Subgradients of Law-Invariant Convex Risk Measures on L^1

G. Svindland*†
EPFL CDM SFI CSF EXTRA 218
CH-1015 Lausanne
Email: gregor.svindland@epfl.ch

16 March 2010

Abstract

We introduce a generalised subgradient for law-invariant closed convex risk measures on L^1 and establish its relationship with optimal risk allocations and equilibria. Our main result gives sufficient conditions ensuring a non-empty generalised subgradient.

Key words: equilibria, generalised subgradients, law-invariant convex risk measures, optimal capital and risk allocations.

1 Introduction

In [17] we established that every law-invariant convex risk measure on L^∞ is $\sigma(L^\infty, L^\infty)$-lower semi-continuous and thus canonically extended to a law-invariant closed convex risk measure on L^1. There are several advantages of the model space L^1: in contrast to L^∞, the model space L^1 includes important risk models such as normally distributed. Moreover, L^1 is in some sense maximal amongst the law-invariant model spaces bearing a locally convex topology and thus allowing for convex duality. Other attempts to extending the model space beyond L^∞ suggest spaces which depend on the risk measure, in terms of being chosen such that some given risk measure stays real valued. But when studying optimal risk allocations and equilibria which involves more than one risk measure, the model space should preferably be independent of these risk measures. Otherwise one would have to shift to some kind of intersection of the respective model spaces, and thus would exclude a lot of possible positions that might be acceptable to at least one of the risk sharing agents, hence reducing the set of potential allocations. In sum, in case of law-invariant convex risk measures the

*We thank Damir Filipović for helpful discussions and much advice throughout the preparation of this paper.
†We thank two anonymous referees for helpful comments and suggestions.

model space L^1 proves suitable, at least when studing optimal capital and risk allocations involving a larger number of risk measures.

In [16] it is shown that any risk in L^1 admits an optimal allocation whenever the preferences of the agents are determined by law-invariant closed convex risk measures. However, the problem of subgradients of law-invariant convex risk measures on L^1 has not been addressed yet. There is a close link between subgradients and optimal risk allocations and equilibria (see [15, 19]). This so-called first order condition guarantees the existence of optimal risk allocations and gives a pricing rule under which the involved agents trade in a state of equilibrium. On L^∞ this relationship is *if and only if* because convex risk measures on L^∞ are automatically continuous and thus everywhere subdifferentiable. In contrast, convex risk measures on L^1 may have empty subgradients. E.g. the entropic risk measure has empty subgradients for any risk which is unbounded from below, although these risks may be acceptable. Hence, for such risks we do have optimal allocations but we do not have a first order condition. But it turns out that if we generalise the notion of a subgradient and if we restrict to law-invariant closed convex risk measures which satisfy certain continuity properties, then we obtain non-empty generalised subgradients for a large class of risks, and we have a first order condition for optimal risk allocations and equilibria similar to the one for ordinary subgradients. Therefore, we are in particular interested in characterising those points at which a law-invariant closed convex risk measure ρ on L^1 is subdifferentiable in that generalised sense. Our main result is theorem 2.9 which states that under a tail continuity condition on ρ, the generalised subgradient at $X \in L^1$ is non-empty whenever there is $\epsilon > 0$ such that $\rho((1+\epsilon)X) < \infty$.

When proving our results we will introduce a class of auxiliary sub-spaces of L^1 which are induced by law-invariant convex risk measures. These Banach spaces are a generalisation of Orlicz spaces. Hence, as a byproduct we also enlighten the connection between law-invariant convex risk measures and Orlicz spaces. On the level of examples, the existence of some relationship between law-invariant convex risk measures and Orlicz spaces has already been observed by several authors (e.g. see [6]).

The structure of the paper is as follows: in section 2 we introduce the generalised subgradient and compare this notion to the ordinary subgradient, as well as stating the existence result theorem 2.9. The proof of this theorem will need some preparation, and in this context we will introduce and study the auxiliary spaces mentioned above in section 3. The proof of theorem 2.9 is then given in section 4. In section 5 we present the connection between generalised subgradients and optimal risk allocations and equilibria. Our results are illustrated by several examples which are collected in section 6. Finally, the appendix A–C collects some auxiliary results which are needed throughout this paper.

2 Subgradients and Generalised Subgradients

Throughout this paper $(\Omega, \mathcal{F}, \mathbb{P})$ is an atom-less probability space, i.e. a probability space supporting a random variable with continuous distribution. (Note that we do not require standardness of the probability space as in [16] and [17]. This is justified in [29].) All equalities and inequalities between random variables are understood in the $\mathbb{P}$-almost sure (a.s.) sense. We write $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \in [0, \infty]$, and $\| \cdot \|_p = \| \cdot \|_{L^p}$ for $p \in [1, \infty]$. The topological dual space of L^p is denoted by L^{p*}. It is well known that $L^{p*} = L^q$ with $q = \frac{p}{p-1}$ for $p < \infty$, and that $L^{\infty *} \supset L^1$ can be identified with *ba*, the space of all bounded finitely additive signed measures μ on $(\Omega, \mathcal{F})$ such that $\mathbb{P}(A) = 0$ implies $\mu(A) = 0$. For any random variable X, we denote by $F_X(x) := \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$, its distribution function and by $q_X(s) := \inf\{x \in \mathbb{R} \mid F_X(x) \geq s\}$, $s \in (0, 1)$, its (left-continuous) quantile function.

We suppose the reader is familiar with standard terminology and basic duality theory for convex functions as outlined in [13] or [24]. Let V be a locally convex vector space such that $\mathbb{R} \subset V \subset L^1$. We call a function $F : V \to [-\infty, \infty]$

(i) *convex* if $F(\lambda X + (1-\lambda)Y) \leq \lambda F(X) + (1-\lambda)F(Y)$ for all $\lambda \in [0, 1]$ (here $\infty - \infty := \infty$),

(ii) *proper* if $F > -\infty$ and the *(effective) domain* $\operatorname{dom} F := \{F < \infty\} \neq \emptyset$,

(iii) *cash-invariant* if $F(0) = 0$ and $F(X + m) = F(X) - m$ for all $m \in \mathbb{R}$,

(iv) *monotone* if $X \geq Y$ implies $F(X) \leq F(Y)$,

(v) *positively homogeneous* if $F(tX) = tF(X)$ for all $t \geq 0$ (here $0 \cdot \infty := 0$),

(vi) *law-invariant* if $F(X) = F(Y)$ for all identically distributed $X \sim Y$,

(vii) *closed* if F is lower semi-continuous (l.s.c.), i.e. the level sets $E_k = \{X \in V \mid F(X) \leq k\}$ are closed for all $k \in \mathbb{R}$, and proper, or if $F \equiv -\infty$ or $F \equiv \infty$.

With some facilitating abuse of notation, we shall write $(X, Z) \mapsto E[XZ]$ for the dual pairing on (V, V^*) even if V does not equal some L^p for $p \in [1, \infty)$, for instance also for $V = L^\infty$. The *dual function*

$$F^*(Z) = \sup_{X \in V} (E[ZX] - F(X)), \quad Z \in V^*,$$

and the *bidual function*

$$F^{**}(X) := \sup_{Z \in V^*} (E[ZX] - F^*(Z)), \quad X \in V,$$

of F are closed convex on V^* or V respectively. The Fenchel–Moreau theorem (proposition 4.1 in [13]) states that $F^{**} = F$ if and only if F is closed convex. The *subgradient* of F at $X \in V$, denoted by $\partial F(X)$, is the set

$$\partial F(X) = \{Z \in V^* \mid F(Y) \geq F(X) + E[Z(Y - X)] \; \forall Y \in V\},$$

and is characterised by

$$Z \in \partial F(X) \quad \Leftrightarrow \quad F(X) = E[ZX] - F^*(Z). \tag{2.1}$$

Definition 2.1. *A* convex risk measure *on V is a convex monotone cash-invariant function $\rho : V \to (-\infty, \infty]$. A* coherent risk measure *is a convex risk measure which in addition is positively homogeneous.*

If ρ is a convex risk measure on L^p, then

$$\operatorname{dom}\rho^* \subset \mathcal{P}^{p*} := \{Z \in L^{p*}_- \mid E[Z1] = -1\} \quad \text{(see e.g. [14])}. \tag{2.2}$$

The set $\mathcal{P}_{\mathcal{R}} := -\mathcal{P}^{\infty*} \cap L^1$ is the set of *pricing rules*. Since we will work with law-invariant convex risk measures, our model space will be L^1. This choice is justified in [17] where it is proved that there is a one-to-one correspondence between law-invariant closed convex risk measures on L^1 and L^∞. This means that every law-invariant convex risk measure ρ_∞ on L^∞ is the restriction to L^∞ of a unique law-invariant closed convex risk measure ρ on L^1, i.e. $\rho_\infty = \rho|_{L^\infty}$. Regarding the dual functions we have

$$\rho^* = \rho^*_\infty \quad \text{on } L^{1*} \tag{2.3}$$

where ρ^* is the dual of ρ in the (L^1, L^{1*})-duality whereas ρ^*_∞ is the dual function of ρ_∞ in the $(L^\infty, L^{\infty*})$-duality. Note that throughout this text we will keep this notational convention, that is given any convex risk measure ρ on L^1 we denote by ρ_∞ its restriction to L^∞.

It is well-known that any proper closed convex function on a Banach space is continuous and subdifferentiable on the interior of its domain (see e.g. [13] corollary 2.5 and proposition 5.2). Moreover, it is proved in [28] that for every convex risk measure ρ on L^1 (which is proper by definition) we have $\operatorname{int}\operatorname{dom}\rho \neq \emptyset$ if and only if ρ is real-valued ($\operatorname{dom}\rho = L^1$) and continuous. We summarise these results on subdifferentiability in the following lemma. A more general version of this lemma is proved in [6].

Lemma 2.2. *Let ρ be a convex risk measure on L^1. Equivalent are:*

(i) *ρ is everywhere subdifferentiable.*

(ii) *ρ is real-valued and continuous.*

(iii) $\operatorname{int}\operatorname{dom}\rho \neq \emptyset$.

An example of a continuous convex risk measure on L^1 is the Average Value at Risk (see example 6.2). But closed convex risk measures are not continuous on L^1 in general. For instance the entropic risk measure

$$\rho(X) = \frac{1}{\beta}\log E[e^{-\beta X}], \quad X \in L^1,$$

where $\beta > 0$, is a closed, but not continuous, convex risk measure on L^1. It is show in lemma 6.1 below (also see [18] lemma 3.29) that

$$\rho(X) = E[ZX] - \rho^*(Z) \quad \Rightarrow \quad Z = \frac{-e^{-\beta X}}{E\left[e^{-\beta X}\right]}.$$

Hence, in view of (2.1), we infer that $\partial\rho(X) = \emptyset$ for every $X \in L^1$ with $\operatorname{essinf} X = -\infty$, even though $\operatorname{dom}\rho$ includes such X (see example 6.4 and [18] example 4.33).

This motivates the following extension of the notion of a subgradient.

Definition 2.3. *The generalised subgradient of a convex risk measure ρ on L^1 at $X \in L^1$ is defined as*

$$\delta\rho(X) := \{Z \in L^1 \mid (XZ) \in L^1, \ \forall Y \in L^\infty : \ \rho(Y) \geq \rho(X) + E[Z(Y - X)]\}.$$

Lemmas 2.4 and 2.7 below show that $\delta\rho$ is indeed a generalisation of $\partial\rho$.

Lemma 2.4. *Let ρ be a convex risk measure on L^1. The following conditions hold:*

(i) *for all $X \in L^1$: $\partial\rho(X) \subset \delta\rho(X) \subset \operatorname{dom}\rho^*_\infty \cap L^1$,*

(ii) *for all $X \in L^\infty$: $\delta\rho(X) = \partial\rho_\infty(X) \cap L^1$,*

(iii) *for all $X \in L^1$: $\delta\rho(X) \neq \emptyset \ \Rightarrow \ X \in \operatorname{dom}\rho$.*

Proof. We only prove the inclusion $\delta\rho(X) \subset \operatorname{dom}\rho^*_\infty \cap L^1$, because the rest is obvious by definition of $\delta\rho(X)$. However, this inclusion follows from the fact that $Z \in \delta\rho(X)$ implies

$$\infty > E[ZX] - \rho(X) \geq \sup_{Y \in L^\infty} E[ZY] - \rho(Y) = \rho^*_\infty(Z).$$

□

We remark that $\delta\rho(X) = \emptyset$ is possible even for $X \in L^\infty$ (see example 6.1). In order to have $\delta\rho(X) \neq \emptyset$ on L^∞ at least, we will have to require that ρ is continuous from below. This property is defined and characterised in the following proposition (see also [20],[18], [10]). It is in fact a property of the restriction ρ_∞ of ρ to L^∞ only. Note that proposition 2.5(iv) shows that continuity from below is satisfied by most law-invariant convex risk measures of interest!

Proposition 2.5. *Let ρ be a law-invariant closed convex risk measure on L^1. Then*

$$\rho_\infty \text{ is } \sigma(L^\infty, L^1)\text{-l.s.c.} \tag{2.4}$$

Moreover, the following conditions are equivalent:

(i) *ρ is* continuous from below, *i.e. for every $X \in L^\infty$ and every sequence $(X_n)_{n\in\mathbb{N}} \subset L^\infty$ with $X_n \uparrow X$ we have $\rho(X_n) \downarrow \rho(X)$.*

(ii) $\operatorname{dom} \rho_\infty^* \subset L^1$.

(iii) *The level sets* $Q_k := \{Z \in L^1 | \rho_\infty^*(Z) \le k\}$, $k \in \mathbb{R}$, *are* $\sigma(L^1, L^\infty)$*-compact.*

(iv) $\{X \in L^1 \mid \operatorname{essinf} X = -\infty\} \cap \operatorname{dom} \rho \neq \emptyset$.

Proof. Property (2.4) and the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are well-known and e.g. proved in [20]. These equivalences are also partially proved in [18] proposition 4.21 and [15] theorem C.1. Moreover, (iv) $\Rightarrow$ (ii) is proved in [10] theorem 3. (i) $\Rightarrow$ (iv): Fix a decreasing sequence of sets $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, such that $\mathbb{P}(A_n) > 0$ and $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$. Since ρ is continuous from below and $-1_{A_n} \uparrow 0$, there is a $n_1 \in \mathbb{N}$ such that $\rho(-1_{A_{n_1}}) \le \frac{1}{2}$. Then again, as $-1_{A_{n_1}} - 1_{A_l} \uparrow -1_{A_{n_1}}$ for $l \to \infty$, there is a $n_2 > n_1$ such that $\rho(-1_{A_{n_1}} - 1_{A_{n_2}}) \le \rho(-1_{A_{n_1}}) + \frac{1}{4}$. Continuing this construction inductively, we find for each $k \in \mathbb{N}$ a $n_{k+1} > n_k$ such that $\rho(\sum_{i=1}^{k+1} -1_{A_{n_i}}) \le \rho(\sum_{i=1}^{k} -1_{A_{n_i}}) + \frac{1}{2^{k+1}}$. The sequence $X_k := \sum_{i=1}^{k} -1_{A_{n_i}}$ converges monotonously to $X := \sum_{i=1}^{\infty} -1_{A_{n_i}}$ which is unbounded from below. By the monotone convergence theorem, and since $\rho^*(-1) = 0$ (see (B.3)), we deduce that

$$E[|X|] = \lim_{k \to \infty} E[-X_k] \le \liminf_{k \to \infty} \rho(X_k) \le \liminf_{k \to \infty} \sum_{i=1}^{k} \frac{1}{2^i} \le 1.$$

Hence, $X \in L^1$ and by l.s.c. of ρ we have that $\rho(X) \le \liminf_{k \to \infty} \rho(X_k) \le 1$, i.e. $X \in \operatorname{dom} \rho$. □

In the following we will often make use of the next lemma.

Lemma 2.6. *Let ρ be a law-invariant closed convex risk measure on L^1. Then $X \in \operatorname{dom} \rho$ if and only if $-X^- \in \operatorname{dom} \rho$.*

Proof. "$\Leftarrow$" follows from $X \ge -X^-$ and monotonicity of ρ. As for "$\Rightarrow$", let $X \in \operatorname{dom} \rho$ and suppose that $\mathbb{P}(X > 0) > 0$, otherwise the assertion is trivial. By (B.1) we know that $E[X | X 1_{\{X<0\}}] \in \operatorname{dom} \rho$. Clearly,

$$E[X | X 1_{\{X<0\}}] = X 1_{\{X<0\}} + \frac{E[X 1_{\{X \ge 0\}}]}{\mathbb{P}(X \ge 0)} 1_{\{X \ge 0\}}.$$

Hence, by cash-invariance and monotonicity we infer that

$$\begin{aligned} \rho(X 1_{\{X<0\}}) &= \rho\left(X 1_{\{X<0\}} + \frac{E[X 1_{\{X \ge 0\}}]}{\mathbb{P}(X \ge 0)} \right) + \frac{E[X 1_{\{X \ge 0\}}]}{\mathbb{P}(X \ge 0)} \\ &\le \rho(E[X | X 1_{\{X<0\}}]) + \frac{E[X 1_{\{X \ge 0\}}]}{\mathbb{P}(X \ge 0)} < \infty. \end{aligned}$$

□

We now establish a characterisation of the generalised subgradient which is analogous to (2.1).

Lemma 2.7. *Let ρ be a law-invariant closed convex risk measure on L^1 which is continuous from below, and let $X \in L^1$. The following conditions are equivalent:*

(i) $\widetilde{Z} \in \delta\rho(X)$,

(ii) $\widetilde{Z} \in \{Z \in L^1 \mid (XZ) \in L^1,\ \forall Y \in L^1 :\ \rho(Y) \geq \rho(X) + E[Z(Y-X)]\}$ *with the convention that* $\infty - \infty := \infty$,

(iii) $\widetilde{Z} \in L^1$ *such that* $(X\widetilde{Z}) \in L^1$ *and* $\rho(X) = E[\widetilde{Z}X] - \rho^*_\infty(\widetilde{Z})$.

Moreover, if $Z \in L^1$ is such that $(XZ) \in L^1$, then $E[XZ] - \rho^*_\infty(Z) \leq \rho(X)$.

Proof. (i) $\Rightarrow$ (ii): Suppose that (i) holds. We will prove that

$$\rho(U) \geq \rho(X) + E[\widetilde{Z}(U-X)] \tag{2.5}$$

for all $U \in L^1$ with the convention that $\infty - \infty = \infty$. Note that lemma 2.4(i) and (2.2) imply $\widetilde{Z} \in L^1_-$. Let $U \in L^1$ such that $E[-\widetilde{Z}U^-] < \infty$ or $E[\widetilde{Z}U^+] > -\infty$ or both, then by (i), monotone convergence and lemma 2.8 below we obtain that

$$\begin{aligned}
\rho(U) &= \lim_{m\to\infty}\lim_{n\to\infty} \rho((U^+ \wedge n) - (U^- \wedge m)) \\
&\geq \lim_{m\to\infty}\lim_{n\to\infty} (\rho(X) - E[\widetilde{Z}X] + E[\widetilde{Z}(U^+ \wedge n)] + E[-\widetilde{Z}(U^- \wedge m)]) \\
&= \rho(X) + E[\widetilde{Z}(U-X)],
\end{aligned}$$

so (2.5) holds. If $U \in L^1$ is such that $E[-\widetilde{Z}U^-] = \infty$ and $E[\widetilde{Z}U^+] = -\infty$, then according to our convention, the right hand side of (2.5) equals ∞, so we have to show that $\rho(U) = \infty$ too. However, this follows from lemma 2.6 and the first case.

(ii) $\Rightarrow$ (iii): Since, in particular, (2.5) is true for all $U \in L^\infty$, we have $E[X\widetilde{Z}] - \rho(X) \geq \rho^*_\infty(\widetilde{Z})$. Moreover, lemma 2.8 below and monotone convergence imply that

$$\begin{aligned}
E[X\widetilde{Z}] - \rho^*_\infty(\widetilde{Z}) &= \lim_{m\to\infty}\lim_{n\to\infty} E[(-m \vee X \wedge n)\widetilde{Z}] - \rho^*_\infty(\widetilde{Z}) \\
&\leq \lim_{m\to\infty}\lim_{n\to\infty} \rho_\infty(-m \vee X \wedge n) \quad = \quad \rho(X).
\end{aligned} \tag{2.6}$$

Hence, we obtain $E[X\widetilde{Z}] - \rho(X) = \rho^*_\infty(\widetilde{Z})$.

(iii) $\Rightarrow$ (i): is obvious.

The final statement of the lemma follows from a computation similar to (2.6). $\square$

The proof of lemma 2.7 relied on the following crucial lemma. We remark that a regularity result similar to (2.7) is stated in [21] for real-valued convex risk measures on Stonean lattices.

Lemma 2.8. *Let ρ be a law-invariant closed convex risk measure on L^1 which is continuous from below and let $H \in L^\infty$, then*

$$\rho(H + X) = \sup_{m\in\mathbb{N}}\inf_{n\in\mathbb{N}} \rho(H + (X^+ \wedge n) - (X^- \wedge m)). \tag{2.7}$$

Proof. Let, $H \in L^\infty_+$, and $X \in L^1$ be bounded from below. Then, $H + (X \wedge n) \in \operatorname{dom} \rho$ for all $n \in \mathbb{N} \cup \{\infty\}$ due to monotonicity of ρ. Again by monotonicity, the sequence $\rho(H + (X \wedge n))$, $n \in \mathbb{N}$, is decreasing and bounded from below by $\rho(H + X)$. We claim that

$$\rho(H + X) = \lim_{n\to\infty} \rho(H + (X \wedge n)). \tag{2.8}$$

In order to prove this, suppose for the moment that there is a $K > 0$ such that $\lim_{n\to\infty} \rho(H+(X\wedge n)) > K > \rho(H+X)$. Note that since $H \geq 0$, we have that

$$\lim_{n\to\infty} \rho(H + X \wedge n) = \lim_{n\to\infty} \rho((H + X) \wedge n).$$

Since $(H + X) \wedge n \in L^\infty$, and as ρ_∞ is everywhere subdifferentiable with $\operatorname{dom} \rho^*_\infty \subset L^1$, we have that for each $n \in \mathbb{N}$ there is a $Z_n \in \partial\rho_\infty((H+X)\wedge n) \subset L^1$, i.e.

$$\rho((H + X) \wedge n) = E[Z_n((H + X) \wedge n)] - \rho^*_\infty(Z_n).$$

By (B.1) we have $\rho^*_\infty(E[Z_n \mid (H + X) \wedge n]) \leq \rho^*_\infty(Z_n)$, so we may assume that Z_n is $\sigma((H+X)\wedge n)$-measurable. Moreover, lemmas C.1, C.2 and law-invariance of ρ^*_∞ imply that

$$\begin{aligned}
\rho_\infty((H + X) \wedge n) &= E[Z_n((H + X) \wedge n)] - \rho^*_\infty(Z_n) \\
&\leq \int_0^1 q_{(H+X)\wedge n}(s) q_{Z_n}(s) ds - \rho^*_\infty(Z_n) \\
&\leq \rho_\infty((H + X) \wedge n)
\end{aligned}$$

which can only hold if

$$E[Z_n((H + X) \wedge n)] = \int_0^1 q_{(H+X)\wedge n}(s) q_{Z_n}(s) ds.$$

According to lemma C.1, we may assume that $Z_n = f_n(X+H)$ for a measurable function $f_n : \mathbb{R} \to \mathbb{R}_-$ which is increasing on $\{F_{H+X} > 0\}$. As $X+H$ is bounded from below we infer that

$$\begin{aligned}
\rho^*_\infty(Z_n) &\leq E[((H + X) \wedge n)Z_n] - \rho((H + X) \wedge n) \\
&\leq -\operatorname{essinf}(H + X) - \rho(H + X) =: r,
\end{aligned}$$

so $Z_n \in Q_r$ for all $n \in \mathbb{N}$. Since Q_r is weakly sequentially compact (proposition 2.5) and $L^1(\Omega, \sigma(H + X), \mathbb{P})$ is weakly complete, we may assume, by considering a subsequence if necessary, that $(Z_n)_{n\in\mathbb{N}}$ converges weakly to some $Z \in Q_r$ and that $Z = f(H + X)$ for a measurable function $f : \mathbb{R} \to \mathbb{R}_-$. Since the Hahn-Banach separation theorem implies that there is a sequence of convex combinations of the Z_n which converges $\mathbb{P}$-a.s. to Z ([30] corollary III.3.9), we may also assume that f is increasing on $\{F_{X+H} > 0\}$. Let $\mathcal{G}_k$, $k \in \mathbb{N}$, be a sequence of sub-σ-algebras of $\mathcal{F}$ such that $E[X + H|\mathcal{G}_k] \in L^\infty$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} E[X + H|\mathcal{G}_k] = X + H$ in L^1 and $\mathbb{P}$-a.s. The following estimation shows that the sequence $(ZE[X+H|\mathcal{G}_k])_{k\in\mathbb{N}}$ is uniformly integrable. To this end let $a \in \mathbb{R}$ such that $F_{X+H}(a) > 0$. Then,

$$|ZE[X+H|\mathcal{G}_k]| \leq |Z| \|(X+H) \wedge a\|_\infty + |f(a)| E[|X+H| \; |\mathcal{G}_k] =: Y_k,$$

because $|f|$ is decreasing on $\{F_{X+H} > 0\}$. Since $(Y_k)_{k\in\mathbb{N}}$ is uniformly integrable, so is $(ZE[X+H|\mathcal{G}_k])_{k\in\mathbb{N}}$. Consequently, we obtain

$$\begin{aligned} E[Z(H+X)] - \rho_\infty^*(Z) &= \lim_{k\to\infty} E[ZE[H+X|\mathcal{G}_k]] - \rho_\infty^*(Z) \\ &\leq \lim_{k\to\infty} \rho_\infty(E[H+X|\mathcal{G}_k]) \\ &= \rho(H+X) < K \end{aligned} \tag{2.9}$$

in which the last equality is due to (B.2). On the other hand, we observe that for all $k \geq n$ we have $E[Z_k((H+X)\wedge n)] - \rho_\infty^*(Z_k) > K$ (because $(H+X)\wedge k \geq (H+X)\wedge n$). Hence, by monotone convergence and l.s.c. of ρ_∞^* we obtain

$$\begin{aligned} E[Z(H+X)] - \rho_\infty^*(Z) &= \lim_{n\to\infty} E[Z((H+X)\wedge n)] - \rho_\infty^*(Z) \\ &\geq \lim_{n\to\infty} \limsup_{k\to\infty} E[Z_k((H+X)\wedge n)] - \rho_\infty^*(Z_k) \\ &\geq K. \end{aligned} \tag{2.10}$$

Clearly, (2.10) contradicts (2.9), and thus (2.8) is proved. For general $H \in L^\infty$, and $X \in L^1$ monotonicity and l.s.c. of ρ imply that $\rho(H+X) = \lim_{m\to\infty} \rho(H + (X \vee -m))$. In conjunction with (2.8) and cash-invariance we obtain

$$\begin{aligned} \rho(H+X) &= \lim_{m\to\infty} \rho((H + \|H\|_\infty) + (X \vee -m)) + \|H\|_\infty \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \rho(H + (n \wedge X \vee -m)). \end{aligned}$$

□

The following theorem gives sufficient conditions ensuring the existence of a non-empty generalised subgradient. It is proved throughout sections 3 and 4.

Theorem 2.9. *Let ρ be a law-invariant closed convex risk measure on L^1 which is continuous from below. If $X \in L^1$ is bounded from below, then $\delta\rho(X) \neq \emptyset$. Otherwise, if ρ and $X \in L^1$ satisfy the following conditions*

$$\text{there is an } \epsilon > 0 \text{ such that } (1+\epsilon)X \in \operatorname{dom}\rho, \tag{2.11}$$

and

$$\lim_{n\to\infty} \rho(X + \epsilon X 1_{\{X \leq -n\}}) = \rho(X), \tag{2.12}$$

then $\delta\rho(X) \neq \emptyset$. In both cases we may assume that $Z \in \delta\rho(X)$ is of type $Z = f(X)$ for a measurable function $f : \mathbb{R} \to \mathbb{R}_-$ which is increasing on $\{F_X > 0\}$.

Remark 2.10. Note that if ρ satisfies the following tail continuity condition

$$\lim_{n\to\infty} \rho(Y + H1_{\{H\leq -n\}}) = \rho(Y) \qquad \text{for all } Y, H \in L^1 \text{ s.t. } (Y - H^-) \in \operatorname{dom}\rho, \tag{2.13}$$

then (2.12) is automatically satisfied, so we obtain $\delta\rho(X) \neq \emptyset$ for every $X \in L^1$ for which there is an $\epsilon > 0$ such that $(1+\epsilon)X \in \operatorname{dom}\rho$. If ρ is coherent, then condition (2.13) is equivalent to

$$\lim_{n\to\infty} \rho(X1_{\{X\leq -n\}}) = 0 \quad \text{for all } X \in \operatorname{dom}\rho$$

and (2.11) is equivalent to $X \in \operatorname{dom}\rho$. $\Diamond$

In examples 6.2, 6.3 and 6.4 we illustrate theorem 2.9 by means of well-known risk measures such as Average Value at Risk, the Semi-deviation Risk Measures, and the Entropic Risk Measure. All these risk measures satisfy the tail continuity condition (2.13). In particular, in example 6.4 we show that we cannot expect any better characterisation of the points at which ρ is generalised subdifferentiable than the one given in theorem 2.9. Moreover, example 6.6 provides a law-invariant closed coherent risk measure ρ on L^1 which is continuous from below and a risk $X \in \operatorname{dom}\rho$ such that (2.12) does not hold, so in particular ρ does not satisfy (2.13).

The proof of theorem 2.9 involves a kind of generalised Orlicz space which is induced by the law-invariant closed convex risk measure examined. These spaces are introduced and studied during section 3. We like to point out that a $(1+\epsilon)$–condition similar to (2.11) appears in other works such as [5] solving optimisation problems by means of Orlicz space theory.

3 The Space L^ρ

Throughout this section let ρ be a law-invariant closed convex risk measure on L^1.

Definition 3.1. *For $C > 0$ let*

$$\|X\|_{C,\rho} := \inf\{\lambda > 0 \mid \rho(-|X|/\lambda) \leq C\}, \quad X \in L^1,$$

with the usual convention that $\inf\emptyset = \infty$*, and define*

$$L^\rho := \{X \in L^1 \mid \|X\|_{C,\rho} < \infty\}.$$

Clearly, we adopted this idea from Orlicz space theory.

Lemma 3.2. (i) $\|\cdot\|_{C,\rho} : L^1 \to [0,\infty]$ *is a law-invariant sub-linear closed function on* $(L^1, \|\cdot\|_1)$.

(ii) *L^ρ is well-defined, i.e. independent of $C > 0$. Moreover, if $C \in (0,1)$, then*
$$C\|\cdot\|_{C,\rho} \leq \|\cdot\|_{1,\rho} \leq \|\cdot\|_{C,\rho}, \tag{3.1}$$
and if $C \geq 1$, then
$$\|\cdot\|_{C,\rho} \leq \|\cdot\|_{1,\rho} \leq C\|\cdot\|_{C,\rho}. \tag{3.2}$$
If ρ is coherent, then for all $C > 0$:
$$C\|\cdot\|_{C,\rho} = \|\cdot\|_{1,\rho} = \rho(-|\cdot|). \tag{3.3}$$

(iii) *$C \cdot \|X\|_{C,\rho} \leq \|X\|_\infty$ for all $X \in L^\infty$ and $C \cdot \|X\|_{C,\rho} \geq \|X\|_1$ for all $X \in L^1$.*

(iv) *$(L^\rho, \|\cdot\|_{C,\rho})$ is a law-invariant Banach space such that $L^\infty \subseteq L^\rho \subseteq L^1$. The inclusion $L^\infty \subset L^\rho$ is strict if and only if ρ is continuous from below. In particular, we have that $\{-X^- \mid X \in \operatorname{dom}\rho\} \subset L^\rho$.*

(v) *If $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ and $X \in L^\rho$, then $E[X|\mathcal{G}] \in L^\rho$.*

Proof. We define $\Lambda_C(X) := \{\lambda > 0 \mid \rho(-|X|/\lambda) \leq C\}$.

(i): The law-invariance of $\|\cdot\|_{C,\rho}$ follows immediately from law-invariance of ρ. Moreover, it is easily verified that $\|tX\|_{C,\rho} = |t| \cdot \|X\|_{C,\rho}$ for all $t \in \mathbb{R}$. In order to show that $\|X+Y\|_{C,\rho} \leq \|X\|_{C,\rho} + \|Y\|_{C,\rho}$ it suffices to consider $X, Y \in L^\rho$ because if either $\|X\|_{C,\rho} = \infty$ or $\|Y\|_{C,\rho} = \infty$ or both, the assertion is trivial. To this end let $\alpha \in \Lambda_C(X)$ and $\beta \in \Lambda_C(Y)$ for some $X, Y \in L^\rho$. Then, by monotonicity and convexity
$$\begin{aligned}
\rho\left(-\frac{|X+Y|}{\alpha+\beta}\right) &\leq \rho\left(-\frac{\alpha}{\alpha+\beta}\frac{|X|}{\alpha} - \frac{\beta}{\alpha+\beta}\frac{|Y|}{\beta}\right) \\
&\leq \frac{\alpha}{\alpha+\beta} \cdot \rho\left(-\frac{|X|}{\alpha}\right) + \frac{\beta}{\alpha+\beta} \cdot \rho\left(-\frac{|Y|}{\beta}\right) \leq C,
\end{aligned}$$
so $\Lambda_C(X) + \Lambda_C(Y) \subset \Lambda_C(X+Y)$ which proves the triangle inequality. We claim that $\|\cdot\|_{C,\rho}$ is l.s.c. on $(L^1, \|\cdot\|_1)$. In order to verify this, denote the level sets of $\|\cdot\|_{C,\rho}$ by $E_k = \{Y \mid \|Y\|_{C,\rho} \leq k\}$, $k \geq 0$, and let $(X_n)_{n\in\mathbb{N}} \subset E_k$ for some $k \geq 0$ be a sequence converging to $X \in L^1$ w.r.t. $\|\cdot\|_1$. Note that $\|Y\|_{C,\rho} \leq k$ if and only if $\rho(-|Y|/(k+\epsilon)) \leq C$ for all $\epsilon > 0$. Since $X_n \in E_k$ for all $n \in \mathbb{N}$, l.s.c. of ρ yields
$$\rho(-|X|/(k+\epsilon)) \leq \liminf_{n\to\infty} \rho(-|X_n|/(k+\epsilon)) \leq C$$
for any $\epsilon > 0$, and thus $X \in E_k$. Hence, E_k is closed in $(L^1, \|\cdot\|_1)$ for every $k \geq 0$, i.e. $\|\cdot\|_{C,\rho}$ is l.s.c. on $(L^1, \|\cdot\|_1)$. Hence, we have proved that $\|\cdot\|_{C,\rho}$ is a law-invariant closed sublinear function on L^1.

(ii): Clearly, if (3.1) and (3.2) hold, then L^ρ is well-defined. We only prove (3.1) since the proof of (3.2) is similar and (3.3) is obvious by positive homogeneity.

To this end, let $C \in (0,1)$, $X \in L^1$ and $\lambda \in \Lambda_1(X)$, i.e. $\rho(-|X|/\lambda) \leq 1$. Then, convexity of ρ yields $\rho(-C|X|/\lambda) \leq C\rho(-|X|/\lambda) \leq C$. Hence, $\frac{1}{C}\Lambda_1(X) \subset \Lambda_C(X)$, so $C\|X\|_{C,\rho} \leq \|X\|_{1,\rho}$. On the other hand, since $C < 1$, we have $\Lambda_C(X) \subset \Lambda_1(X)$ and thus $\|X\|_{1,\rho} \leq \|X\|_{C,\rho}$, and (3.1) is proved.

(iii) and (iv): (i) and (B.1) yield for all $X \in L^1$:

$$\|X\|_{C,\rho} = \||X|\|_{C,\rho} \geq E[|X|] \cdot \|1\|_{C,\rho} = \frac{1}{C}E[|X|] = \frac{1}{C}\|X\|_1. \tag{3.4}$$

Consequently, $\|X\|_{C,\rho} = 0$ if and only if $X = 0$. Apparently, the properties of $\|\cdot\|_{C,\rho}$ ensure that $(L^\rho, \|\cdot\|_{C,\rho})$ is a normed space. In order to prove that this space is complete and thus a Banach space, let $(X_n)_{n\in\mathbb{N}}$ be a Cauchy-sequence in $(L^\rho, \|\cdot\|_{C,\rho})$. Then by (3.4), $(X_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $(L^1, \|\cdot\|_1)$. Let $X \in L^1$ be the unique $\|\cdot\|_1$-limit of $(X_n)_{n\in\mathbb{N}}$. Since $\|\cdot\|_{C,\rho}$ is l.s.c. on $(L^1, \|\cdot\|_1)$, we obtain $\|X\|_{C,\rho} \leq \liminf_{n\to\infty} \|X_n\|_{C,\rho} < \infty$, i.e. $X \in L^\rho$. Let $\epsilon > 0$ and $N(\epsilon) \in \mathbb{N}$ such that $\|X_n - X_k\|_{C,\rho} \leq \epsilon$ for all $k, n \geq N(\epsilon)$. As $(X_n - X_k)$ converges to $X - X_k$ w.r.t. $\|\cdot\|_1$ for $n \to \infty$, we obtain

$$\|X - X_k\|_{C,\rho} \leq \liminf_{n\to\infty} \|X_n - X_k\|_{C,\rho} \leq \epsilon \quad \text{for } k \geq N(\epsilon).$$

Thus we may conclude that X is the $\|\cdot\|_{C,\rho}$-limit of X_n, i.e. $(L^\rho, \|\cdot\|_{C,\rho})$ is complete. For every $0 \neq X \in L^\infty$ we obtain

$$\rho\left(-\frac{C|X|}{\|X\|_\infty}\right) \leq \rho(-C) = C$$

by monotonicity and cash-invariance. Therefore, $\|X\|_{C,\rho} \leq \frac{1}{C}\|X\|_\infty$ and $L^\infty \subset L^\rho$. Now let $X \in \operatorname{dom}\rho$, then $\rho(-X^-) < \infty$ according to lemma 2.6, which implies that $-X^- \in L^\rho$. Hence, if ρ is continuous from below, then, by proposition 2.5, there is a $X \in \operatorname{dom}\rho$ such that $\operatorname{essinf} X = -\infty$, and $-X^- \in L^\rho$, so $L^\rho \setminus L^\infty \neq \emptyset$. Conversely, suppose that $X \in L^\rho \setminus L^\infty$, then, by definition of $\|\cdot\|_{C,\rho}$, there is a $k > 0$ such that $\rho(-k|X|) < \infty$. Since $X \notin L^\infty$, we have either $\operatorname{essinf} X = -\infty$ or $\operatorname{esssup} X = \infty$ or both, which implies that $(-k|X|) \in \{Y \in L^1 \mid \operatorname{essinf} Y = -\infty\} \cap \operatorname{dom}\rho$. But then ρ must be continuous from below (proposition 2.5).

(v): Let $X \in L^\rho$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Then, (i) and (B.1) imply that $\|E[X|\mathcal{G}]\|_{C,\rho} \leq \|X\|_{C,\rho}$, so $E[X|\mathcal{G}] \in L^\rho$. □

The reason for introducing the Banach spaces $(L^\rho, \|\cdot\|_{C,\rho})$ is that we will prove that the domain of $\rho|_{L^\rho}$ has a non-empty interior. Hence, we obtain non-empty subgradients at these interior points. The role of the variable $C > 0$ in the norms $\|\cdot\|_{C,\rho}$ will become clear in (the proof of) lemma 3.4 in which we characterise the interior points of $\operatorname{dom}\rho|_{L^\rho}$.

Lemma 3.3. *Let ρ be continuous from below. Denote by $L^{\rho*}$ the dual space of L^ρ and by $\|\cdot\|_{C,\rho*}$ the operator norm corresponding to $\|\cdot\|_{C,\rho}$.*

(i) *$L^\infty \subset L^{\rho*}$ and $L^{\rho*}_\infty \subset L^1$ where $L^{\rho*}_\infty := \{l|_{L^\infty} \mid l \in L^{\rho*}\}$. In particular, if $l \in L^{\rho*}$ and $Z \in L^1$ such that $l(X) = E[ZX]$ for all $X \in L^\infty$, then $E[Z\,\cdot] \in L^{\rho*}$.*

(ii) *$L^{\rho*} \cap L^1$ and $\|\cdot\|_{C,\rho*}|_{L^1}$ are law-invariant.*

(iii) *For every $Z \in L^{\rho*} \cap L^1$ and any sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$ we have $E[Z|\mathcal{G}] \in L^{\rho*}$.*

Proof. (i): Since $L^\rho \subset L^1$ and $C\|\cdot\|_{C,\rho} \geq \|\cdot\|_1$, every element $Z \in L^\infty = L^{1*}$ defines a continuous linear functional on L^ρ via $X \mapsto E[XZ]$. Thus, we may view L^∞ as a subset of $L^{\rho*}$. By $L^\infty \subset L^\rho$ and $C\|\cdot\|_{C,\rho} \leq \|\cdot\|_\infty$ on L^∞ we must have $L^{\rho*}_\infty \subset L^{\infty*}$. Recall the general property of normed spaces (see e.g. [2] lemma 6.14)

$$\|X\|_{C,\rho} = \sup_{\|Z\|_{C,\rho*}=1} |E[ZX]|. \tag{3.5}$$

Suppose we had $Z_\mu \in L^{\rho*}$ such that $Z_\mu|_{L^\infty} \in L^{\rho*}_\infty \setminus L^1$. W.l.o.g. $\|Z_\mu\|_{C,\rho*} = 1$. This Z_μ viewed as a continuous linear functional on L^∞ corresponds to a finitely additive but not σ-additive bounded signed measure μ on $(\Omega, \mathcal{F})$ such that $\mathbb{P}(A) = 0$ implies $\mu(A) = 0$ (see [18] theorem A.50). Consider the bounded finitely additive measure $|\mu|$ on $(\Omega, \mathcal{F})$ given by

$$|\mu|(A) = \sup\left\{\sum_{i=1}^k |\mu(A_i)| \mid A_1, \ldots, A_k \in \mathcal{F} \text{ are disjoint subsets of } A,\ k \in \mathbb{N}\right\},$$

$A \in \mathcal{F}$ (for details on $|\mu|$ consult e.g. [12] section III.1). Since $\sum_{i=1}^k \pm 1_{A_i} \in L^\rho$ for any (disjoint) sets $A_1, \ldots, A_k \in \mathcal{F}$, we infer from (3.5) that

$$|\mu|(A) \leq \|1_A\|_{C,\rho} \quad \text{for every } A \in \mathcal{F}. \tag{3.6}$$

As $|\mu|$ is not σ-additive, there exists a decreasing sequence of sets $B_n \downarrow \emptyset$ such that $|\mu|(B_n) \downarrow \epsilon > 0$. We will show that

$$\|1_{B_n}\|_{C,\rho} \to 0 \tag{3.7}$$

which contradicts (3.6) and thus shows that $L^{\rho*}_\infty \subset L^1$. To this end, note that for every $\delta > 0$ there is an $N(\delta) \in \mathbb{N}$ such that $\rho(-1_{B_n}/\delta) \leq C$ for $n \geq N(\delta)$ because $-1_{B_n} \uparrow 0$ and ρ is continuous from below. Hence, if $n \geq N(\delta)$, then $\|1_{B_n}\|_{C,\rho} \leq \delta$ and (3.7) is proved.

Let $l \in L^{\rho*}$ and $Z \in L^1$ such that $l(X) = E[ZX]$ for all $X \in L^\infty$. By monotonicity of $\|\cdot\|_\rho$ it follows that for any $Y \in L^\rho$ with $Y \geq 0$ we have

$$E[Z^+(Y \wedge n)] = l((Y \wedge n)1_{\{Z \geq 0\}}) \leq \|l\|_{\rho*}\|Y\|_\rho \quad \forall n \in \mathbb{N}.$$

Hence, $E[Z^+Y] \leq \|l\|_{\rho*}\|Y\|_\rho$, and noting that $Y \in L^\rho$ if and only if $Y^\pm \in L^\rho$, and that $\|Y\|_\rho = \||Y|\|_\rho$, we infer that $E[Z^+\cdot]$ defines a continuous linear functional on L^ρ. Similarly we find that $E[Z^-\cdot] \in L^{\rho*}$, which yields $E[Z\,\cdot] \in L^{\rho*}$.

(ii): We claim that

$$Z \in L^{\rho*} \cap L^1 \text{ if and only if } Z^+, Z^- \in L^{\rho*} \cap L^1. \tag{3.8}$$

To this end, note that for every $A \in \mathcal{F}$ and $X \in L^\rho$ monotonicity of ρ yields $\| \pm 1_A X\|_{C,\rho} \leq \|X\|_{C,\rho}$ and thus $\pm 1_A X \in L^\rho$. Now let $Z \in L^{\rho*} \cap L^1$. Choosing $A = \{Z \geq 0\}$ shows that $Z^+ \in L^{\rho*}$, because $L^\rho \ni X \mapsto E[Z^+X] = E[Z1_{\{Z\geq 0\}}X]$ is a real-valued linear function, and

$$|E[Z^+X]| = |E[Z1_{\{Z\geq 0\}}X]| \leq \|Z\|_{C,\rho*}\|X1_{\{Z\geq 0\}}\|_{C,\rho} \leq \|Z\|_{C,\rho*}\|X\|_{C,\rho}.$$

Similar arguments yield $Z^- \in L^{\rho*}$. The converse implication of (3.8) is trivial.

By (3.8) it suffices to prove the law-invariance property of $L^{\rho*} \cap L^1$ for the positive cone $L^{\rho*}_+ = \{Z \in L^{\rho*} \cap L^1 \mid Z \geq 0\}$ only. Hence, let $Z \in L^{\rho*}_+$. By law-invariance of $\| \cdot \|_{C,\rho}$, lemma C.2, and $\|X\|_{C,\rho} = \||X|\|_{C,\rho}$ we obtain

$$\begin{aligned} \infty > \|Z\|_{C,\rho*} &= \sup_{\|X\|_{C,\rho}=1} |E[ZX]| \\ &= \sup_{\|X\|_{C,\rho}=1} \sup_{Y\sim|X|} E[ZY] \\ &= \sup_{\|X\|_{C,\rho}=1} \int_0^1 q_Z(s)q_{|X|}(s)ds \end{aligned} \tag{3.9}$$

in which the latter expression depends on the distribution of Z only. Now it is easily verified that every $\widetilde{Z}$ such that $\widetilde{Z} \sim Z$ defines a continuous linear functional on L^ρ too. The law-invariance of $\|Z\|_{C,\rho*}$ for general $Z \in L^{\rho*} \cap L^1$ follows from a calculation similar to (3.9), using the fact that $\|X\|_{C,\rho} = \||X|1_{\{Z\geq 0\}} - |X|1_{\{Z<0\}}\|_{C,\rho}$.

(iii): Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and $Z \in L^{\rho*}_+$. Then, lemma 3.2 (v) and (B.1) yield $E[E[Z|\mathcal{G}]X] = E[ZE[X|\mathcal{G}]] \leq \|Z\|_{C,\rho*}\|X\|_{C,\rho}$ for every $X \in L^\rho_+$. Since $X \in L^\rho$ if and only if $X^+, X^- \in L^\rho$, we conclude that $E[Z|\mathcal{G}] \in L^{\rho*}_+$ which, in view of (3.8), completes the proof. □

Lemma 3.4. *Let ρ be continuous from below.*

(i) *The function $\rho|_{L^\rho}$ is a law-invariant closed convex risk measure on L^ρ. Moreover, if ρ is coherent, then $\rho|_{L^\rho}$ is a real-valued continuous coherent risk measure on L^ρ.*

(ii) *For all $X \in \operatorname{int}\operatorname{dom}\rho|_{L^\rho}$, which is for all $X \in L^\rho$ in case ρ is coherent, we have $\partial\rho|_{L^\rho}(X) \neq \emptyset$. In particular,*

$$\textit{if for } X \in L^1 \textit{ there is } \epsilon > 0 \textit{ such that } -(1+\epsilon)|X| \in \operatorname{dom}\rho, \tag{3.10}$$

then $X \in \operatorname{int}\operatorname{dom}\rho|_{L^\rho}$. Moreover, if ρ is coherent and $X \in L^1$, then $X \in \operatorname{int}\operatorname{dom}\rho|_{L^\rho} = L^\rho$ if and only if $-|X| \in \operatorname{dom}\rho$.

Proof. (i): Obviously, $\rho|_{L^\rho}$ is law-invariant, convex, cash-invariant and monotone. Moreover, $\rho|_{L^\rho}$ is l.s.c. because, according to lemma 3.2(iii), if a sequence converges in $(L^\rho, \|\cdot\|_{C,\rho})$, then this sequence converges in $(L^1, \|\cdot\|_1)$ and ρ is l.s.c. on $(L^1, \|\cdot\|_1)$. Suppose that ρ is coherent. Then, $\rho|_{L^\rho}$ is coherent too. Moreover, for every $X \in L^\rho$ there is a $k > 0$ such that $\rho(-|X|/k) \leq 1$. Hence, by positive homogeneity and monotonicity we obtain that $\rho(X) \leq \rho(-|X|) \leq k < \infty$. In other words,

$$\operatorname{dom} \rho|_{L^\rho} = \operatorname{int} \operatorname{dom} \rho|_{L^\rho} = L^\rho.$$

We recall that any real-valued closed convex function on a Banach space is continuous (see [13] corollary 2.5).

(ii) Recall that any closed convex function on a Banach space is subdifferentiable on the interior of its domain ([13] corollary 2.5 and proposition 5.2). If $\|X\|_{C,\rho} < 1$, then there exists a $\lambda \in (0,1)$ such that $\rho(-|X|/\lambda) \leq C$, and by convexity

$$\frac{1}{\lambda}\rho(-|X|) \leq \rho(-|X|/\lambda) \leq C.$$

Thus $\rho(X) \leq \rho(-|X|) \leq \lambda C < \infty$, that is

$$\mathcal{B} := \bigcup_{C>0} \{X \in L^\rho \mid \|X\|_{C,\rho} < 1\} \subset \operatorname{int} \operatorname{dom} \rho|_{L^\rho}.$$

If there is a $\epsilon > 0$ such that $-(1+\epsilon)|X| \in \operatorname{dom} \rho$, then for $\lambda := 1/(1+\epsilon) \in (0,1)$ we have $\rho(-|X|/\lambda) = \rho(-(1+\epsilon)|X|) =: C < \infty$, so $X \in \mathcal{B}$. If ρ is coherent, then by (3.3) $X \in L^\rho$ if and only if $-|X| \in \operatorname{dom} \rho$. $\square$

Remark 3.5. In view of lemma 2.2 the reader might wonder why on the Banach space $(L^\rho, \|\cdot\|_{C,\rho})$ it is possible that $\operatorname{int} \operatorname{dom} \rho|_{L^\rho} \neq \emptyset$ without $\rho|_{L^\rho}$ being real-valued and continuous and thus subdifferentiable on all of L^ρ. The reason is that the proof of lemma 2.2 relies on the fact that L^∞ is dense in $(L^1, \|\cdot\|_1)$ (see [28]). This, however, need not be true for $(L^\rho, \|\cdot\|_{C,\rho})$. In example 6.4 we show that for the entropic risk measure L^ρ corresponds to an Orlicz space for which it is known that L^∞ is not dense. That is one of the reasons why many authors prefer Orlicz hearts (see e.g. [6]) which are closed sub-spaces of Orlicz spaces such that L^∞ is dense. However, Orlicz hearts are in general much smaller than the corresponding Orlicz space. But we can imitate Orlicz hearts, i.e. shift to the subspace $M^\rho \subset L^\rho$ given by

$$M^\rho := \{X \in L^1 \mid \rho(-c|X|) < \infty \quad \forall c > 0\}.$$

Then $\rho|_{M^\rho}$ is a law-invariant real-valued continuous convex risk measure on M^ρ, and thus everywhere subdifferentiable (on M^ρ). $\lozenge$

Remark 3.6. Since $X \in \operatorname{int} \operatorname{dom} \rho|_{L^\rho}$ implies that $(1+\epsilon)X \in \operatorname{dom} \rho|_{L^\rho}$ for small enough $\epsilon > 0$, we have

$$\operatorname{int} \operatorname{dom} \rho|_{L^\rho} \subset \{X \in L^1 \mid X \text{ satisfies condition (2.11)}\}$$

in which the inclusion is strict unless $L^\rho = L^1$. $\lozenge$

4 Proof of Theorem 2.9

Proof of theorem 2.9. Let $X \in L^1$ and suppose that there is an $\epsilon > 0$ such that $(1+\epsilon)X \in \operatorname{dom}\rho$ (which is always satisfied if X is bounded from below). Then, in particular, $-(1+\epsilon)X^- \in \operatorname{dom}\rho$ (lemma 2.6), and thus $-X^- \in \operatorname{int}\operatorname{dom}\rho|_{L^\rho}$ (lemma 3.4). By cash-invariance we may w.l.o.g. assume that $\rho(X) = 0$. Let

$$\rho_{X^+}(U) := \rho(X^+ + U), \quad U \in L^\rho.$$

It is easily verified that ρ_{X^+} is a closed convex risk measure on $(L^\rho, \|\cdot\|_{C,\rho})$. Note that monotonicity implies $\operatorname{dom}\rho|_{L^\rho} \subset \operatorname{dom}\rho_{X^+}$. Hence, $-X^- \in \operatorname{int}\operatorname{dom}\rho_{X^+}$ which implies that $\partial\rho_{X^+}(-X^-) \neq \emptyset$ ([13] corollary 2.5 and proposition 5.2). Let $\mu \in \partial\rho_{X^+}(-X^-)$, i.e.

$$\rho_{X^+}(-X^-) = E[\mu(-X^-)] - \rho^*_{X^+}(\mu), \tag{4.1}$$

and let $Z_\mu \in L^1$ such that $E[\mu Y] = E[Z_\mu Y]$ for all $Y \in L^\infty$ (see lemma 3.3). We claim that

$$(Z_\mu X^+) \in L^1 \quad \text{and} \quad \rho^*_{X^+}(\mu) \geq E[-Z_\mu X^+] + \rho^*_\infty(Z_\mu), \tag{4.2}$$

and

$$E[Z_\mu(-X^-)] \geq E[\mu(-X^-)]. \tag{4.3}$$

Suppose we knew (4.2) and (4.3). Then, (4.1) yields

$$\rho(X) = \rho_{X^+}(-X^-) \leq E[Z_\mu X] - \rho^*_\infty(Z_\mu),$$

or in other words $Z_\mu \in \delta\rho(X)$. In order to verify (4.2), in a first step we compute

$$\begin{aligned}
\sup_{U\in L^\infty} E[Z_\mu U] - \rho(X^+ + U) &= \sup_{U\in L^\infty} \lim_{n\to\infty} E[Z_\mu U] - \rho((X^+ \wedge n) + U) \\
&\leq \liminf_{n\to\infty} \sup_{U\in L^\infty} E[Z_\mu U] - \rho((X^+ \wedge n) + U) \\
&\leq \sup_{U\in L^\infty} E[Z_\mu U] - \rho(X^+ + U)
\end{aligned} \tag{4.4}$$

in which the first equality follows from lemma 2.8. Hence, all inequalities in (4.4) must in fact be equalities. Secondly, we obtain that

$$\begin{aligned}
\rho^*_{X^+}(\mu) &\geq \sup_{U\in L^\infty} E[Z_\mu U] - \rho(X^+ + U) \\
&= \liminf_{n\to\infty} \sup_{U\in L^\infty} E[Z_\mu U] - \rho((X^+ \wedge n) + U) \\
&= \liminf_{n\to\infty} \sup_{U\in L^\infty} E[Z_\mu(U - (X^+ \wedge n))] - \rho(U) \\
&= E[-Z_\mu X^+] + \rho^*_\infty(Z_\mu),
\end{aligned}$$

in which the first equality is due to our first step, and the last equality follows from monotone convergence. Thus, as $\rho^*_{X^+}(\mu) < \infty$, we infer that $E[-Z_\mu X^+] <$

∞ and $\rho^*_\infty(Z_\mu) < \infty$, and (4.2) is proved. As for (4.3), note that if X^- is bounded, then $E[Z_\mu(-X^-)] = E[\mu(-X^-)]$, and we are done. Now suppose that X^- is unbounded and that (2.12) holds, but $\delta := E[(\mu - Z_\mu)(-X^-)] > 0$. This implies

$$E[\mu(-X^-)1_{\{X^- \geq n\}}] \geq E[(\mu - Z_\mu)(-X^-)1_{\{X^- \geq n\}}] = E[(\mu - Z_\mu)(-X^-)] = \delta$$

because monotonicity of ρ implies that

$$\mu, Z_\mu \in \{\nu \in L^{\rho*} \mid \forall Y \in L^\rho_+ : E[\nu Y] \leq 0\},$$

and because $(-X^-)1_{\{X^- < n\}} \in L^\infty$. Since $\rho(X) = 0$ by assumption, (4.1) yields

$$E[\mu(-X^-)] = \rho^*_{X^+}(\mu) = \sup_{Y \in \mathcal{A}_{\rho_{X^+}}} E[\mu Y] \tag{4.5}$$

for $\mathcal{A}_{\rho_{X^+}} = \{Y \in L^\rho \mid \rho_{X^+}(Y) \leq 0\}$ where the last step follows from cash-invariance. Let $X_n := -X^- - \epsilon X^- 1_{\{X^- \geq n\}}$, $n \in \mathbb{N}$. Then $X_n \in \operatorname{dom} \rho_{X^+}$ and $\lim_{n\to\infty} \rho_{X^+}(X_n) = \rho_{X^+}(-X^-) = 0$ due to (2.12). Hence, as cash-invariance implies that $E[\mu 1] = -1$, we obtain by (4.5) that

$$\begin{aligned} E[\mu(-X^-)] &\geq E[\mu(X_n + \rho_{X^+}(X_n))] \\ &= \epsilon E[\mu(-X^-)1_{\{X^- \geq n\}}] + E[\mu(-X^-)] - \rho_{X^+}(X_n) \\ &\geq \epsilon\delta + E[\mu(-X^-)] - \rho_{X^+}(X_n) \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Passing to the limit for $n \to \infty$ yields the desired contradiction to $\delta > 0$. Consequently, (4.3) is proved.

It remains to be shown that Z_μ may be chosen as an increasing function of X. To this end, note that according to lemma 2.7 we have $\rho(X) = E[Z_\mu X] - \rho^*_\infty(Z_\mu)$. By (B.1), which implies that $\rho(X) \leq E[E[Z_\mu|X]X] - \rho^*_\infty(E[Z_\mu|X])$ and thus $E[Z_\mu|X] \in \delta\rho(X)$ (lemma 2.7), we may assume that $Z_\mu = f(X)$ for a measurable function $f : \mathbb{R} \to \mathbb{R}_-$, and still $Z_\mu \in L^{\rho*}$ (lemma 3.3). Moreover, since $-X^- \in L^\rho$, and $L^{\rho*} \cap L^1$ is law-invariant (lemma 3.3) we have that $(-\widetilde{Z}X^-) \in L^1$ for all $\widetilde{Z} \sim Z_\mu$. Consequently, $E[\widetilde{Z}X]$ is well-defined for all $\widetilde{Z} \sim Z_\mu$, so we may apply lemma C.2 in the following. Recalling that $-(q_X)^- = q_{-X^-}$ we obtain

$$-\infty < E[Z_\mu X] \leq \int_0^1 q_{Z_\mu}(s) q_X(s) ds \leq \int_0^1 q_{Z_\mu}(s) q_{-X^-}(s) ds < \infty$$

in which we applied lemmas C.1 and C.2. If $E[Z_\mu X] < \int_0^1 q_{Z_\mu}(s) q_X(s) ds$, then according to lemma C.2 there would be a $\widetilde{Z} \sim Z_\mu$ such that $E[\widetilde{Z}X] > E[Z_\mu X]$. Since $\widetilde{Z}X \in L^1$ (lemma C.1), by law-invariance of ρ^*_∞, and by lemma 2.7, we would have that

$$\rho(X) = E[Z_\mu X] - \rho^*_\infty(Z) < E[\widetilde{Z}X] - \rho^*_\infty(\widetilde{Z}) \leq \rho(X)$$

which is a contradiction. Therefore, $E[XZ_\mu] = \int_0^1 q_Z(s) q_X(s) ds$, so f may be chosen as an increasing function on $\{F_X > 0\}$ (lemma C.1). $\square$

5 Optimal Risk Sharing

In this section we consider n agents with initial endowments (risks) $X_i \in L^1$, whose preferences are represented by law-invariant closed convex risk measures ρ_i on L^1 which are continuous from below, $i = 1, \dots, n$. We write

$$X := X_1 + \dots + X_n$$

for the aggregate endowment. The aim of the agents is to minimise individual and total risk by sharing X optimally. An *allocation* of X is a n-tuple $(Y_1, \dots, Y_n) \in L^1 \times \dots \times L^1$ such that $\sum_{i=1}^n Y_i = X$. The risk minimisation problem is equivalent to finding allocations $(Y_1, \dots, Y_n)$ of X such that

$$\Box_{i=1}^n \rho_i(X) = \sum_{i=1}^n \rho_i(Y_i) \tag{5.1}$$

where $\Box_{i=1}^n \rho_i$ denotes the convolution of the ρ_i as defined in (A.1) (for more details on risk sharing and convolution, please consult [1, 4, 15, 16, 19]). Allocations solving (5.1) are called *optimal*. We are particularly interested in optimal allocations which have the following structure.

Definition 5.1. *An allocation $(Y_1, \dots, Y_n)$ of $X \in L^1$ is called* comonotone *if there exist increasing functions $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$ such that $\sum_{i=1}^n f_i = \mathrm{Id}_{\mathbb{R}}$ and $Y_i = f_i(X)$ for all $i = 1, \dots, n$. These functions f_i are necessarily 1-Lipschitz-continuous.*

The following theorem is our main existence and characterisation result for optimal risk sharing in L^1.

Theorem 5.2. *The convolution $\Box_{i=1}^n \rho_i$ is a law-invariant closed convex risk measure on L^1 which is continuous from below. Its restriction to L^∞ satisfies*

$$(\Box_{i=1}^n \rho_i)_\infty = \Box_{i=1}^n ((\rho_i)_\infty). \tag{5.2}$$

Moreover, for every $X \in L^1$, there exists a comonotone optimal allocation, and the first order condition

$$\delta \Box_{i=1}^n \rho_i(X) = \bigcap_{i=1}^n \delta \rho_i(Y_i) \tag{5.3}$$

holds for every comonotone optimal allocation $(Y_1, \dots, Y_n)$ of X. In particular, if X is bounded from below or if X and $\Box_{i=1}^n \rho_i$ satisfy conditions (2.11) *and* (2.12)*, then $\delta \Box_{i=1}^n \rho_i(X) \neq \emptyset$.*

Proof. It is proved in [16] (and [29]) that $\Box_{i=1}^n \rho_i$ is a law-invariant closed convex risk measure on L^1 admitting a comonotone optimal allocation $(Y_1, \dots, Y_n)$ for any $X \in L^1$. The continuity from below of $\Box_{i=1}^n \rho_i$ follows from proposition 2.5 and lemma A.1. As for (5.2), let $X \in L^\infty$ and $(Y_1, \dots, Y_n)$ be any comonotone allocation of X. Then, due to the 1-Lipschitz-continuity of f_i in definition 5.1, we have $|Y_i| = |f_i(X)| \le |f_i(0)| + |X|$. Hence $Y_i \in L^\infty$, for all $i = 1, \ldots, n$. Now (5.2) follows from the first part of the proof.

As for (5.3), let $(Y_1, \ldots, Y_n)$ be any comonotone optimal allocation of X. Suppose $Z \in \delta \square_{i=1}^n \rho_i(X)$. Then

$$\begin{aligned} \rho_1(Y_1) + \ldots + \rho_n(Y_n) &= \square_{i=1}^n \rho_i(X) = E[ZX] - (\square_{i=1}^n \rho_i)^*_\infty(Z) \\ &= \sum_{i=1}^n E[ZY_i] - (\rho_i)^*_\infty(Z) \end{aligned}$$

by (5.2), lemmas A.1 and 2.7, and the fact that $ZY_i \in L^1$ due to comonotonicity of the allocation. Now lemma 2.7 implies that $Z \in \bigcap_{i=1}^n \delta\rho_i(Y_i)$. Conversely, let $Z \in \bigcap_{i=1}^n \delta\rho_i(Y_i)$, then again by (5.2), and lemmas A.1 and 2.7

$$\begin{aligned} \square_{i=1}^n \rho_i(X) &= \sum_{i=1}^n \rho_i(Y_i) = \sum_{i=1}^n E[ZY_i] - (\rho_i)^*_\infty(Z) \\ &= E[ZX] - (\square_{i=1}^n \rho_i)^*_\infty(Z). \end{aligned}$$

Whence $Z \in \delta \square_{i=1}^n \rho_i(X)$. The final statement of theorem 5.2 is simply an application of theorem 2.9. $\square$

Remark 5.3. Note that the statement (5.3) may be void ($\emptyset = \emptyset$). As mentioned in remark 2.10, if $\square_{i=1}^n \rho_i$ satisfies (2.13), then we only have to check whether there is $\epsilon > 0$ such that (2.11) holds. Condition (2.13) in turn is satisfied by $\square_{i=1}^n \rho_i$ if e.g. $\sum_{i=1}^n \operatorname{dom}\rho_i (= \operatorname{dom}\square_{i=1}^n \rho_i) = L^1$ (lemma 2.2) or if all ρ_i, $i = 1, \ldots, n$, are coherent and satisfy (2.13). In order to verify the latter statement, let $Y \in \operatorname{dom}\square_{i=1}^n \rho_i$ be unbounded from below and let $(f_1(Y), \ldots, f_n(Y)) \in \operatorname{dom}\rho_1 \times \ldots \times \operatorname{dom}\rho_n$ be a comonotone allocation of Y such that $f_i(0) = 0$ for all $i = 1, \ldots, n$. If $f_i(Y)$ is bounded from below, then continuity from below implies that $\lim_{n\to\infty} \rho_i(f_i(Y)1_{\{Y \le -n\}}) = 0$. Otherwise, if $f_i(Y)$ is unbounded from below, then there is an increasing sequence $(m_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that

$$f_i(Y)1_{\{Y \le -n\}} \ge f_i(Y)1_{\{f_i(Y) \le -m_n\}} \quad \text{for all } n \in \mathbb{N},$$

because f_i is increasing and in this case unbounded from below. Hence, we obtain

$$0 \le \lim_{n\to\infty} \rho_i(f_i(Y)1_{\{Y \le -n\}}) \le \lim_{n\to\infty} \rho_i(f_i(Y)1_{\{f_i(Y) \le -m_n\}}) = 0$$

by (2.13). Eventually, as $(f_1(Y)1_{\{Y \le -n\}}, \ldots, f_n(Y)1_{\{Y \le -n\}})$ is an allocation of $Y1_{\{Y \le -n\}}$, we arrive at

$$0 \le \lim_{n\to\infty} \square_{i=1}^n \rho_i(Y1_{\{Y \le -n\}}) \le \lim_{n\to\infty} \sum_{i=1}^n \rho_i(f_i(Y)1_{\{Y \le -n\}}) = 0.$$

$\diamondsuit$

The subgradients $\delta\square_{i=1}^n \rho_i(X)$ induce equilibrium pricing rules as follows. We identify each $Z \in \mathcal{P}_\mathcal{R}$ with the absolutely continuous probability measure $\mathbb{Q} \ll \mathbb{P}$ given by $d\mathbb{Q}/d\mathbb{P} = Z$, and with the corresponding pricing rule

$$L^1(\mathbb{Q}) := L^1(\Omega, \mathcal{F}, \mathbb{Q}) \ni Y \mapsto E_\mathbb{Q}[Y].$$

Definition 5.4. *An allocation* $(\widetilde{Y}_1, \dots, \widetilde{Y}_n)$ *of* X *together with a pricing rule* $\mathbb{Q} \in \mathcal{P}_\mathcal{R}$ *is called an* equilibrium *if* X_i, $\widetilde{Y}_i \in L^1(\mathbb{Q})$, $E_\mathbb{Q}[\widetilde{Y}_i] \le E_\mathbb{Q}[X_i]$, *and*

$$\rho_i(\widetilde{Y}_i) = \inf\{\rho_i(Y_i) \mid Y_i \in L^1(\mathbb{Q}) \cap L^1,\ E_\mathbb{Q}[Y_i] \le E_\mathbb{Q}[X_i]\}$$

for all $i = 1, \dots, n$.

For a thorough discussion of equilibria with respect to convex risk measures we refer to [15]. The following theorem establishes the connection between equilibria, optimal allocations and generalised subgradients.

Theorem 5.5. *The following conditions are equivalent:*

(i) *There exists an equilibrium* $(\widetilde{Y}_1, \dots, \widetilde{Y}_n; \mathbb{Q})$.

(ii) *There exists a comonotone equilibrium* $(\widetilde{Y}_1, \dots, \widetilde{Y}_n; \mathbb{Q})$.

(iii) *There is* $Z \in \delta\square_{i=1}^n \rho_i(X)$ *such that* $ZX_i \in L^1$ *for all* $i = 1, \dots, n$.

Moreover, if $(\widetilde{Y}_1, \dots, \widetilde{Y}_n)$ *is a comonotone optimal allocation of* X *and* (iii) *holds, then* $(\widetilde{Y}_1 + E_\mathbb{Q}[X_1 - \widetilde{Y}_1], \dots, \widetilde{Y}_n + E_\mathbb{Q}[X_n - \widetilde{Y}_n]; \mathbb{Q})$ *where* $d\mathbb{Q}/d\mathbb{P} = -Z$ *is an equilibrium.*

Proof. Let $\mathbb{Q} \ll \mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ such that $X_i \in L^1(\mathbb{Q})$ for all $i = 1, \dots, n$. We claim that

$$\inf_{\substack{Y \in L^1(\mathbb{Q}) \cap L^1 \\ E_\mathbb{Q}[Y] \le E_\mathbb{Q}[X_i]}} \rho_i(Y) = E[ZX_i] - (\rho_i)^*_\infty(Z) \tag{5.4}$$

where $Z := -d\mathbb{Q}/d\mathbb{P}$. In order to verify this, note that by cash-invariance of ρ_i it is obvious that the infimum on the left-hand side of (5.4) equals the infimum taken over those $Y \in L^1(\mathbb{Q}) \cap L^1$ satisfying $E_\mathbb{Q}[Y] = E_\mathbb{Q}[X_i]$. Now for every $Y \in L^1(\mathbb{Q}) \cap L^1$ such that $E_\mathbb{Q}[Y] = E_\mathbb{Q}[X_i]$ lemma 2.8 and monotone convergence imply that

$$\begin{aligned}
\rho_i(Y) &= \lim_{m\to\infty} \lim_{n\to\infty} \rho_i((Y^+ \wedge n) - (Y^- \wedge m)) \\
&\ge \lim_{m\to\infty} \lim_{n\to\infty} E[Z((Y^+ \wedge n) - (Y^- \wedge m))] - (\rho_i)^*_\infty(Z) \\
&= E[ZY] - (\rho_i)^*_\infty(Z) = E[ZX_i] - (\rho_i)^*_\infty(Z).
\end{aligned}$$

Hence, we have established $\geq$ in (5.4). Moreover, since $E_{\mathbb{Q}}[Y + E_{\mathbb{Q}}[X_i - Y]] = E_{\mathbb{Q}}[X_i]$ for every $Y \in L^1(\mathbb{Q})$ and by cash-invariance we obtain

$$\begin{aligned}
\inf_{\substack{Y \in L^1(\mathbb{Q}) \cap L^1 \\ E_{\mathbb{Q}}[Y] \leq E_{\mathbb{Q}}[X_i]}} \rho_i(Y) &= \inf_{Y \in L^1(\mathbb{Q}) \cap L^1} \rho_i(Y + E_{\mathbb{Q}}[X_i - Y]) \\
&= E_{\mathbb{Q}}[-X_i] - \sup_{Y \in L^1(\mathbb{Q}) \cap L^1} \Big(E_{\mathbb{Q}}[-Y] - \rho_i(Y) \Big) \\
&\leq E[ZX_i] - \sup_{Y \in L^\infty} \Big(E[ZY] - \rho_i(Y) \Big) \\
&= E[ZX_i] - (\rho_i)^*_\infty(Z),
\end{aligned}$$

and (5.4) is proved.

(i) $\Leftrightarrow$ (ii): suppose there exists an equilibrium $(\widetilde{Y}_1, \ldots, \widetilde{Y}_n; \mathbb{Q})$. Let $(Y_1, \ldots, Y_n)$ be any comonotone optimal allocation of X, which exists according to theorem 5.2. Then $Y_i \in L^1(\mathbb{Q})$ by comonotonicity and the fact that $X \in L^1(\mathbb{Q})$ by definition of an equilibrium. By rebalancing the cash, this is by adding $c_i = E_{\mathbb{Q}}[X_i - Y_i]$ to each Y_i, we achieve that $E_{\mathbb{Q}}[Y_i + c_i] = E_{\mathbb{Q}}[X_i]$ for all $i = 1, \ldots, n$, and the modified allocation $(Y_1 + c_1, \ldots, Y_n + c_n)$ is still comonotone and optimal due to $\sum_{i=1}^n c_i = 0$ and cash-invariance of the ρ_i. Consequently, we may w.l.o.g. assume that $(Y_1, \ldots, Y_n)$ satisfies $E_{\mathbb{Q}}[Y_i] = E_{\mathbb{Q}}[X_i]$. But then, $\rho_i(\widetilde{Y}_i) \leq \rho_i(Y_i)$ for each $i = 1, \ldots, n$ (definition 5.4), which can only hold if $(\widetilde{Y}_1, \ldots, \widetilde{Y}_n)$ is itself optimal and $\rho_i(\widetilde{Y}_i) = \rho_i(Y_i)$. Hence, $(Y_1, \ldots, Y_n, \mathbb{Q})$ is a comonotone equilibrium. The converse implication is trivial.

(ii) $\Rightarrow$ (iii): in the fist part of the proof, we established that the allocation given by any equilibrium must be optimal. Hence, in view of (5.4), and lemma A.1 we conclude that

$$\begin{aligned}
\square_{i=1}^n \rho_i(X) &= \sum_{i=1}^n \rho_i(\widetilde{Y}_i) = \sum_{i=1}^n E[ZX_i] - (\rho_i)^*_\infty(Z) \\
&= E[ZX] - (\square_{i=1}^n \rho_i)^*_\infty(Z)
\end{aligned}$$

where $Z := -d\mathbb{Q}/d\mathbb{P}$. Consequently, we have proved that $Z \in \delta \square_{i=1}^n \rho_i(X)$ (lemma 2.7).

(iii) $\Rightarrow$ (ii): suppose there is $Z \in \delta \square_{i=1}^n \rho_i(X)$ and let $\mathbb{Q}$ be given by $d\mathbb{Q}/d\mathbb{P} := -Z$. Moreover, let $(Y_1, \ldots, Y_n)$ be any comonotone optimal allocation of X such that $E_{\mathbb{Q}}[Y_i] = E_{\mathbb{Q}}[X_i]$ for all $i = 1, \ldots, n$ (theorem 5.2 and rebalancing the cash). The equality (5.3) implies that $Z \in \delta \rho_i(Y_i)$ for all $i = 1, \ldots, n$. This in conjunction with (5.4) and lemma 2.7 yields

$$\rho_i(Y_i) = E[ZY_i] - (\rho_i)^*_\infty(Z) = E[ZX_i] - (\rho_i)^*_\infty(Z) = \inf_{\substack{Y \in L^1(\mathbb{Q}) \cap L^1 \\ E_{\mathbb{Q}}[Y] \leq E_{\mathbb{Q}}[X_i]}} \rho_i(Y),$$

so we infer that

$$\rho_i(Y_i) = \inf_{\substack{Y \in L^1(\mathbb{Q}) \cap L^1 \\ E_{\mathbb{Q}}[Y] \leq E_{\mathbb{Q}}[X_i]}} \rho_i(Y).$$

Consequently, $(Y_1, \ldots, Y_n; \mathbb{Q})$ is an equilibrium. This also proves the closing statement of the theorem. $\square$

Finally, we provide sufficient conditions for the existence of an equilibrium.

Lemma 5.6. *Suppose that X is bounded from below, or that X and $\square_{i=1}^n \rho_i$ satisfy conditions* (2.11) *and* (2.12)*. If either*

$$\forall\, \widetilde{\mathbb{P}} \in \mathcal{P}_{\mathcal{R}} :\ X \in L^1(\widetilde{\mathbb{P}}) \ \Leftrightarrow\ X_i \in L^1(\widetilde{\mathbb{P}}) \text{ for all } i = 1, \ldots, n \tag{5.5}$$

or

$$X_i \in L^{\square_{i=1}^n \rho_i} \text{ for all } i = 1, \ldots, n, \tag{5.6}$$

then there exists an equilibrium.

Proof. In case of (5.5) combine theorems 5.2 and 5.5. In case of (5.6) we also recall that according to the proof of theorem 2.9, under the stated conditions, we do not only know that there is $Z \in \delta \square_{i=1}^n \rho_i(X)$, but we may also assume that this Z is an element of the dual space of $L^{\square_{i=1}^n \rho_i}$. Hence, if $X_i \in L^{\square_{i=1}^n \rho_i}$, then $ZX_i \in L^1$ for all $i = 1, \ldots, n$, so theorem 5.5 (iii) applies. $\square$

Condition (5.5) is always satisfied if $X_i \in L^\infty$, $i = 1, \ldots, n$, or if the initial risks X_i may be somehow controlled by the aggregate risk X, which should be satisfied in most interesting cases. Condition (5.6) will be applied in example 6.2.

6 Examples

In example 6.1 we show that a law-invariant closed convex risk measure ρ on L^1 which is not continuous from below may have empty generalised subgradients even for bounded risks. Examples 6.2, 6.3, and 6.4 illustrate our main results, in particular theorems 2.9, and 5.5 by means of some well-known risk measures. In examples 6.2 (Average Value at Risk) and 6.3 (L^p-Semi-Deviation Risk Measure) the spaces L^ρ will coincide with some Lebesgue space L^p which are a subclass of Orlicz hearts. Orlicz hearts are proposed as model spaces for convex risk measures in [6]. For a definition of Orlicz spaces/hearts and the details on risk measures on Orlicz hearts please consult [6]. In example 6.4, in which we study the entropic risk measure, we will see that L^ρ corresponds to an Orlicz space which is strictly larger than the corresponding Orlicz heart, and we will find that the set of points at which the entropic risk measure is generalised subdifferentiable is also strictly larger than this Orlicz heart. Example 6.5 then shows that, although the above mentioned prominent examples of law-invariant convex risk measures are all linked to certain Orlicz spaces, the class of L^ρ-spaces covers a far greater variety of law-invariant Banach spaces. This section closes with example 6.6 which is linked to conditions (2.12) and (2.13).

6.1 Essential Infimum

Let $\rho = -\operatorname{essinf}$ and let $X \in L^\infty$ be such that $\mathbb{P}(X = \operatorname{essinf} X) = 0$. Then $\delta\rho(X) = \emptyset$, because for every probability measure $\mathbb{Q} \ll \mathbb{P}$ we have that $\mathbb{Q}(X = \operatorname{essinf} X) = 0$. Supposing we had $-d\mathbb{Q}/d\mathbb{P} \in \delta\rho(X)$, then $E_{\mathbb{Q}}[X - \operatorname{essinf} X] = 0$, which would imply that $X = \operatorname{essinf} X$ $\mathbb{Q}$-a.s., and thus would be a contradiction. Hence, $\delta\rho(X) = \emptyset$, although $\partial\rho_\infty(X) \neq \emptyset$.

6.2 Average Value at Risk

Consider the Average Value at Risk (AVaR_α) at level $\alpha \in (0, 1]$, i.e.

$$\mathrm{AVaR}_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha q_X(s)ds, \quad X \in L^1.$$

It is well-known that AVaR_α is continuous on L^1 (for a proof see e.g. [28]) and thus subdifferentiable by lemma 2.2. Clearly, $L^{\mathrm{AVaR}_\alpha} = L^1$, and in view of lemma 3.2 and continuity w.r.t. $\|\cdot\|_1$ it is easily verified that $\|\cdot\|_{C,\mathrm{AVaR}_\alpha}$ and $\|\cdot\|_1$ are equivalent. Thus $L^{\rho*} = L^\infty$. According to (the proof of) theorem 2.9 for every $X \in L^1$ there is a $f_\alpha : \mathbb{R} \to \mathbb{R}_-$ which is increasing on $\{F_X > 0\}$ such that $f_\alpha(X) \in \partial\,\mathrm{AVaR}_\alpha(X) \subset \delta\,\mathrm{AVaR}_\alpha(X)$. It is proved in [18] theorem 4.47 and remark 4.48 that

$$f_\alpha(X) = -\frac{1}{\alpha}(1_{\{X<q_X(\alpha)\}} - \kappa 1_{\{X=q_X(\alpha)\}})$$

where κ is defined as

$$\kappa := \begin{cases} 0 & \text{if } \mathbb{P}(X = q_X(\alpha)) = 0, \\ \frac{\alpha - \mathbb{P}(X<q_X(\alpha))}{\mathbb{P}(X=q_X(\alpha))} & \text{otherwise} \end{cases}$$

does the job. Note that f_α is indeed increasing, does depend on X, and is not continuous. Let $\beta_i \in (0, 1]$ for $i = 1, \ldots, n$, and let $\gamma := \max_{i=1,\ldots,n} \beta_i$. It is well-known (see e.g. [16]) that

$$\square_{i=1}^n \mathrm{AVaR}_{\beta_i} = \mathrm{AVaR}_\gamma\,.$$

Hence, as we are in the situation of (5.6), and assuming w.l.o.g. that $\beta_1 = \gamma$, we obtain that for any initial risks $X_i \in L^1$ and $X := \sum_{i=1}^n X_i$ an equilibrium is given by $(X + c_1, c_2, \ldots, c_n; \mathbb{Q})$ where $d\mathbb{Q}/d\mathbb{P} = -f_\gamma(X)$ and $c_1 = E_{\mathbb{Q}}[X_1 - X]$, $c_i = E_{\mathbb{Q}}[X_i]$ for $i = 2, \ldots, n$.

6.3 L^p–Semi–Deviation Risk Measure

Let $p \in [1, \infty)$ and $\rho_p(X) = E[-X] + \delta\|(X - E[X])^-\|_p$, $X \in L^1$, for some $\delta \in (0, 1]$. Then ρ_p is a law-invariant closed coherent risk measure on L^1 which is continuous from below (proposition 2.5 (iv)) and satisfies (2.13). In fact ρ_p is continuous if restricted to $(L^p, \|\cdot\|_p)$. It easily verified that $L^{\rho_p} = L^p$, and that

$\| \cdot \|_{C,\rho_p}$ and $\| \cdot \|_p$ are equivalent. Thus we have that $L^{\rho_p *} = L^q$ for $q \in (1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. By theorem 2.9, for every $X \in \operatorname{dom} \rho$ there is a $f : \mathbb{R} \to \mathbb{R}_-$ which is increasing on $\{F_X > 0\}$ such that $f(X) \in \delta\rho_p(X) \cap L^q$ (the fact that $f(X) \in L^{\rho_p *} = L^q$ is shown in the proof of theorem 2.9). It is known (see e.g. [1] or [14]) that

$$f(X) = \begin{cases} -1 & \text{if } X = \text{constant}, \\ -1 + \delta \frac{E[((X-E[X])^-)^{p-1}] - ((X-E[X])^-)^{p-1}}{\|(X-E[X])^-\|_p^{p-1}} & \text{otherwise} \end{cases} \tag{6.1}$$

does the job. Note that for $1 \leq p \leq r < \infty$ we have $\rho_p \leq \rho_r$ due to Hölder's inequality. Consequently, $\operatorname{dom} \rho_p^* \subset \operatorname{dom} \rho_r^*$ and in conjunction with theorem 2.5 in [16] we conclude that $\rho_p \square \rho_r = \rho_p$. Hence, if the initial risks satisfy $X_1, X_2 \in L^p$, then $(X + E_{\mathbb{Q}}[X_1 - X], E_{\mathbb{Q}}[X_2]; \mathbb{Q})$ is an equilibrium provided that $X = X_1 + X_2$ and $-d\mathbb{Q}/d\mathbb{P}$ is given by (6.1). The extension of this two-agent case to the n-agent case is obvious.

6.4 Entropic Risk Measure

The entropic risk measure with parameter $\beta > 0$ is

$$\rho_\beta(X) = \frac{1}{\beta} \log E[e^{-\beta X}], \quad X \in L^1.$$

It is a law-invariant closed convex risk measure on L^1 which is continuous from below (proposition 2.5 (iv)) and satisfies (2.13). For simplicity we consider $\rho := \rho_1$. Then, $\rho_\infty^*(Z) = E[-Z \log(-Z)]$ for every $Z \in \mathcal{P}^{\infty *} \cap L^1$ (see [18] example 4.33). In the following we illustrate the quality of condition (2.11). To this end, assume that $X \in L^1$ satisfies condition (2.11), i.e. there exists $k > 1$ such that $kX \in \operatorname{dom} \rho$. Then we have $Z := -\frac{e^{-X}}{E[e^{-X}]} \in L^1$, and $XZ \in L^1$ too, because $|X|e^{-X} \leq C + e^{-kX}$ for some constant $C > 0$ and $E[e^{-kX}] < \infty$ by assumption. It is proved in [18] lemma 3.29 and example 4.33 that

$$\rho(X) = E[ZX] - \rho_\infty^*(Z),$$

and thus $Z \in \delta\rho(X)$ by lemma 2.7. Obviously, $Z = f(X)$ for an increasing function $f : \mathbb{R} \to \mathbb{R}_-$. Now we show that condition (2.11) is in some sense the best we can expect. For this purpose, consider an $X \in L^1$ being distributed according to

$$F_X(x) = C \cdot \int_{-\infty}^{-1 \wedge x} \frac{e^u}{u^2} du$$

for an appropriate constant $C > 0$. It is easily verified that $X \in \operatorname{dom} \rho$ and $X \in L^\rho$, but $(1 + \epsilon)X \notin \operatorname{dom} \rho$ for all $\epsilon > 0$. We claim that $\delta\rho(X) = \emptyset$. Suppose we had $\delta\rho(X) \neq \emptyset$. Then, according to lemma 6.1 below, this would imply that $Z := -\frac{e^{-X}}{E[e^{-X}]} \in \delta\rho(X)$. But this cannot hold because

$$E[ZX] = \frac{E[-Xe^{-X}]}{E[e^{-X}]} = \infty,$$

so we must have $\delta\rho(X) = \emptyset$.

Next we elaborate on the connection with Orlicz spaces and Orlicz hearts (for a thorough discussion of Orlicz spaces and hearts we refer to [23]). To this end, we let $\Phi(x) = \exp(x) - 1$, $x \geq 0$, and define the spaces

$$L^\Phi := \{X \in L^1 \mid E[\Phi(c|X|)] < \infty \quad \text{for some } c > 0\}$$

and

$$M^\Phi := \{X \in L^1 \mid E[\Phi(c|X|)] < \infty \text{ for all } c > 0\}.$$

The Orlicz space L^Φ is a Banach space if equipped with the Luxemburg norm

$$\|X\|_\Phi := \inf\left\{\lambda > 0 \mid E\left[\Phi\left(\frac{|X|}{\lambda}\right)\right] \leq 1\right\} \text{ (see e.g. [23] theorem 3.3.10).}$$

It is well-known that $L^\infty \subset L^\Phi$ is not dense and that the Orlicz heart $M^\Phi \subsetneqq L^\Phi$ is the $\|\cdot\|_\Phi$-closure of L^∞ in L^Φ, and thus itself a Banach Space. Note that $L^\rho = L^\Phi$, and that $\|\cdot\|_\Phi = \|\cdot\|_{\log 2,\rho}$. In search for subgradients, as an alternative to the space L^ρ, one could think of choosing the Orlicz heart M^Φ, because $\rho|_{M^\Phi}$ is closed and real-valued, and thus continuous and everywhere subdifferentiable ([13] corollary 2.5 and proposition 5.2). However, in doing so, we would neglect a lot of points at which ρ is generalised subdifferentiable. In fact, we have that

$$M^\Phi \subsetneqq \operatorname{int}\operatorname{dom}\rho|_{L^\rho} \subsetneqq \{X \in L^1 \mid X \text{ satisfies condition (2.11)}\}. \tag{6.2}$$

The fact that the first inclusion must be strict is easily verified by considering any X being distributed according to

$$F_X(x) = e^{\lambda x} \wedge 1 \quad \text{for } \lambda > 1.$$

On the one hand, $-k|X| \in \operatorname{dom}\rho$ for every $k \in (1, \lambda)$, so $X \in \operatorname{int}\operatorname{dom}\rho|_{L^\rho}$ by (3.10). On the other hand, for $c \geq \lambda$ we have $E[\Phi(c|X|)] = \infty$, so $X \notin M^\Phi$. The last strict inclusion in (6.2) is justified in remark 3.6.

As for equilibria, it is well-known (see e.g. [4], [16]) that

$$\rho_\beta \Box \rho_\gamma = \rho_\alpha \quad \text{for} \quad \alpha := \frac{\beta\gamma}{\beta + \gamma},$$

and that for every $X \in L^1$ an optimal allocation is given by $\left(\frac{\gamma}{\beta+\gamma}X, \frac{\beta}{\beta+\gamma}X\right)$, which is unique up to rebalancing the cash (which in this case means that all optimal allocations are of type $\left(\frac{\gamma}{\beta+\gamma}X + c, \frac{\beta}{\beta+\gamma}X - c\right)$, $c \in \mathbb{R}$). Moreover, lemma 6.1 shows that $\delta\rho_\alpha$ contains at most one element. Consequently, in view of theorem 5.5, we infer that given any initial risks $X_1, X_2 \in L^1$ and the aggregate endowment $X := X_1 + X_2$ such that there is $Z \in \delta\rho_\alpha(X)$ and $(ZX_1), (ZX_2) \in L^1$, the unique equilibrium is

$$\left(\frac{\gamma}{\beta+\gamma}X + E_\mathbb{Q}[X_1 - \frac{\gamma}{\beta+\gamma}X], \frac{\beta}{\beta+\gamma}X + E_\mathbb{Q}[X_2 - \frac{\beta}{\beta+\gamma}X]; \mathbb{Q}\right)$$

in which $\mathbb{Q}$ is given by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{-\alpha X}}{E[e^{-\alpha X}]}$.

Lemma 6.1. *Let ρ_β be the entropic risk measure with parameter $\beta > 0$. Then,*

$$\forall Y \in L^1: \quad Z \in \delta\rho_\beta(Y) \quad \Rightarrow \quad Z = -\frac{e^{-\beta Y}}{E[e^{-\beta Y}]}.$$

Proof. Let $Y \in \operatorname{dom}\rho_\beta$ and define the probability measure $\widetilde{\mathbb{P}} \approx \mathbb{P}$ by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{-\beta Y}}{E[e^{-\beta Y}]}.$$

Suppose there is a $Z \in \delta\rho_\beta(Y)$. Then, $\frac{d\mathbb{Q}}{d\mathbb{P}} = -Z$ defines a probability measure $\mathbb{Q} \ll \mathbb{P}$, and we have that

$$\rho_\beta(Y) = E_{\mathbb{Q}}[-Y] - \frac{1}{\beta}E_{\mathbb{Q}}\left[\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right] \tag{6.3}$$

in which both $E_{\mathbb{Q}}[-Y] < \infty$ and $E_{\mathbb{Q}}[\log\frac{d\mathbb{Q}}{d\mathbb{P}}] < \infty$ (lemma 2.7). Note that

$$\log\frac{d\mathbb{Q}}{d\mathbb{P}} = \log\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}} + \log\frac{e^{-\beta Y}}{E[e^{-\beta Y}]} = \log\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}} - \beta Y - \beta\rho_\beta(Y),$$

and thus

$$\beta Y + \beta\rho_\beta(Y) + \log\frac{d\mathbb{Q}}{d\mathbb{P}} = \log\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}. \tag{6.4}$$

Since the left hand side of (6.4) is $\mathbb{Q}$-integrable, we obtain $\log\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}} \in L^1(\mathbb{Q})$ and

$$\rho_\beta(Y) - \left(E_{\mathbb{Q}}[-Y] - \frac{1}{\beta}E_{\mathbb{Q}}\left[\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right]\right) = \frac{1}{\beta}E_{\mathbb{Q}}\left[\log\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}\right].$$

By (6.3) we conclude that $E_{\mathbb{Q}}[\log\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}] = 0$ which is equivalent to $\mathbb{Q} = \widetilde{\mathbb{P}}$. $\square$

6.5 The Variety of the $\mathbf{L}^\rho$-spaces

The spaces L^ρ of the preceding examples all corresponded to Orlicz spaces. This is no suprise since the presented risk measures are all closely connected to some Orlicz space generating function. However, as we should expect, this is not the case in general. In this example we will show that L^ρ might almost be any law-invariant Banach space of random variables. To this end, let $(L, \|\cdot\|_L)$ be a Banach space satisfying the following conditions:

(i) $\|\cdot\|_L : L^1 \to [0,\infty]$ is a law-invariant closed (w.r.t. $\|\cdot\|_1$) sublinear function such that $\|X\|_L = \||X|\|_L$ and $|X| \ge |Y| \;\Rightarrow \|X\|_L \ge \|Y\|_L$,

(ii) $\mathbb{R} \subset L = \{X \in L^1 \mid \|X\|_L < \infty\}$.

Consider the law-invariant closed coherent risk measure ρ on L^1 given by

$$\rho(X) = E[-X] + \frac{1}{\|1\|_L}\|(X - E[X])^-\|_L, \quad X \in L^1.$$

It is easily verified that $L^\rho = L$. Note that $(L, \|\cdot\|_L)$ is not necessarily an Orlicz space. Conditions (i) and (ii) are for instance satisfied by any Lorentz space (see e.g. [23] section 10.3 for a definition) which do not coincide with Orlicz spaces in general (see e.g. [23] theorem 10.3.3 and [22]). A concrete example is the space given by

$$\|X\|_L = \frac{1}{2}\int_0^1 \frac{q_{|X|}(s)}{\sqrt{1-s}}ds, \quad X \in L^1.$$

6.6 An Example of a Law-Invariant Closed Coherent Risk Measure which is Continuous from Below but does not satisfy (2.13)

Let $(\Omega, \mathcal{F}, \mathbb{P}) = ((0,1], \mathcal{B}(0,1], \lambda)$ where $\mathcal{B}(0,1]$ is the Borel-σ-algebra over $(0,1]$ and λ denotes the Lebesgue-measure restricted to $\mathcal{B}(0,1]$. Let the probability measures $\mathbb{Q}_n$ be given by

$$\frac{d\mathbb{Q}_n}{d\mathbb{P}} = n 1_{(0,\frac{1}{n^2}]} + \frac{n}{n+1} 1_{(\frac{1}{n^2},1]},$$

and let $\widetilde{Z}_n := -d\mathbb{Q}_n/d\mathbb{P}$, $n \in \mathbb{N}$. Moreover, let

$$\mathcal{Q} := \{Z \in L^\infty \mid \exists n \in \mathbb{N} : Z \sim \widetilde{Z}_n\},$$

and define a law-invariant closed coherent risk measure on L^1 by

$$\rho(X) := \sup_{Z\in\mathcal{Q}} E[ZX] = \sup_{Z\in\mathcal{Q}} \int_0^1 q_X(s) q_Z(s)\, ds, \quad X \in L^1,$$

where the last equality, and thus the law-invariance of ρ, follows from law-invariance of $\mathcal{Q}$ and lemma C.2. Note that the following computations also imply that ρ is continuous from below (proposition 2.5). Consider the point $Y(\omega) := -\frac{1}{\sqrt{\omega}}$, $\omega \in (0,1]$, in L^1. Since the function Y is increasing, it is immediate that

$$\sup_{Z\sim\widetilde{Z}_n} E[ZY] = E[\widetilde{Z}_n Y] = \frac{4n}{n+1}$$

and thus

$$\rho(Y) = \lim_{n\to\infty} \frac{4n}{n+1} = 4.$$

We notice that for every $\epsilon > 0$

$$\lim_{n\to\infty} \rho(Y + \epsilon Y 1_{\{Y\leq -n\}}) \geq \lim_{n\to\infty} E[\widetilde{Z}_n Y] + \epsilon E[\widetilde{Z}_n Y 1_{\{Y\leq -n\}}] = 4 + 2\epsilon > \rho(Y),$$

so ρ and Y do not satisfy (2.12), and in particular ρ does not satisfy (2.13).

A The Convolution

Let $F_1, \dots, F_n : V \to (-\infty, \infty]$ be proper convex functions on some locally convex space V. The *convolution* of $F_1, \dots, F_n$ is the function

$$\square_{i=1}^n F_i(X) := F_1 \square \dots \square F_n(X) := \inf_{\substack{X_i \in V, \\ \sum_{i=1}^n X_i = X}} \sum_{i=1}^n F_i(X_i), \quad X \in V. \tag{A.1}$$

The following properties are well-known (see e.g. [24]).

Lemma A.1. (i) $\square_{i=1}^n F_i : L^p \to [-\infty, \infty]$ *is a convex function,*

(ii) $\operatorname{dom} \square_{i=1}^n F_i = \sum_{i=1}^n \operatorname{dom} F_i$,

(iii) $(\square_{i=1}^n F_i)^* = \sum_{i=1}^n F_i^*$,

(iv) $\operatorname{dom}(\square_{i=1}^n F_i)^* = \bigcap_{i=1}^n \operatorname{dom} F_i^*$.

B Law-invariant Convex Functions

Throughout the paper we draw heavily on the following properties, which are proved in [11] and [28]. Let $p \in [1, \infty]$ and $q := \frac{p}{p-1}$ where $1/0 := \infty$ and $\infty/(\infty - 1) := 1$. For every proper closed convex function $F : L^p \to (-\infty, \infty]$ the following conditions are equivalent:

$$\begin{aligned} F \text{ is law-invariant} &\Leftrightarrow F \text{ is } \sigma(L^p, L^q)\text{-closed and } F^*|_{L^1} \text{ is law-invariant} \\ &\Leftrightarrow X \succeq_c Y \text{ implies } F(X) \leq F(Y) \end{aligned}$$

in which $\succeq_c$ denotes the concave order defined by $X \succeq_c Y$ if and only if $E[u(X)] \geq E[u(X)]$ for all concave functions $u : \mathbb{R} \to \mathbb{R}$. Since $E[X|\mathcal{G}] \succeq_c X$ by Jensen's inequality, if F is law-invariant, then

$$F(E[X|\mathcal{G}]) \leq F(X) \text{ for all sub-}\sigma\text{-algebras } \mathcal{G} \subset \mathcal{F}. \tag{B.1}$$

Hence, if $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is a sequence of sub-σ-algebras of $\mathcal{F}$ and $X \in L^p$ such that $E[X|\mathcal{G}_n]$ converges to X w.r.t. $\|\cdot\|_p$, then (B.1) and l.s.c. of F imply that

$$F(X) = \lim_{n \to \infty} F(E[X|\mathcal{G}_n]). \tag{B.2}$$

Now let $\rho : L^p \to (-\infty, \infty]$ be a law-invariant closed convex risk measure. Then cash-invariance and (B.1) imply that

$$\rho(X) \geq -E[X] \,.$$

Hence, as $0 \in L^p$,

$$\rho^*(-1) = \sup_{X \in L^p} E[-X] - \rho(X) = 0 \,. \tag{B.3}$$

C Hardy-Littlewood Inequalities

Lemma C.1 (theorem A.24 in [18])**.** *For any two random variables X and Z we have*

$$\int_0^1 q_X(1-s)q_Z(s)ds \leq E[XZ] \leq \int_0^1 q_X(s)q_Z(s)ds,$$

provided that the integrals are well-defined. Moreover, if $Z = f(X)$ for a measurable function $f : \mathbb{R} \to \mathbb{R}$ and the upper(lower) bound is finite, then the upper(lower) bound is attained if and only if f can be chosen as an increasing(decreasing) function on either $\{F_X > 0\}$ if Z is bounded from above, or on $\{0 < F_X < 1\}$ else.

The following lemma is an extension of lemma 4.55 in [18]. For the sake of completeness we provide a self-contained proof.

Lemma C.2. *Let $X, Z \in L^1$.*

(i) *If $E[\widetilde{X}Z]$ is well-defined for every $\widetilde{X} \sim X$ and if $\int_0^1 |q_X(s)q_Z(s)|ds < \infty$, then*

$$\sup_{\widetilde{X} \sim X} E[Z\widetilde{X}] = \int_0^1 q_X(s)q_Z(s)ds. \tag{C.1}$$

(ii) *In particular, condition* (i) is satisfied if $(\widetilde{X}Z) \in L^1$ for all $\widetilde{X} \sim X$.

Proof. **step 1.** Suppose the distribution function F_Z of Z is continuous. Then $U := F_Z(Z)$ has a uniform distribution on $(0,1)$ and $Z = q_Z(U)$ $\mathbb{P}$-a.s.. For $\bar{X} := q_X(U) \sim X$ we have that

$$E[|\bar{X}||Z|] = E[|q_X(U)||q_Z(U)|] = \int_0^1 |q_X(s)||q_Z(s)|ds. \tag{C.2}$$

Thus, if $E[\widetilde{X}Z]$ is well-defined for every $\widetilde{X} \sim X$ and if $\int_0^1 |q_X(s)q_Z(s)|ds < \infty$, then (C.1) follows from

$$E[\bar{X}Z] = \int_0^1 q_X(s)q_Z(s)\,ds$$

and lemma C.1. Moreover, if $(\widetilde{X}Z) \in L^1$ for all $\widetilde{X} \sim X$, then $E[\widetilde{X}Z]$ is well-defined for every $\widetilde{X} \sim X$, and $\int_0^1 |q_X(s)q_Z(s)|ds < \infty$ follows from (C.2).

step 2. Now suppose Z has no continuous distribution. Denote by D the countable set of all $z \in \mathbb{R}$ such that $\mathbb{P}[Z = z] > 0$. W.l.o.g. (by adding a constant to Z if necessary) we may assume that $0 \notin D$. Let $A_z := \{Z = z\}$, $z \in D$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ contains no atoms, for each $z \in D$ there is a random variable U_z being uniformly distributed on $(0, \frac{|z|}{2} \wedge 1)$ under the measure $\mathbb{P}(\cdot \mid A_z)$. We claim that the distributions of

$$Z_n := Z - \frac{1}{n}\sum_{z \in D} \operatorname{sgn}(z) U_z 1_{A_z}, \quad n \in \mathbb{N},$$

are continuous. Indeed, for any $y \in \mathbb{R}$

$$\begin{aligned}
\mathbb{P}(Z_n = y) &= \mathbb{P}(Z_n = y, Z \notin D) + \sum_{z \in D} \mathbb{P}(Z = z, U_z = \operatorname{sgn}(z) n(z-y)) \\
&= \mathbb{P}(Z = y, Z \notin D) + \sum_{z \in D} \mathbb{P}(A_z)\mathbb{P}(U_z = \operatorname{sgn}(z) n(z-y) \mid A_z) \\
&= 0\,.
\end{aligned}$$

Note that $Z^{\pm} - 1 \le Z_n^{\pm} \le Z^{\pm}$. Hence, for all $n \in \mathbb{N}$ and for every $\widetilde{X} \sim X$

- $E[\widetilde{X} Z]$ is well-defined if and only if $E[\widetilde{X} Z_n]$ is well-defined,
- $(\widetilde{X} Z) \in L^1$ if and only if $(\widetilde{X} Z_n) \in L^1$, and
- $\int_0^1 |q_Z(s) q_X(s)| ds < \infty$ if and only if $\int_0^1 |q_{Z_n}(s)||q_X(s)| ds < \infty$.

Furthermore, we observe that Z_n converges to Z $\mathbb{P}$-a.s. and in L^1. So in particular, the respective quantile functions converge almost everywhere. Therefore, the sequence $(q_X q_{Z_n})_{n \in \mathbb{N}}$ converges almost everywhere to the integrable function $q_X q_Z$, and we have $|q_X q_{Z_n}| \le |q_X q_Z|$. Consequently, the dominated convergence theorem in combination with step 1 yields

$$\begin{aligned}
\int_0^1 q_X(s) q_Z(s)\, ds &= \lim_{n \to \infty} \int_0^1 q_X(s) q_{Z_n}(s)\, ds \\
&= \lim_{n \to \infty} \sup_{\widetilde{X} \sim X} E[\widetilde{X} Z_n] = \sup_{\widetilde{X} \sim X} E[\widetilde{X} Z]
\end{aligned}$$

where the last equality follows from

$$|E[\widetilde{X} Z_n] - E[\widetilde{X} Z]| \le \frac{1}{n} \|X\|_1 \quad \text{for all } \widetilde{X} \sim X \text{ such that } \widetilde{X} Z \in L^1.$$

Hence, (i) is proved. In order to prove (ii) let

$$\overline{Z} := Z + \sum_{z \in D} \operatorname{sgn}(z) U_z 1_{A_z}$$

and note that $\overline{Z}$ has a continuous distribution and $Z^{\pm} \le \overline{Z}^{\pm} \le Z^{\pm} + 1$. Hence, for all $\widetilde{X} \sim X$ we have $(\widetilde{X} Z) \in L^1$ if and only if $(\widetilde{X}\overline{Z}) \in L^1$, and

$$\int_0^1 |q_Z(s) q_X(s)| ds \le \int_0^1 |q_{\overline{Z}}(s) q_X(s)| ds$$

which, in view of step 1, completes the proof. □

References

[1] Acciaio, B. (2007), Optimal Risk Sharing with Non-monotone Monetary Functionals, *Finance and Stochastics*, 11, 267–289.

[2] Aliprantis, C.D. and Border, K.C. (1999), *Infinite Dimensional Analysis*, Springer.

[3] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D. (1999), Coherent measures of risk, *Mathematical Finance*, 9, 203–228.

[4] Barrieu, P. and El Karoui, N. (2005), Inf-convolution of Risk Measures and Optimal Risk Transfer, *Finance and Stochastics*, 9, 269–298.

[5] Biagini, S. and Frittelli, M. (2008), A unified framework for utility maximization problems: an Orlicz space approach, *Annals of Applied Probability*, Vol. 18/3, pp. 929–966.

[6] Cheridito, P. and Li, T. (2009) Risk Measures on Orlicz Hearts, *Mathematical Finance*, 19, 189-214.

[7] Cherny, A. and Kupper, M. (2008), Divergence Utilities, preprint.

[8] Delbaen, F. (2000), *Coherent Risk Measures*, Cattedra Galileiana. Scuola Normale Superiore di Pisa.

[9] Delbaen, F. (2002), Coherent Risk Measures on General Probability Spaces, *Advances in Finance and Stochastics*, 1–37, Springer.

[10] Delbaen, F. (2009), Risk Measures for Non-integrable Random Variables, *Mathematical Finance*, 19, 329-333

[11] Dana, R-A. (2005), A Representation Result for Concave Schur Concave Functions, *Mathematical Finance*, 15, 613–634.

[12] Dunford, N. and Schwartz, J. (1976), *Linear Operators. Part I: General theory*, Interscience Publishers.

[13] Ekeland, I. and Témam, R. (1999), *Convex Analysis and Variational Problems, Chapter I*, SIAM.

[14] Filipović, D. and Kupper, M. (2007), Monotone and Cash-Invariant Convex Functions and Hulls, *Insurance: Mathematics and Economics*, 41, 1–16.

[15] Filipović, D. and Kupper, M. (2008), Equilibrium Prices for Monetary Utility Functions, *International Journal of Theoretical and Applied Finance*, 11, 325– 343.

[16] Filipović, D. and Svindland, G. (2008), Optimal Capital and Risk Allocations for Law- and Cash-Invariant Convex Functions, *Finance and Stochastics*, 12, 423–439.

[17] Filipović, D. and Svindland, G. (2008), The Canonical Model Space for Law-invariant Convex Risk Measures is L^1, *Mathematical Finance*, forthcoming.

[18] Föllmer, H. and Schied, A. (2004), *Stochastic Finance, An Introduction in Discrete Time, 2nd Edition*, de Gruyter Studies in Mathematics 27.

[19] Jouini, E., Schachermayer, W. and Touzi, N. (2008), Optimal Risk Sharing for Law Invariant Monetary Utility Functions, *Mathematical Finance*, 18, 269–292.

[20] Jouini, E., Schachermayer, W. and Touzi, N. (2006), Law-Invariant Risk Measures have the Fatou Property, *Advances in Mathematical Economics*, 9, 49–71.

[21] Krätschmer, V. (2007), On σ-additive robust representation of convex risk measures for unbounded financial positions in the presence of uncertainty about the market model, SFB 649 Discussion Paper 2007-010, Humboldt University Berlin.

[22] Lorentz, G.G. (1961), Relations Between Function Spaces, Proc. A.M.S, 12, 127–132.

[23] Rao, M.M. and Ren, Z.D. (1991), *Theory of Orlicz Spaces*, Marcel Dekker, Inc.

[24] Rockafellar, R. T. (1997), *Convex Analysis*, Princeton University Press.

[25] Rohlin, V.A. (1962), On the Fundamental Ideas of Measure Theory, *American Mathematical Society: Translations, Series 1*, Vol. 10, 1–54.

[26] Royden, H. L. (1968), *Real Analysis*, Macmillan Publishing, second edition.

[27] Ruszczyński, A. and Shapiro, A. (2006), Optimization of Convex Risk Functions, *Math. Opers. Res.*, 31, 433–451.

[28] Svindland, G. (2008), *Convex Risk Measures Beyond Bounded Risks*, phd-thesis, Ludwig-Maximilians-University Munich.

[29] Svindland, G. (2010), Continuity Properties of Law-Invariant (Quasi-) Convex Risk Functions, *Mathematics and Financial Economics*, forthcoming.

[30] Werner, D. (2002), *Funktionalanalysis*, Springer, 4. Auflage.