Robust preferences and robust portfolio choice

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November 25, 2007

Introduction

Financial markets offer a variety of financial positions. The net result of such a position at the end of the trading period is uncertain, and it may thus be viewed as a real-valued function X on the set of possible scenarios. The problem of portfolio choice consists in choosing, among all the available positions, a position which is affordable given the investor's wealth w, and which is optimal with respect to the investor's preferences.

In its classical form, the problem of portfolio choice involves preferences of von Neumann-Morgenstern type, and a position X is affordable if its price does not exceed the initial capital w. More precisely, preferences are described by a utility functional $E_Q[\,U(X)\,]$, where U is a concave utility function, and where Q is a probability measure on the set of scenarios which models the investor's expectations. The price of a position X is of the form $E^*[\,X\,]$ where P^* is a probability measure equivalent to Q. In this classical case, the optimal solution can be computed explicitly in terms of U, Q, and P^*. Recent research on the problem of portfolio choice has taken a much wider scope. On the one hand, the increasing role of derivatives and of dynamic hedging strategies has led to a more flexible notion of affordability. On the other hand, there is nowadays a much higher awareness of model uncertainty, and this has led to a robust formulation of preferences beyond the von Neuman-Morgenstern paradigm of expected utility.

In Chapter I we review the theory of robust preferences as developed by SCHMEIDLER [1989], GILBOA and SCHMEIDLER [1989], and MACCHERONI et al. [2006]. Such preferences admit a numerical representation in terms of utility functionals $\mathcal{U}$ of the form

$$\mathcal{U}(X) = \inf_{Q\in\mathcal{Q}} \big(E_Q[\,U(X)\,] + \gamma(Q)\big). \tag{0.1}$$

This may be viewed as a *robust* approach to the problem of model uncertainty: The agent considers a whole class of probabilistic models specified by probability measures Q on the given set of scenarios, but different models Q are taken more or less seriously, and this is made precise in terms of the penalty $\gamma(Q)$. In evaluating a given financial position, the agent then takes a *worst case approach* by taking the infimum of expected utilities over the suitably penalized models. There is an obvious analogy between such robust utility functionals and convex risk measures. In fact we show in Section 3 how the representation (1) of preferences which are characterized in terms of a robust extension of the von Neumann-Morgenstern axioms can be reduced to the robust representation of convex risk measures. This is the reason why we begin this chapter with a brief review of the basic properties of convex risk measures.

Suppose that the underlying financial market is modeled by a multi-dimensional semimartingale which describes the price fluctuation of a number of liquid assets. Affordability of a position X given the investor's wealth w can then be defined by the existence of some dynamic trading strategy such that the value of the portfolio generated from the initial capital w up to the final time T is at least equal to X. This is equivalent to the constraint

$$\sup_{P^*\in\mathcal{P}} E^*[\,X\,] \le w,$$

where $\mathcal{P}$ denotes the class of equivalent martingale measures. If the preferences of the investor are given by a robust utility functional of the form (0.1) then the problem of optimal portfolio choice involves the two classes of probability measures $\mathcal{P}$ and $\mathcal{Q}$. In many situations, the solution will consist in identifying two measures $\widehat{P}^* \in \mathcal{P}$ and $\widehat{Q} \in \mathcal{Q}$ such that the solution of the robust problem is given by the solution of the classical problem defined in terms of U, $\widehat{P}^*$, and $\widehat{Q}$.

In Chapter II we consider several approaches to the optimal investment problem for an economic agent who uses a robust utility functional (0.1) and who can choose between risky and riskless investment opportunities in a financial market. In Section 4 we formulate the corresponding optimal investment problem in a general setup and introduce standing assumptions for the subsequent sections. In Section 5 we show how methods from robust statistics can be used to obtain explicit solutions in a complete market model when the robust utility functional is coherent, i.e., the penalty function γ in (0.1) takes only the values 0 and ∞. The relations of this approach to capacity theory are analyzed in Section 6, together with several concrete examples. In Section 7 we develop the general duality theory for robust utility maximization. These duality techniques are then applied in Section 8, where optimal investment strategies for incomplete stochastic factor models are characterized in terms of the unique classical solutions of quasilinear partial differential equations. Instead of this analytical approach one can also use backward stochastic differential equations to characterize optimal strategies, and this technique is briefly discussed in Section 9.

In Chapter III we discuss the problem of portfolio optimization under risk constraints. These constraints have a robust representation, if they are formulated in terms of convex risk measures. Research on optimization problems under risk constraints provides a further perspective on risk measures which are used to regulate financial institutions. The axiomatic theory of risk measures does not take into account their impact on the behavior of financial agents who are subject to regulation and thus does not capture the effect of capital requirements on portfolios, market prices, and volatility. In order to deal with such issues, we discuss static and semi-dynamic risk constraints in an equilibrium setting. In Section 10 we analyze the corresponding partial equilibrium problem. The general equilibrium is discussed in Section 11. The literature on portfolio choice under risk constraints is currently far from complete, and we point to some directions for future research.

I. Robust preferences and monetary risk measures

The goal of this chapter is to characterize investor preferences that are robust in the sense that they account for uncertainty in the underlying models. The main results are presented in Section 3. There it is shown in particular that robust preferences can numerically be represented in terms of robust utility functionals, which involve concave monetary utility functionals. We therefore first provide two preliminary sections on concave monetary utility functionals and convex risk measures. In Section 1 we discuss the dual representation theory in terms of the penalty function of a concave monetary utility functional. In Section 2 we present some standard examples of concave monetary utility functionals.

1 Risk measures and monetary utility functionals

In this section we briefly recall the basic definitions and properties of convex risk measures and monetary utility functionals. We refer to Chapter 4 of FÖLLMER and SCHIED [2004] for a more comprehensive account.

One of the basic tasks in finance is to quantify the risk associated with a given financial position, which is subject to uncertainty. Let Ω be a fixed set of scenarios. The profits and losses (P&L) of such a financial position are described by a mapping $X : \Omega \longrightarrow \mathbb{R}$ where $X(\omega)$ is the discounted net worth of the position at the end of the trading period if the scenario $\omega \in \Omega$ is realized. The goal is to determine a real number $\rho(X)$ that quantifies the risk and can serve as a capital requirement, i.e., as the minimal amount of capital which, if added to the position and invested in a risk-free manner, makes the position acceptable. The following axiomatic approach to such risk measures was initiated in the coherent case by ARTZNER et al. [1990] and later independently extended to the class of convex risk measures by HEATH [2000], FÖLLMER and SCHIED [2002a], and FRITTELLI and ROSAZZA GIANIN [2002].

Definition 1.1. Let $\mathcal{X}$ be a linear space of bounded functions containing the constants. A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is called a *convex risk measure* if it satisfies the following conditions for all $X, Y \in \mathcal{X}$.

- *Monotonicity:* If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- *Cash invariance:* If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.
- *Convexity:* $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$.

The convex risk measure ρ is called a *coherent risk measure* if it satisfies the condition of

- *Positive homogeneity:* If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda\rho(X)$.

The financial meaning of monotonicity is clear. Cash invariance is also called *translation invariance*. It is the basis for the interpretation of $\rho(X)$ as a capital requirement: if the amount m is added to the position and invested in a risk-free manner, the capital requirement is reduced by the same amount. In particular, cash invariance implies $\rho\big(X+\rho(X)\big)=0$, i.e., the accumulate position consisting of X and the risk-free investment $\rho(X)$ is acceptable. While the axiom of cash invariance is best understood in its relation to the interpretation of $\rho(X)$ as a capital requirement for X, it is often convenient to reverse signs and to put emphasis on the *utility* of a position rather than on its risk. This leads to the following concept.

Definition 1.2. A mapping $\phi : \mathcal{X} \to \mathbb{R}$ is called a *concave monetary utility functional* if $\rho(X) := -\phi(X)$ is a convex risk measure. If ρ is coherent then ϕ is called a *coherent monetary utility functional*.

We now assume that P&Ls are described by random variables X on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. More precisely, we consider the case in which $\mathcal{X} = L^\infty$, where for $0 \le p \le \infty$ we denote by L^p the space $L^p(\Omega, \mathcal{F}, \mathbb{P})$. This choice implicitly assumes that concave monetary utility functionals respect $\mathbb{P}$-nullsets in the sense that

$$\phi(X) = \phi(Y) \text{ whenever } X = Y \ \mathbb{P}\text{-a.s.} \tag{1.1}$$

Definition 1.3. The *minimal penalty function* of the concave monetary utility functional ϕ is given for probability measures $Q \ll \mathbb{P}$ by

$$\gamma(Q) := \sup_{X \in L^\infty} \big(\phi(X) - E_Q[\,X\,]\big). \tag{1.2}$$

The following theorem was obtained by DELBAEN [2002] in the coherent case and later extended by FÖLLMER and SCHIED [2002a] to the general concave case. It provides the basic representation for concave monetary utility functionals in terms of probability measures under the condition that certain continuity properties are satisfied. Without these continuity properties one only gets a representation in terms of finitely additive probability measures; see FÖLLMER and SCHIED [2004, Section 4.2].

Theorem 1.1. *For a concave monetary utility functional ϕ with minimal penalty function γ, the following conditions are equivalent.*

(i) For $X \in L^\infty$

$$\phi(X) = \inf_{Q \ll \mathbb{P}} \big(E_Q[\,X\,] + \gamma(Q)\big). \tag{1.3}$$

(ii) ϕ is continuous from above*: if $X_n \searrow X$ $\mathbb{P}$-a.s., then $\phi(X_n) \searrow \phi(X)$.*

(iii) ϕ has the Fatou property*: for any bounded sequence $(X_n) \subset L^\infty$ that converges in probability to some X, we have $\phi(X) \ge \limsup_n \phi(X_n)$.*

Moreover, under these conditions, ϕ is coherent if and only if γ takes only the values 0 *and* ∞*. In this case,* (1.3) *becomes*

$$\phi(X) = \inf_{Q \in \mathcal{Q}} E_Q[\,X\,], \qquad X \in L^\infty, \tag{1.4}$$

where $\mathcal{Q} = \{Q \ll \mathbb{P} \,|\, \gamma(Q) = 0\}$, *called* maximal representing set of ϕ, *is the maximal set of probability measures for which the representation* (1.4) *holds.*

Proof. See, for instance, FÖLLMER and SCHIED [2004, Theorem 4.31 and Corollary 4.34]. □

The theorem shows that every concave monetary utility functional that is continuous from above arises in the following manner. We consider any probabilistic model $Q \ll \mathbb{P}$, but these models are taken more or less seriously according to the size of the penalty $\gamma(Q)$. Thus, the value $\phi(X)$ is computed as the worst-case expectation taken over all models $Q \ll \mathbb{P}$ and penalized by $\gamma(Q)$.

Theorem 1.2. *For a concave monetary utility functional ϕ with minimal penalty function γ, the following conditions are equivalent.*

(i) For any $X \in L^\infty$,

$$\phi(X) = \min_{Q \ll \mathbb{P}} \left(E_Q[\, X\,] + \gamma(Q) \right), \tag{1.5}$$

where the minimum is attained in some $Q \ll \mathbb{P}$.

(ii) ϕ is continuous from below*: if $X_n \nearrow X$ $\mathbb{P}$-a.s., then $\phi(X_n) \nearrow \phi(X)$.*

(iii) ϕ has the Lebesgue property*: for any bounded sequence $(X_n) \subset L^\infty$ that converges in probability to some X, we have $\phi(X) = \lim_n \phi(X_n)$.*

(iv) For each $c \in \mathbb{R}$, the level set $\{dQ/d\mathbb{P} \,|\, \gamma(Q) \le c\}$ is weakly compact in $L^1(\mathbb{P})$.

Proof. The equivalence of (b) and (c) follows from FÖLLMER and SCHIED [2004, Remark 4.23]. That (b) implies (a) follows from FÖLLMER and SCHIED [2004, Proposition 4.21], and that (b) implies (d) follows from FÖLLMER and SCHIED [2004, Lemma 4.22] and the Dunford-Pettis theorem. The proof that (a) implies (d) relies on James' theorem as shown by DELBAEN [2002] in the coherent case. It was recently generalized to the general case by JOUINI et al. [2006]. See also JOUINI et al. [2006, Theorem 5.2] and KRÄTSCHMER [2005] for alternative proofs of the other implications. □

Remark 1.1. Note that it follows from the preceding theorems that continuity from below implies continuity from above. It can be shown that the condition of continuity from above is automatically satisfied if the underlying probability space is standard and ϕ is *law-invariant* in the sense that $\phi(X) = \phi(Y)$ whenever the $\mathbb{P}$-laws of X and Y coincide; see JOUINI et al. [2006]. Several examples for law-invariant concave monetary utility functionals are provided in the next section. Continuity from above also holds as soon as ϕ extends to a concave monetary utility functional on L^p for some $p \in [1, \infty[$; see CHERIDITO et al. [2004, Proposition 3.8].

2 Examples of monetary utility functionals

In this section, we briefly present some popular choices for concave monetary utility functionals on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. One of the best studied examples is the *entropic monetary utility functional*,

$$\phi_\theta^{\text{ent}}(X) = -\frac{1}{\theta}\log \mathbb{E}[\, e^{-\theta X}\,], \tag{2.1}$$

where θ is a positive constant. One easily checks that it satisfies the conditions of Definition 1.2. Moreover, ϕ_θ^{ent} is clearly continuous from below, so that it can be represented as in (1.5) by its minimal penalty function $\gamma_\theta^{\text{ent}}$. Due to standard duality results, this minimal penalty function is given by $\gamma_\theta^{\text{ent}}(Q) = \frac{1}{\theta}H(Q|\mathbb{P})$, where

$$H(Q|\mathbb{P}) = \sup_{X\in L^\infty}\big(E_Q[\,X\,] - \log \mathbb{E}[\,e^X\,]\big) = \mathbb{E}\Big[\,\frac{dQ}{d\mathbb{P}}\log\frac{dQ}{d\mathbb{P}}\,\Big]$$

is the relative entropy of $Q \ll \mathbb{P}$; see, e.g., FÖLLMER and SCHIED [2004, Sections 3.2 and 4.9].

More generally, let $U : \mathbb{R} \to \mathbb{R}$ be concave, increasing, and nonconstant and take x in the interior of $U(\mathbb{R})$. Then

$$\phi_U(X) := \sup\{\, m \in \mathbb{R} \,|\, \mathbb{E}[\,U(X - m)\,] \geq x\,\}, \qquad X \in L^\infty, \tag{2.2}$$

defines a concave monetary utility functional. When considering corresponding risk measure, $\rho := -\phi_U$, the emphasis is on losses rather than on utility, and so it is natural to consider instead of U the convex increasing *loss function* $\ell(x) := -U(-x)$. In terms of ρ, formula (2.2) then becomes

$$\rho(X) := \inf\{\, m \in \mathbb{R} \,|\, \mathbb{E}[\,\ell(-X - m)\,] \leq -x\,\}, \qquad X \in L^\infty. \tag{2.3}$$

The risk measure ρ is called *utility-based shortfall risk measure* and was introduced by FÖLLMER and SCHIED [2002a]. When choosing $U(x) = -e^{-\theta x}$ (or, equivalently, $\ell(x) = e^{\theta x}$), we obtain the entropic monetary utility functional (2.1) as a special case. It is easy to check that ϕ_U is always continuous from below and hence admits the representation (1.5) in terms of its minimal penalty function, which is given by

$$\gamma_U(Q) = \inf_{\lambda>0}\frac{1}{\lambda}\Big(x + \mathbb{E}\Big[\,\widetilde{U}\Big(\lambda\frac{dQ}{d\mathbb{P}}\Big)\,\Big]\Big), \qquad Q \ll \mathbb{P},$$

where $\widetilde{U}(y) = \sup_x(U(x) - xy)$ denotes the convex conjugate function of U; see FÖLLMER and SCHIED [2002a, Theorem 10] or FÖLLMER and SCHIED [2004, Theorem 4.106].

To introduce another closely related class of concave monetary utility functionals, let $g : [0,\infty[\to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous convex function satisfying $g(1) < \infty$ and the superlinear growth condition $g(x)/x \to +\infty$ as $x \uparrow \infty$. Associated to it is the *g-divergence*

$$I_g(Q|\mathbb{P}) := \mathbb{E}\Big[\,g\Big(\frac{dQ}{d\mathbb{P}}\Big)\,\Big], \qquad Q \ll \mathbb{P}, \tag{2.4}$$

as introduced by CSISZAR [1963, 1967]. The g-divergence $I_g(Q|\mathbb{P})$ can be interpreted as a statistical distance between the hypothetical model Q and the reference measure $\mathbb{P}$, so that taking

$\gamma_g(Q) := I_g(Q|\mathbb{P})$ is a natural choice for a penalty function. The level sets $\{dQ/d\mathbb{P} \,|\, I_g(Q|\mathbb{P}) \leq c\}$ are convex and weakly compact in $L^1(\mathbb{P})$, due to the superlinear growth condition. Hence, it follows that $I_g(Q|\mathbb{P})$ is indeed the minimal penalty function of the concave monetary utility functional

$$\phi_g(X) := \inf_{Q \ll \mathbb{P}} \big(E_Q[\,X\,] + I_g(Q|\mathbb{P})\big). \tag{2.5}$$

Moreover, weak compactness of the level sets guarantees that ϕ_g is continuous from below. One can show that ϕ_g satisfies the variational identity

$$\phi_g(X) = \sup_{z \in \mathbb{R}} \big(\mathbb{E}[\,U(X-z)\,] + z\big), \qquad X \in L^\infty, \tag{2.6}$$

where $U(x) = \inf_{z>0}(xz + g(z))$ is the concave conjugate function of g. This formula was obtained by BEN-TAL and TEBOULLE [1987] for $\mathbb{R}$-valued g and extended to the general case by BEN-TAL and TEBOULLE [2007] and SCHIED [2007a]; see also CHERNY and KUPPER [2007] for further properties. The resulting concave monetary utility functionals were called *optimized certainty equivalents* by BEN-TAL and TEBOULLE [2007]. Note that the particular choice $g(x) = x \log x$ corresponds to the relative entropy $I_g(Q|\mathbb{P}) = H(Q|\mathbb{P})$, and so ϕ_g coincides with the entropic monetary utility functional. Another important example is provided by taking $g(x) = 0$ for $x \leq \lambda^{-1}$ and $g(x) = \infty$ otherwise, so that the corresponding coherent monetary utility functional is given by

$$\phi_\lambda(X) := \inf_{Q \in \mathcal{Q}_\lambda} E_Q[\,X\,] \qquad \text{for} \qquad \mathcal{Q}_\lambda := \Big\{Q \ll \mathbb{P} \,\Big|\, \frac{dQ}{d\mathbb{P}} \leq \frac{1}{\lambda}\Big\}. \tag{2.7}$$

This shows that $-\phi_\lambda(X)$ is equal to the coherent risk measure *Average Value at Risk*, $AV@R_\lambda(X)$, which is also called *Expected Shortfall*, *Conditional Value at Risk*, or *Tail Value at Risk*. In this case, we have $U(x) = 0 \wedge x/\lambda$ and hence get the classical duality formula

$$AV@R_\lambda(X) = \frac{1}{\lambda} \inf_{z \in \mathbb{R}} \big(E[\,(z-X)^+\,] - \lambda z\big) \tag{2.8}$$

as a special case of (2.6).

All examples discussed so far in this section are *law-invariant* in the sense that $\phi(X) = \phi(Y)$ whenever the $\mathbb{P}$-laws of X and Y coincide. One can show that every law-invariant concave monetary utility functional ϕ on L^∞ can be represented in the following form:

$$\phi(X) = \inf_{\mu} \Big(\int_{(0,1]} \phi_\lambda(X)\,\mu(d\lambda) + \beta(\mu)\Big), \tag{2.9}$$

where the supremum is taken over all Borel probability measures μ on $]0,1]$, ϕ_λ is as in (2.7), and $\beta(\mu)$ is a penalty for μ. Under the additional assumption of continuity from above, this representation was obtained in the coherent case by KUSUOKA [2001] and later extended by KUNZE [2003], DANA [2005], FRITTELLI and ROSAZZA-GIANIN [2005], and FÖLLMER and SCHIED [2004, Section 4.5]. More recently, JOUINI et al. [2006] showed that the condition of continuity from above can actually be dropped. More examples of concave and coherent monetary utility functionals will be provided at the beginning of Section 8.

3 Robust preferences and their numerical representation

In this section we describe how robust utility functionals appear naturally as numerical representations of investor preferences in the face of model uncertainty as developed by SCHMEIDLER [1989], GILBOA and SCHMEIDLER [1989], and MACCHERONI et al. [2006].

The general aim of the theory of choice is to provide an axiomatic foundation and corresponding representation theory for a normative decision rule by means of which an economic agent can reach decisions when presented with several alternatives. A fundamental example is the von Neumann-Morgenstern theory, in which the agent can choose between several monetary bets with known success probabilities. Such a monetary bet can be regarded as a Borel probability measure on $\mathbb{R}$ and is often called a *lottery*. More specifically, we will consider here the space $\mathcal{M}_{1,c}(S)$ of Borel probability measures with compact support in some given nonempty interval $S \subset \mathbb{R}$. The decision rule is usually taken as a *preference relation* or *preference order* $\succ$ on $\mathcal{M}_{1,c}(S)$, i.e., $\succ$ is a binary relation on $\mathcal{M}_{1,c}(S)$ that is *asymmetric*,

$$\mu \succ \nu \qquad \Rightarrow \qquad \nu \not\succ \mu,$$

and *negatively transitive*,

$$\mu \succ \nu \text{ and } \lambda \in \mathcal{M}_{1,c}(S) \qquad \Rightarrow \qquad \mu \succ \lambda \text{ or } \lambda \succ \nu.$$

The corresponding *weak preference order*, $\mu \succeq \nu$, is defined as the negation of $\nu \succ \mu$. If both $\mu \succeq \nu$ and $\nu \succeq \mu$ hold, we will write $\mu \overset{\circ}{\sim} \nu$. Dealing with a preference order is greatly facilitated if one has a *numerical representation*, i.e., a function $\mathcal{U} : \mathcal{M}_{1,c}(S) \to \mathbb{R}$ such that

$$\mu \succ \nu \qquad \Longleftrightarrow \qquad \mathcal{U}(\mu) > \mathcal{U}(\nu).$$

VON NEUMANN and MORGENSTERN [1944] formulated a set of axioms that are necessary and sufficient for the existence of a numerical representation $\mathcal{U}$ of *von Neumann-Morgenstern form*, that is,

$$\mathcal{U}(\mu) = \int U(x)\,\mu(dx) \tag{3.1}$$

for a function $U : \mathbb{R} \to \mathbb{R}$. The two main axioms are:

• *Archimedean axiom*: for any triple $\mu \succ \lambda \succ \nu$ in $\mathcal{M}_{1,c}(S)$, there are α, $\beta \in]0,1[$ such that $\alpha\mu + (1-\alpha)\nu \succ \lambda \succ \beta\mu + (1-\beta)\nu$.

• *Independence axiom*: for all μ, $\nu \in \mathcal{M}_{1,c}(S)$, the relation $\mu \succ \nu$ implies $\alpha\mu + (1-\alpha)\lambda \succ \alpha\nu + (1-\alpha)\lambda$ for all $\lambda \in \mathcal{M}_{1,c}(S)$ and all $\alpha \in]0,1]$.

These two axioms are equivalent to the existence of an affine numerical representation $\mathcal{U}$. To obtain an integral representation (3.1) for this affine functional on $\mathcal{M}_{1,c}(S)$ one needs some additional regularity condition such as monotonicity with respect to first-order stochastic dominance or topological assumptions on the level sets of $\succ$; see, e.g., KREPS [1988] and FÖLLMER and SCHIED [2004, Chapter 2]. See also HERSTEIN and MILNOR [1953] for a relaxation of these axioms in a generalized setting.

The monetary character of lotteries suggests the further requirement that $\delta_x \succ \delta_y$ for $x > y$, which is equivalent to the fact that U is strictly increasing. In addition, the preference order is called *risk-averse* if for every nontrivial lottery $\mu \in \mathcal{M}_{1,c}(S)$ the certain amount $m(\mu) := \int x\,\mu(dx)$ is strictly preferred over the lottery μ itself, that is, $\delta_{m(\mu)} \succ \mu$. Clearly, risk aversion is equivalent to the fact that U is strictly concave. If U is both increasing and strictly concave, it is called a *utility function*.

In the presence of *model uncertainty* or *ambiguity*, sometimes also called *Knightian uncertainty*, the economic agent only has imperfect knowledge of the success probabilities of a financial bet. Mathematically, such a situation is often modeled by making lotteries contingent on some external source of randomness. Thus, let $(\Omega, \mathcal{F})$ be a given measurable space and consider the class $\widetilde{\mathcal{X}}$ defined as the set of all Markov kernels $\widetilde{X}(\omega, dy)$ from $(\Omega, \mathcal{F})$ to S for which there exists a compact set $K \subset S$ such that $\widetilde{X}(\omega, K) = 1 \quad$ for all $\omega \in \Omega$. In mathematical economics, the elements of $\widetilde{\mathcal{X}}$ are sometimes called *acts* or *horse race lotteries*.

Now consider a given preference order $\succ$ on $\widetilde{\mathcal{X}}$. The space of standard lotteries, $\mathcal{M}_{1,c}(S)$, has a natural embedding into $\widetilde{\mathcal{X}}$ via the identification of $\mu \in \mathcal{M}_{1,c}(S)$ with the constant Markov kernel $\widetilde{X}(\omega) = \mu$, and this embedding induces a preference order on $\mathcal{M}_{1,c}(S)$, which we also denote by $\succ$. We assume that the preference order on $\widetilde{\mathcal{X}}$ is *monotone* with respect to the embedding of $\mathcal{M}_{1,c}(S)$ into $\widetilde{\mathcal{X}}$:

$$\widetilde{Y} \succeq \widetilde{X} \quad \text{if } \widetilde{Y}(\omega) \succeq \widetilde{X}(\omega) \text{ for all } \omega \in \Omega. \tag{3.2}$$

We will furthermore assume the following three axioms, of which the first two are suitable extensions of the two main axioms of classical von Neumann-Morgenstern theory to our present setting.

• *Archimedean axiom:* if $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \widetilde{\mathcal{X}}$ are such that $\widetilde{Z} \succ \widetilde{Y} \succ \widetilde{X}$, then there are $\alpha, \beta \in]0,1[$ with

$$\alpha\widetilde{Z} + (1-\alpha)\widetilde{X} \succ \widetilde{Y} \succ \beta\widetilde{Z} + (1-\beta)\widetilde{X}.$$

• *Weak certainty independence:* if for $\widetilde{X}, \widetilde{Y} \in \widetilde{\mathcal{X}}$ and for some $\nu \in \mathcal{M}_{1,c}(S)$ and $\alpha \in]0,1]$ we have $\alpha\widetilde{X} + (1-\alpha)\nu \succ \alpha\widetilde{Y} + (1-\alpha)\nu$, then

$$\alpha\widetilde{X} + (1-\alpha)\mu \succ \alpha\widetilde{Y} + (1-\alpha)\mu \qquad \text{for all } \mu \in \mathcal{M}_{1,c}(S).$$

• *Uncertainty aversion:* if $\widetilde{X}, \widetilde{Y} \in \widetilde{\mathcal{X}}$ are such that $\widetilde{X} \overset{\circ}{\sim} \widetilde{Y}$, then

$$\alpha\widetilde{X} + (1-\alpha)\widetilde{Y} \succeq \widetilde{X} \quad \text{for all } \alpha \in [0,1].$$

These axioms were formulated by GILBOA and SCHMEIDLER [1989], with the exception that instead of weak certainty independence they originally used the stronger concept of full certainty independence, which we will explain below. The relaxation of full certainty independence to weak certainty independence was suggested by MACCHERONI et al. [2006].

Remark 3.1. In order to motivate the term "uncertainty aversion", consider the following simple example. For $\Omega := \{0,1\}$ define

$$\widetilde{Z}_i(\omega) := \delta_{1000} \cdot \mathbb{1}_{\{i\}}(\omega) + \delta_0 \cdot \mathbb{1}_{\{1-i\}}(\omega), \quad i = 0,1.$$

Suppose that an agent is indifferent between the choices $\widetilde{Z}_0$ and $\widetilde{Z}_1$, which both involve the same kind of uncertainty. In the case of uncertainty aversion, the convex combination $\widetilde{Y} := \alpha\widetilde{Z}_0 + (1-\alpha)\widetilde{Z}_1$ is weakly preferred over both $\widetilde{Z}_0$ and $\widetilde{Z}_1$. It takes the form

$$\widetilde{Y}(\omega) = \begin{cases} \alpha\,\delta_{1000} + (1-\alpha)\delta_0 & \text{for } \omega = 1, \\ \alpha\,\delta_0 + (1-\alpha)\delta_{1000} & \text{for } \omega = 0. \end{cases}$$

This convex combination now allows for upper and lower probability bounds in terms of α, and this means that model uncertainty is reduced in favor of risk. For $\alpha = 1/2$, the resulting lottery $\widetilde{Y}(\omega) \equiv \frac{1}{2}(\delta_{1000} + \delta_0)$ is independent of the scenario ω, i.e., model uncertainty is completely eliminated.

Remark 3.2. The Archimedean axiom and weak certainty independence imply that the restriction of $\succ$ to $\mathcal{M}_{1,c}(S)$ satisfies the independence axiom of von Neumann-Morgenstern theory and hence admits an affine numerical representation $\mathcal{U} : \mathcal{M}_{1,c}(S) \to \mathbb{R}$.

Proof. We need to derive the independence axiom on $\mathcal{M}_{1,c}(S)$. To this end, take $\nu \in \mathcal{M}_{1,c}(S)$ such that $\mu \succ \nu$. We claim that $\mu \succeq \frac{1}{2}\mu + \frac{1}{2}\nu \succeq \nu$. Indeed, otherwise we would, for instance, have that $\frac{1}{2}\mu + \frac{1}{2}\nu \succ \mu = \frac{1}{2}\mu + \frac{1}{2}\mu$. Weak certainty independence now yields $\frac{1}{2}\nu + \frac{1}{2}\nu \succ \frac{1}{2}\nu + \frac{1}{2}\mu$ and in turn $\nu \succ \mu$, a contradiction.

Iterating the preceding argument now yields $\mu \succeq \alpha\mu + (1-\alpha)\nu \succeq \nu$ for every dyadic rational number $\alpha \in [0,1]$. Applying the Archimedean axiom completes the proof. $\square$

In addition to the axioms listed above, we assume henceforth that the affine numerical representation $\mathcal{U} : \mathcal{M}_{1,c}(S) \to \mathbb{R}$ of Remark 3.2 is actually of von Neumann-Morgenstern form (3.1) for some function $U : S \to \mathbb{R}$. For simplicity, we also assume that U is a utility function with unbounded range $U(S)$ containing zero. The following theorem is an extension of the main result in GILBOA and SCHMEIDLER [1989]. In this form, it is due to MACCHERONI et al. [2006].

Theorem 3.1. *Under the above conditions, there exists a unique extension of $\mathcal{U}$ to a numerical representation $\widetilde{\mathcal{U}} : \mathcal{X} \to \mathbb{R}$, and $\widetilde{\mathcal{U}}$ is of the form*

$$\widetilde{\mathcal{U}}(\widetilde{X}) = \phi(\mathcal{U}(\widetilde{X})) = \phi\Big(\int U(x)\,\widetilde{X}(\cdot, dx)\Big)$$

for a concave monetary utility functional ϕ defined on the space of bounded measurable functions on $(\Omega, \mathcal{F})$.

Proof. The proof is a variant of the original proofs in GILBOA and SCHMEIDLER [1989] and MACCHERONI et al. [2006].

Step 1: We prove that there exists a unique extension of $\mathcal{U}$ to a numerical representation $\widetilde{\mathcal{U}} : \mathcal{X} \to \mathbb{R}$. By definition of $\widetilde{\mathcal{X}}$, for every $\widetilde{X} \in \widetilde{\mathcal{X}}$ there exists some real number a such that $[-a, a] \subset S$ and $\widetilde{X}(\omega, [-a, a]) = 1$ for all ω. Monotonicity (3.2) and the fact that U is strictly increasing thus imply $\delta_a \succeq \widetilde{X} \succeq \delta_{-a}$. Standard arguments hence yield the existence of a unique

$\alpha \in [0,1]$ such that $\widetilde{X} \overset{\circ}{\sim} \alpha\delta_a + (1-\alpha)\delta_{-a}$; see, e.g., FÖLLMER and SCHIED [2004, Lemma 2.83]. It follows that

$$\widetilde{\mathcal{U}}(\widetilde{X}) := \widetilde{\mathcal{U}}(\alpha\delta_a + (1-\alpha)\delta_{-a}) = \alpha U(a) + (1-\alpha)U(-a)$$

must be the desired numerical representation.

Step 2: For $\mu \in \mathcal{M}_{1,c}(S)$ let $c(\mu) := U^{-1}(\mathcal{U}(\mu))$ denote the corresponding certainty equivalent. For $\widetilde{X} \in \widetilde{\mathcal{X}}$, monotonicity (3.2) then implies that $\widetilde{X} \overset{\circ}{\sim} \delta_{c(\widetilde{X})}$, where $c(\widetilde{X})$ is the bounded S-valued measurable function defined as the ω-wise certainty equivalent of $\widetilde{X}$. It is therefore enough to show that there exists a concave monetary utility functional ϕ such that

$$\widetilde{\mathcal{U}}(\delta_X) = \phi(U(X)) \tag{3.3}$$

for every S-valued measurable function X with compact range. Indeed, for $\widetilde{X} \in \widetilde{\mathcal{X}}$ we then have

$$\widetilde{\mathcal{U}}(\widetilde{X}) = \widetilde{\mathcal{U}}(\delta_{c(\widetilde{X})}) = \phi(U(c(\widetilde{X}))) = \phi(\mathcal{U}(\widetilde{X})). \tag{3.4}$$

Step 3: We now show that there exists a concave monetary utility functional ϕ such that (3.3) holds for every S-valued measurable function X with compact range. We first note that (3.3) uniquely defines a functional ϕ on the set $\mathcal{X}_U$ of bounded $U(S)$-valued measurable functions. Moreover, ϕ is monotone due to our monotonicity assumptions. We now prove that ϕ satisfies the translation property on $\mathcal{X}_U$. To this end, we first assume that $U(S) = \mathbb{R}$ and take a bounded measurable function X and some $z \in \mathbb{R}$. We then let $X_0 := U^{-1}(2X)$, $z_0 := U^{-1}(2z)$, and $y := U^{-1}(0)$. Taking a such that $a \geq X_0(\omega) \geq -a$ for each ω, we see as in Step 1 that there exists $\beta \in [0,1]$ such that

$$\frac{1}{2}\delta_{X_0} + \frac{1}{2}\delta_y \overset{\circ}{\sim} \frac{\beta}{2}(\delta_a + \delta_y) + \frac{1-\beta}{2}(\delta_{-a} + \delta_y) = \frac{1}{2}\mu + \frac{1}{2}\delta_y,$$

where $\mu = \beta\delta_a + (1-\beta)\delta_{-a}$. Using weak certainty independence, we may replace δ_y by δ_{z_0} and obtain $\frac{1}{2}(\delta_{X_0} + \delta_{z_0}) \overset{\circ}{\sim} \frac{1}{2}(\mu + \delta_{z_0})$. Hence, by using (3.4),

$$\begin{aligned}
\phi(X+z) &= \phi\Big(\frac{1}{2}U(X_0) + \frac{1}{2}U(z_0)\Big) = \phi\Big(\mathcal{U}\Big(\frac{1}{2}\delta_{X_0} + \frac{1}{2}\delta_{z_0}\Big)\Big) \\
&= \phi\Big(\mathcal{U}\Big(\frac{1}{2}\mu + \frac{1}{2}\delta_{z_0}\Big)\Big) = \widetilde{\mathcal{U}}\Big(\frac{1}{2}\mu + \frac{1}{2}\delta_{z_0}\Big) = \mathcal{U}\Big(\frac{1}{2}\mu + \frac{1}{2}\delta_{z_0}\Big) \\
&= \frac{1}{2}\,\mathcal{U}(\mu) + \frac{1}{2}U(z_0).
\end{aligned}$$

The translation property now follows from $U(z_0) = 2z$ and the fact that

$$\begin{aligned}
\frac{1}{2}\mathcal{U}(\mu) &= \frac{1}{2}(\mathcal{U}(\mu) + \mathcal{U}(\delta_y)) = \mathcal{U}\Big(\frac{1}{2}\mu + \frac{1}{2}\delta_y\Big) = \widetilde{\mathcal{U}}\Big(\frac{1}{2}\mu + \frac{1}{2}\delta_y\Big) \\
&= \widetilde{\mathcal{U}}\Big(\frac{1}{2}\delta_{X_0} + \frac{1}{2}\delta_y\Big) = \phi\Big(\frac{1}{2}U(X_0) + \frac{1}{2}U(y)\Big) = \phi(X).
\end{aligned}$$

Here we have again applied (3.4). If $U(S)$ is not equal to $\mathbb{R}$ it is sufficient to consider the cases in which $U(S)$ contains $[0,\infty[$ or $]-\infty,0]$ and to work with positive or negative quantities X

and z, respectively. Then the preceding argument establishes the translation property of ϕ on the spaces of positive or negative bounded measurable functions, and ϕ can be extended by translation to the entire space of bounded measurable functions.

We now prove the concavity of ϕ by showing $\phi(\frac{1}{2}(X+Y)) \geq \frac{1}{2}\phi(X) + \frac{1}{2}\phi(Y)$. This is enough since ϕ is Lipschitz continuous due to monotonicity and the translation property. Let $X_0 := U^{-1}(X)$ and $Y_0 := U^{-1}(Y)$ and suppose first that $\phi(X) = \phi(Y)$. Then $\delta_{X_0} \overset{\circ}{\sim} \delta_{Y_0}$ and uncertainty aversion implies that $\widetilde{Z} := \frac{1}{2}\delta_{X_0} + \frac{1}{2}\delta_{Y_0} \succeq \delta_{X_0}$. Hence, by using (3.4) we get

$$\phi\Big(\frac{1}{2}(X+Y)\Big) = \widetilde{\mathcal{U}}(\widetilde{Z}) \geq \widetilde{\mathcal{U}}(\delta_{X_0}) = \phi(X) = \frac{1}{2}\phi(X) + \frac{1}{2}\phi(Y).$$

If $\phi(X) \neq \phi(Y)$, then we let $z := \phi(X) - \phi(Y)$ so that $Y_z := Y + z$ satisfies $\phi(Y_z) = \phi(X)$. Hence,

$$\phi\Big(\frac{1}{2}(X+Y)\Big) + \frac{1}{2}z = \phi\Big(\frac{1}{2}(X+Y)_z\Big) \geq \frac{1}{2}\phi(X) + \frac{1}{2}\phi(Y_z) = \frac{1}{2}\phi(X) + \frac{1}{2}\phi(Y) + \frac{1}{2}z.$$

□

Instead of weak certainty independence, GILBOA and SCHMEIDLER [1989] consider the stronger axiom of

- *full certainty independence:* for all $\widetilde{X}, \widetilde{Y} \in \widetilde{\mathcal{X}}$, $\mu \in \mathcal{M}_{1,c}(S)$, and $\alpha \in]0,1]$ we have

$$\widetilde{X} \succ \widetilde{Y} \quad \Longrightarrow \quad \alpha\widetilde{X} + (1-\alpha)\mu \succ \alpha\widetilde{Y} + (1-\alpha)\mu.$$

As we have seen in Remark 3.2 and its proof, the axioms of full and weak certainty independence extend the independence axiom for preferences on $\mathcal{M}_{1,c}(S)$ to our present setting, but only under the restriction that the replacing act is *certain*, i.e., it given by a lottery μ that does not depend on the scenario $\omega \in \Omega$. There are good reasons for *not* requiring full independence for all $\widetilde{Z} \in \widetilde{\mathcal{X}}$. As an example, take $\Omega = \{0,1\}$ and define $\widetilde{X}(\omega) = \delta_\omega$, $\widetilde{Y}(\omega) = \delta_{1-\omega}$, and $\widetilde{Z} = \widetilde{X}$. An agent may prefer $\widetilde{X}$ over $\widetilde{Y}$, thus expressing the implicit view that scenario 1 is somewhat more likely than scenario 0. At the same time, the agent may like the idea of *hedging* against the occurrence of scenario 0, and this could mean that the certain lottery

$$\frac{1}{2}(\,\widetilde{Y} + \widetilde{Z}\,) \equiv \frac{1}{2}(\delta_0 + \delta_1)$$

is preferred over the contingent lottery

$$\frac{1}{2}(\,\widetilde{X} + \widetilde{Z}\,) \equiv \widetilde{X},$$

thus violating the independence assumption in its unrestricted form. In general, the role of $\widetilde{Z}$ as a hedge against scenarios unfavorable for $\widetilde{Y}$ requires that $\widetilde{Y}$ and $\widetilde{Z}$ are *not* comonotone, where comonotonicity means:

$$\widetilde{Y}(\omega) \succeq \widetilde{Y}(\tilde{\omega}) \qquad \Longleftrightarrow \qquad \widetilde{Z}(\omega) \succeq \widetilde{Z}(\tilde{\omega}).$$

Thus, the wish to hedge would still be compatible with the following stronger version of certainty independence, called

• *comonotonic independence:* For $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z} \in \widetilde{\mathcal{X}}$ and $\alpha \in]0,1]$

$$\widetilde{X} \succ \widetilde{Y} \quad \Longleftrightarrow \quad \alpha\widetilde{X} + (1-\alpha)\widetilde{Z} \succ \alpha\widetilde{Y} + (1-\alpha)\widetilde{Z}$$

whenever $\widetilde{Y}$ and $\widetilde{Z}$ are comonotone.

Theorem 3.2. *In the setting of Theorem 3.1, full certainty independence holds if and only if ϕ is coherent. Moreover, comonotonic independence is equivalent to the fact that ϕ is* comonotonic, *i.e., $\phi(X+Y) = \phi(X) + \phi(Y)$ whenever X and Y are comonotone.*

Proof. See GILBOA and SCHMEIDLER [1989] and SCHMEIDLER [1989] or FÖLLMER and SCHIED [2004, Sections 2.5 and 4.7]. □

The representation theorem for concave monetary utility functionals, Theorem 1.1, suggests that ϕ from Theorem 3.1 typically admits a representation of the form

$$\phi(X) = \inf_{Q \in \mathcal{Q}} \Big(E_Q[\,X\,] + \gamma(Q) \Big),$$

for some set $\mathcal{Q}$ of probability measures on $(\Omega, \mathcal{F})$ and some penalty function $\gamma : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$. Then the restriction of $\succ$ to bounded measurable functions X on $(\Omega, \mathcal{F})$, regarded as elements of $\widetilde{\mathcal{X}}$ via the identification with δ_X, admits a numerical representation of the form

$$X \longmapsto \inf_{Q \in \mathcal{Q}} \Big(E_Q[\,U(X)\,] + \gamma(Q) \Big). \tag{3.5}$$

It is this representation in which we are really interested. Note, however, that it is necessary to formulate the axiom of uncertainty aversion on the larger space of uncertain lotteries. But even without its axiomatic foundation, the representation of preferences in the face of model uncertainty by a subjective utility assessment (3.5) is highly plausible as it stands. It may in fact be viewed as a *robust* approach to the problem of model uncertainty: The agent penalizes every possible probabilistic view $Q \in \mathcal{Q}$ in terms of the penalty $\gamma(Q)$ and takes a *worst case approach* in evaluating the payoff of a given financial position. The resulting preference structures for entropic penalties $\gamma_\theta^{\mathrm{ent}}(Q) = \frac{1}{\theta} H(Q|\mathbb{P})$ are sometimes called *multiplier preferences* in economics; see, e.g., HANSEN and SARGENT [2001] and MACCHERONI et al. [2006].

II. Robust portfolio choice

In this chapter we consider the optimal investment problem for an economic agent who is averse against both risk and ambiguity and who can choose between risky and riskless investment opportunities in a financial market. Payoffs generated by investment choices are modeled as random variables X defined on the probability space of some underlying market model. By the theory developed in the preceding chapter, it is natural to assume that the utility derived from such a payoff X is given by

$$\inf_{Q\in\mathcal{Q}}\Big(E_Q[\,U(X)\,]+\gamma(Q)\Big) \tag{3.6}$$

for a utility function U, a penalty function γ, and an appropriate set $\mathcal{Q}$ of probability measures. The goal of the investor is thus to maximize this expression over the class of achievable payoffs. If the penalty function vanishes on $\mathcal{Q}$, the expression (3.6) reduces to

$$\inf_{Q\in\mathcal{Q}} E_Q[\,U(X)\,], \tag{3.7}$$

which often greatly simplifies the complexity of the required mathematics. We will therefore often resort to this reduced setting and refer to it as the *coherent case*.

In the next section we formulate the corresponding optimal investment problem in a rather general setup, which we will specify later according to the particular requirements of each method. In the subsequent section we show how methods from robust statistics can be used to obtain explicit solutions for a class of coherent examples in a complete market model. The relations of this approach to capacity theory are analyzed in Section 6 along with several concrete examples. In Section 7 we develop the general duality theory for robust utility maximization. These duality techniques are then applied in Section 8, where optimal investment strategies for incomplete stochastic factor models are characterized in terms of the unique classical solutions of quasilinear partial differential equations. Instead of partial differential equations one can also use backward stochastic differential equations to characterize optimal strategies, and this approach is discussed in Section 9.

4 Problem formulation and standing assumptions

We start by describing the underlying financial market model. The discounted price process of d assets is modeled by a stochastic process $S=(S_t)_{0\leq t\leq T}$, which is assumed to be a d-dimensional semimartingale on a given filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\leq t\leq T},\mathbb{P})$ satisfying the usual conditions. We assume furthermore that $\mathcal{F}_0$ is $\mathbb{P}$-trivial. A self-financing trading strategy can be regarded as a pair (x,ξ), where $x\in\mathbb{R}$ is the initial investment and $\xi=(\xi_t)_{0\leq t\leq T}$ is a d-dimensional predictable and S-integrable process. The value process X associated with (x,ξ) is given by $X_0=x$ and

$$X_t=X_0+\int_0^t \xi_r\,dS_r\,,\qquad 0\leq t\leq T\,.$$

For $x>0$ given, we denote by $\mathcal{X}(x)$ the set of all value processes X that satisfy $X_0\leq x$ and are admissible in the sense that $X_t\geq 0$ for $0\leq t\leq T$. We assume that our model is arbitrage-free in the sense that $\mathcal{P}\neq\emptyset$, where $\mathcal{P}$ denotes the set of measures equivalent to $\mathbb{P}$ under which each $X\in\mathcal{X}(1)$ is a local martingale. If S is locally bounded then a measure $Q\sim\mathbb{P}$ belongs to $\mathcal{P}$ if and only if S is a local Q-supermartingale; see DELBAEN and SCHACHERMAYER [2006].

We now describe the robust utility functional of the investor. The utility function is a strictly increasing and strictly concave function $U:]0,\infty[\to\mathbb{R}$. The utility of a payoff, i.e., of a random variable $X \in L^0(\mathbb{P})$, shall be assessed in terms of a robust utility functional of the form

$$X \longmapsto \inf_Q \big(E_Q[\,U(X)\,] + \gamma(Q)\big). \tag{4.1}$$

Here we assume that γ is bounded from below and equal to the minimal penalty function of the concave monetary utility functional $\phi : L^\infty(\mathbb{P}) \to \mathbb{R}$ that is defined by

$$\phi(Y) := \inf_{Q \ll \mathbb{P}} \big(E_Q[\,Y\,] + \gamma(Q)\big), \qquad Y \in L^\infty(\mathbb{P}), \tag{4.2}$$

and assumed to satisfy the Fatou property. We may suppose without loss of generality that ϕ is normalized in the sense that $\phi(0) = \inf_Q \gamma(Q) = 0$. We also assume that ϕ is *sensitive* in the sense that every nonzero $Y \in L^\infty_+$ satisfies $\phi(Y) > 0$. Sensitivity is also called *relevance*.

Note, however, that the utility functional (4.1) cannot be represented as $\phi(U(X))$ unless the random variable $U(X)$ is bounded, because ϕ is a priori only defined on $L^\infty(\mathbb{P})$. Moreover, if the utility function U is not bounded from below, we must be particularly careful even in defining the expression $\inf_Q \big(E_Q[\,U(X)\,] + \gamma(Q)\big)$. First, it is clear that probabilistic models with an infinite penalty $\gamma(Q)$ should not contribute to the value of the robust utility functional. We therefore restrict the infimum to models Q in the domain

$$\mathcal{Q} := \{Q \ll \mathbb{P} \,|\, \gamma(Q) < \infty\}$$

of γ. That is, we make (4.1) more precise by writing

$$X \longmapsto \inf_{Q \in \mathcal{Q}} \big(E_Q[\,U(X)\,] + \gamma(Q)\big).$$

Second, we have to address the problem that the Q-expectation of $U(X)$ might not be well-defined in the sense that $E_Q[\,U^+(X)\,]$ and $E_Q[\,U^-(X)\,]$ are both infinite. This problem will be resolved by extending the expectation operator $E_Q[\,\cdot\,]$ to the entire set L^0:

$$E_Q[\,F\,] := \sup_n E_Q[\,F \wedge n\,] = \lim_{n \uparrow \infty} E_Q[\,F \wedge n\,] \qquad \text{for arbitrary } F \in L^0.$$

It is easy to see that in doing so we retain the concavity of the functional $X \mapsto E_Q[\,U(X)\,]$ and hence of the robust utility functional.

Thus, our main problem can be stated as follows:

$$\text{Maximize } \inf_{Q \in \mathcal{Q}} \big(E_Q[\,U(X_T)\,] + \gamma(Q)\big) \text{ over all } X \in \mathcal{X}(x). \tag{4.3}$$

Remark 4.1. Let us comment on the assumptions made on the robust utility functional. First, the assumption that ϕ is defined on $L^\infty(\mathbb{P})$ is equivalent to either of the facts that ϕ respects $\mathbb{P}$-nullsets in the sense of (1.1) and that $\gamma(Q)$ is finite only if $Q \ll \mathbb{P}$. Clearly, our problem (4.3) would not be well defined without this assumption as the value of the stochastic integral used to define X_T is only defined $\mathbb{P}$-a.s. (see, however, DENIS and MARTINI [2006]). Second, by Theorem 1.1, the Fatou property of ϕ is equivalent to the fact that ϕ admits a representation of the form (4.2). Third, the assumption of sensitivity is economically natural, since true payoff possibilities should be rewarded with a non-vanishing utility. In the coherent case, sensitivity and the first assumption together are equivalent to the requirement

$$\mathbb{P}[\, A \,] = 0 \qquad \Longleftrightarrow \qquad Q[\, A \,] = 0 \text{ for all } Q \in \mathcal{Q}. \tag{4.4}$$

The fourth assumption is that γ is equal to the minimal penalty function of ϕ. This is a technical assumption, which we can always make without loss of generality.

Example 4.1 (Entropic penalties)**.** As discussed in Section 2, a popular choice for γ is taking $\gamma(Q) = \gamma_\theta^{\text{ent}} = \frac{1}{\theta} H(Q|\mathbb{P})$, where $H(Q|\mathbb{P})$ is the relative entropy of Q with respect to $\mathbb{P}$. According to (2.1), this choice corresponds to the utility functional

$$\inf_{Q \in \mathcal{Q}} \left(E_Q[\, U(X_T) \,] + \gamma(Q) \right) = -\frac{1}{\theta} \log \mathbb{E}[\, e^{-\theta U(X_T)} \,]$$

of the terminal wealth, which clearly satisfies the assumptions made in this section. Its maximization is equivalent to the maximization of the ordinary expected utility $\mathbb{E}[\, \widetilde{U}(X_T) \,]$, where $\widetilde{U}(x) = -e^{-\theta U(x)}$ is strictly concave and increasing. New types of problems appear, however, if instead of the terminal wealth of an investment strategy an intertemporal quantity, such as the intertemporal utility from a consumption-investment strategy, is maximized. The maximization of the corresponding entropic utility functionals is also known as *risk-sensitive control*. We refer, for instance, to FLEMING and SHEU [2000, 2003], HANSEN and SARGENT [2001], BARRIEU and EL KAROUI [2005], BORDIGONI et al. [2005], and the references therein.

5 Projection techniques for coherent utility functionals in a complete market

In this section we assume that the underlying market model is complete in the sense that the set $\mathcal{P}$ consists of the single element P^*. We assume moreover that the monetary utility functional ϕ is coherent with maximal defining set $\mathcal{Q}$ so that (4.3) becomes

$$\text{Maximize } \inf_{Q \in \mathcal{Q}} E_Q[\, U(X_T) \,] \text{ over all } X \in \mathcal{X}(x). \tag{5.1}$$

The following definition has its origins in robust statistical test theory; see HUBER and STRASSEN [1973].

Definition 5.1. $Q_0 \in \mathcal{Q}$ is called a *least favorable measure* with respect to P^* if the density $\pi = dP^*/dQ_0$ (taken in the sense of the Lebesgue decomposition) satisfies

$$Q_0[\, \pi \le t \,] = \inf_{Q \in \mathcal{Q}} Q[\, \pi \le t \,] \qquad \text{for all } t > 0.$$

Remark 5.1. If a least favorable measure Q_0 exists, then it is automatically equivalent to $\mathbb{P}$. To see this, note first that $\mathcal{Q}$ is closed in total variation by our assumption that γ is the minimal penalty function. Hence, due to (4.4) and the Halmos-Savage theorem, $\mathcal{Q}$ contains a measure $Q_1 \sim P^*$. We get

$$1 = Q_0[\, \pi < \infty \,] = \lim_{t \uparrow \infty} Q_0[\, \pi \le t \,] = \lim_{t \uparrow \infty} \inf_{Q \in \mathcal{Q}} Q[\, \pi \le t \,] \le Q_1[\, \pi < \infty \,].$$

Hence, also $P^*[\, \pi < \infty \,] = 1$ and in turn $P^* \ll Q_0$.

A number of examples for least-favorable measures are given in Section 6 below. We next state a characterization of least favorable measures that is a variant of Theorem 6.1 in HUBER and STRASSEN [1973] and in this form taken from SCHIED [2005, Proposition 3.1].

Proposition 5.1. *For $Q_0 \in \mathcal{Q}$ with $Q_0 \sim P^*$ and $\pi := dP^*/dQ_0$, the following conditions are equivalent.*

(a) *Q_0 is a least favorable measure for P^*.*

(b) *For all decreasing functions $f :]0,\infty] \to \mathbb{R}$ such that $\inf_{Q\in\mathcal{Q}} E_Q[\, f(\pi) \wedge 0\,] > -\infty$,*

$$\inf_{Q\in\mathcal{Q}} E_Q[\, f(\pi)\,] = E_{Q_0}[\, f(\pi)\,]\,.$$

(c) *Q_0 minimizes the g-divergence*

$$I_g(Q|P^*) = E_{P^*}\Big[\, g\Big(\frac{dQ}{dP^*}\Big)\,\Big]$$

among all $Q \in \mathcal{Q}$, for all continuous convex functions $g : [0,\infty[\to \mathbb{R}$ such that $I_g(P^*|Q)$ is finite for some $Q \in \mathcal{Q}$.

Sketch of proof. According to the definition, Q_0 is a least favorable measure if and only if $Q_0 \circ \pi^{-1}$ stochastically dominates $Q \circ \pi^{-1}$ for all $Q \in \mathcal{Q}$. Hence, the equivalence of (a) and (b) is just the standard characterization of stochastic dominance (see, e.g., FÖLLMER and SCHIED [2004, Theorem 2.71]). Here and in the next step, some care is needed if f is unbounded or discontinuous.

For showing the equivalence of (b) and (c), let the continuous functions f and g be related by $g(x) = \int_1^x f(1/t)\,dt$. Then g is convex if and only if f is decreasing. For $Q_1 \in \mathcal{Q}$ we let $Q_t := tQ_1 + (1-t)Q_0$ and $h(t) := I_g(Q_t|P^*)$. The right-hand derivative of h is given by $h'_+(0) = E_{Q_1}[f(\pi)] - E_{Q_0}[\, f(\pi)\,]$, which shows hat (b) is the first-order condition for the minimization problem in (c). □

The following result from SCHIED [2005] reduces the robust utility maximization problem to a standard utility maximization problem plus the computation of a least favorable measure, which is *independent* of the utility function.

Theorem 5.1. *Suppose that $\mathcal{Q}$ admits a least favorable measure Q_0. Then the robust utility maximization problem* (4.3) *is equivalent to the standard utility maximization problem with subjective measure Q_0, i.e., to the problem* (5.1) *with the choice $\mathcal{Q} = \{Q_0\}$. More precisely, $X_T^* \in \mathcal{X}(x)$ solves the robust problem* (4.3) *if and only if it solves the standard problem for Q_0, and the corresponding value functions are equal, whether there exists a solution or not:*

$$\sup_{X\in\mathcal{X}(x)} \inf_{Q\in\mathcal{Q}} E_Q[\, U(X_T)\,] = \sup_{X\in\mathcal{X}(x)} E_{Q_0}[\, U(X_T)\,], \quad \textit{for all } x.$$

Idea of proof: For simplicity, we only consider the situation in which the corresponding standard problem for Q_0 admits a unique solution X^0. By standard theory, the final terminal wealth is of the form $X_T^0 = I(\lambda\pi)$, where λ is a positive constant and I is the inverse of the function U'; see, e.g., FÖLLMER and SCHIED [2004, Section 3.3]. We then have for any $X \in \mathcal{X}(x)$ that is not identical to X^0,

$$\inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T)\,] \le E_{Q_0}[\,U(X_T)\,] < E_{Q_0}[\,U(X_T^0)\,] = \inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T^0)\,]. \tag{5.2}$$

Here we have used Proposition 5.1 in the last step. This proves that X^0 is also the unique solution of the robust problem. In the general case one needs additional arguments; see SCHIED [2005]. □

The preceding result has the following economic consequence. Let $\succ$ denote the preference order induced by our robust utility functional, i.e.,

$$X \succ Y \qquad\Longleftrightarrow\qquad \inf_{Q\in\mathcal{Q}} E_Q[\,U(X)\,] > \inf_{Q\in\mathcal{Q}} E_Q[\,U(Y)\,]\,.$$

Then, although $\succ$ does not satisfy the axioms of (subjective) expected utility theory, optimal investment decisions with respect to $\succ$ are still made in accordance with von Neumann-Morgenstern expected utility, provided that we take Q_0 as the subjective probability measure. The surprising part is that this subjective measure neither depends on the initial investment $x = X_0$ nor on the choice of the utility function U. If $\mathcal{Q}$ does not admit a least favorable measure, then it is still possible that the robust problem is equivalent to a standard utility maximization problem with a subjective measure Q, which then, however, will depend on x and U. We also have the following converse to Theorem 5.1:

Theorem 5.2. *Suppose $Q_0 \in \mathcal{Q}$ is such that for all utility functions and all $x > 0$ the robust utility maximization problem* (4.3) *is equivalent to the standard utility maximization problem with respect to Q_0. Then Q_0 is a least favorable measure in the sense of Definition 5.1.*

Proof. See SCHIED [2005]. □

With some additional care, the argument combined in the proofs of Proposition 5.1 and Theorem 5.1 extends to the case in which there exists no least-favorable measure for $\mathcal{Q}$ in the sense of Definition 5.1. To explain this extension, which was carried out by GUNDEL [2005], let us assume for simplicity that each $Q \in \mathcal{Q}$ is equivalent to $\mathbb{P}$ and admits a unique $X^Q \in \mathcal{X}(x)$ that solves the standard problem for Q and is such that $E_{P^*}[\,X_T^Q\,] = x$. The goal is to find some $Q_0 \in \mathcal{Q}$ for which

$$E_{Q_0}[\,U(X_T^{Q_0})\,] = \inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T^{Q_0})\,],$$

for then we can conclude as in (5.2) that X^{Q_0} must be the solution of the robust problem. It is well known that the final terminal wealth of X^Q is of the form $X_T^Q = I(\lambda_Q dP^*/dQ)$, where λ_Q is a positive constant depending on Q and I is the inverse of the function U'; see, e.g., FÖLLMER and SCHIED [2004, Section 3.3]. If U satisfies the Inada conditions, i.e., $U'(0+) = \infty$ and $U'(\infty-) = 0$, then

$$U(X_T^Q) = U\Big(I\Big(\lambda_Q \frac{dP^*}{dQ}\Big)\Big) = \widetilde{U}\Big(\lambda_Q \frac{dP^*}{dQ}\Big) + \lambda_Q \frac{dP^*}{dQ} \cdot X_T^Q,$$

where $\widetilde{U}(y) := \sup_{x>0}\big(U(x) - xy\big)$ denotes the convex conjugate of U. Hence,

$$E_Q[\,U(X_T^Q)\,] = E_Q\Big[\,\widetilde{U}\Big(\lambda_Q \frac{dP^*}{dQ}\Big)\,\Big] + \lambda_Q x.$$

Using the standard fact that λ_Q is the minimizer of the right-hand side when regarded as a function of $\lambda = \lambda_Q$ (see, e.g., KRAMKOV and SCHACHERMAYER [1999, Theorem 2.0]), we thus obtain the following result from GUNDEL [2005]:

Theorem 5.3. *In addition to the preceding assumptions, suppose that Q_0 is a measure in $\mathcal{Q}$ attaining the infimum of the function*

$$Q \longmapsto \inf_{\lambda}\bigg(E_Q\Big[\,\widetilde{U}\Big(\lambda \frac{dP^*}{dQ}\Big)\,\Big] + \lambda x\bigg), \qquad Q \in \mathcal{Q}. \tag{5.3}$$

Then X^{Q_0} solves problem (5.1)*.*

Remark 5.2. Suppose λ_0 is a minimizer of the function

$$\lambda \longmapsto \inf_{Q\in\mathcal{Q}} E_Q\Big[\,\widetilde{U}\Big(\lambda \frac{dP^*}{dQ}\Big)\,\Big] + \lambda x.$$

Then the function (5.3) is equal to

$$E_Q\Big[\,\widetilde{U}\Big(\lambda_0 \frac{dP^*}{dQ}\Big)\,\Big] + \lambda_0 x = E_{P^*}\Big[\,\frac{dQ}{dP^*}\widetilde{U}\Big(\lambda_0 \frac{dP^*}{dQ}\Big)\,\Big] + \lambda_0 x = I_g(Q|P^*) + \lambda_0 x,$$

where $I_g(Q|P^*)$ is the g-divergence associated with the convex function $g(x) = x\widetilde{U}(\lambda_0/x)$; see (2.4). The measure Q_0 in Theorem 5.3 therefore can be characterized as the minimizer in $\mathcal{Q}$ of $I_g(\cdot|P^*)$ for this particular choice of g. This fact and Proposition 5.1 provide the connection to the solution of the problem via least-favorable measures. Note that in the present context, g typically depends on both U and x, and so does Q_0 unless it is a least favorable measure. Theorem 5.3 can be extended to an incomplete market model by considering $P^* \in \mathcal{P}$ as an additional argument in (5.3). For details we refer to GUNDEL [2005]. From a probabilistic point of view, the problem of robust portfolio choice can in fact be regarded as a new version of a classical projection problem: We are looking for a pair $(\widehat{P}^*, \widehat{Q})$ which minimizes a certain divergence functional on the product $\mathcal{P} \times \mathcal{Q}$ of two convex sets of probability measures. For a systematic discussion of this robust projection problem, and of a more flexible version where the class $\mathcal{P}$ of equivalent martingale measures is replaced by a larger class of *extended martingale measures*, we refer to FÖLLMER and GUNDEL [2006], see also Remark 10.2 below.

6 Least-favorable measures and their relation to capacity theory

In the preceding section it was shown that least-favorable measures in the sense of Definition 5.1 provide the solution to the robust utility maximization problem (5.1) in a complete market model. In this section we discuss a general existence result for least-favorable measures in the context of capacity theory, namely the Huber-Strassen theorem. We also provide a number of explicit examples. This connection between Huber-Strassen theory and robust utility maximization was derived in SCHIED [2005].

In Theorem 3.2 we have discussed the assumption of *comonotonic independence*, which is reasonable insofar comonotonic positions cannot act as mutual hedges, and which is equivalent to the fact that ϕ is comonotonic. It is easy to see that every comonotonic concave monetary utility functional is coherent; see FÖLLMER and SCHIED [2004, Lemma 4.77]. Let $\mathcal{Q}$ be the corresponding maximal representing set. Then comonotonicity is equivalent to the fact that the nonadditive set function

$$\widehat{\kappa}(A) := \phi(\mathbb{1}_A) = \inf_{Q \in \mathcal{Q}} Q[\,A\,], \qquad A \in \mathcal{F}_T,$$

is *supermodular* in the sense of Choquet:

$$\widehat{\kappa}(A \cup B) + \widehat{\kappa}(A \cap B) \geq \widehat{\kappa}(A) + \widehat{\kappa}(B) \qquad \text{for } A, B \in \mathcal{F}_T.$$

In this case, $\phi(X)$ can be expressed as the *Choquet integral* of X with respect to $\widehat{\kappa}$, i.e.,

$$\phi(X) = \int_0^\infty \widehat{\kappa}(X > t)\, dt, \qquad \text{for } X \geq 0.$$

These results are due to CHOQUET [1953/54]. We refer to FÖLLMER and SCHIED [2004, Theorem 4.88] for a proof in terms of the set function

$$\kappa(A) := 1 - \widehat{\kappa}(A^c) = \sup_{Q \in \mathcal{Q}} Q[A],$$

which is *submodular* in the sense of Choquet:

$$\kappa(A \cup B) + \kappa(A \cap B) \leq \kappa(A) + \kappa(B) \qquad \text{for } A, B \in \mathcal{F}_T.$$

In fact, it will be convenient to work with κ in the sequel. Let us introduce the following technical assumption:

$$\begin{array}{l}\textit{There exists a Polish topology on } \Omega \textit{ such that} \\ \mathcal{F}_T \textit{ is the corresponding Borel field and } \mathcal{Q} \textit{ is compact.}\end{array} \tag{6.1}$$

It guarantees that κ is a capacity in the sense of Choquet. Assuming that κ is submodular, let us consider the submodular set function

$$w_t(A) := t\kappa(A) - P^*[\,A\,], \qquad A \in \mathcal{F}_T\,. \tag{6.2}$$

It is shown in Lemmas 3.1 and 3.2 of HUBER and STRASSEN [1973] that under condition (6.1) there exists a decreasing family $(A_t)_{t>0} \subset \mathcal{F}_T$ such that A_t minimizes w_t and such that the continuity condition $A_t = \bigcup_{s>t} A_s$ is satisfied.

Definition 6.1. The function

$$\frac{dP^*}{d\kappa}(\omega) = \inf\{\, t \,|\, \omega \notin A_t \,\}\,, \qquad \omega \in \Omega\,,$$

is called the *Radon-Nikodym derivative* of P^* with respect to the Choquet capacity κ.

The terminology "Radon-Nikodym derivative" comes from the fact that $dP^*/d\kappa$ coincides with the usual Radon-Nikodym derivative dP^*/dQ in case where $\mathcal{Q} = \{Q\}$; see HUBER and STRASSEN [1973]. Let us now state the celebrated Huber-Strassen theorem in a form in which it will be needed here.

Theorem 6.1 (Huber-Strassen)**.** *If κ is submodular and* (6.1) *holds, then $\mathcal{Q}$ admits a least favorable measure Q_0 with respect to any probability measure R on $(\Omega, \mathcal{F}_T)$. Moreover, if $R = P^*$ and $\mathcal{Q}$ satisfies* (4.4)*, then Q_0 is equivalent to P^* and given by*

$$dQ_0 = \Big(\frac{dP^*}{d\kappa}\Big)^{-1} dP^*\,.$$

Proof. See HUBER and STRASSEN [1973]. One also needs the fact that $\mathbb{P}[\,0 < dP^*/d\kappa < \infty\,] = 1$; see SCHIED [2005, Lemma 3.1]. □

Together with Theorem 5.1 we get a complete solution of the robust utility maximization problem within the large class of utility functionals that arise from comonotonic coherent monetary utility functionals under assumption (6.1). Before discussing particular examples, let us state the following converse of the Huber-Strassen theorem in order to clarify the role of comonotonicity. For finite probability spaces, Theorem 6.2 is due to HUBER and STRASSEN [1973]. In the form stated above, it was proved by LEMBCKE [1988]. An alternative formulation was given by BEDNARSKI [1982].

Theorem 6.2. *Suppose* (6.1) *is satisfied. If $\mathcal{Q}$ is a convex set of probability measures closed in total variation distance such that every probability measure on $(\Omega, \mathcal{F}_T)$ admits a least favorable measure $Q_0 \in \mathcal{Q}$, then $\kappa(A) = \sup_{Q\in\mathcal{Q}} Q[\,A\,]$ is submodular.*

Proof. See LEMBCKE [1988]. □

In Theorem 6.2 it is crucial to require the existence of a least favorable measure with respect to *every* probability measure on $(\Omega, \mathcal{F}_T)$. Below we will encounter a situation in which least favorable measures exist for certain but not for all probability measures on $(\Omega, \mathcal{F})$, and the corresponding set function κ will not be submodular.

Let us now turn to the discussion of particular examples. The following example class was first studied by BEDNARSKI [1981] under slightly different conditions than here. These examples also play a role in the theory of law-invariant risk measures; see KUSUOKA [2001] and Sections 4.4 through 4.7 in FÖLLMER and SCHIED [2004].

Example 6.1. The following class of submodular set functions arises in the "dual theory of choice under risk" as proposed by YAARI [1987]. Let $\psi : [0,1] \to [0,1]$ be an increasing concave function with $\psi(0) = 0$ and $\psi(1) = 1$. In particular, ψ is continuous on $]0,1]$. We define κ by

$$\kappa(A) := \psi(P[\,A\,])\,, \qquad A \in \mathcal{F}\,.$$

Then κ is submodular, gives rise to a comonotonic monetary utility functional defined as the Choquet integral of $\widehat{\kappa}(A) := 1 - \kappa(A)$, and the corresponding maximal representing set $\mathcal{Q}$ can be described in terms of ψ; see CARLIER and DANA [2003] for the case in which ψ is C^1 and FÖLLMER and SCHIED [2004, Theorem 4.73 and Corollary 4.74] for the general case. If $(\Omega, \mathcal{F}_T)$ is a standard Borel space, then there exists a compact metric topology on Ω whose Borel field is $\mathcal{F}_T$. For such a topology, $\mathcal{Q}$ is weakly compact, and so (6.1) is satisfied. Consequently, $\mathcal{Q}$ admits a least favorable measure Q_0. It can be explicitly determined in the case in which $\psi(t) = (t\lambda^{-1}) \wedge 1$ for some $\lambda \in]0,1[$, which corresponds to (2.5) and hence to the risk measure $AV@R_\lambda$. To state this result, we assume that the price density $Z := P^*/d\mathbb{P}$ has a continuous distribution $F_Z(x) = \mathbb{P}[\,Z \leq x\,]$. By q_Z we denote a corresponding quantile function, i.e., a generalized inverse of the increasing function F_Z. With this notation, the Radon-Nikodym derivative of P^* with respect to κ is given by

$$\pi = \frac{dP^*}{d\kappa} = c \cdot (Z \vee q_Z(t_\lambda))\,,$$

where c is the normalizing constant and t_λ is the unique maximizer of the function

$$t \mapsto \frac{(t - 1 + \lambda)^+}{\int_0^t q_Z(s)\,ds};$$

see SCHIED [2004, 2005].

Example 6.2 (Weak information)**.** Let Y be a measurable function on $(\Omega, \mathcal{F}_T)$, and denote by μ its law under P^*. For $\nu \sim \mu$ given, let

$$\mathcal{Q} := \Big\{\, Q \ll P^* \,\Big|\, Q \circ Y^{-1} = \nu \,\Big\}\,.$$

The robust utility maximization problem for this set $\mathcal{Q}$ was studied by BAUDOIN [2002], who coined the terminology "weak information". The interpretation behind the set $\mathcal{Q}$ is that an investor has full knowledge about the pricing measure P^* but is uncertain about the true distribution P of market prices and only knows that a certain functional Y of the stock price has distribution ν. Define Q_0 by

$$dQ_0 = \frac{d\nu}{d\mu}(Y)\,dP^*.$$

Then $Q_0 \in \mathcal{Q}$ and the law of $\pi := dQ_0/dP^* = d\mu/d\nu(Y)$ is the same for all $Q \in \mathcal{Q}$. Hence, Q_0 satisfies the definition of a least favorable measure. The same procedure can be applied to *any* measure $R \sim P^*$. Using this fact and Theorem 6.2 one can show that $\mathcal{Q}$ fits into the framework of the Huber-Strassen theory, i.e., that $\kappa(A) := \sup_{Q \in \mathcal{Q}} Q[\,A\,]$ is submodular; see SCHIED [2005, Proposition 3.4].

In the 1970's and 1980's, explicit formulas for Radon-Nikodym derivatives with respect to capacities were found in a number of examples such as sets $\mathcal{Q}$ defined in terms of ε-contamination or via probability metrics like total variation or Prohorov distance; we refer to Chapter 10 in the book by HUBER [1981] and the references therein. But, unless Ω is finite, these examples fail to satisfy either implication in (4.4). Nevertheless, they are still interesting for discrete-time market models.

We now study a situation in which a least-favorable measure exists although the Huber-Strassen theorem does not apply. To this end, we consider a Black-Scholes market model with d risky assets $S_t = (S_t^1, \dots, S_t^d)$ that satisfy an SDE of the form

$$dS_t^i = S_t^i \sum_{j=1}^{d} \sigma_t^{ij}\, dW_t^j + \alpha_t^i S_t^i\, dt \tag{6.3}$$

for a d-dimensional Brownian motion $W = (W^1, \dots, W^d)$ and a volatility matrix σ_t that has full rank. Now suppose the investor is uncertain about the future drift $\alpha_t = (\alpha_t^1, \dots, \alpha_t^d)$ in the market: any drift α is possible that is adapted to the filtration generated by W and satisfies $\alpha_t \in C_t$, where C_t is a nonrandom bounded closed convex subset of $\mathbb{R}^d$. Let us denote by $\mathcal{A}$ the set of all such processes α. This uncertainty in the choice of the drift can be expressed by the set

$$\mathcal{Q} := \Big\{\, Q \,\Big|\, S \text{ has drift } \alpha^Q \in \mathcal{A} \text{ under } Q \,\Big\}\,.$$

Under P^* the drift α in (6.3) vanishes. We denote by α_t^0 the element in C_t that minimizes the norm $|\sigma_t^{-1}x|$ among all $x \in C_t$.

Theorem 6.3. *Suppose that σ_t is deterministic and that both α_t^0 and σ_t are continuous in t. Then $\mathcal{Q}$ admits a least favorable measure Q_0 with respect to P^*, which is characterized by having the drift α^0.*

Proof. In SCHIED [2005, Propositon 3.2] the problem is solved by transforming it into a problem for *uncertain volatility* as studied by EL KAROUI et al. [1998]. $\square$

An obvious question is whether the strong condition that the volatility σ_t and the drift α^0 are deterministic can be relaxed. One case of interest is, for instance, a local volatility model in which the equation (6.3) is replaced by the one-dimensional SDE

$$dS_t = \sigma(t, S_t) S_t\, dW_t + \alpha_t S_t\, dt\,. \tag{6.4}$$

In this case, however, it may occur that it is no longer optimal to take the drift that is closest to the riskneutral case $\alpha \equiv 0$. The reason is that the utility of an investment can be reduced both by a small drift and by a large volatility, and these two requirements may be competing with each other. This effect may also destroy the existence of a least favorable measure; see HERNÁNDEZ-HERNÁNDEZ and SCHIED [2006, Example 2.7] for the discussion of a related tradeoff effect. Furthermore, SCHIED [2005, Proposition 3.3] discusses examples in which no least-favorable measure exists, due to the fact that either the coefficient σ or the least-favorable drift α_t^0 are not deterministic.

7 Duality techniques in incomplete markets

In this section we discuss the general duality theory for robust portfolio choice in a very general setting and under rather weak assumptions. The results presented here build on the corresponding results for ordinary utility maximization as obtained by KRAMKOV and SCHACHERMAYER [1999, 2003]. The duality theory for coherent robust utility functionals was first developed by QUENEZ [2004] and later extended by SCHIED and WU [2005] and SCHIED [2007a]. Our exposition follows the latter article. Recently, WITTMÜSS [2006] further extended these results to cover also the cases of consumption-investment strategies and random endowment; see also BURGERT and RÜSCHENDORF [2005] for some earlier results in that direction. Related questions arise for the problem of efficiently hedging a contingent claim when risk is measured in terms of a convex risk measure; see, e.g., CVITANIC and KARATZAS [2001], KIRCH [2000], KIRCH and RUNGGALDIER [2005], FAVERO [2001], FAVERO and RUNGGALDIER [2002], SCHIED [2004, 2006], RUDLOFF [2006], SEKINE [2004], KLÖPPEL and SCHWEIZER [2007].

The main importance of the duality method lies in the fact that the dual problem is often simpler than the primal one. Therefore it can be advantageous to combine duality with another optimization technique such as optimal stochastic control. This is already true for the maximization of classical von Neumann-Morgenstern utility. But for robust utility maximization the duality method has the additional advantage that the dual problem simply involves the minimization of a convex functional. The primal problem, on the other hand, requires to find a saddlepoint of a functional that is concave in one argument and convex in the other. This fact will become important in Section 8, where stochastic control techniques are used to solve the dual rather than the primal problem.

In addition to the assumptions stated in Section 4, we assume that the utility function $U :]0,\infty[\to \mathbb{R}$ is continuously differentiable and satisfies the Inada conditions

$$U'(0+) = +\infty \qquad \text{and} \qquad U'(\infty-) = 0.$$

We also assume that the concave monetary utility functional,

$$\phi(Y) = \inf_{Q\in\mathcal{Q}}(E_Q[\,Y\,] + \gamma(Q)), \qquad Y \in L^\infty(\mathbb{P}), \tag{7.1}$$

is continuous from below as defined in Theorem 1.2.

The *value function of the robust problem* is defined as

$$u(x) := \sup_{X\in\mathcal{X}(x)} \inf_{Q\in\mathcal{Q}} \left(E_Q[\,U(X_T)\,] + \gamma(Q)\right).$$

We also need the value function of the optimal investment problem for an investor with subjective measure $Q \in \mathcal{Q}$:

$$u_Q(x) := \sup_{X\in\mathcal{X}(x)} E_Q[\,U(X_T)\,].$$

Next, we define as usual the convex conjugate function $\widetilde{U}$ of U by

$$\widetilde{U}(y) := \sup_{x>0} \left(U(x) - xy\right), \qquad y > 0.$$

With this notation, KRAMKOV and SCHACHERMAYER [1999, Theorem 3.1] states that, for $Q \sim \mathbb{P}$ with finite value function u_Q,

$$u_Q(x) = \inf_{y>0} \big(\widetilde{u}_Q(y) + xy\big) \qquad \text{and} \qquad \widetilde{u}_Q(y) = \sup_{x>0}(u_Q(x) - xy), \tag{7.2}$$

where the dual value function $\widetilde{u}_Q$ is given by

$$\widetilde{u}_Q(y) = \inf_{Y \in \mathcal{Y}_Q(y)} E_Q[\,\widetilde{U}(Y_T)\,],$$

and the space $\mathcal{Y}_Q(y)$ is defined as the set of all positive Q-supermartingales such that $Y_0 = y$ and XY is a Q-supermartingale for all $X \in \mathcal{X}(1)$. Note that this definition also makes sense for measures $Q \ll \mathbb{P}$ that are not equivalent to $\mathbb{P}$, although in this case the duality relations (7.2) need not hold. We next define the *dual value function of the robust problem* by

$$\widetilde{u}(y) := \inf_{Q \in \mathcal{Q}} \big(\widetilde{u}_Q(y) + \gamma(Q)\big) = \inf_{Q \in \mathcal{Q}} \inf_{Y \in \mathcal{Y}_Q(y)} \big(E_Q[\,\widetilde{U}(Y_T)\,] + \gamma(Q)\big).$$

Definition 7.1. Let $y > 0$ be such that $\widetilde{u}(y) < \infty$. A pair (Q, Y) is a *solution of the dual problem* if $Q \in \mathcal{Q}$, $Y \in \mathcal{Y}_Q(y)$, and $\widetilde{u}(y) = E_Q[\,\widetilde{U}(Y_T)\,] + \gamma(Q)$.

Let us finally introduce the set $\mathcal{Q}_e$ of measures in $\mathcal{Q}$ that are equivalent to $\mathbb{P}$:

$$\mathcal{Q}_e := \{Q \in \mathcal{Q} \,|\, Q \sim \mathbb{P}\}.$$

The facts that γ is the minimal penalty function of ϕ and that ϕ is sensitive guarantee that $\mathcal{Q}_e$ is always nonempty. This follows from the Halmos-Savage theorem similarly to the argument in Remark 5.1.

Theorem 7.1. *In addition to the above assumptions, let us assume that*

$$u_{Q_0}(x) < \infty \textit{ for some } x > 0 \textit{ and some } Q_0 \in \mathcal{Q}_e \tag{7.3}$$

and that

$$\widetilde{u}(y) < \infty \textit{ implies } \widetilde{u}_{Q_1}(y) < \infty \textit{ for some } Q_1 \in \mathcal{Q}_e. \tag{7.4}$$

Then the robust value function u is concave, takes only finite values, and satisfies

$$u(x) = \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} \big(E_Q[\,U(X_T)\,] + \gamma(Q)\big) = \inf_{Q \in \mathcal{Q}} \sup_{X \in \mathcal{X}(x)} \big(E_Q[\,U(X_T)\,] + \gamma(Q)\big).$$

Moreover, the two robust value functions u and $\widetilde{u}$ are conjugate to another:

$$u(x) = \inf_{y>0} \big(\widetilde{u}(y) + xy\big) \qquad \textit{and} \qquad \widetilde{u}(y) = \sup_{x>0} \big(u(x) - xy\big). \tag{7.5}$$

In particular, $\widetilde{u}$ is convex. The derivatives of u and $\widetilde{u}$ satisfy

$$u'(0+) = \infty \qquad \textit{and} \qquad \widetilde{u}'(\infty-) = 0. \tag{7.6}$$

If furthermore $\widetilde{u}(y) < \infty$, then the dual problem admits a solution $(\widehat{Q}, \widehat{Y})$ that is maximal *in the sense that any other solution (Q, Y) satisfies $Q \ll \widehat{Q}$ and $Y_T = \widehat{Y}_T$ Q-a.s.*

Proof. See SCHIED [2007a, Theorem 2.3]. □

It is possible that the maximal $\widehat{Q}$ is *not* equivalent to $\mathbb{P}$; see SCHIED [2007a, Example 3.2]. If this happens, then $\widehat{Q}$ considered as a financial market model on its own may admit arbitrage opportunities. In this light, one also has to understand the conditions (7.3) and (7.4): they exclude the possibility that the value functions u_Q and $\widetilde{u}_Q$ are only finite for some degenerate model $Q \in \mathcal{Q}$, for which the duality relations (7.2) need not hold.

The situation simplifies considerably if we assume that *all* measures in $\mathcal{Q}$ are equivalent to $\mathbb{P}$. In this case, condition (7.4) is always satisfied and (7.3) can be replaced by the assumption that $u(x) < \infty$ for some $x > 0$. Moreover, the optimal $\widehat{Y}$ is then $\mathbb{P}$-almost surely unique. Despite this fact, however, and in contrast to the situation in standard utility maximization, it can happen that the dual value function $\widetilde{u}$ is not strictly convex — even if all measures in $\mathcal{Q}$ are equivalent to $\mathbb{P}$; see SCHIED [2007a, Example 3.1]. Equivalently, the value function u may fail to be continuously differentiable. A sufficient condition for the strict convexity of $\widetilde{u}$ and the continuous differentiability of u is given in the next result. It applies in particular to entropic penalties and to penalty functions defined in terms of many other statistical distance functions as described in Section 2.

Proposition 7.1. *Suppose that the assumptions of Theorem 7.1 are satisfied and γ is strictly convex on $\mathcal{Q}$. Then u is continuously differentiable and $\widetilde{u}$ is strictly convex on its domain.*

Proof. See SCHIED [2007a Proposition 2.4]. □

Our next aim is to get existence results for optimal strategies. In the classical case $\mathcal{Q} = \{\mathbb{P}\}$, it was shown in KRAMKOV and SCHACHERMAYER [2003] that a necessary and sufficient condition for the existence of optimal strategies at each initial capital is the finiteness of the dual value function $\widetilde{u}_{\mathbb{P}}$. This condition translates as follows to our robust setting:

$$\widetilde{u}_Q(y) < \infty \qquad \text{for all } y > 0 \text{ and each } Q \in \mathcal{Q}_e. \tag{7.7}$$

It was shown in KRAMKOV and SCHACHERMAYER [2003, Note 2] that (7.7) holds as soon as u_Q is finite for all $Q \in \mathcal{Q}_e$ and the asymptotic elasticity of the utility function U is strictly less than one:

$$AE(U) = \limsup_{x \uparrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

Theorem 7.2. *In addition to the assumptions of Theorem 7.1, let us assume* (7.7)*. Then both value functions u and $\widetilde{u}$ take only finite values and satisfy*

$$u'(\infty-) = 0 \qquad \textit{and} \qquad \widetilde{u}'(0+) = -\infty. \tag{7.8}$$

The robust value function u is strictly concave, and the dual value function $\widetilde{u}$ is continuously differentiable. Moreover, for any $x > 0$ there exist an optimal strategy $\widehat{X} \in \mathcal{X}(x)$ for the robust problem. If $y > 0$ is such that $\widetilde{u}'(y) = -x$ and $(\widehat{Q}, \widehat{Y})$ is a solution of the dual problem, then

$$\widehat{X}_T = I(\widehat{Y}_T) \qquad \widehat{Q}\text{-a.s.} \tag{7.9}$$

for $I := -\widetilde{U}'$, and $(\widehat{Q}, \widehat{X})$ is a saddlepoint for the robust problem:

$$u(x) = \inf_{Q \in \mathcal{Q}} \left(E_Q[\, U(\widehat{X}_T)\,] + \gamma(Q) \right) = E_{\widehat{Q}}[\, U(\widehat{X}_T)\,] + \gamma(\widehat{Q}) = u_{\widehat{Q}}(x) + \gamma(\widehat{Q}).$$

Furthermore, $\widehat{X}\widehat{Y}\widehat{Z}$ is a martingale under $\mathbb{P}$, where $(\widehat{Z}_t)_{0 \le t \le T}$ is the density process of $\widehat{Q}$ with respect to $\mathbb{P}$.

Proof. See SCHIED [2007a, Theorem 2.5]. □

Remark 7.1. In the preceding theorem, let us take $(\widehat{Q}, \widehat{Y})$ as a maximal solution of the dual problem as constructed in Theorem 7.1. Then the solution $\widehat{X}_T$ will be $\mathbb{P}$-a.s. unique as soon as $\widehat{Q} \sim \mathbb{P}$. This equivalence holds trivially if all measures in $\mathcal{Q}$ are equivalent to $\mathbb{P}$. In the general case, however, $\widehat{Q}$ need not be equivalent to $\mathbb{P}$, so that (7.9) cannot guarantee the $\mathbb{P}$-a.s. uniqueness of $\widehat{X}_T$; see SCHIED [2007a, Example 3.2]. Nevertheless, we can construct an optimal strategy from a given solution of the dual problem by superhedging an appropriate contingent claim $H \ge 0$. To this end, suppose the assumptions of Theorem 7.2 hold. Let $(\widehat{Q}, \widehat{Y})$ be a solution of the dual problem at level $y > 0$ and consider the contingent claim

$$H := I(\widehat{Y}_T)\mathbb{1}_{\{\widehat{Z} > 0\}},$$

where $d\widehat{Q} = \widehat{Z}\, d\mathbb{P}$. Then $x = -\widetilde{u}'(y)$ is the minimal initial investment $x' > 0$ for which there exists some $X \in \mathcal{X}(x')$ such that $X_T \ge H$ $\mathbb{P}$-a.s. If furthermore $\widehat{X} \in \mathcal{X}(x)$ is such a strategy, then it is a solution for the robust utility maximization problem at initial capital x; see SCHIED [2007a, Corollary 2.6].

Remark 7.2. Instead of working with the terminal values of processes in the space $\mathcal{Y}_Q(y)$, it is sometimes more convenient to work with the densities of measures in the set $\mathcal{P}$ of equivalent local martingale measures. In fact, one can show that the dual value function satisfies

$$\widetilde{u}(y) = \inf_{P^* \in \mathcal{P}} \inf_{Q \in \mathcal{Q}_e} \Big(E_Q\Big[\, \widetilde{U}\Big(y \frac{dP^*}{dQ} \Big) \Big] + \gamma(Q) \Big); \tag{7.10}$$

see SCHIED [2007a, Remark 2.7] Since the infimum in (7.10) need not be attained, it is often not possible to represent the optimal solution $\widehat{X}_T$ in terms of the density of an equivalent martingale measure. Nevertheless, FÖLLMER and GUNDEL [2007] recently observed that the elements of $\mathcal{Y}_Q(1)$ can be interpreted as density processes of 'extended martingale measures'; see Remark 10.2 below.

8 Solution with stochastic control techniques

Stochastic control techniques for solving robust utility maximization problems were used, e.g., by HANSEN and SARGENT [2001], TALAY and ZHENG [2002], KORN and WILMOTT, P. [2002], KORN and MENKENS [2005], KORN and STEFFENSEN [2006], HERNÁNDEZ-HERNÁNDEZ and SCHIED [2006, 2007a, 2007b], SCHIED [2007b], and DOKUCHAEV [2007].

Here we consider an incomplete market model with a risky asset, whose volatility and long-term trend are driven by an external stochastic factor process. The robust utility functional is defined in terms of a HARA utility function with risk aversion parameter $\alpha \in \mathbb{R}$ and a dynamically consistent concave or coherent monetary utility functional, which allows for model uncertainty in the distributions of both the asset price dynamics and the factor process. The exposition follows HERNÁNDEZ-HERNÁNDEZ and SCHIED [2006, 2007a], and SCHIED [2007b], and the main idea is to apply stochastic control techniques to the *dual* rather than the *primal* problem. This has the advantage that the dual problem is a pure minimization problem, while the original primal problem is a minimax problem, so that the associated nonlinear PDE would be of Hamilton-Jacobi-Bellman-Isaacs-type. This idea is well known in nonlinear optimization. In the context of robust utility maximization, it was first employed by QUENEZ [2004] to facilitate the use of BSDE techniques; cf. Section 9. See CASTAÑEDA-LEYVA and HERNÁNDEZ-HERNÁNDEZ [2005] for a related control approach to the dual problem of a standard utility maximization problem. We refer to FLEMING and SONER [1993] for an introduction to stochastic control.

We first describe the financial market model. Under the reference measure $\mathbb{P}$ the risky asset is defined through the SDE of the following factor model

$$dS_t = S_t b(Y_t)\, dt + S_t \sigma(Y_t)\, dW_t^1 \tag{8.1}$$

with deterministic initial condition S_0. Here W^1 is a standard $\mathbb{P}$-Brownian motion and Y denotes an external economic factor process modeled by the SDE

$$dY_t = g(Y_t)\, dt + \rho_1(Y_t)\, dW_t^1 + \rho_2(Y_t)\, dW_t^2 \tag{8.2}$$

for a standard $\mathbb{P}$-Brownian motion W^2, which is independent of W^1 under $\mathbb{P}$. We suppose that the economic factor can be observed but cannot be traded directly so that the market model is typically incomplete. Models of this type have been widely used in finance and economics, the case of a mean-reverting factor process with the choice $g(y) := -\kappa(\mu - y)$ being particularly popular; see, e.g., FOUQUE et al. [2000], FLEMING and HERNÁNDEZ-HERNÁNDEZ [2003], and the references therein. We assume that g belongs to $C^2(\mathbb{R})$, with derivative $g' \in C_b^1(\mathbb{R})$, and b, σ, ρ_1, and ρ_2 belong to $C_b^2(\mathbb{R})$, where $C_b^k(\mathbb{R})$ denotes the class of bounded functions with bounded derivatives up to order k. We will also assume that

$$\sigma(y) \geq \sigma_0 \text{ and } a(y) := \frac{1}{2}(\rho_1^2(y) + \rho_2^2(y)) \geq \sigma_1^2 \text{ for some constants } \sigma_0, \sigma_1 > 0. \tag{8.3}$$

The market price of risk with respect to the reference measure $\mathbb{P}$ is defined via the function

$$\theta(y) := \frac{b(y)}{\sigma(y)}.$$

The assumption of time-independent coefficients is for convenience only. It is also easy to extend our results to a d-dimensional stock market model replacing the one-dimensional SDE (8.1).

Remark 8.1. By taking $\rho_2 \equiv 0$, $\rho_1(y) = \sigma(y)$, $g(y) = b(y) - \frac{1}{2}\sigma^2(y)$, and $Y_0 = \log S_0$ it follows that Y coincides with $\log S$. Hence, S solves the SDE of a local volatility model:

$$dS_t = S_t \widetilde{b}(S_t)\, dt + S_t \widetilde{\sigma}(S_t)\, dW_t^1, \tag{8.4}$$

where $\widetilde{b}(x) = b(\log x)$ and $\widetilde{\sigma}(x) = \sigma(\log x)$. Thus, our analysis includes the study of the robust optimal investment problem for local volatility models given by (8.4).

To define $\gamma(Q)$, we assume henceforth that $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ is the canonical path space of $W = (W^1, W^2)$. Then every probability measure $Q \ll \mathbb{P}$ admits a progressively measurable process $\eta = (\eta_1, \eta_2)$ such that

$$\frac{dQ}{d\mathbb{P}} = \mathcal{E}\Big(\int_0 \eta_{1t}\, dW_t^1 + \int_0 \eta_{2t}\, dW_t^2\Big)_T \qquad Q\text{-a.s.},$$

where $\mathcal{E}(M)_t = \exp(M_t - \langle M \rangle_t/2)$ denotes the Doleans-Dade exponential of a continuous semimartingale M. Such a measure Q will receive a penalty

$$\gamma(Q) := E_Q\Big[\int_0^T h(\eta_t)\, dt\Big], \tag{8.5}$$

where $h : \mathbb{R}^2 \to [0, \infty]$ is convex and lower semicontinuous. For simplicity, we suppose that $h(0) = 0$ so that $\gamma(\mathbb{P}) = 0$. We also assume that h is continuously differentiable on its effective domain $\operatorname{dom} h := \{\eta \in \mathbb{R}^2 \mid h(\eta) < \infty\}$ and satisfies the coercivity condition

$$h(x) \geq \kappa_1 |x|^2 - \kappa_2 \qquad \text{for some constants } \kappa_1,\, \kappa_2 > 0. \tag{8.6}$$

Again, our assumption that h does not depend on time is for notational convenience only. Let us also introduce the concave monetary utility functional

$$\phi(X) := \inf_{Q \ll \mathbb{P}} \big(E_Q[\,X\,] + \gamma(Q)\big), \qquad X \in L^\infty.$$

Remark 8.2. The choice $h(x) = |x|^2/2$ corresponds to the entropic penalty function $\gamma(Q) = \gamma_1^{\text{ent}}(Q) = H(Q|\mathbb{P})$; see also Section 2. Hence, the coercivity condition (8.6) implies that also in the general case γ can be bounded by the relative entropy $H(\cdot|\mathbb{P})$. This easily yields that ϕ is sensitive in the sense that $\phi(X) > 0$ for any nonzero $X \in L_+^\infty$, because the entropic monetary utility functional (2.1) is obviously sensitive. Moreover, since the level sets $\{dQ/d\mathbb{P} \mid H(Q|\mathbb{P}) \leq c\}$ are weakly compact (this follows, e.g., by combining Theorem 1.2 with the straightforward fact that the entropic monetary utility functional (2.1) is continuous from below), also γ must have weakly relatively compact level sets. In fact, one can show that the level sets of γ are weakly closed (see DELBAEN [2006] for the coherent and HERNÁNDEZ-HERNÁNDEZ and SCHIED [2007a, Lemma 4.1] for the general case), so that γ is the minimal penalty function of ϕ, and ϕ is continuous from below. In particular, ϕ and γ satisfy the assumptions of Sections 4 and 7. F. Delbaen recently showed that the coercivity condition (8.6) is not only sufficient but also necessary for ϕ to be continuous from below.

An important particular case occurs if, for some compact convex set $\Gamma \subset \mathbb{R}^2$,

$$h(x) = \begin{cases} 0 & \text{if } x \in \Gamma, \\ \infty & \text{if } x \notin \Gamma. \end{cases} \tag{8.7}$$

In this case, ϕ is coherent with maximal representing set

$$\mathcal{Q} := \Big\{ Q \sim \mathbb{P} \,\Big|\, \frac{dQ}{d\mathbb{P}} = \mathcal{E}\Big(\int_0 \eta_{1t}\, dW_t^1 + \int_0 \eta_{2t}\, dW_t^2 \Big)_T,\ \eta = (\eta_1, \eta_2) \in \mathcal{C} \Big\}, \tag{8.8}$$

where $\mathcal{C}$ denotes the set of all progressively measurable processes $\eta = (\eta_1, \eta_2)$ such that, $dt \otimes d\mathbb{P}$-almost everywhere, $\eta_t \in \Gamma$. Note that due to Novikov's theorem we have a one-to-one correspondence between measures $Q \in \mathcal{Q}$ and processes $\eta \in \mathcal{C}$ (up to $dt \otimes d\mathbb{P}$-nullsets).

Remark 8.3. Let us introduce the *conditional penalty functions*

$$\gamma_t(Q) := E_Q\Big[\int_t^T h(\eta_u)\, du \,\Big|\, \mathcal{F}_t \Big], \qquad t \geq 0,$$

and the corresponding family of *conditional concave monetary utility functionals*,

$$\phi_t(X) := \operatorname*{ess\,inf}_{Q \ll \mathbb{P}} \big(E_Q[\,X\,] + \gamma_t(Q) \big), \qquad t \geq 0 \text{ and } X \in L^\infty.$$

This family is *dynamically consistent* in the sense that

$$\phi_0(\phi_t(X)) = \phi_0(X) \qquad \text{for all } X \in L^\infty, \tag{8.9}$$

and this property greatly facilitates the use of our class of concave monetary utility functionals. Indeed, dynamic consistency corresponds to the Bellman principle in dynamic programming and is the essential ingredient for the application of control methods. Recently, the dynamic consistency (8.9) of risk measures has been the subject of ongoing research; see, e.g., ARTZNER et al. [2007], RIEDEL [2004], CHERIDITO et al. [2004, 2005, 2006], DETLEFSEN and SCANDOLO [2005], FRITTELLI and ROSAZZA GIANIN [2003], WEBER [2006], TUTSCH [2006], FÖLLMER and PENNER [2006]. Note, however, that with the exception of the entropic monetary utility functional, the conditional versions of most of the examples in Section 2 are *not* dynamically consistent; see SCHIED [2007a, Section 3] for some examples and discussion.

Let $\mathcal{A}$ denote the set of all progressively measurable process π such that $\int_0^T \pi_s^2\, ds < \infty$ $\mathbb{P}$-a.s. For $\pi \in \mathcal{A}$ we define

$$X_t^{x,\pi} := x \cdot \exp\Big(\int_0^t \pi_s \sigma(Y_s)\, dW_s^1 + \int_0^t \Big[\pi_s b(Y_s) - \frac{1}{2}\sigma^2(Y_s)\pi_s^2 \Big]\, ds \Big). \tag{8.10}$$

Then $X^{x,\pi}$ satisfies

$$X_t^{x,\pi} = x + \int_0^t \frac{X_s^{x,\pi}\pi_s}{S_s}\, dS_s$$

and thus describes the evolution of the wealth process $X^{x,\pi}$ of an investor with initial endowment $x > 0$ investing the fraction π_s of the current wealth into the risky asset at time $s \in [0, T]$. That is, $X^{x,\pi}$ can be represented as value process of the admissible strategy $\xi_s = X_s^{x,\pi}\pi_s / S_s$ and hence belongs to the set $\mathcal{X}(x)$. Conversely, any strictly positive process in $\mathcal{X}(x)$ can be described as in (8.10).

The objective of the investor consists in

$$\text{maximizing} \quad \inf_{Q \ll \mathbb{P}} \big(E_Q[\,U(X_T^{x,\pi})\,] + \gamma(Q) \big) \quad \text{over} \quad \pi \in \mathcal{A}, \tag{8.11}$$

where the utility function $U :]0,\infty[\to \mathbb{R}$ is henceforth specified as a HARA utility function with constant relative risk aversion $\alpha \in \mathbb{R}$, i.e.,

$$U(x) = \begin{cases} \frac{x^\alpha}{\alpha} & \text{if } \alpha \neq 0, \\ \log x & \text{if } \alpha = 0. \end{cases} \tag{8.12}$$

Such utility functions are also called CRRA utility functions. For $\alpha \neq 0$ we define the conjugate exponent β by

$$\beta := \frac{\alpha}{1-\alpha}.$$

The following theorem combines the main results from HERNÁNDEZ-HERNÁNDEZ and SCHIED [2006] and SCHIED [2007b] into a single statement. It can be extended to cover also the optimization of consumption-investment strategies; see SCHIED [2007b] for the case $\alpha > 0$. Recall that $a = \frac{1}{2}(\rho_1^2 + \rho_2^2)$.

Theorem 8.1 (Coherent case, $\alpha \neq 0$)**.** *Suppose $\alpha \neq 0$ and h is given by* (8.7) *so that ϕ is coherent with maximal representing set* (8.8)*. Then there exists a unique strictly positive and bounded solution $v \in C^{1,2}(]0,T] \times \mathbb{R}) \cap C([0,T] \times \mathbb{R})$ of the quasilinear PDE*

$$\begin{aligned} w_t = a w_{yy} + (g + \beta\rho_1\theta) w_y + \frac{1}{2}(1 - \alpha\rho_2^2) w_y^2 + \\ + \inf_{\eta \in \Gamma} \Big[(\rho_1(1+\beta)\eta_1 + \beta\rho_2\eta_2) w_y + \frac{\beta(1+\beta)}{2}(\eta_1 + \theta)^2 \Big] \end{aligned} \tag{8.13}$$

with initial condition

$$w(0,\cdot) \equiv 0, \tag{8.14}$$

and the value function of the robust utility maximization problem (8.11) *can then be expressed as*

$$u(x) = \sup_{\pi \in \mathcal{A}} \inf_{Q \in \mathcal{Q}} E_Q\Big[\, U(X_T^{x,\pi}) \,\Big] = \frac{x^\alpha}{\alpha} e^{(1-\alpha) w(T, Y_0)}. \tag{8.15}$$

If $\eta^*(t,y)$ is a measurable Γ-valued function that realizes the maximum in (8.13)*, then an optimal strategy $\widehat{\pi} \in \mathcal{A}$ can be obtained by letting $\widehat{\pi}_t = \pi^*(T-t, Y_t)$ for*

$$\pi^*(t,y) = \frac{1}{\sigma(y)} \Big[(1+\beta)(\eta_1^*(t,y) + \theta(y)) + \rho_1(y) \frac{v_y(t,y)}{v(t,y)} \Big].$$

Moreover, by defining a measure $\widehat{Q} \in \mathcal{Q}$ via

$$\frac{d\widehat{Q}}{d\mathbb{P}} = \mathcal{E}\Big(\int_0 \eta_1^*(T-t, Y_t)\, dW_t^1 + \int_0 \eta_2^*(T-t, Y_t)\, dW_t^2 \Big)_T,$$

we obtain a saddlepoint $(\widehat{\pi}, \widehat{Q})$ for the maximin problem (8.11)*.*

Idea of Proof. The theorem was obtained by HERNÁNDEZ-HERNÁNDEZ and SCHIED [2006] for $\alpha < 0$ and deterministic coefficients ρ_1, ρ_2, and by SCHIED [2007b] in the general case with $\alpha > 0$. In both cases, the main idea is to apply stochastic control techniques to the dual rather than the primal problem. First, it follows from Remark 8.2 that the results in Section 7 are applicable. Let us denote by $\mathcal{M}$ the set of all progressively measurable processes ν such that $\int_0^T \nu_t^2\,dt < \infty$ $\mathbb{P}$-a.s., and define

$$Z_t^\nu := \mathcal{E}\Big(-\int \theta(Y_s)\,dW_s^1 - \int \nu_s\,dW_s^2\Big)_t.$$

Then Z^ν belongs to the space $\mathcal{Y}_{\mathbb{P}}(1)$ as defined in Section 7, and the density process of every $P^* \in \mathcal{P}$ is of this form. As before, we denote by $\widetilde{U}(z) = \sup_{x\geq 0}(U(x) - zx)$ the convex conjugate function of U. By (7.10), the dual value function of the robust utility maximization problem is given by

$$\widetilde{u}(z) := \inf_{\eta\in\mathcal{C}}\inf_{\nu\in\mathcal{M}} \mathbb{E}\Big[D_T^\eta \widetilde{U}\Big(\frac{zZ_T^\nu}{D_T^\eta}\Big)\Big], \tag{8.16}$$

where $D_t^\eta = \mathcal{E}(\int_0 \eta_s\,dW_s)_t$. Due to (7.5), the primal value function u can then be obtained as

$$u(x) = \min_{z>0}(\widetilde{u}(z) + zx). \tag{8.17}$$

Moreover, Theorem 7.2 yields that if $\widehat{z} > 0$ minimizes (8.17) and there are control processes $(\widehat{\eta}, \widehat{\nu})$ minimizing (8.16) for $z = \widehat{z}$, then $X_T^{x,\widehat{\pi}} = I(\widehat{z}Z_T^{\widehat{\nu}}/D_T^{\widehat{\eta}})$ is the terminal wealth of an optimal strategy $\widehat{\pi}$. In our specific setting (8.12), we have $\widetilde{U}(z) = z^{-\beta}/\beta$. Thus, we can simplify the duality formula (8.17) as follows. First, the expectation in (8.16) equals

$$\mathbb{E}\Big[D_T^\eta \widetilde{U}\Big(\frac{zZ_T^\nu}{D_T^\eta}\Big)\Big] = \frac{z^{-\beta}}{\beta}\mathbb{E}\big[\,(D_T^\eta)^{1+\beta}(Z_T^\nu)^{-\beta}\,\big] =: \frac{z^{-\beta}}{\beta}\Lambda_{\eta,\nu}.$$

Optimizing over $z > 0$ then yields that

$$\min_{z>0}\Big(\frac{z^{-\beta}}{\beta}\Lambda_{\eta,\nu} + zx\Big) = \frac{1+\beta}{\beta}x^{\beta/(1+\beta)}\Lambda_{\eta,\nu}^{1/(1+\beta)} = \frac{x^\alpha}{\alpha}\Lambda_{\eta,\nu}^{1-\alpha},$$

where the optimal z is given by $\widehat{z} = (\Lambda_{\eta,\nu}/x)^{1-\alpha}$. Using (8.16) and (8.17) now yields

$$u(x) = \frac{x^\alpha}{\alpha}\big(\inf_{\nu\in\mathcal{M}}\inf_{\eta\in\mathcal{C}}\Lambda_{\eta,\nu}\big)^{1-\alpha}. \tag{8.18}$$

Our next aim is to further simplify $\Lambda_{\eta,\nu}$. To this end, note that

$$\begin{aligned}
&(D_T^\eta)^{1+\beta}(Z_T^\nu)^{-\beta}\\
&= \mathcal{E}\Big(\int ((1+\beta)\eta_{1s} + \beta\theta(Y_s))\,dW_s^1 + \int ((1+\beta)\eta_{2s} + \beta\nu_s)\,dW_s^2\Big)_T \qquad (8.19)\\
&\quad\times \exp\Big(\int_0^T q(Y_s,\eta_s,\nu_s)\,ds\Big),
\end{aligned}$$

where the function $q : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to [0, \infty[$ is given by

$$q(y, \eta, \nu) = \frac{\beta(1+\beta)}{2}\big[(\eta_1 + \theta(y))^2 + (\eta_2 + \nu)^2\big] + \beta r(y).$$

The Doleans-Dade exponential in (8.19) will be denoted by $\Delta_t^{\eta,\nu}$. If $\int_0^T \nu_t^2 \, dt$ is bounded, then $\mathbb{E}[\, \Delta_T^{\eta,\nu} \,] = 1$. In general, however, we may have $\mathbb{E}[\, \Delta_T^{\eta,\nu} \,] < 1$ and this fact creates some technical difficulties. Our aim is to minimize $\Lambda_{\eta,\nu}$ over $\eta \in \mathcal{C}$ and $\nu \in \mathcal{M}_0$. To this end, we introduce the function

$$J(t, y, \eta, \nu) := \mathbb{E}\Big[\, (D_t^\eta)^{1+\beta}(Z_t^\nu)^{-\beta} \,\Big] = \mathbb{E}\Big[\, \Delta_t^{\eta,\nu} \exp\Big(\int_0^t q(Y_r, \eta_r, \nu_r) \, dr \Big) \,\Big],$$

so that $J(T, Y_0, \eta, \nu) = \Lambda_{\eta,\nu}$. The minimization of $J(t, y, \eta, \nu)$ is now carried out by stochastic control methods. Let us denote

$$\widetilde{g}(y) := g(y) + \beta\rho_1(y)\theta(y).$$

If we have a (sufficiently bounded) classical solution v to the HJB equation

$$\begin{aligned} v_t &= a v_{yy} + \widetilde{g}(y) v_y + \inf_{\nu \in \mathbb{R}} \inf_{\eta \in \Gamma} \Big(\big[\rho_1(1+\beta)\eta_1 + \rho_2\big((1+\beta)\eta_2 + \beta\nu\big)\big] v_y, + q(\cdot, \eta, \nu) v \Big), \\ v(0, y) &= 1, \end{aligned}$$

then standard verification arguments yield that $v(t, y) = \inf_{\nu \in \mathcal{M}} \inf_{\eta \in \mathcal{C}} J(t, y, \eta, \nu)$. Moreover, $w := \log w$ solves (8.13). It remains to prove existence of classical solutions to the preceding HJB equation. This is carried out by using *a priori* estimates in conjunction with approximation arguments. The details are beyond the scope of this survey, and we refer to HERNÁNDEZ-HERNÁNDEZ and SCHIED [2006] for the case $\alpha < 0$ and to SCHIED [2007b] for the case $\alpha > 0$. It should be noted that the methods for obtaining classical solutions in these two cases are rather different. □

We now turn to the case of a general penalty function γ given by (8.5). We also specify the risk aversion parameter α as zero, i.e.,

$$U(x) = \log x.$$

This choice has the advantage that the portfolio optimization no longer depends on the initial capital x, resulting in a dimension reduction. Our goal is to characterize the value function

$$u(x) = \sup_{\pi \in \mathcal{A}} \inf_{Q \ll \mathbb{P}} \big(E_Q[\, \log X_T^{x,\pi} \,] + \gamma(Q) \big)$$

of the robust utility maximization problem (8.11) in terms of the solution v of the quasi-linear parabolic initial value problem

$$\begin{cases} v_t = a v_{yy} + \Psi(v_y) + g v_y \\ v(0, \cdot) = 0, \end{cases} \tag{8.20}$$

where the nonlinearity $\Psi(v_y) = \Psi(y, v_y(t, y))$ is given by

$$\Psi(y, z) := \psi(y, (\rho_1(y), \rho_2(y))z) \qquad y, z \in \mathbb{R}.$$

for the function

$$\psi(y, x) := \inf_{\eta\in\mathbb{R}^2} \Big\{\eta \cdot x + \frac{1}{2}(\eta_1 + \theta(y))^2 + h(\eta)\Big\}, \qquad y \in \mathbb{R},\ x \in \mathbb{R}^2.$$

Here, $\eta \cdot x$ denotes the inner product of η and x. We note that similar results as in Theorems 8.2 and 8.3 below hold also for the robust optimization of consumption-investment strategies; see HERNÁNDEZ-HERNÁNDEZ and SCHIED [2007b].

Theorem 8.2. *Suppose that* $\operatorname{dom} h$ *is compact. Then there exists a unique classical solution* v *to* (8.20) *within the class of functions in* $C^{1,2}(]0, T[\times\mathbb{R}) \cap C([0, T] \times \mathbb{R})$ *satisfying a polynomial growth condition. The value function* u *of the robust utility maximization problem is given by*

$$u(x) = \log x + v(T, Y_0).$$

Suppose furthermore that $\eta^* : [0, T] \times \mathbb{R} \to \mathbb{R}$ *is a measurable function such that* $\eta^*(t, y)$ *belongs to the supergradient of the concave function* $x \mapsto \psi(y, x)$ *at* $x = (\rho_1(y), \rho_2(y))v_y(t, y)$*. Then an optimal strategy* $\widehat{\pi}$ *for the robust problem can be obtained by letting*

$$\widehat{\pi}_t = \frac{\eta_1^*(T - t, Y_t) + \theta(Y_t)}{\sigma(Y_t)}, \qquad 0 \le t \le T.$$

Moreover, by defining a measure $\widehat{Q} \sim \mathbb{P}$ *via*

$$\frac{d\widehat{Q}}{d\mathbb{P}} = \mathcal{E}\Big(\int_0 \eta^*(T - t, Y_t)\, dW_t\Big)_T, \tag{8.21}$$

we obtain a saddlepoint $(\widehat{\pi}, \widehat{Q})$ *for the maximin problem* (8.11).

Proof. The strategy of the proof is similar to the one of Theorem 8.1. See HERNÁNDEZ-HERNÁNDEZ and SCHIED [2007a]. □

The problem becomes more difficult when $\operatorname{dom} h$ is noncompact, because then we can no longer apply standard theorems on the existence of classical solutions to (8.20). Other problems appear when $\operatorname{dom} h$ is not only noncompact but also unbounded. For instance, we then may have $\gamma(Q) < \infty$ even if Q is not equivalent but merely absolutely continuous with respect to $\mathbb{P}$, and this can lead to difficulties as pointed out in Section 7. Moreover, since the optimal η^* takes values in the unbounded set $\operatorname{dom} h$, one needs an additional argument to ensure that the stochastic exponential in (8.21) is a true martingale and so defines a probability measure $\widehat{Q} \ll \mathbb{P}$. To deal with this case, we assume for simplicity that ρ_1 and ρ_2 are constant. We also need an additional condition on the shape of the function ψ. Note that g is unbounded if, e.g., Y is an Ornstein-Uhlenbeck process.

Definition 8.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an upper semicontinuous concave function. We say that f satisfies a *radial growth condition in direction* $x \in \mathbb{R}^2 \backslash \{0\}$ if there exist positive constants p_0 and C such that

$$\max\{|z| \,|\, z \in \partial f(px)\} \leq C(1 + |\partial_p^+ f(px)| \vee |\partial_p^- f(px)|) \qquad \text{for } p \in \mathbb{R},\ |p| \geq p_0,$$

where $\partial f(px)$ denotes the supergradient of f at px and $\partial_p^+ f(px)$ and $\partial_p^- f(px)$ are the right-hand and left-hand derivatives of the concave function $p \mapsto f(px)$.

Note that if f is of the form $f(x) = f_0(|x|)$ for some convex increasing function f_0, then the radial growth condition is satisfied in any direction $x \neq 0$ with constant $C = 1/|x|$.

Theorem 8.3. *Suppose that ρ_1 and ρ_2 are constant, $|\Psi(y,p)/p| \to \infty$ as $|p| \to \infty$, and assume that $\psi(y,\cdot)$ satisfies a radial growth condition in direction (ρ_1, ρ_2), uniformly in y. Then there exists a unique classical solution v to (8.20) within the class of polynomially growing functions in $C^{1,2}(]0,T[\times\mathbb{R}) \cap C([0,T] \times \mathbb{R})$ whose gradient satisfies a growth condition of the form*

$$\big|\partial_p^- \Psi(y; v_y(t,y))\big| \vee \big|\partial_p^+ \Psi(y; v_y(t,y))\big| \leq C_1(1 + |y|)$$

for some constant C_1. The value function u of the robust utility maximization problem satisfies $u(x) = \log x + v(T, Y_0)$, and also the conclusions on the optimal strategy $\widehat{\pi}$ and the measure $\widehat{Q}$ in Theorem 8.1 remain true.

Proof. The proof relies on Theorem 8.2 and PDE arguments. See HERNÁNDEZ-HERNÁNDEZ and SCHIED [2007a]. □

Remark 8.4. For numerical solutions of the HJB equations in this section one can use, e.g., a multigrid Howard algorithm as explained in AKIAN [1990] and KUSHNER and DUPUIS [2001]. For convergence results of such numerical schemes see KUSHNER and DUPUIS [2001], KRYLOV [2000], BARLES and JAKOBSEN [2005], and the references therein.

9 BSDE approach

In the preceding section we used stochastic control methods to characterize the solution of our optimization problem in terms of a quasilinear partial differential equation, which then can be solved numerically. Instead of partial differential equations one can also use backward stochastic differential equations (BSDEs), and in this section we discuss some possible approaches. An early result in this direction is due to QUENEZ [2004], where, as in Section 8, BSDE techniques are applied to the dual rather than the primal problem. A direct BSDE approach to the primal problem was given by MÜLLER [2005]. Related problems arise in the maximization of recursive utilities in the sense of DUFFIE and EPSTEIN [1992]; see, e.g., EL KAROUI et al. [2001], LAZRAK and QUENEZ [2003], and the references therein. For the general notion of a BSDE and its applications to finance, we refer to EL KAROUI et al. [1997].

The market model we consider in this section is similar to the ones used at the end of Section 6 and in Section 8. It consists of m risky assets $S_t = (S_t^1, \dots, S_t^m)$ that satisfy an SDE of the form

$$dS_t^i = S_t^i \sum_{j=1}^d \sigma_t^{ij}\, dW_t^j + b_t^i S_t^i\, dt, \qquad i = 1 \dots, m,$$

for a d-dimensional Brownian motion $W = (W^1, \dots, W^d)$, a drift vector process $b = (b^1, \dots, b^m)$, and a volatility matrix process σ. Both b and σ are assumed to be bounded and adapted to the natural filtration $(\mathcal{F}_t)$ of W. In addition, we suppose that $d \geq m$, that σ has full rank $dt \otimes d\mathbb{P}$-a.e., and that

$$\theta_t := \sigma_t'(\sigma_t \sigma_t')^{-1} b_t$$

is bounded. Here and in the sequel, a' denotes the transpose of a vector or a matrix a. Similarly as in (8.8), model uncertainty is described in terms of the set

$$\mathcal{Q} := \Big\{ Q \ll \mathbb{P} \,\Big|\, \frac{dQ}{d\mathbb{P}} = D_T^\eta,\ \eta \in \mathcal{C} \Big\},$$

where, for a predictable family (C_t) of uniformly bounded closed convex subsets of $\mathbb{R}^d$,

$$\mathcal{C} = \{ \eta \,|\, \eta \text{ is predictable and } \eta_t \in C_t\ dt \otimes d\mathbb{P}\text{-a.e.} \}$$

and

$$D_t^\eta = \mathcal{E}\Big(\int_0 \eta_s'\, dW_s \Big)_t, \qquad 0 \leq t \leq T.$$

The utility function U is assumed to be a logarithmic utility function. To formulate the dual problem, we introduce the set

$$\mathcal{M} := \{ \nu \,|\, \nu \text{ is predictable, } \mathbb{R}^d\text{-valued, and } \sigma_t \nu_t = 0\ dt \otimes d\mathbb{P}\text{-a.e.} \}$$

and the local martingales

$$Z_t^\nu = \mathcal{E}\Big(-\int_0 (\theta_s + \nu_s)'\, dW_s \Big)_t, \qquad 0 \leq t \leq T,\ \nu \in \mathcal{M}.$$

Then Z^ν belongs to the space $\mathcal{Y}_{\mathbb{P}}(1)$ as defined in Section 7, and the density process of every $P^* \in \mathcal{P}$ is of this form. As before, we denote by $\widetilde{U}(z) = \sup_{x \geq 0}(U(x) - zx)$ the convex conjugate function of U. By (7.10), the dual value function of the robust utility maximization problem is given by

$$\widetilde{u}(z) := \inf_{\eta \in \mathcal{C}} \inf_{\nu \in \mathcal{M}} \mathbb{E}\Big[D_T^\eta \widetilde{U}\Big(\frac{z Z_T^\nu}{D_T^\eta} \Big) \Big], \tag{9.1}$$

where $D_t^\eta = \mathcal{E}(\int_0 \eta_s\, dW_s)_t$. Due to (7.5), the primal value function u can then be obtained as

$$u(x) = \min_{z > 0}(\widetilde{u}(z) + zx). \tag{9.2}$$

Moreover, Theorem 7.2 yields that if $\widehat{z} > 0$ minimizes (9.2) and there are control processes $(\widehat{\eta}, \widehat{\nu})$ minimizing (9.1) for $z = \widehat{z}$, then $X_T^{x,\widehat{\pi}} = I(\widehat{z} Z_T^{\widehat{\nu}} / D_T^{\widehat{\eta}})$ is the terminal wealth of an optimal strategy $\widehat{\pi}$. We therefore concentrate on solving the dual problem of finding minimizers $(\widehat{\eta}, \widehat{\nu})$ in (9.1). The following result is taken from QUENEZ [2004].

Theorem 9.1. *Suppose $U(x) = \log x$ and, for*

$$f(t,z) := \operatorname*{ess\,inf}_{\eta\in\mathcal{C},\,\nu\in\mathcal{M}} \Big(\eta_t' z + \frac{1}{2}|\theta_t + \eta_t + \nu_t|^2\Big), \qquad z \in \mathbb{R}^d, \tag{9.3}$$

let (Y,Z) be the solution to the BSDE $-dY_t = f(t,Z_t)\,dt - Z_t'\,dW_t$ with terminal condition $Y_T = 0$. Then there exists a pair $(\widehat{\eta},\widehat{\nu}) \in \mathcal{C}\times\mathcal{M}$ such that $\widehat{\nu}$ is bounded, $f(t,Z_t) = \widehat{\eta}_t' Z_t + \frac{1}{2}|\theta_t + \widehat{\eta}_t + \widehat{\nu}_t|^2$, and $(\widehat{\eta},\widehat{\nu})$ solves the dual problem (9.1) for any z.

Sketch of Proof. In the logarithmic case we have $\widetilde{U}(z) = -1 - \log z$ and hence

$$\mathbb{E}\Big[\, D_T^\eta \widetilde{U}\Big(\frac{zZ_T^\nu}{D_T^\eta}\Big)\Big] = -1 - \log z + \mathbb{E}\Big[\, D_T^\eta \log \frac{D_T^\eta}{Z_T^\nu}\,\Big].$$

It is possible to show that the rightmost expectation is equal to

$$\frac{1}{2} E_Q\Big[\int_0^T |\theta_s + \eta_s + \nu_s|^2\,ds\Big];$$

see HERNÁNDEZ-HERNÁNDEZ and SCHIED [2007a, Lemma 3.4]. Letting

$$J_t^{\eta,\nu} := \frac{1}{2}\mathbb{E}\Big[\int_t^T \frac{D_s^\eta}{D_t^\eta}\cdot|\theta_s + \eta_s + \nu_s|^2\,ds \,\Big|\, \mathcal{F}_t\Big]$$

there exists a square-integrable process $Z^{\eta,\nu}$ such that $(J^{\eta,\nu}, Z^{\eta,\nu})$ solves the BSDE

$$-dJ_t^{\eta,\nu} = \big(\eta_t' Z_t^{\eta,\nu} + \frac{1}{2}|\theta_t + \eta_t + \nu_t|^2\big)\,dt - (Z_t^{\eta,\nu})'\,dW_t, \qquad dJ_T^{\eta,\nu} = 0.$$

Once the existence of $(\widehat{\eta},\widehat{\nu})$ as minimizer in (9.3) has been established, the result follows; see QUENEZ [2004, Section 7.4] for details. □

III. Portfolio choice under robust constraints

The measurement and management of the downside risk of portfolios is a key issue for financial institutions and regulatory authorities. The regulator is concerned with the stability of the financial system and intends to minimize the risk of financial crises by imposing rules on financial institutions. An important regulatory tool are capital constraints, as has often been stressed by regulatory authorities: "Capital regulation is the cornerstone of bank regulators' efforts to maintain a safe and sound banking system, a critical element of overall financial stability" (BERNANKE [2006]). Capital constraints restrict the risk which banks can take on. The rules specify the amount of capital that banks need to hold to safeguard their solvency and long-run viability.

Regulatory rules for financial institutions have been revised in recent years: new minimum standards for capital adequacy are described in the Basel II framework that national supervisory authorities are currently implementing. This new regulatory framework seeks to improve the previous rules and to provide at the same time a more flexible framework that can better adjust to the evolution of financial markets. The goal of regulation is to maintain overall financial stability. While the aims of the new Basel II framework are well justified, it remains an open question to what extent and in which circumstances the new rules will actually enhance the stability of financial markets. Recent research hints that capital constraints can also lead to adverse effects in certain economic situations, see Section 10. While it is an important first step to better understand the impact of the Basel II framework on financial markets, ultimately only the design, evaluation and implementation of alternative risk measurement techniques and associated capital constraints can lead to better and possibly even optimal regulatory standards.

Regulation requires appropriate ways to measure risk. The properties of the risk measures which are used for this purpose directly influence the impact of regulation on the economic stability of individual banks and the overall financial system. It is therefore important to thoroughly understand risk measurement schemes and the corresponding capital constraints. Different methodologies can be applied for this purpose. From a mathematical point of view, risk measures are either functionals on spaces of random variables, stochastic processes, or more general measurable functions which model financial positions. The recently very popular axiomatic approach to risk measurement first specifies desirable features and then characterizes functionals which satisfy these properties, see Section 1 for a detailed discussion. The foundation to this systematic investigation of risk measures was provided in the seminal paper ARTZNER, DELBAEN, EBER and HEATH [1999]. Their work was motivated by the serious deficiencies of the industry standard Value at Risk (VaR) as a measure of the downside risk. VaR penalizes diversification in many situations and does not take into account the size of very large losses exceeding the value at risk.

While such axiomatic results are an important first step towards better risk management, an analysis of the economic implications of different approaches to risk measurement is indispensable. If risk measures are used as the basis of regulatory capital constraints, they distort the incentives of financial institutions that are subject to regulation. This impact of capital requirements on portfolio holdings cannot be inferred from the axiomatic theory on risk measures. The resulting feedback effects on portfolios, market prices, and volatility need to be taken into account.

The analysis of the virtues and drawbacks of risk measurement schemes requires models in which the investment decisions of financial agents be explicitly modeled. Regulatory authorities force financial institutions to abide by risk constraints whose formulation is based on risk measurement procedures. Financial institutions try to optimize their portfolios under these constraints. Different modeling approaches are available to capture these economic realities. A first approach consists in the analysis of the portfolio optimization problem for a single agent in a financial market in which primary security prices are modeled as exogenous stochastic processes (partial equilibrium). A second approach focuses on market equilibrium models with multiple agents in which prices are formed endogenously under risk constraints (general equilibrium).

Since specific models can only be caricatures of reality, good risk management techniques should work well for a large number of models, that is, they should be robust. This includes general stochastic market state processes as well as general classes of preferences. In the sections below, we will review relevant contributions to the theory of portfolio choice under risk constraints. This will illustrate that the current understanding of optimal regulation is far from complete. Partial equilibrium models are considered in Section 10, and general equilibrium models in Section 11.

10 Partial equilibrium

The formulation of risk constraints requires the specification of risk measurement procedures. So far, the literature has considered two measurement schemes which were suggested by BASAK and SHAPIRO [2001] and CUOCO, HE and ISAENKO [2007], respectively. A third approach could use dynamic risk measures, but has not been investigated in models of portfolio choice so far.

10.1 Static risk constraints

The first risk measurement scheme, suggested by BASAK and SHAPIRO [2001], works as follows. Consider a financial institution that intends to maximize its wealth at a finite time horizon T. The institution has to respect its budget constraint. In addition, BASAK and SHAPIRO [2001] assume that final wealth at time T needs to satisfy some risk constraint which can be specified in terms of a risk measure or another *ad hoc* risk measurement functional.

To be more specific, consider a market over a finite time horizon T which consists of $d+1$ assets, one bond and d stocks. We suppose that the bond price is constant. The price processes of the stocks are given by an $\mathbb{R}^d$-valued semimartingale S on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, R)$ with $\mathcal{F} = \mathcal{F}_T$ satisfying the usual conditions. An $\mathcal{F}$-measurable random variable will be interpreted as the value of a *financial position* at maturity T or, equivalently, as the *terminal wealth* of an agent. Positions which are R-almost surely equal can be identified.

An investor with initial capital x intends to maximize her utility from terminal wealth at time horizon T by choosing an optimal admissible strategy. A trading strategy with initial value x is a d-dimensional predictable, S-integrable process $(\xi_t)_{0\leq t\leq T}$ which specifies the amount of each asset in the portfolio. In order to exclude doubling strategies, it is usually required for admissible trading strategies that the corresponding value process

$$X_t := x + \int_0^t \xi_s dS_s \qquad (0 \leq t \leq T). \tag{10.1}$$

is bounded from below by some constant (which may depend on ξ). $\mathcal{X}(x)$ denotes the set of admissible wealth processes with initial value less or equal to x. In the absence of a risk constraint, the investor can choose a self-financing admissible trading strategy with corresponding wealth process $X \in \mathcal{X}(x)$ to optimize terminal wealth X_T according to her preferences.

Preferences are commonly represented in terms of a utility functional $\mathcal{U}$; see Section 3. Particular examples include expected and robust expected utility. Letting $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$

be a utility function, the robust expected utility of wealth X_T at maturity T is given by

$$\mathcal{U}(X_T) = \inf_{Q \in \mathcal{Q}_0} E_Q[U(X_T)], \tag{10.2}$$

where $\mathcal{Q}_0$ is a set of subjective probability measures. If the cardinality of $\mathcal{Q}_0$ is one, the utility functional reduces to classical expected utility. For a discussions of numerical representations of preference orders see Section 3.

A risk constraint in the sense of BASAK and SHAPIRO [2001] amounts to requiring the agent to satisfy $\rho(X_T) \leq z$ for some risk measurement functional ρ, for example a risk measure (see Section 1), and a threshold level z. The agent's optimization problem is in this case:

$$\begin{aligned} &\text{Maximize } \mathcal{U}(X_T) \text{ over all } X \in \mathcal{X}(x) \\ &\text{which satisfy } \rho(X_T) \leq z. \end{aligned} \tag{10.3}$$

Observe that the risk constraint is imposed at the initial date and not reevaluated later. This is a serious disadvantage of the risk measurement procedure in (10.3). In addition, optimal stochastic dynamic trading strategies and portfolio wealth processes need to be interpreted as commitment solutions which are specified at the initial date for all future contingencies by the optimizing financial agent.

The partial equilibrium behavior of the single agent problem (10.3) has been discussed on different levels of generality. A general model framework is important to ensure the robustness of the results. BASAK and SHAPIRO [2001], GABIH, GRECKSCH and WUNDERLICH [2004], and GABIH, GRECKSCH and WUNDERLICH [2005] analyze the economic impact of the risk constraints in a complete financial market which is driven by Brownian motions. Risk constraints are formulated in terms of VaR and an additional risk functional. Solutions are conjectured by duality considerations, but these articles do not verify that these satisfy the constraints and hence exist.

In contrast to the one-dimensional case involving only a budget constraint, precise conditions for existence constitute the most difficult part of the analysis. This gap in the literature is closed by GUNDEL and WEBER [2008] who, in addition, formulate the risk constraint in terms of convex risk measures and do not stick to a Brownian world. Instead GUNDEL and WEBER [2008] and GUNDEL and WEBER [2007] provide a complete solution to the problem in a semimartingale setting. GUNDEL and WEBER [2007] investigates the problem of portfolio choice under robust risk constraints in an incomplete market for agents whose preferences can be represented by general robust utility functionals, see Section 3. We will, first, review the general results and techniques of GUNDEL and WEBER [2007], and then discuss the economic implications which are investigated for specific examples in BASAK and SHAPIRO [2001], GABIH, GRECKSCH and WUNDERLICH [2005], GABIH, GRECKSCH and WUNDERLICH [2004] and GUNDEL and WEBER [2008].

GUNDEL and WEBER [2007] focus on the optimization problem (10.3) for a *robust utility functional*

$$\mathcal{U}(X_T) := \inf_{Q_0 \in \mathcal{Q}_0} E_Q[U(X_T)]. \tag{10.4}$$

Downside risk is measured by *utility-based shortfall risk*, a convex risk measure in the sense of Definition 1.1 which was already introduced in Section 2. Let $\ell : \mathbb{R} \to [0, \infty]$ be a loss function, i.e., an increasing function that is not constant. The level x_1 shall be a point in the interior of the range of ℓ. Let Q_1 be a fixed subjective probability measure equivalent to R, which we will use for the purpose of risk management. The space of financial positions $\mathcal{D}$ consists of random variables X for which the integral $\int \ell(-X)dQ_1$ is well defined. The *utility-based shortfall risk* (*UBSR* in the following) ρ_{Q_1} of a position X is defined by

$$\rho_{Q_1}(X) = \inf\{m \in \mathbb{R} : \; E_{Q_1}[\ell\,(-X - m)] \le x_1\}; \tag{10.5}$$

see also (2.3). If there is no model uncertainty, the shortfall risk constraint is given by $\rho_{Q_1}(X) \le 0$. A financial position X which satisfies this constraint is acceptable from the point of view of the risk measure ρ_{Q_1}. This is equivalent to $E_{Q_1}[\ell(-X)] \le x_1$. In the case of model uncertainty the probability measure Q_1 is unknown and one considers a whole set $\mathcal{Q}_1$ of subjective measures which are equivalent to the reference measure R. The corresponding *robust UBSR constraint* is given by $\sup_{Q_1 \in \mathcal{Q}_1} \rho_{Q_1}(X) \le 0$. That is, any financial position must be acceptable from the point of view of *all* risk measures ρ_{Q_1} $(Q_1 \in \mathcal{Q}_1)$. This corresponds to choosing $\rho = \sup_{Q_1 \in \mathcal{Q}_1} \rho_{Q_1}$ and $z = 0$ in problem (10.3) and is equivalent to

$$\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X)] \le x_1. \tag{10.6}$$

GUNDEL and WEBER [2007] show that the dynamic robust optimization problem (10.3) with robust risk constraint (10.6) can be reduced to a static optimization problem. Letting $\mathcal{P}$ be the set of equivalent martingale measures and

$$\mathcal{I} = \Big\{ X \ge 0 : \;\; X \in L^1(P) \text{ for all } P \in \mathcal{P} \text{ and } U(X)^- \in L^1(Q_0) \text{ for all } Q_0 \in \mathcal{Q}_0 \Big\} \tag{10.7}$$

be the set of terminal financial positions with well defined utility and prices, the corresponding static problem is given by

$$\begin{aligned} &\text{Maximize } \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[U(X)] \text{ over all } X \in \mathcal{I} \\ &\text{that satisfy } \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X)] \le x_1 \text{ and } \sup_{P \in \mathcal{P}} E_P[X] \le x. \end{aligned} \tag{10.8}$$

Theorem 10.1. *Let S be locally bounded, and assume that the essential domain of the utility function U is bounded from below.*

The optimization problem (10.8) *admits a solution if and only if the optimization problem* (10.3) *with risk constraint* (10.6) *admits a solution.*

If X^* is a solution to problem (10.8)*, then there exists a solution $(\widehat{X}_t) \in \mathcal{X}(x_0)$ to* (10.3) *with risk constraint* (10.6) *with $\widehat{X}_T \ge X^*$ R-almost surely. In this case, $\widehat{X}_T = X^*$ R-almost surely, if the solution to* (10.8) *is R-almost surely unique. If, conversely, $(\widehat{X}_t) \in \mathcal{X}(x_0)$ is a solution to* (10.3) *with risk constraint* (10.6)*, then $\widehat{X}_T$ is a solution to* (10.8).

Theorem 10.1 reduces the original dynamic problem to the static problem (10.8) and a replication problem. Observe that under the conditions of this theorem the optimal solution can always be replicated by an admissible trading strategy. GUNDEL and WEBER [2008], and GUNDEL and WEBER [2007] characterize the optimal solution of problem (10.8).

GUNDEL and WEBER [2008] provides the solution to an auxiliary problem without model uncertainty. This provides the basis for the complete solution of problem (10.8) in the general case. Consider first the special case that the set of subjective probability measures $\mathcal{Q}_0$ and $\mathcal{Q}_1$ as well as the set of martingale measures $\mathcal{P}$ are singletons. Under suitable integrability assumptions the unique solution to the constrained maximization problem (10.8) can be written in the form

$$x^* \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right), \tag{10.9}$$

where $x^* : [0, \infty[\times]0, \infty[\rightarrow \mathbb{R}$ is a continuous deterministic function. λ_1^*, λ_2^* are suitable real parameters which need to be chosen in such a way that the budget and risk constraint are satisfied. $\frac{dQ_1}{dQ_0}$ and $\frac{dP}{dQ_0}$ signify the Radon-Nikodym densities of Q_1 and P with respect to Q_0. The function x^* is obtained as the solution of a family of deterministic maximization problems and can explicitly be characterized.

The solution to the auxiliary problem corresponds to a dual problem which is also key to characterization of the optimal solution in the general case. Consider the function

$$(\lambda_1, \lambda_2) \mapsto \widetilde{U}_{\lambda_1,\lambda_2}(P|Q_1|Q_0) = E_R \left[\widetilde{U} \left(\lambda_2 \frac{dP}{dR}, \lambda_1 \frac{dQ_1}{dR}, \frac{dQ_0}{dR} \right) \right]$$

with $\widetilde{U}(p, q_1, q_0) = \sup_{x \in \mathbb{R}} (q_0 U(x) - q_1 \ell(-x) - xp)$. The parameters $(\lambda_1^*, \lambda_2^*)$ in (10.9) can be identified as the minimizers of the function

$$(\lambda_1, \lambda_2) \mapsto \widetilde{U}_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_2.$$

In the general case of an incomplete market and model uncertainty, under technical conditions described in GUNDEL and WEBER [2007], the optimal solution takes the same form as before:

$$X^* \ := \ x^* \left(\lambda_1^* \frac{dQ_1^*}{dQ_0^*}, \lambda_2^* \frac{dP^*}{dQ_0^*} \right).$$

However, the subjective probability measures $Q_0^* \in \mathcal{Q}_0$, $Q_1^* \in \mathcal{Q}_1$, the real parameters λ_1^*, λ_2^*, and a finite measure P^*, which is equivalent to the reference measure R, need to be chosen appropriately. It is interesting to observe that the positive measure P^* is not necessarily a probability measure, but might have total mass strictly less than 1.

The quantities Q_0^*, Q_1^*, P^*, and λ_1^*, λ_2^* can be characterized through the dual formulation of the original problem. Letting

$$\widetilde{U}_{\lambda_1,\lambda_2}(P|Q_1|Q_0) = E_R \left[\widetilde{U} \left(\lambda_2 \frac{dP}{dR}, \lambda_1 \frac{dQ_1}{dR}, \frac{dQ_0}{dR} \right) \right],$$

there exists a minimizer $(\lambda_1^*, \lambda_2^*, Q_0^*, Q_1^*, P^*) \in (\mathbb{R}_+)^2 \times \mathcal{Q}_0 \times \mathcal{Q}_1 \times \mathcal{P}^T$ of

$$\widetilde{U}_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_2.$$

In the dual problem the set of martingale measures $\mathcal{P}$ is replaced by appropriate projections $\mathcal{P}^T$ of extended martingale measures which are introduced in Remark 10.2. The utility of the optimal claim X^* is given by

$$\inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[U(X^*)] = \widetilde{U}_{\lambda_1^*,\lambda_2^*}(P^*|Q_1^*|Q_0^*) + \lambda_1^* x_1 + \lambda_2^* x_2$$

The measures Q_0^*, Q_1^*, and P^*, which are obtained from the solution of the dual problem, can be characterized as worst case measures. If the expectation of the optimal wealth or claim X^* with respect to a measure $P \in \mathcal{P}^T$ is interpreted as the "P-price" of X^*, then X^* is most expensive under the pricing measure P^*, i.e.,

$$E_{P^*}[X^*] \quad = \quad \sup_{P \in \mathcal{P}^T} E_P[X^*].$$

At the same time the subjective probability measures Q_1^* and Q_0^* assign to the optimal claim X^* the highest risk and the lowest von Neumann-Morgenstern utility among all measures in $\mathcal{Q}_1$ and $\mathcal{Q}_0$, respectively:

$$\begin{aligned} E_{Q_1^*}\left[\ell(-X^*)\right] &= \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}\left[\ell(-X^*)\right], \\ E_{Q_0^*}[U(X^*)] &= \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[U(X^*)]. \end{aligned}$$

The robust solution X^* turns out to be the classical solution under these worst case measures.

Observe, however, that P^* is not necessarily a probability measure, but could also have mass strictly less than 1. It is therefore useful to formulate the solution to the robust utility maximization problem under a joint budget and risk constraint in terms of the dual set of nonnegative supermartingales

$$\mathcal{Y}_R(1) = \{Y \geq 0 : \ Y_0 = 1, \ XY \ R\text{-supermartingale} \quad \forall\, X \in \mathcal{X}(1)\},$$

see Section 7.

Remark 10.2. FÖLLMER and GUNDEL [2006] show that the elements $Y \in \mathcal{Y}_R(1)$ can be identified with extended martingale measures $\bar{P}^Y$ on the product space $\bar{\Omega} = \Omega \times]0,\infty]$ endowed with the predictable sigma-algebra $\bar{\mathcal{F}}$. More precisely, under suitable regularity assumptions on the underlying filtration any non-negative supermartingale Y with $Y_0 = 1$ induces a unique probability measure $\bar{P}^Y$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ such that

$$\bar{P}^Y\left[\, A \times]t, \infty]\, \right] = E_R[Y_t; A] \qquad (A \in \mathcal{F}_t, \ t \geq 0),$$

in analogy to Doob's classical construction of conditional Brownian motions induced by superharmonic functions; cf. FÖLLMER [1973]. The property $Y \in \mathcal{Y}_R(1)$ translates into the condition that the value process (X_t) of any admissible trading strategy, viewed as a process $\bar{X}_t(\omega) = X_t(\omega)\mathbf{1}_{]t,\infty]}(s)$ on the product space, is a supermartingale with respect to $\bar{P}^Y$ and the predictable filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$. This condition defines the class of extended martingale measures, introduced in FÖLLMER and GUNDEL [2006].

Let us now discuss economic implications of downside risk constraints. Specific examples suggests that VaR might actually increase extreme risks in comparison to the unconstrained optimal strategy. This has first been pointed out in the seminal paper by BASAK and SHAPIRO [2001]. For a detailed mathematical derivation of the results, the reader is referred to GABIH, GRECKSCH and WUNDERLICH [2005] and to GABIH, GRECKSCH and WUNDERLICH [2004]. BASAK and SHAPIRO [2001] consider a model with just one risky asset in a Black-Scholes market, i.e., the price S of the single stock is modeled by a geometric Brownian motion. Economic agents solve the maximization problem (10.3) under a Value at Risk constraint for $\rho = VaR_p,\ p \in]0,1[$. The utility functional takes the form $\mathcal{U}(X_T) = E_R[U(X_T)]$ where R denotes the statistical measure and $U(x) = \frac{x^\alpha}{\alpha}$, $\alpha < 1$, is a utility function for agents with constant relative risk aversion (CRRA). These functions are also called HARA utility functions which refers to hyperbolic absolute risk aversion. The case $\alpha = 0$ corresponds to logarithmic utility. In comparison to an unconstrained portfolio, a VaR constraint reduces, of course, the overall utility an investor can achieve – positive gains of the optimal claim decrease for good states of the economy. For intermediate states of the economy, a VaR-investor behaves like a portfolio insurer to keep the final wealth level above $-z$. However, in those worst states of the world which occur with probability p the losses of the VaR-investor are larger than for an investor who does not face any constraint. In comparison to no constraint, the VaR-investor reduces her holding of the stock for large stock prices S. However, for small values of S which correspond to low wealth the VaR-investor adopts a gambling strategy and increase her exposure to the risky asset. It has been pointed out by BERKELAAR, CUMPERAYOT and KOUWENBERG [2002] that this behavior resembles strategies of investors who choose their investments according to prospect theory, see KAHNEMAN and TVERSKY [1979], and KAHNEMAN and TVERSKY [1992]. These exhibit risk-averse behavior over gains, but are risk-seeking over losses.

In contrast to VaR, in the simple Black-Scholes market setting of BASAK and SHAPIRO [2001] alternative risk constraints lead to a significant reduction of the downside risk. This has been verified for *utility-based shortfall risk* in GUNDEL and WEBER [2008]. Properties of this risk measure are discussed in FÖLLMER and SCHIED [2004], WEBER [2006], DUNKEL and WEBER [2007], and GIESECKE, SCHMIDT and WEBER [2005]. BASAK and SHAPIRO [2001] and GABIH, GRECKSCH and WUNDERLICH [2005] choose $\rho : L^1 \to \mathbb{R},\ X \mapsto E_{\tilde{R}}[(X-q)^-]$ to define the risk constraint in (10.3). Here, $q \in \mathbb{R}$ and $\tilde{R}$ is either chosen as the unique equivalent martingale measure (BASAK and SHAPIRO [2001]) or as the statistical measure (GABIH, GRECKSCH and WUNDERLICH [2005]). Observe that ρ is not cash-invariant and thus not a risk measure in the sense of Definition 1.1. But their risk constraint can be reformulated in terms of a utility-based shortfall risk measure which can be interpreted as a limiting case of GUNDEL and WEBER [2008], see GABIH, SASS and WUNDERLICH [2007].

Although the specific examples above already hint at which risk measures can successfully be employed to contain risk, more case studies are necessary to obtain robust characterization results. However, there are more fundamental reasons why one needs to move away from the setup of BASAK and SHAPIRO [2001]. While GUNDEL and WEBER [2007] provide a very general solution to the portfolio optimization problem (10.3) under risk constraints, all five articles, BASAK and SHAPIRO [2001], GABIH, GRECKSCH and WUNDERLICH [2005], GABIH, GRECKSCH and WUNDERLICH [2004], GUNDEL and WEBER [2008], and GUNDEL and WEBER [2007], use the risk measurement scheme (10.3) which is imposed at the initial date and not reevaluated later. These papers might be an important first step in understanding the behavioral impact of regulatory capital requirements. However, they need to be complemented by models which incorporates fully dynamic risk measurement techniques. Risk measurement values should be revised as additional information becomes available.

10.2 Semi-dynamic risk constraints

An alternative risk measurement scheme has been suggested by CUOCO, HE and ISAENKO [2007]. It provides a more realistic and semi-dynamic model of risk constraints. The scope of the original paper by CUOCO, HE and ISAENKO [2007] is limited. It investigates a complete financial market whose primary security price processes follow a geometric Brownian motion, and focuses on only a few risk constraint specifications. However, the basic modeling idea – which resembles current industry practice in the special case of VaR – can be extended. In combination with results from the axiomatic theory of risk measures, the approach of CUOCO, HE and ISAENKO [2007] has significant potential as a starting-point for future research in general market settings. Since CUOCO, HE and ISAENKO [2007] focus only on the simplest special cases, we give here a stylized description generalizing their approach.

At each point in time t investors assess their risk on the basis of all available information. Risk is measured for the time window $[t, t+\tau]$ with $\tau > 0$ using a distribution-invariant static risk measure ρ (or other risk measurement functional). The risk measure is applied to the conditional distribution of *projected* changes in wealth.

In this context, projected wealth is an auxiliary quantity in the risk measurement procedure. Given a portfolio strategy at time t of the investor, wealth is projected to time $t+\tau$ under the counter to fact assumption that the proportion of wealth invested in each asset in the portfolio (relative exposure) as well as the market coefficients do not change in the time interval $[t, t+\tau]$. The dynamic risk measurement at time t is obtained by applying the static risk measure ρ to the conditional distribution of the projected change in wealth.

Let us emphasize that this quantity does not represent the risk of the *true* change of wealth over the time window from t to $t+\tau$ in terms of the risk measure ρ. First, market coefficients change over time. Second, investors are allowed to modify their trading strategies continuously. The dynamic risk measurement procedure is rather a scheme which is easily implementable and, at the same time, sensitive to new information.

Consider, for example, a financial market with d primary assets $S^1, \ldots, S^d$ which are modeled by a d-dimensional Itô-process and a money market account S^0 with constant interest rate r:

$$\begin{aligned} dS_t^0 &= S_t^0 r dt \\ dS_t^i &= S_t^i \left(\mu_t^i dt + \sum_{j=1}^m \sigma_t^{ij} dW_t^i \right), \quad i = 1, 2, \ldots, d \end{aligned}$$

with mean rate of return process μ and variance-covariance process σ. Letting $\pi = (\pi_t)_{t \in [0,\infty)}$

be the fraction of current wealth X_t^π invested in each of the d assets, $t \in [0,\infty[$, the SDE of the wealth process is given by

$$dX_t^\pi = X_t^\pi \left((r + \pi_t^* \mu_t)dt + \pi_t^* \sigma_t dW_t \right),$$

where v^* denotes the transpose of a vector $v \in \mathbb{R}^d$.

The fictitious projected change in wealth at time t for the time interval $[t, t+\tau]$ is given by

$$P_t^\pi = X_t^\pi \cdot \exp\left((r + \pi_t^* \mu_t - \frac{1}{2}|\pi_t^* \sigma_t|)\tau + \pi_t^* \sigma_t (W_{t+\tau} - W_t) \right) - X_t^\pi. \tag{10.10}$$

The risk measurement at time t is obtained by applying ρ to the conditional distribution $\mathcal{L}\left(P_t^\pi | \mathcal{F}_t\right)$ of P_t^π given the information $\mathcal{F}_t$ at time t.

The risk constraint is now specified as follows. A trading strategy is feasible at time t, if the risk of the projected change of wealth (10.10) measured by the risk measure ρ does not exceed a fixed threshold level. The objective of the financial investor is to invest optimally according to some criterion while at the same time satisfying the risk constraint.

There are certain variants of the latter model which focus on relative instead of absolute quantities. Alternatively, when projected wealth changes are calculated, one could assume that instead of wealth proportions the number of shares is fictitiously held constant, or that market coefficients are not fixed but vary stochastically. In any case, given such a model the optimal trading strategy and wealth process need to be characterized and the impact on the downside risk needs to be evaluated.

CUOCO, HE and ISAENKO [2007] investigate a complete market model where asset price processes follow geometric Brownian motions. The objective of the investors is to maximize the von Neumann-Morgenstern utility of terminal wealth in the finite time-horizon economy. Absolute and relative risk constraints are specified in terms of VaR and *average value at risk* (AVaR). CUOCO, HE and ISAENKO [2007] characterize the optimal trading strategy and terminal wealth in terms of a Hamilton-Jacobi-Bellman equation. The optimal trading strategy is a multiple of the classical Merton proportion (the unconstrained optimal strategy) with a factor of at most 1. The equivalence of VaR and AVaR are demonstrated, and numerical case studies for CRRA/HARA utility illustrate the model. CUOCO, HE and ISAENKO [2007] claim that a dynamic version of VaR can successfully be used for regulation in a market driven by a multidimensional geometric Brownian motion. Similar results have also been obtained by PIRVU and ZITKOVIC [2007] who investigate growth-optimal investment in a market driven by Itô-processes under dynamic risk constraints when projected wealth is calculated under the assumption of fixed market coefficients. However, it remains open whether these findings are robust. In alternative or more general settings, different risk measures might be appropriate, but this issue requires substantial further investigation.

10.3 Further contributions

GUNDY [2005] investigates the problem (10.3) under risk constraints which are specified in terms of value at risk, expected shortfall, and average value at risk. Under certain conditions, the dynamic problem corresponds to a static utility maximization under risk constraints.

GUNDY [2005] characterizes the existence, uniqueness, and structure of the solutions in the static case. The dynamic problem is studied for a complete financial market which is driven by Brownian motion. EMMER, KORN and KLÜPPELBERG [2001] investigate the optimal portfolio problem (10.3) in a complete multidimensional Black-Scholes market under a capital at risk constraint. The capital at risk at level $p \in]0,1[$ of a random variable is the difference between the mean and the value at risk at level p. EMMER, KORN and KLÜPPELBERG [2001] solve the optimization problem under the strong assumption that the fraction of wealth invested in each asset is held constant over time. KLÜPPELBERG and PERGAMENCHTCHIKOV [2007] investigate optimal utility of consumption and terminal wealth for investors with power utility functions under downside risk constraints in a generalized complete multidimensional Black-Scholes market where the interest-rate, the mean rate of return process and the variance-covariance process must be deterministic, but may be time-dependent. Downside risk constraints are uniform versions of value at risk and average value at risk constraints. As in BASAK and SHAPIRO [2001], these are imposed at time 0 and not reevaluated later. BOYLE and TIAN [2007], GABIH, GRECKSCH, RICHTER and WUNDERLICH [2006], and BASAK, SHAPIRO and TEPLA [2006] solve versions of problem (10.3) if investors compare their performance to a random benchmark at the time horizon T. GABIH, GRECKSCH, RICHTER and WUNDERLICH [2006] focus on a Black-Scholes market with limits on the expected utility loss and derive explicit results. Generalizing value at risk, BOYLE and TIAN [2007] impose limits on the probability that terminal wealth lies below the benchmark. For a complete market driven by Brownian motion, the existence and structure of the solution are characterized, and special cases are discussed explicitly. For a Black-Scholes market the portfolio optimization problem of an investor with CRRA/HARA utility is considered in BASAK, SHAPIRO and TEPLA [2006]. Economic implications for special cases are discussed in detail. For contributions to strict portfolio insurance we refer to BRENNAN and SCHWARTZ [1989], BASAK [1995], GROSSMAN and ZHOU [1996], JENSEN and SORENSEN [2001], and LAKNER and NYGREN [2006].

CUOCO and LIU [2006] emphasize that the actual values of risk measures cannot be observed by regulators. Instead, the Basel Committee's Internal Model Approach (IMA) requires financial institutions to self-report VaR measurements. Capital constraints are based on these self-reported numbers. The IMA mechanism creates an adverse selection problem, since banks have an incentive to underreport the true value at risk to reduce capital constraints. The Basel Committee suggested to address this problem by "backtesting": regulators should record actual profit and loss distributions and evaluate the frequency of "exceptions" which exceed the reported value at risk; banks should be penalized, if inconsistencies are observed. CUOCO and LIU [2006] provide a model for IMA and investigate the optimal reporting and portfolio selection problem in a complete, multidimensional Black-Scholes market. The optimal trading strategy can be recovered from the dual value function which is characterized in terms of a Hamilton-Jacobi-Bellman equation. Based on numerical case studies, CUOCO and LIU [2006] claim that IMA effectively bounds portfolio risk and induces risk revelation in their model framework.

11 General equilibrium

Single agent models (partial equilibrium) specify prices exogenously and constitute one possible approach to analyze the impact of downside risk constraints on the behavior of economic agents and to assess the virtues of risk measures; another are market equilibrium models with multiple agents in which prices are formed endogenously under risk constraints (general equilibrium). General equilibrium models provide a framework to study feedback effects of regulation on prices which are neglected in the partial equilibrium case.

11.1 Static risk constraints

So far, the literature on general equilibrium models which incorporate risk constraints is very limited, and only special cases have been studied. BASAK and SHAPIRO [2001] provide a first characterization of general equilibrium effects in their risk management setting for agents with intertemporal consumption and logarithmic utility for the case that instantaneous aggregate consumption follows a geometric Brownian motion. BERKELAAR, CUMPERAYOT and KOUWENBERG [2002] base their analysis on BASAK and SHAPIRO [2001] and LUCAS [1978], and provide a more detailed analysis in a model with economic agents with constant relative risk aversion.

In their model, agents maximize utility of consumption and terminal wealth over a finite time horizon T. The risk constraint is imposed at time 0 on terminal wealth at time T. The total consumption rate in the economy equals an exogenous dividend rate process which is modeled as a geometric Brownian motion. The equilibrium price and consumption processes are derived for an economy with two types of traders: unregulated and VaR-constrained traders.

To be more specific, BERKELAAR, CUMPERAYOT and KOUWENBERG [2002] consider a pure exchange economy in a finite horizon $[0, T]$ with agents with CRRA/HARA utility. The utility functions of all agents are assumed to be identical. Agents consume a single perishable consumption good. The aggregate endowment of the economy with this good is modeled by a geometric Brownian motion:

$$d\delta_t = \mu_\delta \delta_t dt + \sigma_\delta \delta_t dB_t,$$

where μ_δ and σ_δ are constant drift and volatility coefficients and B is Brownian motion. The information is modeled by the augmented Brownian filtration generated by B. All processes are assumed to be adapted. Two financial assets are traded in the financial market, a money market account with price process β which is in zero supply and a stock with price S which is in constant net supply of 1. These processes follow the stochastic differential equations:

$$\begin{aligned} d\beta_t &= r_t \beta_t dt, \\ d(S_t + \delta_t) &= S_t \left(\mu_t dt + \sigma_t dB_t \right), \end{aligned}$$

where the interest rate process r, the drift process μ, and the volatility process σ are not exogenously given, but determined in equilibrium. The dividends δ of the stock correspond to the perishable consumption good.

For agents $i = 1, 2$, let $H^i, U^i : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ denote appropriate utility functions. At time t, agents of type i hold ξ^i_t stocks and ψ^i_t bonds such that their wealth equals $W^i_t = \xi^i_t S_t + \psi^i_t \beta_t$. Their consumption rate process is denoted by $(c^i_t)_{t\in[0,T]}$. All agents are assumed to be small and act as price takers.

Unregulated agents solve the standard optimization problem under a budget constraint, i.e.,

$$\begin{aligned} \max_{c^i,\ \xi^i,\ \psi^i} \quad & E\left[\int_0^T U^i(c^i_s)ds + \rho^i H^i(W^i_T)\right] \\ \text{s.t.} \quad & W^i_0 = w^i \\ & dW^i_t = \xi^i_t d(S_t + \delta_t) + \psi^i_t d\beta_t - c^i_t dt \\ & W^i_t \geq 0, \quad \text{for } \forall t \in [0,T], \end{aligned}$$

where $i = 1$ denotes the type of the agents, $\rho^i > 0$ is the weight of the relative importance of consumption and final wealth at time T in the utility functional, and $w^i > 0$ denotes initial wealth of agents of type i.

Regulated agents solve the same problem for $i = 2$ under an additional VaR-constraint at level p with threshold q, i.e.,

$$P[W^2_T \geq q] \geq 1 - p.$$

In order to determine the price processes in equilibrium, the following conditions are imposed on the optimal solutions $\hat{c}^i$, $\hat{\xi}^i$, $\hat{\psi}^i$, $i = 1, 2$, of the utility maximization problems:

(i) *Clearing of the commodity market:*

$$\hat{c}^1_t + \hat{c}^2_t = \delta_t, \qquad 0 \leq t \leq T.$$

(ii) *Clearing of the stock market:*

$$\hat{\xi}^1_t + \hat{\xi}^2_t = 1, \qquad 0 \leq t \leq T.$$

(iii) *Clearing of the money market:*

$$\hat{\psi}^1_t + \hat{\psi}^2_t = 0, \qquad 0 \leq t \leq T.$$

For an introduction to the equilibrium problem for small investors in financial markets and mathematical solution techniques we refer to Chapter 4 in KARATZAS and SHREVE [1998].

BERKELAAR, CUMPERAYOT and KOUWENBERG [2002] find that the results of BASAK and SHAPIRO [2001] derived in partial equilibrium still hold in general equilibrium. In addition, the presence of VaR-risk managers typically reduces stock volatility in general equilibrium, but may increase it in bad states of the economy, i.e. for high values of the state price density. In some cases it can also increase the probability of extremely negative returns. BERKELAAR, CUMPERAYOT and KOUWENBERG [2002] conclude that VaR-risk management has a stabilizing effect on the economy for normal and good states. It might, however, worsen catastrophic states that occur with small probability, since VaR-managers adopt gambling strategies and increase their stock holdings in these circumstances.

While BERKELAAR, CUMPERAYOT and KOUWENBERG [2002] provide many interesting insights for risk management in a general equilibrium framework, they restrict attention to VaR-constraints, CRRA-utility and dividend rates which follow a geometric Brownian motion. At the same time, BERKELAAR, CUMPERAYOT and KOUWENBERG [2002] stick to the risk measurement setup (10.3) of BASAK and SHAPIRO [2001], GABIH, GRECKSCH and WUNDERLICH [2005], GABIH, GRECKSCH and WUNDERLICH [2004], GUNDEL and WEBER [2008], and GUNDEL and WEBER [2007] in which the risk constraint on terminal wealth is imposed at time 0 and not reevaluated later. Future research needs to incorporate general dynamic risk measure constraints, utility functionals and dividend rate processes.

11.2 Semi-dynamic risk constraints

LEIPPOLD, TROJANI and VANINI [2006] investigate a general equilibrium model similar to BERKELAAR, CUMPERAYOT and KOUWENBERG [2002]. In contrast to the latter article, they impose dynamic wealth-dependent VaR-limits which are similar to those in CUOCO, HE and ISAENKO [2007]. Instantaneous aggregate consumption does not necessarily follow a geometric Brownian motion, but is driven by a stochastic factor process; the risk aversion of agents is heterogeneous. When analyzing the model, LEIPPOLD, TROJANI and VANINI [2006] use a perturbation approximation.

Their analysis suggests that VaR constraints have ambiguous effects on equity volatility and equity expected returns. The consequences of VaR regulation on economic variables are hardly predictable. Their article and the literature review above demonstrate that the design of robust regulatory standards with an unambiguous and desirable impact across a large number of economic models is an important open problem; the current regulatory standard VaR seems deficient in many respects.

11.3 Further contributions

General equilibrium model of portfolio insurance are provided in BRENNAN and SCHWARTZ [1989], BASAK [1995], GROSSMAN and ZHOU [1996], and VANDEN [2006]. BARRIEU and EL KAROUI [2005] investigate optimal risk transfer and the design of financial instruments aimed to hedge risk which is not traded on financial markets. The issuer minimizes a risk measure under the constraint imposed by the buyer who enters the transaction only if her risk level remains below a given threshold. The problem is reduced to an inf-convolution problem involving a transformation of the risk measure.

References

AKIAN, M. [1990]: Méthodes multigrilles en contrôle stochastique. Thesis, Université de Paris IX (Paris-Dauphine), Paris, 1990. Institut National de Recherche en Informatique et en Automatique (INRIA), Rocquencourt.

ANSCOMBE, F. J.; AUMANN, R. J. [1963]: A definition of subjective probability. Ann. Math. Statistics 34, 199–205.

ARTZNER, P.; DELBAEN, F.; EBER, J.-M.; HEATH, D. [1999]: Coherent measures of risk, Mathematical Finance 9(3), 203–228.

ARTZNER, P.; DELBAEN, F.; EBER, J.-M.; HEATH, D.; KU, H. [2007]: Coherent multiperiod risk adjusted values and Bellman's principle. Ann. Oper. Res. 152, 5–22.

BARILLAS, F.; HANSEN, L.; SARGENT, T. [2007]: Doubts or variability? Working paper, NYU.

BARLES, G.; JAKOBSEN, E. [2005]: Error bounds for monotone approximation schemes for parabolic Hamilton-Jacobi-Bellman equations. SIAM J. Numer. Anal. 43, no. 2, 540–558.

BARRIEU, P.; EL KAROUI, N. [2005]: Inf-convolution of risk measures and optimal risk transfer, Finance and Stochastics 9(2), 269–298.

BASAK, S. [1995]: A general equilibrium model of portfolio insurance, The Review of Financial Studies 8(4), 1059–1090.

BASAK, S.; SHAPIRO, A.; TEPLA, L. [2006]: Risk management with benchmarking, Management Science 52(4), 542–557.

BASAK, S.; SHAPIRO, A. [2001]: Value-at-risk based risk management: Optimal policies and asset prices, The Review of Financial Studies 14, 371–405.

BAUDOIN, F. [2002]: Conditioned stochastic differential equations: theory, examples and application to finance. Stochastic Process. Appl. 100, 109–145.

BEDNARSKI, T. [1981]: On solutions of minimax test problems for special capacities, Z. Wahrsch. Verw. Gebiete 58 (1981) 397–405.

BEDNARSKI, T.[1982]: Binary experiments, minimax tests and 2-alternating capacities, Ann. Statist. **10**, 226–232.

BEN-TAL, A.; TEBOULLE, M. [1987]: Penalty functions and duality in stochastic programming via ϕ-divergence functionals. Math. Oper. Research **12**, 224-240.

BEN-TAL, A.; TEBOULLE, M. [2007]: An old-new concept of convex risk measures: the optimized certainty equivalent. Math. Finance 17, no. 3, 449–476.

BENSOUSSAN, A. [1984]: On the theory of option pricing. Acta Appl. Math. 2, no. 2, 139–158.

BERKELAAR, A.; CUMPERAYOT, P.; KOUWENBERG, R. [2002]: The effect of VaR-based risk management on asset prices and volatility smile, European Financial Management 8(2), 139–164.

BERNANKE, B.S. [2006]: Banking regulation and supervision: Balancing benefits and costs. Remarks before the Annual Convention of the American Bankers Association, Phoenix, Arizona.

BORDIGONI, G.; MATOUSSI, A.; SCHWEIZER, M. [2005]: A stochastic control approach to a robust utility maximization problem. To appear in Proceedings of Abel Symposium 2005, Springer.

BOYLE, P.; TIAN, W. [2007]: Portfolio management with constraints, Mathematical Finance 17(3), 319–343.

BRENNAN, M.J.; SCHWARTZ, E.S. [1989]: Portfolio insurance and financial market equilibrium, The Journal of Business 62(4), 455–472.

BURGERT, C.; RÜSCHENDORF, L. [2005] Optimal consumption strategies under model uncertainty. Statist. Decisions 23, no. 1, 1–14.

CARLIER, G.; DANA, R. A. [2003]: Core of convex distortions of a probability. J. Econom. Theory 113, no. 2, 199–222.

CARR, P.; GEMAN, H.; MADAN, D. [2001]: Pricing and hedging in incomplete markets. J. Financial Econom. **62**, 131–167.

CHERNY, A. [2006]: Weighted V@R and its properties. Finance Stoch. 10, no. 3, 367–393.

CHERNY, A. [2007a]: Equilibrium with coherent risk. Theory of Probability and Its Applications, 52, No.4, 34p.

CHERNY, A. [2007b]: Pricing and hedging European options with discrete-time coherent risk. Finance and Stochastics, 11, No.4, p.537-569.

CHERNY, A.; GRIGORIEV, P. [2007]: Dilatation monotone risk measures are law invariant. Finance Stoch. 11, no. 2, 291–298.

CHERNY, A.; KUPPER, M. [2007]: Divergence utilities. Preprint, Moscow State University.

CSISZAR, I. [1963]: Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten. Magyar Tud. Akad. Mat. Kutató Int. Közl. **8**, 85–108.

CSISZAR, I. [1967]: On topological properties of f-divergences. Studia. Sci. Math. Hungarica **2**, 329-339.

CASTAÑEDA-LEYVA, N.; HERNÁNDEZ-HERNÁNDEZ D. [2005] Optimal consumption-investment problems in incomplete markets with stochastic coefficients. SIAM J. Control Optim. 44, No. 4, pp. 1322–1344.

CHERIDITO, P.; DELBAEN, F.; KUPPER, M. [2004]: Coherent and convex monetary risk measures for bounded càdlàg processes. Stoch. Proc. Appl. 112, 1-22.

CHERIDITO, P.; DELBAEN, F.; KUPPER, M. [2005]: Coherent and convex monetary risk measures for unbounded càdlàg processes. Finance Stoch. 9, 1713-1732.

CHERIDITO, P.; DELBAEN, F.; KUPPER, M. [2006]: Dynamic monetary risk measures for bounded discrete-time processes. Electronic J. Probab. 11, 57-106.

CHOQUET, G. [1953/54] Theory of capacities. Ann. Inst. Fourier 5, 131-295.

CONT, R. [2006]: Model uncertainty and its impact on the pricing of derivative instruments. Math. Finance 16, 519-542, (2006).

CUOCO, D.; LIU, H. [2006]: An analysis of VaR-based capital requirements, Journal of Financial Intermediation 15, 362–394.

CUOCO, D.; HE, H; ISAENKO, S. [2007]: Optimal dynamic trading strategies with risk limits, Operations Research. To appear.

CVITANIC, J.; KARATZAS, I. [2001]: Generalized Neyman-Pearson lemma via convex duality. Bernoulli 7, 79–97.

DANA, R.-A. [2005] A representation result for concave Schur concave functions. Math. Finance 15, 613-634.

DELBAEN, F. [2002]: Coherent measures of risk on general probability spaces. In: Advances in Finance and Stochastics. Essays in Honour of Dieter Sondermann, Springer-Verlag, 1–37.

DELBAEN, F. [2000]: Coherent risk measures. Cattedra Galileiana. Scuola Normale Superiore, Classe di Scienze, Pisa.

DELBAEN, F. [2006]: The structure of m-stable sets and in particular of the set of riskneutral measures. In: Yor, M., Émery, M. (eds.): In Memoriam Paul-André Meyer - Séminaire de Probabilités XXXIX. Berlin Heidelberg New York: Springer, pp. 215-258.

DELBAEN, F.; SCHACHERMAYER, W. [2006]: The mathematics of arbitrage. Springer Finance. Springer-Verlag, Berlin.

DENIS, L., MARTINI, C. [2006] A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. Ann. Appl. Probab. 16, no. 2, 827–852.

DENNEBERG, D. [1994]: Non-additive measure and integral. Theory Decis. Lib. Ser. B Math. Statist. Meth. 27, Kluwer Academic Publishers, Dordrecht.

DETLEFSEN, K.; SCANDOLO, G. [2005]: Conditional and dynamic convex risk measures. Finance Stoch. 9, no. 4, 539–561.

DOKUCHAEV, N. [2007]: Maximin investment problems for discounted and total wealth. To appear in IMA Journal of Management Mathematics.

DUFFIE, D.; EPSTEIN, L. [1992]: Stochastic differential utility. With an appendix by the authors and C. Skiadas. Econometrica 60, no. 2, 353–394.

DUNKEL, J.; WEBER, S. [2007]: Efficient Monte Carlo methods for convex risk measures in portfolio credit risk models, WSC 2007 Proceedings. To appear.

EICHHORN, A.; RÖMISCH, W. [2005]: Polyhedral risk measures in stochastic programming. SIAM J. Optim. 16, no. 1, 69–95.

EMMER, S. ; KORN, R.; KLÜPPELBERG, C. [2001]: Optimal portfolios with bounded capital at risk, Mathematical Finance 11(4), 365–384.

EL KAROUI, N.; JENBLANC-PICQUÉ, M.; SHREVE, S. [1998]: Robustness of the Black and Scholes formula, Math. Finance 8, 93–126.

EL KAROUI, N.; PENG, S.; QUENEZ, M. C. [1997]: Backward stochastic differential equations in finance. Math. Finance 7, no. 1, 1–71.

EL KAROUI, N.; PENG, S.; QUENEZ, M. C. [2001]: A dynamic maximum principle for the optimization of recursive utilities under constraints. Ann. Appl. Probab. 11, no. 3, 664–693.

FAVERO, G. [2001]: Shortfall risk minimization under model uncertainty in the binomial case: adaptive and robust approaches. Math. Methods Oper. Res. 53, no. 3, 493–503.

FAVERO, G.; RUNGGALDIER, W. [2002]: A robustness result for stochastic control. Systems control Lett. 46, no. 2, 91–97.

FLEMING, W.; HERNÁNDEZ-HERNÁNDEZ, D. [2003]: An optimal consumption model with stochastic volatility. Finance Stoch. 7, no. 2, 245–262.

FLEMING, W. H.; SHEU, S. J. [2000]: Risk-sensitive control and an optimal investment model. INFORMS Applied Probability Conference (Ulm, 1999). Math. Finance 10, no. 2, 197–213.

FLEMING, W. H.; SHEU, S. J. [2002]: Risk-sensitive control and an optimal investment model. II. Ann. Appl. Probab. 12, no. 2, 730–767.

FLEMING, W.; SONER, M. [1993]: Controlled Markov processes and viscosity solutions. Springer-Verlag, New York.

FÖLLMER, H. [1973]: On the representation of semimartingales, Annals of Probability, 1(4), 580–589.

FÖLLMER, H. [2001]: Probabilistic Aspects of Financial Risk. Plenary Lecture at the Third European Congress of Mathematics. Proceedings of the European Congress of Mathematics, Barcelona 2000, Birkhäuser, Basel.

FÖLLMER, H.; GUNDEL, A. [2006] Robust projections in the class of martingale measures, Illinois Journal of Mathematics 50(2), 439–472.

FÖLLMER, H.; LEUKERT, P. [2000]: Efficient hedging: cost versus shortfall risk. Finance Stoch. 4, 117–146.

FÖLLMER, H.; PENNER, I. [2006]: Convex risk measures and the dynamics of their penalty functions. Stat. Decisions 24(1),61-96.

FÖLLMER, H.; SCHIED, A. [2002a]: Convex measures of risk and trading constraints. Finance Stoch. 6, 429–447.

FÖLLMER, H.; SCHIED, A. [2002b]: Robust preferences and convex measures of risk. Advances in Finance and Stochastics, 39–56, Springer, Berlin.

FÖLLMER, H.; SCHIED, A. [2004]: Stochastic Finance: An Introduction in Discrete Time, 2nd revised and extended edition. Walter de Gruyter & Co., Berlin, de Gruyter Studies in Mathematics 27, 2004.

FOUQUE, J.-P.; PAPANICOLAOU, G.; SIRCAR, K. R. [2000] Derivatives in financial markets with stochastic volatility Cambridge University Press, Cambridge.

FRITTELLI, M.; ROSAZZA GIANIN, E. [2002]: Putting order in risk measures. J. Banking & Finance 26, 1473-1486.

FRITTELLI, M.; ROSAZZA GIANIN, E. [2003]: Dynamic convex risk measures. In: New Risk Measures in Investment and Regulation, G. Szegö ed., John Wiley & Sons, New York.

FRITTELLI, M.; ROSAZZA GIANIN, E. [2005]: Law-invariant convex risk measures. Advances in Mathematical Economics 7, 33-46.

GABIH, A.; SASS, J.; WUNDERLICH, R. [2007]: Utility maximization under bounded expected loss, RICAM report. `http://www.ricam.oeaw.ac.at/publications/reports/06/rep06-24.pdf`

GABIH, A.; GRECKSCH, W.; RICHTER, M.; WUNDERLICH, R. [2006]: Optimal portfolio strategies benchmarking the stock market, Mathematical Methods of Operations Research 64, 211–225.

GABIH, A.; GRECKSCH, W.; WUNDERLICH, R. [2004]: Optimal portfolios with bounded shortfall risks, in 'Tagungsband zum Workshop Stochastic Analysis', TU Chemnitz, Chemnitz, pp. 21–41. (`http://archiv.tu-chemnitz.de/pub/2004/0120`).

GABIH, A.; GRECKSCH, W.; WUNDERLICH, R. [2005]: Dynamic portfolio optimization with bounded shortfall risks, Stochastic Analysis and Applications 3(23), 579–594.

GIESECKE, K.; SCHMIDT, T.; WEBER, S. [2005]: Measuring the risk of extreme events. Working Paper.

GILBOA, I.; SCHMEIDLER, D. [1989]: Maxmin expected utility with non-unique prior, Journal of Mathematical Economics 18, 141–153.

GROSSMAN, S.J.; ZHOU,Z. [1996]: Equilibrium analysis of portfolio insurance, The Journal of Finance 51(4), 1379–1403.

GUNDEL, A. [2005]: Robust utility maximization in complete and incomplete market models, Finance Stoch. 9, no. 2, 151–176.

GUNDEL, A. [2006]: Robust Utility Maximization, f-Projections, and Risk Constraints. Ph.D. thesis, Humboldt University Berlin.

GUNDEL, A.; WEBER, S. [2007]: Robust utility maximization with limited downside risk in incomplete markets, Stochastic Processes and Their Applications 117(11), 1663–1688.

GUNDEL, A.; WEBER, S. [2008]: Utility maximization under a shortfall risk constraint. To appear in Journal of Mathematical Economics.

GUNDY, R. [2005]: Portfolio Optimization with Risk Constraints. PhD thesis, Universität Ulm. `http://vts.uni-ulm.de/doc.asp?id=5427`.

HANSEN, L.; SARGENT, T. [2001]: Robust control and model uncertainty. American Economic Review 91, 60-66.

HERNÁNDEZ-HERNÁNDEZ, D.; SCHIED, A. [2006]: Robust utility maximization in a stochastic factor model. Statistics & Decisions 24, No. 3, 109-125.

HERNÁNDEZ-HERNÁNDEZ D.; SCHIED, A. [2007a]: A control approach to robust utility maximization with logarithmic utility and time-consistent penalties. Stoch. Proc. Appl. 117, No. 8, 980-1000 (2007).

HERNÁNDEZ-HERNÁNDEZ D.; SCHIED, A. [2007b]: Robust maximization of consumption with logarithmic utility. Proceedings of the 2007 American Control Conference, 1120-1123 (2007).

HEATH, D. [2000]: Back to the future. Plenary lecture, First World Congress of the Bachelier Finance Society, Paris.

HEATH, D.; KU, H. [2004]: Pareto equilibria with coherent measures of risk, Mathematical Finance 14(2), 163–172.

HERSTEIN, I.; MILNOR, J. [1953]: An axiomatic approach to measurable utility. Econometrica 21, 291–297.

HU, Y.; IMKELLER, P.; MÜLLER, M. [2005]: Utility maximization in incomplete markets. Ann. Appl. Probab. 15, no. 3, 1691–1712.

HUBER, P. [1981]: Robust statistics. Wiley Ser. Probab. Math. Statist., Wiley, New York.

HUBER, P.; STRASSEN, V. [1973]: Minimax tests and the Neyman-Pearson lemma for capacities. Ann. Statist. 1, 251–263.

JENSEN, B.A.; SORENSEN, C. [2001]: Paying for minimum interest rate guarantees: who should compensate whom, European Financial Management 7(2), 183–211.

JOUINI, E.; SCHACHERMAYER, W.; TOUZI, N. [2006]: Law invariant risk measures have the Fatou property. Advances in Mathematical Economics. Vol. 9, 49–71, Adv. Math. Econ., 9, Springer, Tokyo.

KAHNEMAN, D.; TVERSKY, A. [1979]: Prospect theory: An analysis of decision under risk, Econometrica 47, 263–291.

KAHNEMAN, D.; TVERSKY, A. [1992]: Advances in prospect theory: Cumulative representation of uncertainty, Journal of Risk and Uncertainty 5, 297–323.

KARATZAS, I.; SHREVE, S.E. [1998]: Methods of Mathematical Finance, Springer, New York.

KARATZAS, I.; ŽITKOVIĆ, G. [2003]: Optimal consumption from investment and random endowment in incomplete semimartingale markets. Ann. Probab. 31, no. 4, 1821–1858.

KIRCH, M. [2000]: Efficient Hedging in Incomplete Markets under Model Uncertainty. PhD thesis, Humboldt University Berlin.

KIRCH, M.; RUNGGALDIER, W. [2005]: Efficient hedging when asset prices follow a geometric Poisson process with unknown intensities. SIAM J. Control Optim. 43, no. 4, 1174–1195

KLÖPPEL, S.; SCHWEIZER, M. [2007]: Dynamic indifference valuation via convex risk measures. To appear in Math. Finance.

KLÜPPELBERG, C.; PERGAMENCHTCHIKOV, S. [2007]: Optimal consumption and investment with bounded capital-at-risk for power utility functions. Preprint, TU München.

KORN, R.; MENKENS, O. [2005]: Worst-case scenario portfolio optimization: a new stochastic control approach. Math. Methods Oper. Res. 62, no. 1, 123–140.

KORN, R.; STEFFENSEN, M. [2006]: On worst case portfolio optimization. Preprint, TU Kaiserslautern.

KORN, R.; WILMOTT, P. [2002]: Optimal portfolios under the threat of a crash. Int. J. Theor. Appl. Finance 5, no. 2, 171–187.

KRAMKOV, D.; SCHACHERMAYER, W. [1999]: The asymptotic elasticity of utility functions and optimal investment in incomplete markets. Ann. Appl. Probab. 9, no. 3, 904–950.

KRAMKOV, D.; SCHACHERMAYER, W. [2003]: Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. Ann. Appl. Probab., Vol. 13, no. 4.

KRÄTSCHMER, V. [2005]: Robust representation of convex risk measures by probability measures. Finance Stochast. **9**, 597-608.

KREPS, D. [1988]: Notes on the theory of choice. Westview Press, Boulder, CO.

KRYLOV, N. V. [2000]: On the rate of convergence of finite-difference approximations for Bellman's equations with variable coefficients. Probab. Theory Ralat. Fields, 117, 1-16.

KUNZE, M. [2003]: Verteilungsinvariante konvexe Risikomaße. Diplomarbeit, Humboldt-Universität zu Berlin.

KUSHNER, H.; DUPUIS, P. [2001]: Numerical methods for stochastic control problems in continuous time. Second edition. Applications of Mathematics (New York), 24. Stochastic Modelling and Applied Probability. Springer-Verlag, New York.

KUSUOKA, S. [2001]: On law invariant coherent risk measures. Adv. Math. Econ. 3, 83–95.

LAKNER, P.; NYGREN, L.M [2006]: Portfolio optimization with downside risk constraints, Mathematical Finance 16(2), 283–299.

LAZRAK, A.; QUENEZ, M.-C. [2003]: A generalized stochastic differential utility. Math. Oper. Res. 28, no. 1, 154–180.

LEIPPOLD, M.; TROJANI, F.; VANINI, P. [2006]: Equilibrium impact of value-at-risk regulation, Journal of Economic Dynamics & Control 30, 1277–1313.

LEMBCKE, J. [1988]: The necessity of strongly subadditive capacities for Neyman-Pearson minimax tests, Monatsh. Math. 105 , 113–126.

LUCAS, R.E. [1978]: Asset pricing in an exchange economy, Econometrica 46(6), 1429–1445.

MACCHERONI, F.; RUSTICHINI, A.; MARINACCI, M. [2006]: Ambiguity aversion, robustness, and the variational representation of preferences, Econometrica 74, 1447-1498.

MÜLLER, M. [2005]: Market completion and robust utility maximization. PhD Thesis, Humboldt-Universität zu Berlin 2005. `http://edoc.hu-berlin.de/docviews/abstract.php?id=26287`

VON NEUMANN, J.; MORGENSTERN, O. [1944]: Theory of Games and Economic Behavior. Princeton University Press, Princeton, New Jersey.

PIRVU, T.; ZITKOVIC, G. [2007]: Maximizing the growth rate under risk constraints. Working Paper.

QUENEZ, M. [2004]: Optimal portfolio in a multiple-priors model. Seminar on Stochastic Analysis, Random Fields and Applications IV, 291–321, Progr. Probab., 58, Birkhäuser, Basel.

RIEDEL, F. [2004]: Dynamic coherent risk measures. Stochastic Process. Appl. 112, no. 2, 185–200.

RUDLOFF, B. [2006]: Hedging in incomplete markets and testing compound hypotheses via convex duality. Ph.D. thesis, University of Halle-Wittenberg.

RUNGGALDIER, W. [2001]: Adaptive and robust control procedures for risk minimization under uncertainty. In: Optimal control and Partial Differential Equations (J.L.Menaldi, E.Rofman, A.Sulem, eds.). Volume in Honour of Prof. Alain Bensoussan's 60th Birthday, IOS Press, 2001, pp.549-557.

RUNGGALDIER, W. [2003]: On stochastic control in finance. Mathematical systems theory in biology, communications, computation, and finance (Notre Dame, IN, 2002), 317–344, IMA Vol. Math. Appl., 134, Springer, New York.

RUSZCZYŃSKI, A.; SHAPIRO, A.[2006]: Conditional risk mappings. Math. Oper. Res. 31, no. 3, 544–561.

RUSZCZYŃSKI, A.; SHAPIRO, A. [2006]: Optimization of convex risk functions. Math. Oper. Res. 31, no. 3, 433–452.

SAVAGE, L. J. [1954]: The foundations of statistics. Wiley Publ. Stat., John Wiley and Sons, New York.

SCHIED, A. [2004]: On the Neyman-Pearson problem for law-invariant risk measures and robust utility functionals, Ann. Appl. Probab. 14, 1398-1423.

SCHIED, A. [2005]: Optimal investments for robust utility functionals in complete market models. Math. Oper. Research. 30, No. 3, 750-764.

SCHIED, A. [2006]: Risk measures and robust optimization problems. Stochastic Models 22, 753-831.

SCHIED, A. [2007a]: Optimal investments for risk- and ambiguity-averse preferences: a duality approach. Finance Stoch. 11, No. 1, 107-129.

SCHIED, A. [2007b]: Robust optimal control for a consumption-investment problem. To appear in Mathematical Methods of Operations Research.

SCHIED, A.; STADJE, M. [2007]: Robustness of Delta hedging for path-dependent options in local volatility models. To appear in Journal of Applied Probability 44, no. 4.

SCHIED, A.; WU, C.-T. [2005]: Duality theory for optimal investments under model uncertainty, Stat. Decisions 23, No. 3, 199-217.

SCHMEIDLER, D. [1989]: Subjective probability and expected utility without additivity. Econometrica 57 , no. 3, 571–587.

SEKINE, J. [2004]: Dynamic minimization of worst conditional expectation of shortfall. Math. Finance. 14, pp.605-618.

TALAY, D.; ZHENG, Z. [2002]: Worst case model risk management. Finance Stochast. 6, 517-537.

TUTSCH, S. [2006]: Konsistente und konsequente dynamische Risikomasse und das Problem der Aktualisierung. Ph.D. thesis, Humboldt-Universität zu Berlin.

VANDEN, J.M. [2006]: Portfolio insurance and volatility regime switching, Mathematical Finance 16(2), 387–417.

WEBER, S. [2006], Distribution-invariant risk measures, information, and dynamic consistency, Mathematical Finance 16, 419–442.

WITTMÜSS, W. [2006]: Robust optimization of consumption with random endowment. To appear in Stochastics.

YAARI, M. [1987]: The dual theory of choice under risk. Econometrica 55, 95–116.

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