b^\alpha(\mathbb{R}) \to C_b^\gamma(\mathbb{R})$ and

$$||G^j(\tau, s)f||_{C_b^\gamma(\mathbb{R})} \leq c(\tau - s)^{-(\gamma-\alpha)/2}||f||_{C_b^\alpha(\mathbb{R})}$$

for $f \in C_b^\alpha(\mathbb{R})$ and any $s \leq \tau \leq T$.

This allows us to prove the first key theorem on existence and uniqueness, details are in the appendix.

Proof of Theorem 3 We seek a solution in the Banach space $C^{0,\alpha}([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n$ explicitly in terms of a Neumann series (Dyson series in physics and Peano series in matrix analysis). This series converges for short times, thanks to uniform a-priori estimates obtained by Gronwall's lemma and can be extended for all times. Uniqueness follows easily by another application of

Gronwall. If we do not assume Hypothesis 2, then $\alpha = 0$, otherwise α is the Hölder class of the coefficients of $\mathcal{A}$. □

The proofs of the other results are now simple applications of operator estimates.

Proof of Theorem 4 The results are simple consequences of Gronwall's lemma.

(i) This is the a-priori estimate of the proof of Theorem 3, see Equation (8).
(ii) Easy corollary of (i).
(iii) Consider the difference in the market reserves

$$\begin{aligned}\mathbf{V}_1(\tau) - \mathbf{V}_2(\tau) &= \int_0^\tau \mathbf{G}(\tau, s)\left(\mathbf{T}_1(s)\mathbf{V}_1(s) - \mathbf{T}_2(s)\mathbf{V}_2(s)\right) ds \\ &= \int_0^\tau \mathbf{G}(\tau, s)\mathbf{T}_1(s)\left(\mathbf{V}_1(s) - \mathbf{V}_2(s)\right) ds \\ &\quad + \int \mathbf{G}(\tau, s)\left(\mathbf{T}_1(s) - \mathbf{T}_2(s)\right)\mathbf{V}_2(s) ds. \end{aligned} \tag{7}$$

The second term can be bounded by the uniform estimates (8). Then apply Gronwall's inequality on the first term to bound $||\mathbf{V}_1(\tau) - \mathbf{V}_2(\tau)||_{Y_1}$.

The same method applies for the general case $Y_2 = C_b^{0,\alpha}([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n$. □

Proof of Theorem 6 Before proving the pointwise estimates, we note that the simplified solution (6) does indeed solve the PDE system (3) in the case $c^{jk} - \delta^{jk} = 0$ as $\mathcal{A}$ and $\mathbf{M}$ commute. The proof is then a simple application of Theorem 4.5 of Kunze *et al.* (2010), technical details are in the appendix. □

5. Concluding remarks

As indicated in the introduction, this exposition can serve as a stepping stone towards the treatment of more general but economically relevant situations:

(i) an easy generalization is to include variable (deterministic or stochastic) interest rates and stochastic transition intensities;
(ii) also the illiquidity of the market for the risky asset can be taken into account. Under standard liquidity models this would introduce a non-linear (e.g. quadratic) term in the PDE that depends on the gradient of the reserve with respect to the risky asset;
(iii) the risky asset could be driven by Lévy, affine or polynomial processes. This has drawn significant recent attention in derivative pricing and also reflects the need to assess model risk in the light of the ongoing financial crisis. The semigroup methods set out in this paper lend themselves easily to the relevant analysis. Technically, this involves boundary-value problems in Sobolev spaces with the differential operator $\mathcal{A}$ replaced by a pseudodifferential operator.

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A Computational details of proofs

We first collect properties of the operators **T**.

LEMMA 8 *Assume Hypothesis 1* (ii)(b) *and let* α *be the Hölder class of* δ^{jk}. *Then there is a constant* κ *such that* **T** *is a bounded operator preserving the space* $C^{0,\alpha}([0,T]\times\mathbb{R})\otimes\mathbb{R}^n$ *with operator norm less than or equal to* $n(n-1)||\mu||\cdot\kappa$. *The constant* κ *depends only on* θ, c^{jk}, δ^{jk} *and their derivatives.*

Proof The the action of **T** in the (τ, y)-coordinates is given as $T^{jk}f(\tau,y) = f\circ\chi^{jk}(\tau,y)$, i.e. by composition with a transformation χ^{jk} where

$$\chi^{jk}: \quad (\tau,y)\mapsto\left(\tau,\log\left(e^y + c^{jk}(\tau) - \delta^{jk}(\tau,y)\right)\right).$$

We treat the cases $\theta = 0$ and $\theta\in(0,1)$ separately.

(1) For the case $\theta = 0$ fix $\tau, y\in[0,T]\times\mathbb{R}$ we arrive at

$$\begin{aligned}|\mathbf{TV}(\tau,y)|^2 \leq & n\left(\left|\sum_{k\neq 1}\mu^{1k}\right|^2 ||V^1||_\infty^2 + |\mu^{12}|^2||V^2||_\infty^2 + \ldots + |\mu^{1n}|^2||V^n||_\infty^2\right) \\ & + \ldots + n\left(|\mu^{n1}|^2||V^1||_\infty^2 + \ldots + \left|\sum_{k\neq n}\mu^{nk}\right|^2 ||V^n||_\infty^2\right),\end{aligned}$$

by Hölder's inequality and the fact that each T^{jk} has operator norm bounded by 1. By the definition of $||\mu||$ we have $\left|\sum_{k\neq j}\mu^{jk}(\tau)\right|^2 \leq (n-1)||\mu||^2$. Hence

$$\begin{aligned}||\mathbf{TV}||_\infty^2 \leq & n\left((n-1)^2||\mu||^2||V^1||_\infty^2 + ||\mu||^2||V^2||_\infty^2 + \ldots + ||\mu||^2||V^n||_\infty^2\right) + \cdots \\ \leq & n\left(||\mu||^2||V^1||_\infty^2 + \ldots + (n-1)^2||\mu||^2||V^n||_\infty^2\right) \\ \leq & n^2(n-1)^2||\mu||^2||(V^1,\ldots,V^n)^\top||_\infty^2\end{aligned}$$

whence the result follows.

(2) For $\theta\in(0,1)$ choose $f\in C^{0,\theta}([0,T]\times\mathbb{R})$ and let $y_1, y_2\in\mathbb{R}$. Then

$$\frac{|T^{jk}f(\tau,y_1) - T^{jk}f(\tau,y_2)|}{|y_1-y_2|^\theta} = \frac{|f\circ\chi^{jk}(y_1) - f\circ\chi^{jk}(y_2)|}{|\chi^{jk}(y_1)-\chi^{jk}(y_2)|^\theta}\cdot\frac{|\chi^{jk}(y_1)-\chi^{jk}(y_2)|^\theta}{|y_1-y_2|^\theta}.$$

This shows that the Hölder norm $||T^{jk}f(\tau)||_{C^\theta(\mathbb{R})}$ is finite and bounded in terms of the norms $||f(\tau)||_{C^\theta(\mathbb{R})}$ and $||c^j(\tau) - \delta^{jk}(\tau)||_{C^1(\mathbb{R})}$. The result follows by extension to $C^{0,\theta}([0,T]\times\mathbb{R})\otimes\mathbb{R}^n$. □

Proof of Proposition 7 The aim is to apply Theorem 2.4 of Lorenzi (2011) in a simplified form which says that under the certain hypotheses, for any $0\leq\alpha\leq\beta\leq 1$ there is a constant c such that

$$||G(t,s)f||_{C_b^\beta(\mathbb{R})} \le c(t-s)^{-(\beta-\alpha)/2}||f||_{C_b^\alpha(\mathbb{R})}$$

for $f \in C_b^\alpha(\mathbb{R})$ and any $s \le t \le T$. We check that the required Hypotheses 6.1.1 (i) (ii) (iii) (iv-1) of Lorenzi and Bertoldi (2007) are satisfied; the proof for the time-dependent case as exhibited in Lorenzi (2011) carries over. For the reader's convenience, we use the numbering and almost identical notation as in Lorenzi and Bertoldi (2007): let $Q = \frac{1}{2}\pi^2\sigma^2$ and $B = r + (c^j - \delta^j)e^{-y} - Q$, all other variables have the same meaning.

(i) satisfied by Hypothesis 1 (i)(a)

(ii) To this end define

$$\varphi_0(y) = \begin{cases} e^y & \text{for } y < 0, \\ 1+y & \text{for } y \ge 0. \end{cases}$$

and let ψ be a smooth bump function supported on $[-1,1]$ and equal to 1 on $[-1/2, 1/2]$. Setting $\varphi = (1-\psi)\varphi_0$ yields $\mathcal{A}\varphi$ bounded above so that for any $\lambda \in \mathbb{R}_+$ we find that $\mathcal{A}\varphi - \lambda\varphi$ is also bounded above on $[0,T] \times \mathbb{R}$.

(iii) Assume that $\nu(\tau, y) = \nu_0$ constant. Then

(a) we have that $|Q(\tau,y)y| = \frac{1}{2}\Theta^2\sigma^2|y| \le c_1\nu_0(1+y^2)$ for some $c_1 > 0$ as Q is bounded by assumption Hypothesis 1 (i)(c);

(b) we find that $Q(\tau,y) \le c_2\nu_0(1+|y|^2)$ for some $c_2 > 0$ as Q is bounded;

(c) We distinguish the cases $y \ge 0$ and $y < 0$. First of all, note that

$$B(\tau,y)y = \left(r - \frac{1}{2}\pi^2\sigma^2\right)y + \left(c^j - \delta^j\right)e^{-y}y$$

Now for $y \ge 0$, the first term is bounded above by $(r + c_1\nu_0)(1+y^2)$ by Hypothesis 1 (i)(a). Boundedness for $y < 0$ is obvious. The term $c^j e^{-y} y$ is also uniformly bounded for any y by inspection. The final term $-\delta^j e^{-y} y$ is bounded due to Hypothesis 1 (iii)(b) since this translates to $-y\delta^j(\tau,y)e^{-y} \le k(1+y^2)$ for some $k > 0$.

(iv-1) This concerns the dissipativity of the operator $\mathcal{A}(\tau)$. Here, consider

$$\partial_y B = -\left(c^j - \delta^j\right)e^{-y} - \left(\partial_y\delta^j\right)e^{-y} - \partial_y Q.$$

The second summand is bounded as $|\partial_x\delta^j|$ is bounded since this translates into boundedness of $e^{-y}\partial_y\delta^j(\tau,y)$. The third summand is bounded above by Hypothesis 2 2. In the first summand, the term $-c^j e^{-y}$ is bounded above by 0 for any y since the contribution functions c^j are non-negative. It remains to show that $\delta^j e^{-y}$ is bounded for any y. This, however, follows directly from Hypothesis 1 (iii)(a). All other assumptions are satisfied by Hypothesis 1 (i) and Hypothesis 2 (i)–(iii) and step (iv-1) above.

□

Proof of Theorem 3 We first prove a-priori estimates showing that if a solution V exists then it must be uniformly bounded in terms of global data. The bounds follow easily using the differential form of Gronwall's lemma (cf. Appendix B.j. of Evans (2010)). Taking uniform estimates on both sides of (4) yields

$$||\mathbf{V}(\tau)||_{C^\alpha(\mathbb{R})\otimes\mathbb{R}^n} \leq c\hat{T}\int_0^\tau ||\mathbf{V}(s)||_{C^\alpha(\mathbb{R})\otimes\mathbb{R}^n} + c||\beta||_{C^{0,\alpha}([0,T]\times\mathbb{R})\otimes\mathbb{R}^n}\varphi(\tau),$$

where $\hat{T} = \sup_s ||T(s)||$ is the supremum of the operator norms of T and $\varphi(\tau) = \int_0^\tau e^{-r(\tau-s)}ds$. The constant c is from Proposition 7. Gronwall's lemma implies

$$||\mathbf{V}(\tau)||_{C^\alpha(\mathbb{R})\otimes\mathbb{R}^n} \leq c||\beta||_{C^{0,\alpha}([0,T]\times\mathbb{R})\otimes\mathbb{R}^n}\left(c\hat{T}e^{c\hat{T}\tau}\int_0^\tau \varphi(s)ds + \varphi(\tau)\right), \tag{8}$$

proving the claim.

Now set $\mathbf{V} = e^{-r\tau}\mathbf{U}$ leading to a Volterra-type equation (cf. Kress (1999))

$$\mathbf{U}(\tau) = \int_0^\tau \mathbf{G}(\tau,s)\mathbf{T}(s)\mathbf{U}(s)ds + \mathbf{f}(\tau), \tag{9}$$

where $\mathbf{f}(\tau) = \int_0^\tau e^{rs}\beta(s)\,ds$. Under the operation

$$(\mathbf{GT}\#\xi)(\tau) = \int_0^\tau \mathbf{G}(\tau,s)\mathbf{T}(s)\xi(s)ds,$$

mapping $C^\alpha(\mathbb{R})\otimes\mathbb{R}^n$ to itself, the solution of this equation can be written as

$$\mathbf{U} = \mathbf{f} + \mathbf{GT}\#\mathbf{f} + \mathbf{GT}\#\mathbf{GT}\#\mathbf{f} + \cdots. \tag{10}$$

This Neumann series converges in the norm of Z for τ sufficiently small, i.e. on a time interval $[0, T_0]$ for some $T_0 > 0$.

To extend the solution along the timeline, we iterate the step 2 finitely many times with new initial conditions to obtain a solution on the full interval $[0, T]$. The new PDE to be solved is again of the form (9) where this time $\mathbf{f}(\tau) = \mathbf{G}(\tau, 0)\mathbf{U}(T_0) + \int_0^\tau e^{rs}\beta(s)\,ds$. Again, the solution is explicitly given by (10). This series converges for small values of τ so that we have now extended the solution to $[0, T_1]$ with $T_1 > T_0$. Note that thanks to (8) the time extension by $T_1 - T_0$ only depends on global constants and not on T_0 so that we may repeat the process finitely many times to obtain a solution on the interval $[0, T]$.

Uniqueness follows easily: suppose that $\mathbf{U}_1$ and $\mathbf{U}_2$ were two solutions of (9). Then $\mathbf{U}_1 - \mathbf{U}_2$ also satisfies an equation of this type, however, with $f = 0$. Gronwall's lemma then implies $\mathbf{U}_1 - \mathbf{U}_2 = 0$. □

Proof of Theorem 6 We note that the Hypothesis 1.3 of Kunze *et al.* (2010) are satisfied by the proof of Proposition 7. Hence by Theorem 4.5 of Kunze *et al.* (2010) for every $p > 1$ we have for $f \in C_b^1(\mathbb{R})$ that

$$|\left(\partial_x G^j(\tau, s) f\right)(x)|^p \leq e^{\sigma(p)(\tau-s)} \left(G^j(\tau, s)|\partial_x f|^p\right)(x) \tag{11}$$

for $s \leq \tau$ and $x \in \mathbb{R}$, where

$$\sigma(p) = p \cdot \left(||\partial_y B||_\infty + \frac{||\partial_y Q||_\infty}{4\pi_0 \min\{p-1, 1\}}\right)$$

with $Q = \frac{1}{2}\pi^2\sigma^2$ and $B = r + (c^j - \delta^j)e^{-y} - Q$. The norms are finite by Hypothesis 2 (ii) on Q and by step (iv-1) of the proof of Proposition 7 for B. Thus applying (11) to (3) and using Hölder's inequality for functions we find

$$\begin{aligned}
|(\partial_y V^j)(\tau, y)|^p &\leq \tau^{p-1} \int_0^\tau \left|\left[\partial_y G^j(\tau, s)\beta'^j(s, \cdot)\right](y) e^{-r(\tau-s)}\right|^p ds \\
&\leq \tau^{p-1} \int_0^\tau e^{-rp(\tau-s)+\sigma(\tau-s)} G^j(\tau, s) \left|\partial_y \beta'^j(s, \cdot)\right|^p (y)\, ds
\end{aligned}$$

so that

$$|(\partial_y V)(\tau, y)|^p \leq \int_0^\tau e^{-(rp-\sigma)(\tau-s)} G^j(\tau, s)|T^{1-\frac{1}{p}} \partial_y \beta'^j(s, \cdot)|^p\, ds \tag{12}$$

as $0 \leq \tau \leq T$. Now note that the right-hand side of (12) is the solution of the PDE (7) given in Duhamel form. $\square$