

STABILITY RESULTS FOR LÉVY TERM STRUCTURE MODELS

BARBARA RÜDIGER AND STEFAN TAPPE

ABSTRACT. In this note, we study term structure models driven by Lévy processes and provide stability results for them. In reality, we can never be sure of the accuracy of a proposed model. With this motivation, we present sufficient conditions which ensure that the model has the tendency to recover from perturbations. Our results include stability conditions for the forward rates, yield curves and option prices.

1. INTRODUCTION

The value at time t of one monetary unit to be paid at time $T \geq t$ is expressed by a *Zero Coupon Bond*. A Zero Coupon Bond is a contract which guarantees the holder one monetary unit at the maturity date T . The corresponding bond prices till maturity can be written as the continuous discounting of one unit of cash

$$P(t, T) = \exp \left(- \int_t^T f(t, s) ds \right),$$

where $f(t, T)$ is the rate prevailing at time t for instantaneous borrowing at time T , the so-called the forward rate for date T .

The classical continuous time framework for the evolution of the forward rates goes back to Heath, Jarrow and Morton (HJM) [14]. They assume that, for every date T , the forward rates $f(t, T)$ follow an Itô process of the form

$$(1.1) \quad f(t, T) = f(0, T) + \int_0^t \alpha_{\text{HJM}}(s, T) ds + \int_0^t \sigma(s, T) dW_s, \quad t \in [0, T]$$

where W is a Wiener process.

In this paper, we consider Lévy term structure models, which generalize the classical HJM framework by replacing the Wiener process W in (1.1) by a more general Lévy process X , also taking into account the occurrence of jumps. This extension has been proposed by Eberlein et al. [8, 7, 3, 4, 5, 6]. In the sequel, we therefore assume that, for every date T , the forward rates $f(t, T)$ follow an Itô process

$$f(t, T) = f(0, T) + \int_0^t \alpha_{\text{HJM}}(s, T) ds + \int_0^t \sigma(s, T) dX_s, \quad t \in [0, T]$$

with X being a Lévy process.

In reality, we can never be sure of the accuracy of a proposed model. Therefore, we are interested to know how much its corresponding quantities (forward rates, option prices, etc.) would change if we perturb the model – i.e. the volatility $\sigma(t, T)$ and the initial forward curve $f(0, T)$ – a bit. In order to approach this stability problem, we will switch to the Musiela parametrization of forward curves $r_t(x) = f(t, t+x)$ (see

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[17]), which allows us to consider the forward rates as the solution of a stochastic partial differential equation (SPDE), the so-called HJMM (Heath–Jarrow–Morton–Musielà) equation

$$(1.2) \quad \begin{cases} dr_t &= \left(\frac{d}{dx}r_t + \alpha_{\text{HJM}}(r_t)\right)dt + \sigma(r_{t-})dX_t \\ r_0 &= h_0, \end{cases}$$

and to apply stability results for Lévy driven SPDEs, which can, e.g., be found in [1, 12, 16]. Existence and uniqueness of the Lévy driven HJMM equation (1.2) has been investigated in [2, 11, 15, 18, 19].

In order to ensure that the implied bond market $P(t, T)$ is free of arbitrage opportunities, we assume the existence of an equivalent martingale measure. Under such a measure, the drift $\alpha_{\text{HJM}} : H \rightarrow H$ in (1.2) is given by the HJM drift condition

$$(1.3) \quad \alpha_{\text{HJM}}(h) = \frac{d}{dx}\Psi\left(-\int_0^\bullet \sigma(h)(\eta)d\eta\right) = -\sigma(h)\Psi'\left(-\int_0^\bullet \sigma(h)(\eta)d\eta\right),$$

where Ψ denotes the cumulant generating function of the Lévy process, see [7, Sec. 2.1].

Therefore, the principal difficulty when applying stability results for SPDEs is to assure that not only the volatility σ , but also the corresponding drift term α_{HJM} which depends on σ , satisfy appropriate regularity conditions.

The remainder of the note is organized as follows. In Section 2 we introduce the term structure model, and in Section 3 we present the announced stability results.

2. PRESENTATION OF THE TERM STRUCTURE MODEL

In this section, we introduce the Lévy term structure model. From now on, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, and let $X = (X_t)_{t \geq 0}$ be a real-valued Lévy process with drift $b \in \mathbb{R}$, Gaussian part $c \geq 0$ and Lévy measure ν , that is, the characteristic function of X_1 is given by

$$\varphi_{X_1}(u) = \exp\left(ibu - \frac{c}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{[-1,1]}(x))\nu(dx)\right), \quad u \in \mathbb{R}.$$

In what follows, we assume the existence of constants $N, \epsilon > 0$ such that

$$\int_{\{|x|>1\}} e^{zx}\nu(dx) < \infty, \quad z \in [-(1+\epsilon)N, (1+\epsilon)N].$$

Then, the Lévy process X possesses moments of arbitrary order. The cumulant generating function

$$\Psi(z) := \ln \mathbb{E}[e^{zX_1}]$$

exists on $[-(1+\epsilon)N, (1+\epsilon)N]$, and belongs to class C^∞ on the open interval $(-(1+\epsilon)N, (1+\epsilon)N)$. We fix an arbitrary constant $\beta > 0$ and denote by H_β the space of all absolutely continuous functions $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$(2.1) \quad \|h\|_\beta := \left(|h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta x} dx\right)^{1/2} < \infty.$$

Spaces of this kind have been introduced in [9]. According to [13, Thm. 2.1], the space H_β is a separable Hilbert space, the shift semigroup $(S_t)_{t \geq 0}$ defined by $S_t h := h(t + \cdot)$ is a C_0 -semigroup on H_β , there are constants $C_1, C_2 > 0$ such that

$$(2.2) \quad \|h\|_{L^\infty(\mathbb{R}_+)} \leq C_1 \|h\|_\beta, \quad h \in H_\beta,$$

$$(2.3) \quad \|h - h(\infty)\|_{L^1(\mathbb{R}_+)} \leq C_2 \|h\|_\beta, \quad h \in H_\beta,$$

and there exist another separable Hilbert space \mathcal{H}_β , a C_0 -group $(U_t)_{t \in \mathbb{R}}$ on \mathcal{H}_β and continuous linear operators $\ell \in L(H_\beta, \mathcal{H}_\beta)$, $\pi \in L(\mathcal{H}_\beta, H_\beta)$ such that

$$\pi U_t \ell = S_t \quad \text{for all } t \in \mathbb{R}_+.$$

The latter result allows us to apply the stability results from [12], where SPDEs are understood as time-dependent transformations of SDEs. The particular representation of the Hilbert space \mathcal{H}_β is not required in the sequel. Let H_β^0 be the subspace

$$(2.4) \quad H_\beta^0 := \left\{ h \in H_\beta : \lim_{x \rightarrow \infty} h(x) = 0 \right\},$$

and let $U \subset H_\beta^0$ be the set

$$(2.5) \quad U := \left\{ h \in H_\beta^0 : \left\| \int_0^\bullet h(\eta) d\eta \right\|_{L^\infty(\mathbb{R}_+)} \leq N \right\}.$$

For each $h \in U$ we define the function $\Sigma(h) : \mathbb{R}_+ \rightarrow \mathbb{R}$ as

$$\Sigma(h) := h \cdot \Psi' \left(- \int_0^\bullet h(\eta) d\eta \right).$$

Let $C_b^{\text{lip}} = C_b^{\text{lip}}(H_\beta; H_\beta^0)$ be the linear space of all bounded Lipschitz functions $\sigma : H_\beta \rightarrow H_\beta^0$. The linear space C_b^{lip} equipped with the norm

$$\|\sigma\|_{\text{lip}} = \sup_{h \in H_\beta} \|\sigma(h)\|_\beta + \sup_{\substack{h_1, h_2 \in H_\beta \\ h_1 \neq h_2}} \frac{\|\sigma(h_1) - \sigma(h_2)\|_\beta}{\|h_1 - h_2\|_\beta}$$

is a Banach space. We define the subset $F \subset C_b^{\text{lip}}$ as

$$F := \{ \sigma \in C_b^{\text{lip}} : \sigma(H_\beta) \subset U \}.$$

2.1. Lemma. *The following statements are valid:*

- (1) *We have $\Sigma(U) \subset H_\beta^0$, and $\Sigma : U \rightarrow H_\beta^0$ is locally Lipschitz continuous.*
- (2) *For each $\sigma \in F$ we have $\Sigma \circ \sigma \in C_b^{\text{lip}}$, and for each $n \in \mathbb{N}$ there exists a constant $M = M(n) > 0$ such that*

$$\|\Sigma \circ \sigma\|_{\text{lip}} \leq M$$

for all $\sigma \in F$ with $\|\sigma\|_{\text{lip}} \leq n$.

Proof. By [11, Prop. 4.5] there exists a constant $C_3 > 0$ such that for all $h, g \in U$ we have

$$\|\Sigma h - \Sigma g\|_\beta \leq C_3(1 + \|h\|_\beta + \|g\|_\beta + \|g\|_\beta^2) \|h - g\|_\beta,$$

which provides both assertions. \square

Now let $\sigma \in F$ be a volatility. Note that we can write the HJM drift term (1.3) – which ensures the absence of arbitrage – as $\alpha_{\text{HJM}} = \Sigma \circ \sigma$. Hence, by Lemma 2.1 we have $\alpha_{\text{HJM}} \in C_b^{\text{lip}}$. Consequently, for each $h_0 \in H_\beta$ there exists a unique mild solution for (1.2) with $r_0 = h_0$ on the state space H_β of forward curves with càdlàg sample paths satisfying

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|r_t\|^2 \right] < \infty \quad \text{for all } T \geq 0,$$

see, e.g., [11, Thm. C.1].

3. STABILITY RESULTS

In this section, we present the announced stability results for the Lévy term structure model presented in the previous section.

Let volatilities $\sigma \in F$ and $(\sigma^n)_{n \in \mathbb{N}} \subset F$ be given. Furthermore, let initial conditions $h_0 \in H_\beta$ and $(h_0^n)_{n \in \mathbb{N}} \subset H_\beta$ be given. In addition to the HJMM equation (1.2), we also consider the sequence of HJMM equations

$$(3.1) \quad \begin{cases} dr_t^n &= \left(\frac{d}{dx} r_t^n + \alpha_{\text{HJM}}^n(r_t^n) \right) dt + \sigma^n(r_{t-}^n) dX_t \\ r_0^n &= h_0^n, \end{cases}$$

where – in order to ensure the absence of arbitrage – the corresponding drift terms are given by $\alpha_{\text{HJM}}^n = \Sigma \circ \sigma^n$. The following standing assumption assumptions prevail throughout this section:

- $h_0^n \rightarrow h_0$ in H_β ;
- $\sigma^n(h) \rightarrow \sigma(h)$ in H_β for all $h \in H_\beta$;
- $\sup_{n \in \mathbb{N}} \|\sigma^n\|_{\text{lip}} < \infty$.

Then, by Lemma 2.1 we have

- $\alpha_{\text{HJM}}^n(h) \rightarrow \alpha_{\text{HJM}}(h)$ in H_β for all $h \in H_\beta$;
- $\sup_{n \in \mathbb{N}} \|\alpha_{\text{HJM}}^n\|_{\text{lip}} < \infty$.

For what follows, we define the joint Lipschitz constant

$$(3.2) \quad L := \sup_{n \in \mathbb{N}} \max\{\|\alpha_{\text{HJM}}^n\|_{\text{lip}}, \|\sigma^n\|_{\text{lip}}\} < \infty.$$

Denoting by $(r_t)_{t \geq 0}$ the mild solution for (1.2), for every $T \geq 0$ we have

$$(3.3) \quad \begin{aligned} \epsilon_n(T, r) &:= \left(\mathbb{E} \left[\int_0^T \|\alpha_{\text{HJM}}(r_s) - \alpha_{\text{HJM}}^n(r_s)\|_\beta^2 ds \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_0^T \|\sigma(r_s) - \sigma^n(r_s)\|_\beta^2 ds \right] \right)^{1/2} \rightarrow 0, \end{aligned}$$

which follows from Lebesgue's dominated convergence theorem. Now, we are ready to prove stability of the forward curves under the previous conditions.

3.1. Proposition. *For all $T \geq 0$ there exists a constant $K_1 = K_1(T, L) > 0$ such that*

$$(3.4) \quad \mathbb{E} \left[\sup_{t \in [0, T]} \|r_t - r_t^n\|_\beta^2 \right]^{1/2} \leq K_1 \sqrt{\|h_0 - h_0^n\|_\beta^2 + \epsilon_n^2} \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

where $\epsilon_n = \epsilon_n(T, r)$ was defined in (3.3).

Proof. Since we have the joint Lipschitz constant (3.2), the assertion follows from [12, Prop. 9.1]. \square

Besides the forward curve, the yield curve is another measurement of the bond market. Given a bond price $P(t, T)$, the yield $Y(t, T)$ is the quantity

$$Y(t, T) := -\frac{\ln P(t, T)}{T-t} = \frac{1}{T-t} \int_t^T f(t, s) ds.$$

Switching to the Musiela parametrization, we thus define the *yield curve operator* $y : H_\beta \rightarrow C(\mathbb{R}_+)$ as the linear operator

$$y(h)(x) := \begin{cases} h(0), & \text{if } x = 0, \\ \frac{1}{x} \int_0^x h(\eta) d\eta, & \text{if } x > 0. \end{cases}$$

3.2. Lemma. *We have $y(H_\beta) \subset C_b(\mathbb{R}_+)$ and $y \in L(H_\beta, C_b(\mathbb{R}_+))$, that is, y is a continuous linear operator from H_β to $C_b(\mathbb{R}_+)$.*

Proof. For $h \in H_\beta$ we estimate, by using (2.2),

$$\begin{aligned} \|y(h)\|_{L^\infty(\mathbb{R}_+)} &= \sup_{x \in (0, \infty)} \left| \frac{1}{x} \int_0^x h(\eta) d\eta \right| \leq \sup_{x \in (0, \infty)} \frac{1}{x} \int_0^x |h(\eta)| d\eta \\ &\leq \|h\|_{L^\infty(\mathbb{R}_+)} \leq C_1 \|h\|_\beta, \end{aligned}$$

finishing the proof. \square

We define $(y_t)_{t \geq 0}$ as the $C_b(\mathbb{R}_+)$ -valued process $y_t(x) := y(r_t(x))$, where $(r_t)_{t \geq 0}$ denotes the mild solution for the HJMM equation (1.2). Then, the time t yield curve is given by

$$Y(t, T) = y_t(T - t), \quad T \geq t.$$

3.3. Proposition. *For all $T \geq 0$ there exists a constant $K_2 = K_2(T, L) > 0$ such that*

$$(3.5) \quad \mathbb{E} \left[\sup_{t \in [0, T]} \|y_t - y_t^n\|_{L^\infty(\mathbb{R}_+)}^2 \right]^{1/2} \leq K_2 \sqrt{\|h_0 - h_0^n\|_\beta^2 + \epsilon_n^2} \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

where $\epsilon_n = \epsilon_n(T, r)$ was defined in (3.3).

Proof. This is a consequence of Proposition 3.1 and Lemma 3.2. \square

Next, we analyze the stability of option prices under perturbations of the interest rate model (1.2). For this purpose, we will assume that the HJMM equation (1.2) only produces positive forward curves. This is a reasonable condition, as negative forward rates are rarely observed at the market. More precisely, from now on we suppose that the HJMM equation (1.2) is positivity preserving, that is, for all $h_0 \in P$ we have

$$\mathbb{P}(r_t \in P) = 1, \quad t \geq 0$$

where $(r_t)_{t \geq 0}$ denotes the mild solution for (1.2) with $r_0 = h_0$, and where $P \subset H_\beta$ denotes the subset

$$P = \{h \in H_\beta : h \geq 0\}$$

consisting of all nonnegative forward curves. We note that the conditions

$$\begin{aligned} \sigma(h)(\xi) &= 0, \quad \xi \in (0, \infty) \text{ and } h \in H_\beta \text{ with } h(\xi) = 0, \\ h + \sigma(h)x &\in P, \quad h \in P \text{ and } \nu\text{-almost all } x \in \mathbb{R}. \end{aligned}$$

are necessary and sufficient for the positivity preserving property of the HJMM equation (1.2), see [13, Cor. 4.23].

Now, let us fix a future date $T > 0$ and a payoff profile $\phi : P \rightarrow \mathbb{R}$ depending on the forward curve r_T . Since we model the HJMM equation (1.2) under a risk-neutral probability measure, the time t price of ϕ is given by

$$\pi_t(\phi) = \mathbb{E} \left[e^{-\int_t^T r_s(0) ds} \phi(r_T) \mid \mathcal{F}_t \right], \quad t \in [0, T]$$

where $r_t(0)$ denotes the short rate at time t .

3.4. Examples. *Let us consider the following examples:*

- (1) $\phi \equiv 1$ is the payoff profile of a T -bond.
- (2) We fix another future date S with $T < S$ and a strike rate K , and set

$$(3.6) \quad \phi(h) = \left(\exp \left(- \int_0^{S-T} h(\eta) d\eta \right) - K \right)^+, \quad h \in P.$$

Then we have

$$\phi(r_T) = \left(\exp \left(- \int_0^{S-T} r_T(\eta) d\eta \right) - K \right)^+ = (P(T, S) - K)^+,$$

and hence ϕ is the payoff profile of a call option on a S -bond. Note that the cash flow of a floor is (up to a constant) equivalent to the cash flow of a call option on a bond, see [10, Chap. 2].

(3) Accordingly, the function

$$\phi(h) = \left(K - \exp \left(- \int_0^{S-T} h(\eta) d\eta \right) \right)^+, \quad h \in P$$

is the payoff profile of a put option on an S -bond, and its cash flow is (up to a constant) equivalent to the cash flow of a cap.

3.5. Proposition. For all $T \geq 0$ and all Lipschitz continuous payoff profiles $\phi : P \rightarrow \mathbb{R}$ there exists a constant $K_3 = K_3(T, \phi, L, r) > 0$ such that

$$(3.7) \quad \sup_{t \in [0, T]} \mathbb{E}[|\pi_t(\phi) - \pi_t^n(\phi)|] \leq K_3 \sqrt{\|h_0 - h_0^n\|_\beta^2 + \epsilon_n^2} \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

where $\epsilon_n = \epsilon_n(T, r)$ was defined in (3.3).

Proof. By the Lipschitz continuity of ϕ , there exist constants $L_\phi, K_\phi > 0$ such that

$$\begin{aligned} |\phi(h) - \phi(g)| &\leq L_\phi \|h - g\|_\beta, \quad h, g \in H_\beta \\ |\phi(h)| &\leq K_\phi (1 + \|h\|_\beta), \quad h \in H_\beta. \end{aligned}$$

Note that $\text{ev}_0 : H_\beta \rightarrow \mathbb{R}$, $\text{ev}_0(h) = h(0)$ is a continuous linear operator by [13, Thm. 2.1]. Hence, using the Cauchy-Schwarz inequality we calculate

$$\begin{aligned} \sup_{t \in [0, T]} \mathbb{E}[|\pi_t(\phi) - \pi_t^n(\phi)|] &\leq \sup_{t \in [0, T]} \mathbb{E} \left[\left| e^{-\int_t^T r_s(0) ds} \phi(r_T) - e^{-\int_t^T r_s^n(0) ds} \phi(r_T^n) \right| \right] \\ &\leq \sup_{t \in [0, T]} \mathbb{E} \left[\left| \left(e^{-\int_t^T r_s(0) ds} - e^{-\int_t^T r_s^n(0) ds} \right) \phi(r_T) \right| \right] \\ &\quad + \sup_{t \in [0, T]} \mathbb{E} \left[e^{-\int_t^T r_s^n(0) ds} |\phi(r_T) - \phi(r_T^n)| \right] \\ &\leq K_\phi \sup_{t \in [0, T]} \mathbb{E} \left[(1 + \|r_T\|_\beta) \int_t^T |r_s(0) - r_s^n(0)| ds \right] + L_\phi \mathbb{E}[\|r_T - r_T^n\|_\beta] \\ &\leq K_\phi \mathbb{E}[(1 + \|r_T\|_\beta)^2]^{1/2} \mathbb{E} \left[\left(\int_0^T |r_s(0) - r_s^n(0)| ds \right)^2 \right]^{1/2} + L_\phi \mathbb{E}[\|r_T - r_T^n\|_\beta^2]^{1/2} \\ &\leq \sqrt{2T} K_\phi (1 + \mathbb{E}[\|r_T\|_\beta^2])^{1/2} \mathbb{E} \left[\int_0^T |r_t(0) - r_t^n(0)|^2 dt \right]^{1/2} + L_\phi \mathbb{E}[\|r_T - r_T^n\|_\beta^2]^{1/2} \\ &\leq \left(\sqrt{2T} K_\phi (1 + \mathbb{E}[\|r_T\|_\beta^2])^{1/2} \|\text{ev}_0\| + L_\phi \right) \mathbb{E} \left[\sup_{t \in [0, T]} \|r_t - r_t^n\|_\beta^2 \right]^{1/2}. \end{aligned}$$

Now, applying Proposition 3.1 completes the proof. \square

3.6. Remark. Note that Proposition 3.5 applies to all payoff profiles presented in Examples 3.4. For instance, the payoff profile (3.6) of a call option on an S -bond can be written as

$$\phi(h) = (\psi(h) - K)^+, \quad h \in P$$

with $\psi : P \rightarrow \mathbb{R}$ being defined as

$$\psi(h) = \exp \left(- \int_0^{S-T} h(\eta) d\eta \right), \quad h \in P.$$

Hence, it suffices to prove Lipschitz continuity of ψ . For arbitrary $h, g \in P$ this is established, using estimate (2.2), by the calculation

$$\begin{aligned} |\psi(h) - \psi(g)| &= \left| \exp\left(-\int_0^{S-T} h(\eta) d\eta\right) - \exp\left(-\int_0^{S-T} g(\eta) d\eta\right) \right| \\ &\leq \left| \int_0^{S-T} h(\eta) d\eta - \int_0^{S-T} g(\eta) d\eta \right| \leq \int_0^{S-T} |h(\eta) - g(\eta)| d\eta \\ &\leq (S-T) \|h - g\|_{L^\infty(\mathbb{R}_+)} \leq (S-T) C_1 \|h - g\|_\beta. \end{aligned}$$

Consequently, the prices of caps and floors are stable under perturbations of the term structure model.

A Zero Coupon Bond $P(t, T)$ has the payoff profile $\phi \equiv 1$. Applying Proposition 3.5 yields for every $T \geq 0$ the estimate

$$\sup_{t \in [0, T]} \mathbb{E}[|P(t, T) - P^n(t, T)|] \leq K_3 \sqrt{\|h_0 - h_0^n\|_\beta^2 + \epsilon_n^2} \rightarrow 0 \quad \text{for } n \rightarrow \infty$$

with a constant $K_3 = K_3(T, L, r) > 0$. Now, we shall improve this result by considering the bond curve $T \mapsto P(t, T)$ at time t . Note that we can express the bond prices as

$$P(t, T) = \exp\left(-\int_t^T f(t, s) ds\right).$$

Switching to the Musiela parametrization, we thus introduce the *bond curve operator* $p : H_\beta \rightarrow C(\mathbb{R}_+)$ by

$$p(h) := \exp\left(-\int_0^\bullet h(\eta) d\eta\right).$$

3.7. Lemma. *The following statements are valid:*

- (1) We have $p(P) \subset C_b(\mathbb{R}_+)$.
- (2) There exists a constant $L_1 > 0$ such that

$$\|p(h_1) - p(h_2)\|_{L^\infty(\mathbb{R}_+)} \leq L_1 \|h_1 - h_2\|_\beta, \quad h_1, h_2 \in P \text{ with } h_1(\infty) = h_2(\infty).$$

- (3) For every $x_0 \in \mathbb{R}_+$ there exists a constant $L_2 = L_2(x_0) > 0$ such that

$$\|p(h_1) - p(h_2)\|_{L^\infty[0, x_0]} \leq L_2 \|h_1 - h_2\|_\beta, \quad h_1, h_2 \in P.$$

Proof. It is clear that $p(h) \in C_b(\mathbb{R}_+)$ for all $h \in P$. For $h_1, h_2 \in P$ with $h_1(\infty) = h_2(\infty)$ we obtain, by using estimate (2.3),

$$\begin{aligned} \|p(h_1) - p(h_2)\|_{L^\infty(\mathbb{R}_+)} &= \sup_{x \in \mathbb{R}_+} \left| e^{-\int_0^x h_1(\eta) d\eta} - e^{-\int_0^x h_2(\eta) d\eta} \right| \\ &\leq \sup_{x \in \mathbb{R}_+} \left| \int_0^x (h_1(\eta) - h_2(\eta)) d\eta \right| \leq \|h_1 - h_2\|_{L^1(\mathbb{R}_+)} \leq C_2 \|h_1 - h_2\|_\beta. \end{aligned}$$

Similarly, for $x_0 \in \mathbb{R}_+$ and $h_1, h_2 \in P$, by (2.2) we get

$$\begin{aligned} \|p(h_1) - p(h_2)\|_{L^\infty[0, x_0]} &= \sup_{x \in [0, x_0]} \left| e^{-\int_0^x h_1(\eta) d\eta} - e^{-\int_0^x h_2(\eta) d\eta} \right| \\ &\leq \sup_{x \in [0, x_0]} \left| \int_0^x (h_1(\eta) - h_2(\eta)) d\eta \right| \leq \int_0^{x_0} |h_1(\eta) - h_2(\eta)| d\eta \\ &\leq x_0 \|h_1 - h_2\|_{L^\infty(\mathbb{R}_+)} \leq x_0 C_1 \|h_1 - h_2\|_\beta. \end{aligned}$$

This completes the proof. \square

We define $(p_t)_{t \geq 0}$ as the $C_b(\mathbb{R}_+)$ -valued process $p_t(x) := p(r_t(x))$, where $(r_t)_{t \geq 0}$ denotes the mild solution for the HJMM equation (1.2). Then, the time t bond curve is given by

$$P(t, T) = p_t(T - t), \quad T \geq t.$$

3.8. Proposition. *For all $T, x_0 \geq 0$ there exists a constant $K_4 = K_4(T, x_0, L)$ such that*

$$(3.8) \quad \mathbb{E} \left[\sup_{t \in [0, T]} \|p_t - p_t^n\|_{L^\infty[0, x_0]}^2 \right]^{1/2} \leq K_4 \sqrt{\|h_0 - h_0^n\|_\beta^2 + \epsilon_n^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where $\epsilon_n = \epsilon_n(T, r)$ was defined in (3.3).

Proof. This is a consequence of Proposition 3.1 and Lemma 3.7. \square

For the rest of this section, we suppose that the following stronger conditions are satisfied:

- $h_0^n \rightarrow h_0$ in H_β ;
- $\sigma^n \rightarrow \sigma$ in C_b^{lip} .

Then, for all $h \in H_\beta$ we have

$$\|\sigma(h) - \sigma^n(h)\|_\beta \leq \|\sigma - \sigma^n\|_{\text{lip}} \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

and, by Lemma 2.1, there exists a constant $L_1 > 0$ (depending on σ) such that for all $h \in H_\beta$ we have

$$\begin{aligned} \|\alpha_{\text{HJM}}(h) - \alpha_{\text{HJM}}^n(h)\|_\beta &= \|\Sigma\sigma(h) - \Sigma\sigma^n(h)\|_\beta \\ &\leq L_1 \|\sigma(h) - \sigma^n(h)\|_\beta \leq L_1 \|\sigma - \sigma^n\|_{\text{lip}}. \end{aligned}$$

Consequently, for any $T \geq 0$ there is a constant $L_2 = L_2(T)$ such that we can estimate $\epsilon_n = \epsilon_n(T, r)$ defined in (3.3) by

$$\epsilon_n \leq L_2 \|\sigma - \sigma^n\|_{\text{lip}} \rightarrow 0.$$

Therefore, we can improve the estimates (3.4)–(3.8) by replacing the right-hand sides for $i = 1, 2, 3, 4$ by

$$(3.9) \quad K_i \sqrt{\|h_0 - h_0^n\|_\beta^2 + \|\sigma - \sigma^n\|_{\text{lip}}^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

showing that the dependence of the considered quantities on the initial curves and on the volatilities is locally Lipschitz.

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BERGISCHE UNIVERSITÄT WUPPERTAL, FACHBEREICH C – MATHEMATIK UND INFORMATIK,
GAUSSSTRASSE 20, D-42097 WUPPERTAL, GERMANY
E-mail address: ruediger@uni-wuppertal.de

LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR MATHEMATISCHE STOCHASTIK, WELFEN-
GARTEN 1, 30167 HANNOVER, GERMANY
E-mail address: tappe@stochastik.uni-hannover.de