

Utility Maximization Under a Shortfall Risk Constraint

Anne Gundel

Stefan Weber

*Humboldt-Universität
zu Berlin*

Cornell University

First version: November 8, 2005; this version: January 2, 2008*

Abstract

The article analyzes optimal portfolio choice of utility maximizing agents in a general continuous-time financial market model under a joint budget and downside risk constraint. The risk constraint is given in terms of a class of convex risk measures. We do not impose any specific assumptions on the price processes of the underlying assets. We analyze under which circumstances the risk constraint is binding. We provide a closed-form solution to the optimization problem in a general semimartingale framework. For a complete market, the wealth maximization problem is equivalent to a dynamic portfolio optimization problem.

Key words: Utility maximization, optimal portfolio choice, utility-based shortfall risk, convex risk measures, semimartingales

JEL Classification: G11, D81

Mathematics Subject Classification (2000): 91B16, 91B28, 62C, 62P05

*Corresponding author: Stefan Weber, School of Operations Research and Information Engineering, Cornell University, 279 Rhodes Hall, Ithaca, NY 14853, USA, Phone (607) 254-4825, Fax (607) 255-9129, email: sweber@orie.cornell.edu, web: people.orie.cornell.edu/~sweber. For useful comments we would like thank participants of seminars at Cornell University, ETH Zürich, the University of Virginia, Universität Bonn, Universität Halle-Wittenberg, Universität Frankfurt, the Workshop Stochastic Analysis and Applications in Finance in Leipzig, and the INFORMS 2005 meeting in Ottawa. Useful remarks of an anonymous referee are gratefully acknowledged.

1 Introduction

For financial institutions the measurement and management of downside risk is a key issue. Value at Risk (VaR) has emerged as the industry standard for risk measurement but shows serious deficiencies as a measure of downside risk. It penalizes diversification in many situations and does not take into account the size of very large losses exceeding the value at risk. These problems motivated intense research on alternative risk measures whose foundation was provided by Artzner, Delbaen, Eber & Heath (1999).

A good *risk measure* needs to have several virtues. First, it should measure risk on a monetary scale: the notion of risk entails the amount of capital we need to set aside in order to make a position acceptable from a risk management perspective. Second, a risk measure should penalize concentrations and encourage diversification. Third, a risk measure should be sensitive to the size of losses. Taking a more practical perspective, a risk measure should also be easily estimated from simulations of profit and loss distributions. Many characterization theorems for alternative families of risk measures are now available. An excellent summary of recent results can be found in Föllmer & Schied (2004).

While these results are an important first step towards better risk management, an analysis of the economic implications of different approaches to risk measurement is indispensable. In the current article we retain the standard financial economics paradigm of rational choice, and investigate the agent's optimal wealth under a joint budget and risk measure constraint for arbitrary utility functions. Regulators, for example, might impose a risk constraint to certain companies, a manager of a firm might require his traders to stay within some risk limit, or an investor might wish to bound his own risk exposure. We provide a complete solution for the utility maximization problem under these constraints in a general semimartingale framework. For the market we neither have to assume absence of arbitrage nor completeness. Instead, for unbounded utility functions the absence of arbitrage can be seen as a consequence of the existence of a solution to the utility maximization problem. In order to analyze the impact of the downside risk constraint we discuss two examples and compare the solutions to both utility maximization without risk constraint and under a value at risk constraint. While the risk measure VaR limits the probability of a loss, it actually leads to large losses in these events. This deficiency is not shared by the family of *utility-based shortfall risk measures* (UBSR) on which we focus in the current article. In fact,

UBSR measures possess all the virtues which we discussed above. For a detailed description of their properties we refer to Föllmer & Schied (2004), Weber (2006), Dunkel & Weber (2005), and Giesecke, Schmidt & Weber (2005).

Utility maximization under a budget constraint is a fundamental problem in Mathematical Finance and has been studied in many articles, see, e.g., Aumann & Perles (1965), Merton (1969), Merton (1971), Pliska (1986), Cox & Huang (1989), Cox & Huang (1991), Karatzas, Lehoczky & Shreve (1987), He & Pearson (1991), Karatzas, Lehoczky, Shreve & Xu (1991), Kramkov & Schachermayer (1999), Goll & Rüschendorf (2001), and Bellini & Frittelli (2002). Optimal investment policies under joint budget and downside risk constraints in terms of value at risk and a second risk functional¹ have been studied in a Brownian setting by Basak & Shapiro (2001) and Gabih et al. (2005). For these special cases, Basak & Shapiro (2001) and Gabih et al. (2005) analyze the economic impact of the risk constraints. Solutions are suggested by duality methods, but both articles do not verify that these actually satisfy the constraints and hence exist. In contrast to the one-dimensional case of a budget constraint only, this verification together with precise conditions for existence provides the most difficult part of the analysis. To the best of our knowledge, we are the first who close this gap in the literature. In addition, we formulate the risk constraint in terms of convex risk measures and do not stick to a Brownian world but provide a complete solution to the problem in a general semimartingale setting.

In the case of a utility function which is not bounded from above existence of a solution to the utility maximization problem has an interesting implication. If its utility is finite, the market has the *no free lunch with vanishing risk property* (NFLVR) in the sense of Delbaen and Schachermayer, see, e.g., Delbaen & Schachermayer (1997) and the references therein. This parallels related results of Ankircher & Imkeller (2005) who investigate utility maximization without risk constraint and the role of asymmetric information.

Our article is also the basis for a solution of a robust utility maximization problem under a joint robust budget and risk constraint in an incomplete market if an economic agent is faced with

¹It should be pointed out that the alternative risk functional in Basak & Shapiro (2001) and Gabih, Grecksch & Wunderlich (2005) is not a risk measure in the sense of Artzner et al. (1999) or Föllmer & Schied (2004), since it is not translation-invariant. Nevertheless, it can be shown that their risk constraint can be reformulated in terms of a risk measure. Conceptually, the ideas in Section 2.3 need to be applied to a particular loss function which is piecewise linear.

model uncertainty. This problem is solved in Gundel & Weber (2007). For an axiomatic theory of robust utility we refer to Gilboa & Schmeidler (1989) and Föllmer & Schied (2004). Solutions to robust utility maximization problems without a risk constraint can be found in Quenez (2004), Gundel (2005), Föllmer & Gundel (2006), Schied (2005), and Schied & Wu (2005) .

The paper is organized as follows. In Section 2 we present the constrained maximization problem. Section 3 is devoted to the solution. The proofs are collected in Section 6. In Section 4 we discuss specific examples of price processes, namely geometric Brownian motion and a geometric Poisson process. Section 5 concludes the article. An appendix contains auxiliary results.

2 The Constrained Maximization Problem

We consider a market over a finite time horizon T which consists of $d + 1$ assets, one bond and d stocks. We suppose that the bond price is constant. The price processes of the stocks are given by an \mathbb{R}^d -valued semimartingale S on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, R)$ satisfying the usual conditions, where $\mathcal{F} = \mathcal{F}_T$; see Protter (2004), page 3.

An \mathcal{F} -measurable random variable will be interpreted as the value of a *financial position* at maturity T or, equivalently, as the *terminal wealth* of an agent. Positions which are R -almost surely equal can be identified. The set of all terminal financial positions is denoted by L^0 .

A self-financing portfolio with initial value x is a d -dimensional predictable, S -integrable process $(\xi_t)_{0 \leq t \leq T}$ which specifies the amount of each asset in the portfolio. The corresponding value process of the portfolio is given by

$$V_t := x + \int_0^t \xi_s dS_s \quad (0 \leq t \leq T). \quad (1)$$

The family $\mathcal{V}(x)$ denotes all non-negative value processes of self-financing portfolios with initial value equal to x .

Definition 2.1. A probability measure P which is absolutely continuous with respect to R is called an *absolutely continuous martingale measure* if any $V \in \mathcal{V}(1)$ is a local martingale under P . The family of these measures is denoted by \mathcal{P} . Any $P \in \mathcal{P}$ which is equivalent to R is called an *equivalent local martingale measure*. The family of these measures will be denoted by \mathcal{P}_e .

We interpret measures in the set \mathcal{P} as pricing measures and assume throughout that

$$\mathcal{P} \neq \emptyset. \quad (2)$$

The stronger statement $\mathcal{P}_e \neq \emptyset$ is related to the absence of arbitrage opportunities (namely *no free lunch with vanishing risk*), and will actually follow from the existence of a solution of the utility-maximization problem if the utility function is not bounded from above.

2.1 Utility Functionals

We are interested in maximizing the utility from terminal wealth given a joint budget and risk measure constraint. The problem consists in finding a maximal element $W \in \mathcal{F}$ with respect to a given preference order \succeq on some set of admissible financial positions. Under mild conditions such a preference order admits a numerical representation

$$W \succeq \tilde{W} \iff U(W) \geq U(\tilde{W})$$

with some utility functional U . If the preference order satisfies the von-Neumann-Morgenstern or Savage axioms, then U can be expressed in terms of a Bernoulli utility function $u : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ and a probability measure Q_0 which is equivalent to the reference measure R , i.e.,

$$U(W) = E_{Q_0}[u(W)]. \quad (3)$$

In the current paper we consider the case of preference orders that admit a numerical representation (3). This analysis is also the basis for an extension to robust expected utility in the sense of Gilboa & Schmeidler (1989) which is considered in Gundel & Weber (2007).

We always assume that the utility function $u : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ is strictly increasing, strictly concave, continuously differentiable in the interior of the domain $\text{dom } u := \{x \in \mathbb{R} : u(x) > -\infty\}$, and satisfies the Inada conditions

$$u'(\infty) := \lim_{x \rightarrow \infty} u'(x) = 0, \quad (\text{U1})$$

$$u'(\bar{x}_u) := \lim_{x \searrow \bar{x}_u} u'(x) = \infty \quad (\text{U2})$$

for $\bar{x}_u := \inf\{x \in \mathbb{R} : u(x) > -\infty\}$. It follows that the interior of the essential domain of u is given by the open interval $\text{dom } u = (\bar{x}_u, \infty)$. Note that \bar{x}_u might actually take the value $-\infty$.

The inverse of the derivative of u will play a crucial role in the analysis and is denoted by $I := (u')^{-1}$.

2.2 The Budget Constraint

Let us fix an initial wealth $x_2 \in \mathbb{R}$, and let $P \in \mathcal{P}$ be an absolutely continuous martingale measure. We will consider financial positions $W \in L^1(P)$ that satisfy the following budget constraint:

$$E_P[W] \leq x_2. \quad (4)$$

Intuitively, (4) signifies that the P -price of W is bounded by x_2 . W.l.o.g. we may and will always choose $x_2 > \bar{x}_u$, since otherwise no terminal wealth with finite utility satisfies the budget constraint (4).

Remark 2.2. If the set of equivalent martingale measures \mathcal{P}_e contains only one element, the market is complete. For any financial position $W \in L^1(P)$ there exists a trading strategy in the underlying assets, described by a d -dimensional predictable, S -integrable process $(\xi_t)_{0 \leq t \leq T}$, such that P -almost surely

$$E_P[W|\mathcal{F}_t] = E_P[W] + \int_0^t \xi_s dS_s \quad (0 \leq t \leq T),$$

i.e., any financial position is attainable by a self-financing strategy, see Jacod (1975), Theorem 5.4. The budget constraint (4) guarantees that the cost of replication is bounded by x_2 . Hence, requiring the terminal wealth to satisfy the budget constraint (4) is exactly the same as considering only those financial positions that can be obtained as the terminal wealth from a self-financing trading strategy if the initial wealth is x_2 . The wealth maximization problem is thus equivalent to the dynamic portfolio optimization problem of maximizing $E_{Q_0} \left[u \left(x_2 + \int_0^T \xi_s dS_s \right) \right]$ over some set of admissible strategies ξ .

Remark 2.3. If the set of equivalent martingale measures \mathcal{P}_e contains more than one element, the market is incomplete and the terminal wealth can in general not be replicated by a self-financing strategy. In addition, the budget constraint (4) does not even guarantee that solutions to the optimization problem which we are going to consider can be super-hedged by portfolios with an initial value of x_2 . That is, if $W \in L^1(P)$ is an optimal solution to the constrained optimization problem (9) below, there might not exist a trading strategy $(\xi_t)_{0 \leq t \leq T}$ in the underlying assets such that P -almost surely

$$E_P[W|\mathcal{F}_t] \leq x_2 + \int_0^t \xi_s dS_s \quad (0 \leq t \leq T).$$

Thus, P is not directly linked to replication costs. Nevertheless, if $\mathcal{P}_e \neq \emptyset$, thus the market being free of arbitrage, then $P \in \mathcal{P}$ could be interpreted as both a pricing measure which is used by a particular financial institution or the pricing measure in market equilibrium.

In addition, the solution of the utility maximization problem (9) below for an incomplete market (i.e., $|\mathcal{P}_e| > 1$) provides the basis for the analysis of the robust utility maximization problem in Gundel & Weber (2007).

2.3 The Risk Constraint

We will investigate the problem of utility maximization in presence of both a budget and a risk constraint. The risk of a financial position can be quantified by appropriate risk measures. We let \mathcal{D} be some vector space of random variables that contains the constants.

Definition 2.4. A mapping $\rho : \mathcal{D} \rightarrow \mathbb{R}$ is called a *risk measure* (on \mathcal{D}) if it satisfies the following conditions for all $W_1, W_2 \in \mathcal{D}$:

- *Inverse Monotonicity:* If $W_1 \leq W_2$, then $\rho(W_1) \geq \rho(W_2)$.
- *Translation Invariance:* If $m \in \mathbb{R}$, then $\rho(W + m) = \rho(W) - m$.

VaR is a risk measure according to the above definition, but it does in general not encourage diversification of positions, since it is not a *convex* risk measure if $L^\infty \subseteq \mathcal{D}$. In the current article we focus on a particular example of a convex risk measure, namely *utility-based shortfall risk*. Utility-based shortfall risk is most easily defined as a *capital requirement*, i.e., the smallest monetary amount that has to be added to a position to make it acceptable.² We will now give the definition of utility-based shortfall risk.

Let $\ell : \mathbb{R} \rightarrow [0, \infty]$ be a convex loss function, i.e., an increasing function that is not constant. The level x_1 shall be a point in the interior of the range of ℓ . Let Q_1 be a fixed subjective probability measure equivalent to R , which we will use for the purpose of risk management.³ The space of financial positions \mathcal{D} is chosen in such a way that for $W \in \mathcal{D}$ the integral $\int \ell(-W)dQ_1$ is well defined.

²Note that every static risk measure can be defined as a capital requirement, see Föllmer & Schied (2004). To be more precise, if ρ is a risk measure, then $\mathcal{A} = \{W \in \mathcal{D} : \rho(W) \leq 0\}$ defines its acceptance set, i.e., the set of positions with non-positive risk. ρ is then recovered as $\rho(W) = \inf\{m \in \mathbb{R} : W + m \in \mathcal{A}\}$.

³For example, in our model one could suppose that both Q_1 and Q_0 signify the empirical real world measure.

Define an acceptance set

$$\mathcal{A} = \{W \in \mathcal{D} : E_{Q_1}[\ell(-W)] \leq x_1\}. \quad (5)$$

A financial position is thus acceptable if the expected value of $\ell(-W)$ under the subjective probability measure Q_1 , i.e., the expected loss $E_{Q_1}[\ell(-W)]$, is not more than x_1 .⁴

The acceptance set \mathcal{A} induces the risk measure *utility-based shortfall risk* (*UBSR* in the following) ρ as the associated capital requirement

$$\rho(W) = \inf\{m \in \mathbb{R} : W + m \in \mathcal{A}\}. \quad (6)$$

Utility-based shortfall risk is convex and does therefore encourage diversification. Examples of loss functions ℓ include exponentials $\exp(\alpha x)$, $\alpha > 0$, which leads to the so-called *entropic risk measure*, for which a simple explicit formula is available; see Föllmer & Schied (2004), Example 4.105. Alternatively, one-sided loss functions can be used to measure downside risk only. These risk measures look at losses only and do not consider tradeoffs between gains and losses. Examples include $x^\alpha \cdot \mathbf{1}_{(0,\infty)}(x)$, $\alpha > 1$, or exponentials $\exp(\alpha x) \cdot \mathbf{1}_{(0,\infty)}(x)$, $\alpha > 0$, where $\mathbf{1}_{(0,\infty)}$ denotes the indicator function of $(0, \infty)$.

In the current article we investigate utility maximization under a risk measure constraint. The shortfall risk constraint (*UBSR constraint* in the following) shall be given by

$$\rho(W) \leq 0. \quad (7)$$

A financial position W which satisfies (7) is acceptable from the point of view of the risk measure ρ . This is equivalent to

$$E_{Q_1}[\ell(-W)] \leq x_1. \quad (8)$$

We require the loss function ℓ to satisfy additional technical conditions. We assume that ℓ is strictly convex, strictly increasing, and continuously differentiable on the interval $(-\bar{x}_\ell, \infty)$ for some $\bar{x}_\ell \in \mathbb{R} \cup \{\infty\}$, that $\ell(x) = 0$ for $x \leq -\bar{x}_\ell$, and that ℓ is continuous on the whole real line and $\lim_{x \rightarrow \infty} \ell'(x) = \infty$.

⁴We have defined acceptability in terms of a loss function ℓ . Alternatively, we could define $u_\ell(x) = -\ell(-x)$ and interpret u_ℓ as a *Bernoulli utility function*. $U_\ell(W) = E_{Q_1}[u_\ell(W)]$ defines in this case a *von Neumann Morgenstern utility*. W is thus acceptable if its utility is at least $-x_1$. This explains why the risk measure is called *utility-based*.

2.4 The Maximization Problem

Let us denote the set of terminal financial positions with well defined utility by

$$\mathcal{I} = \{W \in L^1(\Omega, \mathcal{F}, P) : W \geq \bar{x}_u \text{ and } u(W)^- \in L^1(\Omega, \mathcal{F}, Q_0)\}.$$

We are now able to formulate the optimization problem under a joint budget and UBSR constraint:

$$\text{Maximize } E_{Q_0}[u(W)] \text{ over all } W \in \mathcal{I} \text{ that satisfy } E_{Q_1}[\ell(-W)] \leq x_1 \text{ and } E_P[W] \leq x_2. \quad (9)$$

The set of all financial positions in \mathcal{I} that satisfy the constraints is denoted by \mathcal{W} , i.e.,

$$\mathcal{W} := \{W \in L^1(\Omega, \mathcal{F}, P) : W \geq \bar{x}_u, u(W)^- \in L^1(\Omega, \mathcal{F}, Q_0), E_{Q_1}[\ell(-W)] \leq x_1, \text{ and } E_P[W] \leq x_2\}.$$

If $\bar{x}_u > -\infty$, then we may and will always assume w.l.o.g. that $\bar{x}_\ell \in (\bar{x}_u, \infty]$. Since any terminal wealth with utility larger than $-\infty$ does not take any values below \bar{x}_u with positive probability, any loss constraint with $\bar{x}_\ell \leq \bar{x}_u$ is trivially satisfied, and we are back in the classical case without any risk constraint.

3 The Solution

We will show that under suitable integrability assumptions the unique solution to the constrained maximization problem (9) can be written in the form

$$w^* \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right),$$

where $w^* : [0, \infty) \times (0, \infty) \rightarrow (\bar{x}_u, \infty)$ is a continuous deterministic function, and λ_1^*, λ_2^* are suitable real parameters. $\frac{dQ_1}{dQ_0}$ and $\frac{dP}{dQ_0}$ signify the Radon-Nikodym densities of Q_1 and P with respect to Q_0 . w^* is obtained as the solution of a family of deterministic maximization problems. We state the solution first and postpone all proofs to later sections.

We define a family of functions g_{y_1, y_2} with $y_1, y_2 \geq 0$ by

$$g_{y_1, y_2}(x) := u(x) - y_1 \ell(-x) - y_2 x.$$

For each pair $y_1 \geq 0, y_2 > 0$ the maximizer of g_{y_1, y_2} is unique and equals

$$w^*(y_1, y_2) := \begin{cases} J(y_1, y_2) & \text{if } y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell+), \\ \bar{x}_\ell & \text{if } u'(\bar{x}_\ell) \leq y_2 \leq u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell+), \\ I(y_2) & \text{if } y_2 < u'(\bar{x}_\ell), \end{cases} \quad (10)$$

see Appendix A. Here, $J(y_1, y_2)$ denotes the unique solution to the equation $u'(x) + y_1 \ell'(-x) = y_2$ for the case that $y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell+)$, and $I = (u')^{-1}$. Note that $w^*(0, y_2) = I(y_2) = J(0, y_2)$.

The derivation of the solution of (9) requires as prerequisite the solution of a related problem. We need to determine a financial position $Y^* \geq \bar{x}_u$ that minimizes the expected loss under the budget constraint (4). That is, we have to solve the problem

$$\text{Minimize } E_{Q_1}[\ell(-W)] \text{ over all financial positions } W \geq \bar{x}_u \text{ with } E_P[W] \leq x_2. \quad (11)$$

We will see that the solution to this problem is of the form

$$-L \left(c^* \frac{dP}{dQ_1} \right).$$

Here $L : \mathbb{R} \rightarrow [-\bar{x}_\ell, -\bar{x}_u]$ is defined as the generalized inverse of the derivative of the loss function ℓ , i.e.,

$$L(y) := \begin{cases} -\bar{x}_u & \text{if } y \geq \ell'(-\bar{x}_u), \\ (\ell')^{-1}(y) & \text{if } \ell'(-\bar{x}_\ell+) < y < \ell'(-\bar{x}_u), \\ -\bar{x}_\ell & \text{if } y \leq \ell'(-\bar{x}_\ell+). \end{cases} \quad (12)$$

L is a continuous function which is strictly increasing on $[\ell'(-\bar{x}_\ell+), \ell'(-\bar{x}_u)]$. Properties of the functions w^* and L are collected in Appendix A.

In order to guarantee the existence of our solution to the optimization problem (9), we have to make the following technical assumptions.

Assumption 3.1. Let the functions w^* and L be defined as in (10) and (12). We impose the following integrability assumptions for all $\lambda_1 \geq 0, \lambda_2 > 0$, and $c > 0$:

- (a) $w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \in L^1(P)$,
- (b) $\ell \left(-w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \in L^1(Q_1)$,
- (c) $u \left(w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right)^- \in L^1(Q_0)$,
- (d) $L \left(c \frac{dP}{dQ_1} \right) \in L^1(P)$,
- (e) $\ell \left(L \left(c \frac{dP}{dQ_1} \right) \right) \in L^1(Q_1)$.

Assumption 3.1 extends standard integrability conditions to the case of utility maximization under a joint budget and UBSR constraint. Assumptions (a)-(c) guarantee that price, expected loss and utility of the solution are well defined. Assumptions (d) and (e) impose integrability of

the solution to the loss minimization problem (11), which is an intermediate step in the analysis of problem (9). In contrast to the utility maximization problem without risk constraint, the existence of a solution to (9) is not immediate from Assumption 3.1, but requires a sophisticated analysis of the constraints, see Lemma 6.1 and Section 6.4 below.

Before we proceed with our main results, let us discuss sufficient conditions for Assumption 3.1. If the essential domain of the utility function u is bounded from below, Assumption 3.1 follows from a growth condition on u and a moment condition on $\frac{dQ_0}{dP}$. This parallels the assumptions made in Aumann & Perles (1965), Cox & Huang (1991), and Bank & Riedel (2001).

Proposition 3.2. *Assume that $\bar{x}_u = 0$, $\mathcal{W} \neq \emptyset$ and that u has regular asymptotic elasticity (RAE) in the sense of Kramkov & Schachermayer (1999), i.e.,*

$$\limsup_{x \rightarrow \infty} \frac{xu'(x)}{u(x)} < 1$$

Suppose moreover that there exist $b \in (0, 1)$ and $C > 0$ such that

$$\limsup_{x \rightarrow \infty} \frac{u(x)}{x^b} \leq C, \tag{13}$$

$$\frac{dQ_0}{dP} \in L^{\frac{b}{1-b}}(Q_0). \tag{14}$$

Then Assumption 3.1 holds, and condition Assumption 3.1(c) can be replaced by the stronger statement that for all $\lambda_1 \geq 0$, $\lambda_2 > 0$,

$$u \left(w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \in L^1(Q_0).$$

Proof. See Section 6.1. □

Let us now state the solution to the loss minimization problem (11).

Lemma 3.3. *Suppose that Assumptions 3.1(d)&(e) hold and let $x_2 < \bar{x}_\ell$. Then the equation*

$$x_2 = -E_P \left[L \left(c \frac{dP}{dQ_1} \right) \right] \tag{15}$$

has a solution $c^ > 0$. A solution to problem (11) is given by*

$$Y^* := -L \left(c^* \frac{dP}{dQ_1} \right),$$

and the budget constraint is binding. On the set $\{\frac{dP}{dR} > 0\}$, the loss minimizing terminal wealth is R -almost surely unique, i.e., $Y^ 1_{\{dP/dR > 0\}} = \tilde{Y}^* 1_{\{dP/dR > 0\}}$ R -almost surely for any other solution \tilde{Y} to (11).*

Proof. See Section 6.2. □

Suppose now that Assumption 3.1 holds. Before stating the main result of the article, we have to introduce some notation. We assume that $x_2 > \bar{x}_u$, since otherwise there is no solution to the maximization problem. We let $x_1 > 0$, according to the definition of UBSR. There exists a unique solution $\tilde{\lambda}_2 > 0$ of the equation⁵

$$x_2 = E_P \left[I \left(\lambda_2 \frac{dP}{dQ_0} \right) \right]. \quad (16)$$

The terminal wealth $I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right)$ is actually the unique optimal solution to the utility maximization problem if we remove the risk constraint, see, e.g., Cox & Huang (1989) or Kramkov & Schachermayer (1999).

We will now state the solution to the original problem (9) which we will prove in Section 6.

Theorem 3.4. *Suppose that Assumption 3.1 holds. Let $x_1 > 0$, $x_2 > \bar{x}_u$, and let c^* and $\tilde{\lambda}_2$ be defined as in (15) and (16). There are three cases:*

(i) *We have $x_2 < \bar{x}_\ell$ and $x_1 \leq E_{Q_1} \left[\ell \left(L \left(c^* \frac{dP}{dQ_1} \right) \right) \right]$. Then there exists either no or P -almost surely just one financial position which satisfies both constraints.*

(ii) *We have $E_{Q_1} \left[\ell \left(-I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right) \right) \right] < x_1$. This implies that either $x_2 \geq \bar{x}_\ell$ or, if $x_2 < \bar{x}_\ell$, $x_1 > E_{Q_1} \left[\ell \left(L \left(c^* \frac{dP}{dQ_1} \right) \right) \right]$.*

Then

$$W^* := I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right)$$

is a solution to the maximization problem (9), and the UBSR constraint is not binding. If $u(W^) \in L^1(Q_0)$, then it is the unique solution.*

(iii) *We have either $x_2 \geq \bar{x}_\ell$ or, if $x_2 < \bar{x}_\ell$, $x_1 > E_{Q_1} \left[\ell \left(L \left(c^* \frac{dP}{dQ_1} \right) \right) \right]$, and in both cases $E_{Q_1} \left[\ell \left(-I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right) \right) \right] \geq x_1$.*

Then a solution to the maximization problem (9) exists and both constraints are binding. A

⁵This follows from Assumption 3.1 (a) for $\lambda_1 = 0$, the Inada conditions (U1) and (U2), and the monotone convergence theorem.

solution is given by

$$W^* := w^* \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right) = \begin{cases} J \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right) & \text{on } \left\{ \lambda_2^* \frac{dP}{dQ_0} > u'(\bar{x}_\ell) + \lambda_1^* \frac{dQ_1}{dQ_0} \ell'(-\bar{x}_\ell+) \right\}, \\ \bar{x}_\ell & \text{on } \left\{ u'(\bar{x}_\ell) \leq \lambda_2^* \frac{dP}{dQ_0} \leq u'(\bar{x}_\ell) + \lambda_1^* \frac{dQ_1}{dQ_0} \ell'(-\bar{x}_\ell+) \right\}, \\ I \left(\lambda_2^* \frac{dP}{dQ_0} \right) & \text{on } \left\{ \lambda_2^* \frac{dP}{dQ_0} < u'(\bar{x}_\ell) \right\}, \end{cases}$$

where w^* and J are defined as in (10), and $\lambda_1^* \geq 0$, $\lambda_2^* > 0$ satisfy

$$x_1 = E_{Q_1} [\ell(-W^*)] \quad (17)$$

and

$$x_2 = E_P [W^*]. \quad (18)$$

If $u(W^*) \in L^1(Q_0)$, then W^* is the unique solution.

Proof. See Section 6.3. □

This theorem provides a complete solution to the utility maximization problem (9) in all possible cases. Case (i) includes two subcases. The first is the irrelevant situation in which the constraints are too strict and there exists no terminal wealth that satisfies both constraints. The second subcase is highly nongeneric with the loss of the loss-minimizing terminal wealth being equal to the loss threshold x_1 . On the subset of Ω where P is equivalent to R the only possible investment is the one in the loss-minimizing position, and on the complement we should take W^* as large as possible. For case (ii) observe that $I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right)$ is the solution to the utility maximization problem without risk constraint. If this position satisfies the UBSR constraint, then it must also be a solution of the optimization problem with UBSR constraint.

Finally, (iii) covers all the remaining cases. In this case, the solution can be interpreted as a portfolio of an unconstrained solution under a modified budget constraint and two puts with strike \bar{x}_ℓ , i.e.,

$$W^* = I \left(\lambda_2^* \frac{dP}{dQ_0} \right) + \left(\bar{x}_\ell - I \left(\lambda_2^* \frac{dP}{dQ_0} \right) \right)^+ - \left(\bar{x}_\ell - J \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right) \right)^+. \quad (19)$$

The portfolio contains a long position in the asset $I \left(\lambda_2^* \frac{dP}{dQ_0} \right)$ which is the solution for a tighter budget constraint, but no risk constraint. Because of the UBSR constraint, an optimizing agent

needs to buy insurance against portfolio values below \bar{x}_ℓ . Here, a very conservative strategy would be the approach of a portfolio insurer who buys protection against any shortfall below the threshold \bar{x}_ℓ . Such an agent goes long in a put on $I\left(\lambda_2^* \frac{dP}{dQ_0}\right)$ with strike \bar{x}_ℓ , which guarantees full protection. In the maximization problem (9) the UBSR constraint is, however, not that tight. The agent can still short a put on the asset $J\left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0}\right)$ with strike \bar{x}_ℓ and gain some additional profit from selling this put. Since $J(y_1, y_2) \geq I(y_2)$ for all $(y_1, y_2) \in [0, \infty) \times (0, \infty)$, the second put in (19) will only be exercised if the first put is exercised. In this case, the gains from the first put are larger than the losses from the second put. Hence, going short in the second put makes our investment less costly, but we are still partly insured against losses. The final payoff is in general not bounded from below, unless the domain of the utility function is bounded from below, that is, $\bar{x}_u > -\infty$.

Corollary 3.5. *Suppose that Assumption 3.1 holds. Let $x_1 > 0$, $x_2 > \bar{x}_u$, and assume that the maximization problem (9) admits a solution W^* . If u is not bounded from above and the optimal wealth has finite utility, i.e., $E_{Q_0}[u(W^*)] < \infty$, then P is equivalent to R . In particular, $\mathcal{P}_e \neq \emptyset$, and the financial market satisfies the no free lunch with vanishing risk property (NFLVR).*

Proof. According to Theorem 3.4, the optimal solution is given by W^* . Suppose P is not equivalent to R . Then $Q_0\left\{\frac{dP}{dQ_0} = 0\right\} > 0$, and $W^* = \infty$ on $\left\{\frac{dP}{dQ_0} = 0\right\}$ R -almost surely according to Remark A.2 and since $I(0) := \lim_{y_2 \rightarrow 0} I(y_2) = \infty$. Thus, $E_{Q_0}[u(W^*)] = \infty$, a contradiction. This implies that P is equivalent to R . NFLVR follows from the first fundamental theorem of asset pricing. \square

4 Examples

In the current section we focus on two examples of a financial market with a single risky stock and a bond. We assume that the bond price is constant. The stock price is modelled either as a geometric Brownian motion or a geometric Poisson process. For an exponential utility function, we compare the optimal wealth in the UBSR-constrained problem with a binding risk constraint to two benchmark cases: the solution to the classical problem without risk constraint, and the solution to the utility maximization problem if the risk constraint is defined in terms of Value at Risk.⁶

⁶The solution to the utility maximization problem under a VaR constraint can be found in Basak & Shapiro (2001).

As Bernoulli utility function we choose $u(x) = 1 - e^{-x}$. The loss function shall be given by $\ell(x) = (e^x - e^{-\bar{x}\ell}) \vee 0$, and we set $\bar{x}\ell = 0$. The deterministic function w^* can then easily be calculated as

$$w^*(y_1, y_2) = \begin{cases} -\log(y_2) + \log(1 + y_1) & \text{if } y_2 > e^{-\bar{x}\ell} + y_1 e^{-\bar{x}\ell}, \\ \bar{x}\ell & \text{if } e^{-\bar{x}\ell} \leq y_2 \leq e^{-\bar{x}\ell} + y_1 e^{-\bar{x}\ell}, \\ -\log(y_2) & \text{if } y_2 < e^{-\bar{x}\ell}. \end{cases}$$

4.1 A Geometric Brownian Motion Model

In our first example we assume that the stock price $(S_t)_{0 \leq t \leq T}$ can be described by a generalized geometric Brownian motion under the subjective measure Q_0 . To be precise, we assume that $B^0 = (B_t^0)_{0 \leq t \leq T}$ is a Brownian motion under the measure Q_0 . The information filtration shall be generated by B^0 . The dynamics of S is described by the stochastic differential equation

$$dS_t = S_t(\sigma_t dB_t^0 + \mu_t^0 dt) \quad (0 \leq t \leq T),$$

where the stochastic mean $\mu^0 = (\mu_t^0)_{0 \leq t \leq T}$ and the volatility $\sigma = (\sigma_t)_{0 \leq t \leq T}$ with $\sigma_t > 0$ are suitable stochastic processes.

In this case, the financial market is complete, and the unique absolutely continuous and equivalent martingale measure P is given by the stochastic exponential

$$\frac{dP}{dQ_0} = \mathcal{E} \left(- \int_0^T \alpha_s^0 B_s^0 \right) = \exp \left(- \int_0^T \alpha_s^0 B_s^0 - \frac{1}{2} \int_0^T (\alpha_s^0)^2 ds \right),$$

where $\alpha^0 := \mu^0 / \sigma$.

Let us now define the subjective measure Q_1 , which is used for the risk constraint of the utility maximization problem. Let $\mu_1 = (\mu_t^1)_{0 \leq t \leq T}$ be a suitable stochastic process. Setting $\alpha^1 := \mu^1 / \sigma$ and

$$B_t^1 := B_t^0 + (\alpha_t^0 - \alpha_t^1)t \quad (0 \leq t \leq T),$$

we assume that B^1 is a Brownian motion under the measure Q_1 . By Girsanov's theorem, this holds true if and only if the Radon-Nikodym density of Q_1 with respect to Q_0 is given by the stochastic exponential

$$\frac{dQ_1}{dQ_0} = \mathcal{E} \left(\int_0^T (\alpha_s^1 - \alpha_s^0) dB_s^0 \right) = \exp \left(\int_0^T (\alpha_s^1 - \alpha_s^0) dB_s^0 - \frac{1}{2} \int_0^T (\alpha_s^1 - \alpha_s^0)^2 ds \right).$$

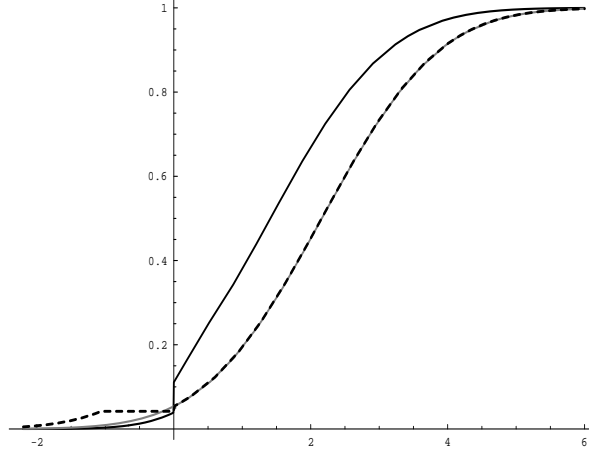


Figure 1: Distribution function of the optimal terminal wealth for a stock price driven by geometric Brownian motion. Black line: with UBSR constraint; gray line: without risk constraint; dashed line: with VaR constraint.

In terms of the Q_1 -Brownian motion B^1 , the stock price S can be rewritten as

$$dS_t = S_t(\sigma_t dB_t^1 + \mu_t^1 dt) \quad (0 \leq t \leq T).$$

The measures Q_0 , Q_1 , and P completely specify the financial market model. By Theorem 3.4 we obtain as optimal terminal wealth

$$W^* = w^* \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right) = \begin{cases} -\log \left(\lambda_2^* \frac{dP}{dQ_0} \right) + \log \left(1 + \lambda_1^* \frac{dQ_1}{dQ_0} \right) & \text{on } \left\{ \lambda_2^* \frac{dP}{dQ_0} > e^{-\bar{x}_\ell} + \lambda_1^* \frac{dQ_1}{dQ_0} e^{-\bar{x}_\ell} \right\}, \\ \bar{x}_\ell & \text{on } \left\{ e^{-\bar{x}_\ell} \leq \lambda_2^* \frac{dP}{dQ_0} \leq e^{-\bar{x}_\ell} + \lambda_1^* \frac{dQ_1}{dQ_0} e^{-\bar{x}_\ell} \right\}, \\ -\log \left(\lambda_2^* \frac{dP}{dQ_0} \right) & \text{on } \left\{ \lambda_2^* \frac{dP}{dQ_0} < e^{-\bar{x}_\ell} \right\}. \end{cases}$$

Here, λ_1^* and λ_2^* have to be chosen such that W^* satisfies the budget constraint. Since the densities of P and Q_1 with respect to Q_0 are known, the agent's optimal wealth can explicitly be rewritten in terms of stochastic integrals.

For $\alpha^0 \equiv 0.3$, $\alpha^1 \equiv 0.2$, $T = 20$, $x_1 = 0.18$, $x_2 = 0.36$, and a VaR-level at 0.1 under Q_1 ,⁷ Figures 1 and 2 show the cumulative distribution functions and densities of the agent's optimal wealth under

⁷Observe that VaR is calculated under the measure Q_1 while the distribution and density functions are plotted under the measure Q_0 .

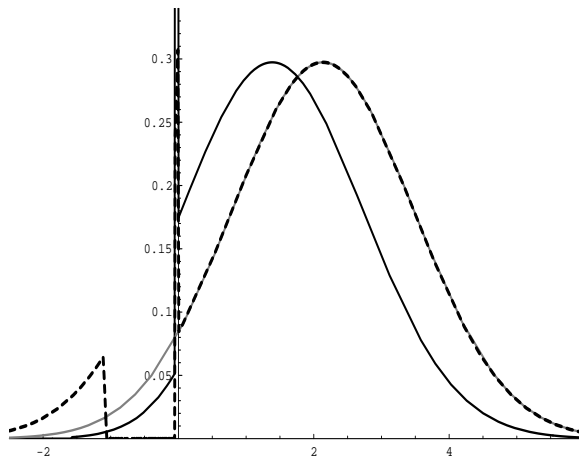


Figure 2: Density function of the optimal terminal wealth for a stock price driven by geometric Brownian motion. Black line: with UBSR constraint; gray line: without risk constraint; dashed line: with VaR constraint.

different constraints under the measure Q_0 . We compare the solution W^* of the maximization problem under the UBSR constraint (black line) with the solutions of the problem without risk constraint (gray line) and with a VaR constraint (dashed line). Both risk constraints limit the probability of losses considerably. The VaR constraint, however, leads to a higher probability of very large losses compared to the solution without any risk constraint. In this case there is only a slight change in the distribution of the VaR-optimal wealth for positive values, the main shift takes place on the negative side where the probability of small losses is decreased whereas the probability of very large losses is increased. Risk management based on VaR encourages insurance against medium size losses, but favors high losses. UBSR, in contrast, reduces the risk of very high losses. In the context of the current model, regulators and managers should hence better use UBSR measures instead of VaR in order to prevent high losses.

4.2 A Pure Jump Model

In the second example we will investigate what happens if the stock price is driven by a pure jump process instead of geometric Brownian motion. We restrict our attention to a stock price which is driven by a Poisson process $N = (N_t)_{0 \leq t \leq T}$ with jump rate λ under the measure Q_0 . We assume

that N generates the filtration. The process M defined by $M_t^0 := N_t - \lambda t$ ($0 \leq t \leq T$) is a Q_0 -martingale. We assume that the stock price S is a geometric Poisson process whose dynamics can be described by the following stochastic differential equation,

$$dS_t = \mu^0 S_t dt + \sigma S_{t-} dM_t^0 \quad (0 \leq t \leq T)$$

for some $\mu^0 \in \mathbb{R}$ and $\sigma > 0$ such that $\mu^0/\sigma < \lambda$. Then the financial market is complete, and the unique absolutely continuous and equivalent martingale measure is given by the Radon-Nikodym density

$$\frac{dP}{dQ_0} = \exp(\alpha^0 T) \left(1 - \frac{\alpha^0}{\lambda}\right)^{N_T},$$

where $\alpha^0 := \mu^0/\sigma$.

For simplicity, we assume that the subjective probability measure Q_1 is specified in the following way. Let $\mu^1 \in \mathbb{R}$ be given such that $\mu^1/\sigma < \lambda$. With $\alpha^1 := \mu^1/\sigma$, we let Q_1 be the measure under which M^1 with $M_t^1 := M_t^0 + (\alpha^0 - \alpha^1)t$ is a martingale. Then the density of Q_1 with respect to Q_0 is given by

$$\frac{dQ_1}{dQ_0} = \exp((\alpha^0 - \alpha^1)T) \left(1 - \frac{\alpha^0 - \alpha^1}{\lambda}\right)^{N_T}.$$

The dynamics of the stock price can be rewritten in terms of M^1 :

$$dS_t = \mu^1 S_t dt + \sigma S_{t-} dM_t^1 \quad (0 \leq t \leq T).$$

Letting

$$\begin{aligned} A &:= \left\{ \lambda_2^* \frac{dP}{dQ_0} > e^{-\bar{x}_\ell} + \lambda_1^* \frac{dQ_1}{dQ_0} e^{-\bar{x}_\ell} \right\}, \\ B &:= \left\{ e^{-\bar{x}_\ell} \leq \lambda_2^* \frac{dP}{dQ_0} \leq e^{-\bar{x}_\ell} + \lambda_1^* \frac{dQ_1}{dQ_0} e^{-\bar{x}_\ell} \right\}, \\ C &:= \left\{ \lambda_2^* \frac{dP}{dQ_0} < e^{-\bar{x}_\ell} \right\}, \end{aligned}$$

the optimal terminal wealth is given by

$$\begin{aligned} W^* &= w^* \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right) \\ &= \begin{cases} -\log(\lambda_2^*) - \alpha^0 T - N_T \log\left(1 - \frac{\alpha^0}{\lambda}\right) + \log\left(1 + \lambda_1^* \exp((\alpha^0 - \alpha^1)T) \left(1 - \frac{\alpha^0 - \alpha^1}{\lambda}\right)^{N_T}\right) & \text{on } A, \\ \bar{x}_\ell & \text{on } B, \\ -\log(\lambda_2^*) - \alpha^0 T - N_T \log\left(1 - \frac{\alpha^0}{\lambda}\right) & \text{on } C, \end{cases} \end{aligned}$$

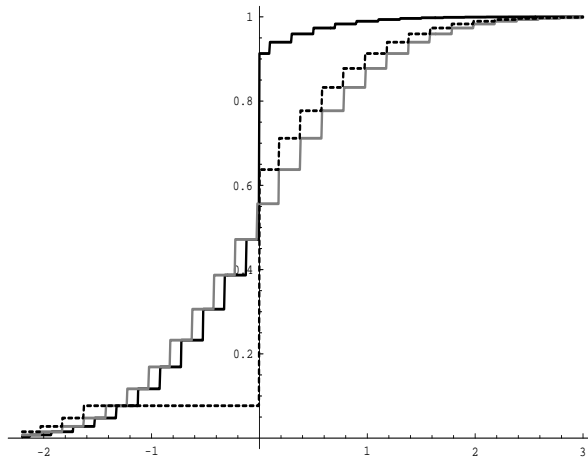


Figure 3: Distribution function of the terminal wealth for a stock price driven by a pure jump process. Black line: with UBSR constraint, gray line: without risk constraint, dashed line: with VaR constraint

where λ_1^* and λ_2^* have to be chosen such that W^* satisfies the budget constraint.

For $\alpha^0 = \alpha^1 \equiv 0.2$, $T = 20$, $x_1 = 0.6$, $x_2 = -0.9$, and a VaR-level at 0.12, Figure 4 shows the cumulative distribution functions for the optimal solutions under different constraints under the measure Q_0 . We compare the solution W^* of the maximization problem under the UBSR constraint (black line) with the solutions of the problem without risk constraint (gray line) and with a VaR constraint (dashed line). The results in the case of a pure jump stock price resemble the effects which we have already observed in the continuous model above. However, the jump of the dashed line in zero is much larger here than in the previous example since here the VaR constraint is strong enough to also shift mass from the negative to the positive side.

5 Conclusion

We provide a complete solution for the utility maximization problem under a joint budget and UBSR constraint in a financial market. We do not impose any specific assumptions on the price processes of the underlying assets and solve the problem in a general semimartingale framework. For a complete market, the wealth maximization problem is equivalent to a dynamic portfolio optimization problem. We characterize precisely under which conditions the budget constraint is too

strict and no solution can be obtained. Otherwise, there exists a solution to the maximization problem. In the latter case, the solution is explicitly determined in our article. The derivation requires a careful analysis of the constraints. The solution is represented as a deterministic function of two random variables. These random variables are given as multiples of Radon-Nikodym-derivatives. These involve the pricing measure, and the subjective probability measures which are used to assess losses and utility.

We then compare our solution to two benchmark portfolios: the optimal solutions of the utility maximization problems without risk constraint, and with a VaR constraint, respectively. We find that from a regulator's or manager's point of view one should favor the UBSR constraint over a VaR constraint. A VaR constraint leads to large losses in the worst states. It is actually worse than no constraint at all. Compared to both a VaR constraint and no risk constraint, the UBSR constraint decreases the size of the losses considerably. Thus, the convex risk measure UBSR is not only superior to VaR from the perspective of the axiomatic theory of risk measures, but also influences investments of rational agents in a desirable way.

An interesting extension of the problems discussed in the current article would be to include consumption in the analysis. In this case, utility depends on both terminal wealth as well as consumption. Without risk constraint, this problem has been solved for the Hindy-Huang-Kreps preferences in Bank & Riedel (2000) and Bank & Riedel (2001). With a risk constraint, the extended utility maximization problem is still open. For a suitable choice of the risk constraint one should impose dynamic consistency on risk measurements, see, e.g., Weber (2006) and Schied (2007).

6 Proofs

In this section we prove Proposition 3.2, Lemma 3.3 and Theorem 3.4. The proof of the theorem relies on another lemma whose proof is postponed to the subsection 6.4. We remark that the subjective measures Q_0 and Q_1 are equivalent to the reference measure R . Hence, a statement holds Q_i -almost surely ($i = 0, 1$) if and only if it holds R -almost surely.

6.1 Proof of Proposition 3.2

Proof of Proposition 3.2. We write “const” for arbitrary constants.

(a) For x large enough, we have

$$0 \leq u'(x) \leq \frac{u(x)}{x} \leq \text{const} \cdot x^{b-1}$$

This implies that for some constant d , we have

$$0 \leq I(y) \leq \begin{cases} \text{const} \cdot y^{\frac{1}{b-1}}, & y \text{ small,} \\ d, & \text{otherwise.} \end{cases}$$

Thus, $0 \leq E_P \left(I \left(\lambda \frac{dP}{dQ_0} \right) \right) \leq d + \text{const} \cdot E_{Q_0} \left(\left(\frac{dP}{dQ_0} \right)^{\frac{1}{b-1}} \right) < \infty$. We obtain (a), observing that $0 \leq w^*(y_1, y_2) \leq I(y_2)$ for all y_1, y_2 .

(b) We have $-w^* \leq 0$, $0 \leq \ell$, and ℓ is increasing. From this it follows that

$$0 \leq \ell \left(-w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \leq \ell(0).$$

This implies (b).

(c) Letting $W \in \mathcal{W}$ with $E_{Q_0}(u(W)) > -\infty$, we get from (21) using (a) & (b) that

$$\begin{aligned} -\infty &< E_{Q_0}(u(W)) \\ &- \lambda_1 \left(x_1 - E_{Q_1} \left[\ell \left(-w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \right] \right) \\ &- \lambda_2 \left(x_2 - E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right] \right) \\ &\leq E_{Q_0} \left[u \left(w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \right] \end{aligned}$$

This implies (c).

To show that we even have $u \left(w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \in L^1(Q_0)$, observe that $0 \leq w^*(y_1, y_2) \leq I(y_2)$ for all y_1, y_2 . This implies with some constant e ,

$$u(w^*(y_1, y_2)) \leq u(I(y_2)) \leq \begin{cases} \text{const} \cdot y_2^{\frac{b}{b-1}}, & y_2 \text{ small,} \\ e, & \text{otherwise.} \end{cases}$$

Thus,

$$E_{Q_0} \left[u \left(w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \right] \leq e + \text{const} \cdot E_{Q_0} \left[\left(\frac{dQ_0}{dP} \right)^{\frac{b}{1-b}} \right] < \infty$$

(e) We have $L : \mathbb{R} \rightarrow [-\bar{x}_\ell, 0]$. Since ℓ is nonnegative and increasing, we obtain that

$$0 \leq \ell \left(L \left(c \frac{dP}{dQ_1} \right) \right) \leq \ell(0).$$

This implies (e).

(d) $L\left(c\frac{dP}{dQ_1}\right)$ is bounded from above by 0. Thus, we need to show that $E_P\left(L\left(c\frac{dP}{dQ_1}\right)\right) > -\infty$. For $W \in \mathcal{W}$ the left hand side of (20) is bounded below by $-x_1$. Since $E_{Q_1}\left(\ell\left(L\left(c\frac{dP}{dQ_1}\right)\right)\right)$ is finite by (e), we obtain (d). \square

6.2 Proof of Lemma 3.3

Proof of Lemma 3.3. For any terminal wealth $W \geq \bar{x}_u$ with $E_P[W] \leq x_2$ and any $c > 0$ we have

$$\begin{aligned} E_{Q_1}[-\ell(-W)] &\leq E_{Q_1}[-\ell(-W)] + c(x_2 - E_P[W]) \\ &\leq E_R\left[\sup_{x > \bar{x}_u} \left(-\frac{dQ_1}{dR}\ell(-x) - c\frac{dP}{dR}x\right)\right] + cx_2 \\ &= -E_{Q_1}\left[\ell\left(L\left(c\frac{dP}{dQ_1}\right)\right)\right] + c\left(x_2 + E_P\left[L\left(c\frac{dP}{dQ_1}\right)\right]\right), \end{aligned} \quad (20)$$

where the final equality follows Lemma A.1(x).

$L\left(c\frac{dP}{dQ_1}\right)$ converges by definition to $-\bar{x}_\ell$ P -almost surely as $c \rightarrow 0$ and to $-\bar{x}_u$ as $c \rightarrow \infty$. Hence by Assumption 3.1(d) and monotone convergence, for any $\bar{x}_u < x_2 < \bar{x}_\ell$ we can find $c > 0$ such that

$$x_2 = -E_P\left[L\left(c\frac{dP}{dQ_1}\right)\right].$$

Let c^* be such a solution. Then $E_{Q_1}[-\ell(-W)] \leq E_{Q_1}\left[-\ell\left(L\left(c^*\frac{dP}{dQ_1}\right)\right)\right] = E_{Q_1}[-\ell(-Y^*)]$ for any $W \geq \bar{x}_u$ that satisfies the budget constraint. Y^* satisfies the budget constraint and is thus a solution to (15).

In order to show the uniqueness part, let $\tilde{Y} \geq \bar{x}_u$ be any other loss-minimizing position that satisfies the budget constraint. Since $\ell(-x) = 0$ for $x \geq \bar{x}_\ell$, also $\tilde{Y}1_{\{\tilde{Y} \leq \bar{x}_\ell\}}$ is a loss-minimizing position. Since ℓ is strictly convex on $[-\bar{x}_\ell, -\bar{x}_u]$ by assumption, we have $\tilde{Y}1_{\{\tilde{Y} \leq \bar{x}_\ell\}} = Y^*1_{\{Y^* \leq \bar{x}_\ell\}} = Y^*$ R -almost surely. From the budget constraint $x_2 = E_P[-\tilde{Y}]$ and $x_2 = E_P[-Y^*] = E_P[-\tilde{Y}1_{\{\tilde{Y} \leq \bar{x}_\ell\}}]$ it now follows that $\tilde{Y} \leq \bar{x}_\ell$ and hence $\tilde{Y} = Y^*$ P -almost surely. Thus \tilde{Y} may differ from Y^* only on the set $\left\{\frac{dP}{dR} = 0\right\}$. \square

6.3 Proof of Theorem 3.4

Proof of Theorem 3.4. The functional $W \mapsto E_{Q_0}[u(W)]$ is strictly concave on the convex subset of \mathcal{W} of financial positions with finite utility. Thus, there is at most one solution to problem (9) if the utility of the optimal terminal wealth is finite.

For any $W \in \mathcal{W}$ and $\lambda_1 \geq 0, \lambda_2 > 0$, we have

$$\begin{aligned}
E_{Q_0}[u(W)] &\leq E_{Q_0}u[(W)] + \lambda_1(x_1 - E_{Q_1}[\ell(-W)]) + \lambda_2(x_2 - E_P[W]) \\
&\leq E_{Q_0} \left[\sup_{x \in \mathbb{R}} \left\{ u(x) - \lambda_1 \frac{dQ_1}{dQ_0} \ell(-x) - \lambda_2 \frac{dP}{dQ_0} x \right\} \right] + \lambda_1 x_1 + \lambda_2 x_2 \\
&= E_{Q_0} \left[u \left(w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \right] \\
&\quad + \lambda_1 \left(x_1 - E_{Q_1} \left[\ell \left(-w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \right] \right) \\
&\quad + \lambda_2 \left(x_2 - E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right] \right),
\end{aligned} \tag{21}$$

where the equality follows from Lemma A.1(ii). Observe that $w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \in \mathcal{W}$ for any $\lambda_1 \geq 0$ and $\lambda_2 > 0$ by Assumption 3.1.

(i) This follows from Lemma 3.3. If $x_1 < E_{Q_1} \left[\ell \left(L \left(c^* \frac{dP}{dQ_1} \right) \right) \right]$, then the constraint set is empty.

Otherwise,

$$W^* := -L \left(c^* \frac{dP}{dQ_1} \right) 1_{\{\frac{dP}{dR} > 0\}} + \infty \cdot 1_{\{\frac{dP}{dR} = 0\}}$$

solves the loss minimization problem (11). Hence it satisfies both constraints, and any other terminal wealth satisfying both constraints equals W^* on the set $\{\frac{dP}{dR} > 0\}$. On $\{\frac{dP}{dR} = 0\}$ we cannot do any better than setting W^* equal to ∞ . Hence, W^* solves the utility maximization problem (9).

(ii) First note that $E_{Q_1} \left[\ell \left(L \left(c^* \frac{dP}{dQ_1} \right) \right) \right] \leq E_{Q_1} \left[\ell \left(-I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right) \right) \right]$ due to Lemma 3.3. If $E_{Q_1} \left[\ell \left(-I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right) \right) \right] < x_1$, then the latter two summands in (21) are equal to zero for $\lambda_1 = 0$, $\lambda_2 = \tilde{\lambda}_2$. Since $w^*(0, y_2) = I(y_2)$, this implies

$$\sup_{W \in \mathcal{W}} E_{Q_0}[u(W)] \leq E_{Q_0} \left[u \left(I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right) \right) \right],$$

Thus, $I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right)$ is a solution, and the UBSR constraint is not binding.

(iii) By Lemma 6.1 below there exist $\lambda_1^* \geq 0$ and $\lambda_2^* > 0$ such that the latter two summands in (21) are equal to zero. This implies that

$$\sup_{W \in \mathcal{W}} E_{Q_0}[u(W)] \leq E_{Q_0} \left[u \left(w^* \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right) \right) \right].$$

Hence, $w^* \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right)$ is a solution to problem (9), and both constraints are binding. \square

The following lemma was used in the preceding proof of Theorem 3.4. Section 6.4 will be devoted to its proof.

Lemma 6.1. *Suppose that Assumption 3.1 holds. Let $x_1 > 0$, $x_2 > \bar{x}_u$, and let $\tilde{\lambda}_2$ be the unique solution to the equation $x_2 = E_P \left[I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right) \right]$. For $x_2 < \bar{x}_\ell$, let $c^* > 0$ be defined as in Lemma 3.3.*

Assume that either $x_2 \geq \bar{x}_\ell$ or, if $x_2 < \bar{x}_\ell$, $x_1 > E_{Q_1} \left[\ell \left(L \left(c^ \frac{dP}{dQ_1} \right) \right) \right]$. If $E_{Q_1} \left[\ell \left(-I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right) \right) \right] \geq x_1$, then there exist $\lambda_1 \geq 0$, $\lambda_2 > 0$ such that*

$$x_1 = E_{Q_1} \left[\ell \left(-w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \right]$$

and

$$x_2 = E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right].$$

6.4 Proof of Lemma 6.1

In the current section we will prove Lemma 6.1. We will always suppose that Assumption 3.1 holds, and fix a level $x_2 \in (\bar{x}_u, \infty)$ for the budget constraint. For $\lambda_1 \geq 0$, we let

$$W^*(\lambda_1) := w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right), \quad (22)$$

where λ_2 is chosen such that the budget constraint $x_2 = E_P [W^*(\lambda_1)]$ is satisfied.

Lemma 6.2. *For each $\lambda_1 \geq 0$ the random variable $W^*(\lambda_1)$ is R -almost surely well defined.*

Proof. On $\left\{ \frac{dP}{dR} = 0 \right\} = \left\{ \frac{dP}{dQ_0} = 0 \right\}$ we have $W^*(\lambda_1) = +\infty$ according to Remark A.2. Thus, it suffices to show that $W^*(\lambda_1)$ is P -almost surely well defined.

Let $\lambda_1 \geq 0$ be fixed. The existence of a $\lambda_2 > 0$ for which $x_2 = E_P [W^*(\lambda_1)]$ follows from Lemma A.1(v) and (vi), the continuity of w^* , Assumption 3.1(a) and monotone convergence.

Suppose now that

$$x_2 = E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right] = E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \hat{\lambda}_2 \frac{dP}{dQ_0} \right) \right]. \quad (23)$$

for $\lambda_2 \geq \hat{\lambda}_2$. For fixed first argument, the function w^* is decreasing in its second argument by Lemma A.1(v), thus

$$w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \leq w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \hat{\lambda}_2 \frac{dP}{dQ_0} \right). \quad (24)$$

Because of condition (23), the preceding inequality (24) must P -almost surely be an equality. This implies that W^* is P -almost surely well defined. \square

Lemma 6.3. *For each $\lambda_1 \geq 0$ we let $\lambda(\lambda_1)$ be the supremum of all $\lambda_2 > 0$ such that the budget constraint*

$$x_2 = E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right]$$

is satisfied. Then $\lambda(\lambda_1) \in (0, \infty)$ and the supremum is attained. Moreover, the function $\lambda(\lambda_1)/\lambda_1$ is decreasing for $\lambda_1 \in (0, \infty)$. In particular,

$$\lim_{\lambda_1 \rightarrow \infty} \frac{\lambda(\lambda_1)}{\lambda_1} \in [0, \infty)$$

exists.

Proof. By Lemma A.1(vi), $w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right)$ converges to \bar{x}_u as $\lambda_2 \rightarrow \infty$ and diverges to infinity as $\lambda_2 \rightarrow 0$. Moreover, $E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right]$ is continuous in λ_2 by monotone convergence and Assumption 3.1(a). This implies the first claim, since $\bar{x}_u < x_2 < \infty$. Furthermore, $\lambda(\lambda_1)$ is indeed a maximum, since $E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right]$ is continuous in λ_2 .

In order to show that $\lambda(\lambda_1)/\lambda_1$ is decreasing, let $\lambda'_1 > \lambda_1 > 0$. Let $\alpha := \lambda'_1/\lambda_1 > 1$. It follows from Lemma A.1 (vii) that

$$w^* \left(\lambda'_1 \frac{dQ_1}{dQ_0}, \lambda(\lambda'_1) \frac{dP}{dQ_0} \right) = w^* \left(\alpha \lambda_1 \frac{dQ_1}{dQ_0}, \alpha \lambda_1 \frac{\lambda(\lambda'_1)}{\lambda'_1} \frac{dP}{dQ_0} \right) \leq w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 \frac{\lambda(\lambda'_1)}{\lambda'_1} \frac{dP}{dQ_0} \right).$$

This implies that

$$x_2 = E_P \left[w^* \left(\lambda'_1 \frac{dQ_1}{dQ_0}, \lambda(\lambda'_1) \frac{dP}{dQ_0} \right) \right] \leq E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 \frac{\lambda(\lambda'_1)}{\lambda'_1} \frac{dP}{dQ_0} \right) \right].$$

Suppose now that $\lambda(\lambda'_1)/\lambda'_1 > \lambda(\lambda_1)/\lambda_1$. Since w^* is decreasing in its second argument with first argument fixed, there exists $\lambda_2 \geq \lambda_1 \frac{\lambda(\lambda'_1)}{\lambda'_1} > \lambda(\lambda_1)$ such that

$$x_2 = E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right],$$

contradicting the maximality of $\lambda(\lambda_1)$. □

In order to avoid any ambiguity, we will always work with the following version of the stochastic process $(W^*(\lambda_1))_{\lambda_1 \geq 0}$:

$$W^*(\lambda_1) := w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda(\lambda_1) \frac{dP}{dQ_0} \right) \quad (\lambda_1 \geq 0).$$

Lemma 6.4. *Let $\lambda_1 \geq 0$. If $(\lambda_1^{(n)})_{n \in \mathbb{N}}$ is a sequence with $\lambda_1^{(n)} \rightarrow \lambda_1$, then there exists a subsequence $(\lambda_1^{(n_j)})_{n \in \mathbb{N}}$ such that $W^*(\lambda_1^{(n_j)}) \rightarrow W^*(\lambda_1)$ R -almost surely.*

Proof. For $n \in \mathbb{N}$ we choose $\lambda_2^{(n)} = \lambda(\lambda_1^{(n)}) > 0$. In a first step we show that this sequence is both bounded and bounded away from zero.

Suppose that the sequence $(\lambda_2^{(n)})_n$ is unbounded. Then there exists an increasing subsequence $\lambda_2^{(n_j)}$ which diverges to infinity as $j \rightarrow \infty$. Let $\hat{\lambda} := \max_{n \in \mathbb{N}} \lambda_1^{(n)}$. By Lemma A.1(v)&(vi),

$$W^* \left(\lambda_1^{(n_j)} \right) \leq w^* \left(\hat{\lambda}_1 \frac{dQ_1}{dQ_0}, \lambda_2^{(n_j)} \frac{dP}{dQ_0} \right) \xrightarrow{j \rightarrow \infty} \bar{x}_u.$$

Due to Assumption 3.1(a), the monotone convergence theorem implies that $x_2 \leq \bar{x}_u$, a contradiction. Thus, $(\lambda_2^{(n)})_n$ is bounded.

Suppose that zero is an accumulation point of $(\lambda_2^{(n)})_n$. Then there exists a decreasing subsequence $\lambda_2^{(n_j)}$ which converges to zero as $j \rightarrow \infty$. Let $\hat{\lambda} := \min_{n \in \mathbb{N}} \lambda_1^{(n)}$. By Lemma A.1(v)&(vi),

$$W^* \left(\lambda_1^{(n_j)} \right) \geq w^* \left(\hat{\lambda}_1 \frac{dQ_1}{dQ_0}, \lambda_2^{(n_j)} \frac{dP}{dQ_0} \right) \xrightarrow{j \rightarrow \infty} \infty.$$

The monotone convergence theorem implies that $x_2 = \infty$, a contradiction. Thus, $(\lambda_2^{(n)})_n$ is bounded away from zero.

For any convergent sequence $(\lambda_1^{(n)})_n$ with limit λ_1 we can now find a subsequence $(\lambda_1^{(n_j)})$ such that $(\lambda_2^{(n_j)})$ is convergent with limit, say, $\lambda_2 \in (0, \infty)$. Hence, $\lim_{j \rightarrow \infty} W^* \left(\lambda_1^{(n_j)} \right) = w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right)$ R -almost surely on $\{dP/dQ_0 > 0\}$. But on $\{dP/dQ_0 = 0\}$ we have $W^* \left(\lambda_1^{(n_j)} \right) = \infty = w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right)$. Thus we obtain R -almost sure convergence on Ω .

Furthermore, choosing $\tilde{\lambda}_1 := \min_{j \in \mathbb{N}} \lambda_1^{(n_j)}$, $\hat{\lambda}_1 := \max_{j \in \mathbb{N}} \lambda_1^{(n_j)} \in [0, \infty)$, and $\tilde{\lambda}_2 := \max_{j \in \mathbb{N}} \lambda_2^{(n_j)}$, $\hat{\lambda}_2 := \min_{j \in \mathbb{N}} \lambda_2^{(n_j)} \in (0, \infty)$, we have the bounds

$$w^* \left(\tilde{\lambda}_1 \frac{dQ_1}{dQ_0}, \tilde{\lambda}_2 \frac{dP}{dQ_0} \right) \leq W^*(\lambda_1^{(n_j)}) \leq w^* \left(\hat{\lambda}_1 \frac{dQ_1}{dQ_0}, \hat{\lambda}_2 \frac{dP}{dQ_0} \right).$$

By Lebesgue's dominated convergence theorem and Assumption 3.1(a) we obtain therefore,

$$x_2 = \lim_{j \rightarrow \infty} E_P \left[W^* \left(\lambda_1^{(n_j)} \right) \right] = E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right].$$

By Lemma 6.2, this implies $W^*(\lambda_1) = w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right)$.

□

For the proof of the main result we will need to investigate the function

$$k : \lambda_1 \mapsto E_{Q_1} [\ell(-W^*(\lambda_1))].$$

Lemma 6.5. *The function k is continuous.*

Proof. Let $(\lambda_1^{(n)})_n$ be a sequence of non-negative reals converging to λ_1 . We need to show that any accumulation point k^* of $(k(\lambda_1^{(n)}))_n$ is equal to $k(\lambda_1)$. By Lemma 6.4 we can choose a subsequence $(\lambda_1^{(n_j)})$ such that both $k(\lambda_1^{(n_j)}) \rightarrow k^*$ and $W^*(\lambda_1^{(n_j)}) \rightarrow W^*(\lambda_1)$ R -almost surely. We have

$$\lim_{j \rightarrow \infty} k(\lambda_1^{(n_j)}) = \lim_{j \rightarrow \infty} E_{Q_1} \left[\ell \left(-W^* \left(\lambda_1^{(n_j)} \right) \right) \right] \stackrel{(*)}{=} E_{Q_1} [\ell(-W^*(\lambda_1))] = k(\lambda_1).$$

Equality $(*)$ follows from the dominated convergence theorem, since for all $j \in \mathbb{N}$ we have the inequality

$$0 \leq \ell \left(-W^*(\lambda_1^{(n_j)}) \right) \leq \ell \left(-w^* \left(\hat{\lambda}_1 \frac{dQ_1}{dQ_0}, \hat{\lambda}_2 \frac{dP}{dQ_0} \right) \right) \quad (25)$$

with $\hat{\lambda}_1 = \min_j \lambda_1^{(n_j)}$, $\hat{\lambda}_2 = \max_j \lambda_2^{(n_j)}$. The upper bound in (25) is Q_1 -integrable by Assumption 3.1(b). \square

Recall that L is the generalized inverse of the derivative of the loss function ℓ , see equation (12). L is a continuous function which is strictly increasing on $[\ell'(-\bar{x}_\ell+), \ell'(-\bar{x}_u)]$. With this function we can characterize the asymptotic behavior of $W^*(\lambda_1)$, $\ell(-W^*(\lambda_1))$, and of the expectations of these quantities for $\lambda_1 \rightarrow \infty$.

Lemma 6.6. *Let $c^* := \lim_{\lambda_1 \rightarrow \infty} \lambda(\lambda_1)/\lambda_1$.*

(i) *Suppose $x_2 > \bar{x}_\ell$. In this case, we have $c^* = 0$ and $\lim_{\lambda_1 \rightarrow \infty} k(\lambda_1) = 0$.*

(ii) *Suppose $\bar{x}_u < x_2 < \bar{x}_\ell$.*

In this case, we have $c^ > 0$ and*

$$\lim_{\lambda_1 \rightarrow \infty} W^*(\lambda_1) = -L \left(c^* \frac{dP}{dQ_1} \right) \quad P - \text{almost surely.}$$

Furthermore, c^ is a solution to the equation*

$$x_2 = -E_P \left[L \left(c \frac{dP}{dQ_1} \right) \right], \quad (26)$$

and

$$\lim_{\lambda_1 \rightarrow \infty} k(\lambda_1) = E_{Q_1} \left[\ell \left(L \left(c^* \frac{dP}{dQ_1} \right) \right) \right]$$

(iii) *If $x_2 = \bar{x}_\ell$, then $\lim_{\lambda_1 \rightarrow \infty} k(\lambda_1) = 0$.*

Proof. (o) Let $\tilde{c} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be decreasing with $\lim_{y_1 \rightarrow \infty} \tilde{c}(y_1) = c > 0$. We will repeatedly use the fact that

$$w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 \tilde{c}(\lambda_1) \frac{dP}{dQ_0} \right) \xrightarrow{\lambda_1 \rightarrow \infty} -L \left(c \frac{dP}{dQ_1} \right) \quad P - a.s. \quad (27)$$

and

$$\ell \left(-w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 \tilde{c}(\lambda_1) \frac{dP}{dQ_0} \right) \right) \xrightarrow{\lambda_1 \rightarrow \infty} \ell \left(L \left(c \frac{dP}{dQ_1} \right) \right) \quad R - a.s. \quad (28)$$

If $\tilde{c}(\lambda_1) \equiv c$, then the convergence is monotone by Lemma A.1(vii) and

$$E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 c \frac{dP}{dQ_0} \right) \right] \xrightarrow{\lambda_1 \rightarrow \infty} -E_P \left[L \left(c \frac{dP}{dQ_1} \right) \right] \quad (29)$$

and

$$E_{Q_1} \left[\ell \left(-w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 c \frac{dP}{dQ_0} \right) \right) \right] \xrightarrow{\lambda_1 \rightarrow \infty} E_{Q_1} \left[\ell \left(L \left(c \frac{dP}{dQ_1} \right) \right) \right]. \quad (30)$$

The statements (27)-(30) follow from Lemma A.1(ix) in the following way: Since $Q_1 \sim Q_0 \sim R$, we have R -almost surely

$$w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 \tilde{c}(\lambda_1) \frac{dP}{dQ_0} \right) = w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 \tilde{c}(\lambda_1) \frac{dQ_1}{dQ_0} \frac{dP}{dQ_1} \right).$$

This expression converges to $-L \left(c \frac{dP}{dQ_1} \right)$ R -almost surely on $\left\{ \frac{dP}{dQ_1} > 0 \right\}$ due to Lemma A.1(ix), which implies (27). Furthermore, on $\left\{ \frac{dP}{dQ_1} = 0 \right\}$ we have $w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 \tilde{c}(\lambda_1) \frac{dQ_1}{dQ_0} \frac{dP}{dQ_1} \right) = \infty$ and $-L \left(c \frac{dP}{dQ_1} \right) = \bar{x}_\ell$, hence $\ell \left(-w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 \tilde{c}(\lambda_1) \frac{dP}{dQ_0} \right) \right) = 0 = \ell \left(L \left(c \frac{dP}{dQ_1} \right) \right)$, and (28) follows. (29) and (30) follow now from Lemma A.1(ix), Assumption 3.1(a)&(b), and the monotone convergence theorem.

We will now prove part (i). Let $x_2 > \bar{x}_\ell$.

(i-a) Suppose $c^* > 0$. By Lemma 6.3 we have $\lambda(\lambda_1)/\lambda_1 > c^*/2 > 0$ for $\lambda_1 > 0$. Thus by Lemma A.1 (v)&(vii), for $\lambda_1 \geq \lambda'_1 > 0$:

$$W^*(\lambda_1) \leq w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \frac{c^* \lambda_1}{2} \frac{dP}{dQ_0} \right) \leq w^* \left(\lambda'_1 \frac{dQ_1}{dQ_0}, \frac{c^* \lambda'_1}{2} \frac{dP}{dQ_0} \right).$$

From (29) we obtain

$$x_2 = E_P[W^*(\lambda_1)] \leq E_P \left[w^* \left(\lambda'_1 \frac{dQ_1}{dQ_0}, \frac{c^* \lambda'_1}{2} \frac{dP}{dQ_0} \right) \right] \xrightarrow{\lambda'_1 \rightarrow \infty} E_P \left[-L \left(\frac{c^*}{2} \cdot \frac{dP}{dQ_1} \right) \right] \leq \bar{x}_\ell,$$

a contradiction. Thus, $c^* = 0$.

(i-b) Since $c^* = 0$, it follows from Lemma A.1(v) that for any $\epsilon > 0$ and λ_1 large enough, $W^*(\lambda_1) \geq w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \epsilon \lambda_1 \frac{dP}{dQ_0} \right)$. With $k(\lambda_1) = E_{Q_1}[\ell(-W^*(\lambda_1))]$, this implies

$$\begin{aligned} 0 &\leq \liminf_{\lambda_1 \rightarrow \infty} k(\lambda_1) \leq \limsup_{\lambda_1 \rightarrow \infty} k(\lambda_1) \\ &\leq \lim_{\lambda_1 \rightarrow \infty} E_{Q_1} \left[\ell \left(-w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \epsilon \lambda_1 \frac{dP}{dQ_0} \right) \right) \right] = E_{Q_1} \left[\ell \left(L \left(\epsilon \frac{dP}{dQ_1} \right) \right) \right] \end{aligned}$$

due to (30). Furthermore the dominated convergence theorem and Assumption 3.1(e) imply

$$\lim_{\epsilon \rightarrow 0} E_{Q_1} \left[\ell \left(L \left(\epsilon \frac{dP}{dQ_1} \right) \right) \right] = 0,$$

since ℓ and L are increasing. Thus, $\lim_{\lambda_1 \rightarrow \infty} k(\lambda_1) = 0$.

We will now prove part (ii). Let $x_2 < \bar{x}_\ell$.

(ii-a) Let us first show that $x_2 < \bar{x}_\ell$ implies $c^* > 0$. Suppose $c^* = 0$. Then for every $\epsilon > 0$ there exists $\lambda'_1 > 0$ such that for $\lambda_1 \geq \lambda'_1$

$$W^*(\lambda_1) \geq w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 \epsilon \frac{dP}{dQ_0} \right). \quad (31)$$

From (29) we obtain for $\lambda_1 \geq \lambda'_1$

$$x_2 = E_P[W^*(\lambda_1)] \geq E_P \left[w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_1 \epsilon \frac{dP}{dQ_0} \right) \right] \stackrel{\lambda_1 \rightarrow \infty}{\searrow} E_P \left[-L \left(\epsilon \frac{dP}{dQ_1} \right) \right].$$

Thus, the monotone convergence theorem and Assumption 3.1(e) imply

$$x_2 \geq \lim_{\epsilon \rightarrow 0} E_P \left[-L \left(\epsilon \frac{dP}{dQ_1} \right) \right] = \bar{x}_\ell,$$

a contradiction. This implies $c^* > 0$. The first result now follows from (27).

(ii-b) We will now show that $c^* = \lim_{\lambda_1 \rightarrow \infty} \lambda(\lambda_1)/\lambda_1$ is a solution of equation (26). It is not difficult to see that for $\lambda_1 > n \in \mathbb{N}$ we have

$$\begin{aligned} -L \left(\frac{\lambda(n)}{n} \cdot \frac{dP}{dQ_1} \right) &\stackrel{(1)}{\leq} w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \frac{\lambda(n)}{n} \cdot \lambda_1 \cdot \frac{dP}{dQ_0} \right) \stackrel{(2)}{\leq} W^*(\lambda_1) \\ &\stackrel{(3)}{\leq} w^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, c \lambda_1 \frac{dP}{dQ_0} \right) \stackrel{(4)}{\leq} w^* \left(\frac{dQ_1}{dQ_0}, c \frac{dP}{dQ_0} \right). \end{aligned} \quad (32)$$

Inequality (1) follows from Lemma A.1(ix). Inequalities (2) and (3) follow from Lemma A.1(v) and the fact that $\frac{\lambda(n)}{n} \cdot \lambda_1 \geq \lambda(\lambda_1) \geq c \lambda_1$ for $\lambda_1 > n$, since $\lambda(\lambda_1)/\lambda_1$ decreases to c as $\lambda_1 \rightarrow \infty$.

Inequality (4) follows from Lemma A.1 (vii).

Due to (27) and Assumption 3.1(a)&(d) we may apply the dominated convergence theorem to obtain

$$x_2 = E_P [W^*(\lambda_1)] \rightarrow E_P \left[-L \left(c^* \frac{dP}{dQ_1} \right) \right].$$

Thus, c^* is a solution to equation (26).

Analogously, due to (28) and Assumption 3.1(b)&(e) we may apply the dominated convergence theorem to obtain

$$\lim_{\lambda_1 \rightarrow \infty} k(\lambda_1) = \lim_{\lambda_1 \rightarrow \infty} E_{Q_1} [\ell(-W^*(\lambda_1))] = E_{Q_1} \left[\ell \left(L \left(c^* \frac{dP}{dQ_1} \right) \right) \right].$$

(iii) Let $x_2 = \bar{x}_\ell$. If $c^* = 0$, argue as in part (i-b) to verify the claim. If $c^* > 0$, argue as in part (ii-b) to show that $x_2 = E_P \left[-L \left(c^* \frac{dP}{dQ_1} \right) \right]$. Since $-L \left(c^* \frac{dP}{dQ_1} \right) \leq \bar{x}_\ell = x_2$, this implies $-L \left(c^* \frac{dP}{dQ_1} \right) = \bar{x}_\ell$ P -almost surely and hence Q_1 -almost surely on $\left\{ \frac{dP}{dQ_1} > 0 \right\}$. But since $-L(0) = \bar{x}_\ell$, it holds Q_1 -almost surely on Ω .

Analogously to part (ii-b), we obtain finally

$$\lim_{\lambda_1 \rightarrow \infty} k(\lambda_1) = E_{Q_1} \left[\ell \left(L \left(c^* \frac{dP}{dQ_1} \right) \right) \right] = E_{Q_1} [\ell(-\bar{x}_\ell)] = 0.$$

□

We summarize the asymptotic behavior of k in the following corollary.

Corollary 6.7. *Suppose that Assumption 3.1 holds and let $x_2 > \bar{x}_u$. Let $\tilde{\lambda}_2$ be the unique solution to the equation*

$$x_2 = E_P \left[I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right) \right].$$

The asymptotic behavior of k can be characterized in the following way.

$$\begin{aligned} \lim_{\lambda_1 \rightarrow 0} k(\lambda_1) &= E_{Q_1} \left[\ell \left(-I \left(\tilde{\lambda}_2 \frac{dP}{dQ_1} \right) \right) \right], \\ \lim_{\lambda_1 \rightarrow \infty} k(\lambda_1) &= \begin{cases} 0 & \text{if } x_2 \geq \bar{x}_\ell, \\ E_{Q_1} \left[\ell \left(L \left(c^* \frac{dP}{dQ_1} \right) \right) \right] & \text{if } x_2 < \bar{x}_\ell, \end{cases} \end{aligned}$$

where c^* is a solution of (26).

Proof. Note that $W^*(0) = I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right)$. Hence the first claim follows from Lemma 6.5. The second one is only a reformulation of Lemma 6.6. □

Finally, we arrive at the following conclusion, which finishes the proof of Lemma 6.1.

Corollary 6.8. *Suppose that Assumption 3.1 holds and let $x_2 > \bar{x}_u$. By $\mathcal{R}(k)$ we denote the range of k . It holds $(a, b) \subseteq \mathcal{R}(k)$ with*

$$a = \begin{cases} 0 & \text{if } x_2 \geq \bar{x}_\ell, \\ E_{Q_1} \left[\ell \left(L \left(c^* \frac{dP}{dQ_1} \right) \right) \right], & \text{if } x_2 < \bar{x}_\ell, \end{cases} \quad b = E_{Q_1} \left[\ell \left(-I \left(\tilde{\lambda}_2 \frac{dP}{dQ_0} \right) \right) \right],$$

where $\tilde{\lambda}_2$ and c^* are chosen as in Corollary 6.7.

Proof. The proof is immediate from Lemma 6.5 and Corollary 6.7. □

A Properties of the deterministic function w^*

Here we will discuss how the function w^* , that gives us the optimal terminal wealth, can be obtained and describe its properties. For this purpose we consider a family of functions g_{y_1, y_2} with $y_1, y_2 \geq 0$, defined by

$$g_{y_1, y_2}(x) := u(x) - y_1 \ell(-x) - y_2 x.$$

In the following we will sometimes drop the indices y_1, y_2 if there is no danger of confusion.

Lemma A.1.

- (i) g_{y_1, y_2} is strictly concave and thus continuous on its essential domain $\text{dom } g_{y_1, y_2} = \text{dom } u$.
- (ii) g_{y_1, y_2} attains its supremum on \mathbb{R} if and only if $y_2 > 0$. In this case, the maximizer is unique and equals

$$w^*(y_1, y_2) = \begin{cases} J(y_1, y_2) & \text{if } y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +), \\ \bar{x}_\ell & \text{if } u'(\bar{x}_\ell) \leq y_2 \leq u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +), \\ I(y_2) & \text{if } y_2 < u'(\bar{x}_\ell). \end{cases} \quad (33)$$

Here, $J(y_1, y_2)$ denotes the unique solution to the equation $u'(x) + y_1 \ell'(-x) = y_2$ for the case that $y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +)$, and $I := (u')^{-1}$.

- (iii) If $\bar{x}_\ell = \infty$, (33) simplifies to

$$w^*(y_1, y_2) = J(y_1, y_2).$$

- (iv) The function $w^* : [0, \infty) \times (0, \infty) \rightarrow (\bar{x}_u, \infty)$, defined in (33), is continuous.
- (v) $w^*(y_1, y_2)$ is decreasing in y_2 for $y_1 \geq 0$ fixed, and increasing in y_1 for $y_2 > 0$ fixed.
- (vi) $w^*(y_1, y_2)$ converges to $\bar{x}_u \in \mathbb{R} \cup \{-\infty\}$ as $y_2 \rightarrow \infty$ and to infinity as $y_2 \rightarrow 0$ for fixed $y_1 \geq 0$.
- (vii) If $\alpha \geq 1$, then $w^*(\alpha y_1, \alpha y_2) \leq w^*(y_1, y_2)$.

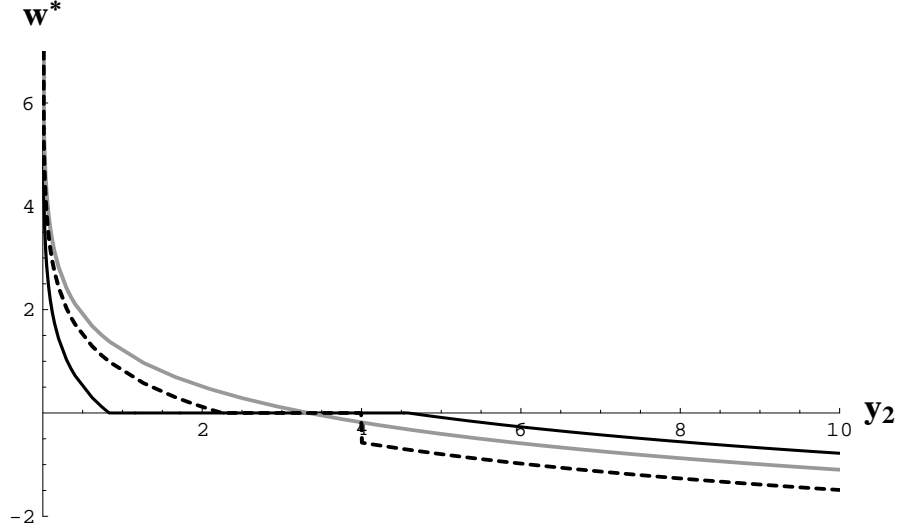


Figure 4: w^* as a function of y_2

(viii) Let $L : \mathbb{R} \rightarrow [-\bar{x}_\ell, \infty)$ be the generalized inverse of the derivative of the loss function ℓ , i.e.,

$$L(y) = \begin{cases} -\bar{x}_u & \text{if } y \geq \ell'(-\bar{x}_u), \\ (\ell')^{-1}(y) & \text{if } \ell'(-\bar{x}_\ell+) < y < \ell'(-\bar{x}_u), \\ -\bar{x}_\ell & \text{if } y \leq \ell'(-\bar{x}_\ell+). \end{cases} \quad (34)$$

L is a continuous function which is strictly increasing on $[\ell'(-\bar{x}_\ell+), \ell'(-\bar{x}_u)]$.

If $e > 0$ is such that $\ell'(-\bar{x}_\ell+) < e < \ell'(-\bar{x}_u)$, and $\mu := u'(-L(e))$, then we have for all $y_1 \geq 0$,

$$w^*(0, \mu) = w^*(y_1, \mu + y_1 e).$$

(ix) Let $\tilde{c} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be decreasing with $\lim_{y_1 \rightarrow \infty} \tilde{c}(y_1) = c > 0$. Then

$$\lim_{y_1 \rightarrow \infty} w^*(y_1, \tilde{c}(y_1) \cdot y_1) = -L(c) \in [\bar{x}_u, \bar{x}_\ell].$$

Moreover, $w^*(y_1, c y_1)$ converges for $y_1 \rightarrow \infty$ to $-L(c)$ monotonously from above.

(x) We have

$$\sup_{x > \bar{x}_u} \{-y_1 \ell(-x) - y_2 x\} = -y_1 \ell\left(L\left(\frac{y_2}{y_1}\right)\right) + y_2 L\left(\frac{y_2}{y_1}\right).$$

Figure 4 shows an example of $w^*(\lambda_1^* y_1, \lambda_2^* y_2)$ as a function of y_2 , where λ_1^* and λ_2^* are the parameters from Theorem 3.4 such that W^* satisfies the constraints. We chose again the exponential utility function $u(x) = 1 - e^{-x}$. The black line shows the terminal wealth with the USBR constraint, where $\ell(x) = (e^x - e^{-\bar{x}_\ell}) \vee 0$. The gray line shows $w^*(0, \tilde{\lambda}_2 y_2) = I(\tilde{\lambda}_2)$, which gives the optimal terminal wealth without risk constraint. The dashed line shows the optimal terminal wealth with a VaR constraint. For the latter case the solution can be found in Basak & Shapiro (2001).

Proof of Lemma A.1. (i) The sum of the strictly concave function u and the concave function $-\ell(\cdot)$ is strictly concave, and $\text{dom } g = \text{dom } u \cap \text{dom } \ell(\cdot) = \text{dom } u$.

(ii)&(iii) Suppose first that $y_2 = 0$. Then $g(x) = u(x) - y_1\ell(-x)$, and g does not attain its supremum on \mathbb{R} . If conversely $y_2 > 0$, then

$$g'(x) = \begin{cases} u'(x) + y_1\ell'(-x) - y_2 & \text{if } x < \bar{x}_\ell, \\ u'(x) - y_2 & \text{if } x > \bar{x}_\ell. \end{cases}$$

Hence by the Inada conditions (U1) and (U2) we have

$$\lim_{x \searrow \bar{x}_u} g'(x) = \infty > 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} g'(x) = -y_2 < 0$$

because ℓ is convex, continuous, and increasing and hence $\ell'(-\infty) = 0$ also in the case $\bar{x}_\ell = \infty$. Since g is strictly concave on its essential domain, this implies that g has a unique maximum.

Next we prove that the maximizer of g is given by w^* as defined in (33). Suppose first that $\bar{x}_\ell < \infty$.

If $y_2 > u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$, then $g'(\bar{x}_\ell-) < 0$. It follows that g is decreasing in a neighborhood of \bar{x}_ℓ . Thus, $w^*(y_1, y_2) < \bar{x}_\ell$. Since g is strictly concave and continuously differentiable on the interval $(\bar{x}_u, \bar{x}_\ell)$, $w^*(y_1, y_2)$ is characterized as the unique solution of $g'(x) = 0$ with $x \in (\bar{x}_u, \bar{x}_\ell)$. This implies that $w^*(y_1, y_2) = J(y_1, y_2)$.

If $y_2 < u'(\bar{x}_\ell)$, then $g'(\bar{x}_\ell+) > 0$. It follows that g is increasing in a neighborhood of \bar{x}_ℓ . Thus, $w^*(y_1, y_2) > \bar{x}_\ell$. In this case, the first order condition implies that $w^*(y_1, y_2) = I(y_2)$.

If $u'(\bar{x}_\ell) \leq y_2 \leq u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$, then $g'(\bar{x}_\ell-) \geq 0 \geq g'(\bar{x}_\ell+)$. Since g is strictly concave, we obtain that $w^*(y_1, y_2) = \bar{x}_\ell$.

Next, let us assume that $\bar{x}_\ell = \infty$. Then $g(x) = u(x) - y_1\ell(-x) - y_2x$ for all $x \in \text{dom } g$, thus $w^*(y_1, y_2) = J(y_1, y_2)$ by the first order condition which proves (iii). Moreover, by our assumptions on u and ℓ , the condition $y_2 > u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$ is trivially satisfied in this case.

(iv) In equation (33) we distinguish three regions. First, we demonstrate that w^* is continuous on

$$D := \{(y_1, y_2) : y_2 \geq u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+), y_1 \geq 0, y_2 > 0\}.$$

The function $h : [u'(\bar{x}_\ell), \infty) \rightarrow \mathbb{R}_+$, $\mu \mapsto \ell'(-I(\mu))$ is continuous and strictly increasing. For each $\mu \in [u'(\bar{x}_\ell), \infty)$ the graph Γ_μ of the linear function $y_1 \mapsto \mu + h(\mu)y_1$, $y_1 \in \mathbb{R}_+$, defines a ray in D . D equals the disjoint union of the rays Γ_μ , $\mu \in [u'(\bar{x}_\ell), \infty)$, and we may continuously project D along these rays onto $\{0\} \times [u'(\bar{x}_\ell), \infty)$. By $\pi(y_1, y_2)$ we denote the projection of $(y_1, y_2) \in D$, i.e., $\mu \in [u'(\bar{x}_\ell), \infty)$ such that $y_2 = \mu + h(\mu)y_1$. Since π and I are continuous, the mapping $(y_1, y_2) \mapsto I(\pi(y_1, y_2))$ is continuous. Observe that

$$u'(I(\pi(y_1, y_2)) + \ell'(-I(\pi(y_1, y_2))) \cdot y_1 = \pi(y_1, y_2) + h(\pi(y_1, y_2)) \cdot y_1 = y_2.$$

Thus, $I(\pi(y_1, y_2)) = w^*(y_1, y_2)$.

Second, observe that $J(y_1, y_2) = \bar{x}_\ell$ if $y_2 = u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$, and that $I(y_2) = \bar{x}_\ell$ if $y_2 = u'(\bar{x}_\ell)$. Altogether, it follows that w^* is a continuous function.

(v) Simply note that both $u'(x)$ and $\ell'(-x)$ are decreasing in x .

(vi) For $y_1 \geq 0$ fixed, $u'(x) + y_1\ell'(-x)$ is strictly decreasing and continuous in x on the interval $(\bar{x}_u, \bar{x}_\ell)$ with $\lim_{x \searrow \bar{x}_u} (u'(x) + y_1\ell'(-x)) = \infty$. This implies $w^*(y_1, y_2) \rightarrow \bar{x}_u$ as $y_2 \rightarrow \infty$.

Moreover, $\lim_{y_2 \rightarrow 0} w^*(y_1, y_2) = \lim_{y_2 \rightarrow 0} I(y_2) = \infty$.

(vii) We first show the claim for $y_2 \geq u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +)$. Since $w^*(y_1, y_2) = \bar{x}_\ell$ for $y_2 = u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +)$ and $w^*(y_1, y_2) \leq \bar{x}_\ell$ for $y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +)$, we may restrict our attention to $y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +)$. Then

$$\alpha y_2 > \alpha u'(\bar{x}_\ell) + \alpha y_1 \ell'(-\bar{x}_\ell +) \geq u'(\bar{x}_\ell) + \alpha y_1 \ell'(-\bar{x}_\ell +).$$

Thus, $w^*(y_1, y_2)$ is the unique solution of $u'(x) + y_1 \ell'(-x) = y_2$, and $w^*(\alpha y_1, \alpha y_2)$ is the unique solution of $u'(x) + \alpha y_1 \ell'(-x) = \alpha y_2$. This implies

$$\alpha y_2 = \alpha u'(w^*(y_1, y_2)) + \alpha y_1 \ell'(-w^*(y_1, y_2)) > u'(w^*(y_1, y_2)) + \alpha y_1 \ell'(-w^*(y_1, y_2)).$$

Since $u'(x)$ and $\ell'(-x)$ are decreasing in x on $(\bar{x}_u, \bar{x}_\ell)$, we obtain $w^*(\alpha y_1, \alpha y_2) \leq w^*(y_1, y_2)$.

If $y_2 \leq u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +)$, $w^*(y_1, y_2)$ depends on y_2 only and is decreasing in y_2 . Now the result follows easily.

(viii) The properties of L follow immediately from our assumptions on ℓ .

In order to derive the last claim, observe that $w^*(0, \mu)$ is the unique solution of $u'(x) = \mu$ or, equivalently, $x = I(\mu)$. If $e > \ell'(-\bar{x}_\ell +)$, then $\mu = u'(-L(e)) > u'(\bar{x}_\ell)$. Thus, $\mu + y_1 e > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +)$. This implies that $w^*(y_1, \mu + y_1 e)$ is the unique solution to $u'(x) + y_1 \ell'(-x) = \mu + y_1 e$. On the other hand, since $e < \ell'(-\bar{x}_u)$,

$$u'(w^*(0, \mu)) + y_1 \ell'(-w^*(0, \mu)) = \mu + y_1 \ell'(-I(\mu)) = \mu + y_1 \ell'[-I(u'(-L(e)))] = \mu + y_1 e,$$

and $w^*(0, \mu)$ is also the unique solution to $u'(x) + y_1 \ell'(-x) = \mu + y_1 e$. Thus, $w^*(0, \mu) = w^*(y_1, \mu + y_1 e)$.

(ix) If $c \geq \ell'(-\bar{x}_u)$, then $\tilde{c}(y_1) \geq c > u'(\bar{x}_\ell)/y_1 + \ell'(-\bar{x}_\ell +)$ for y_1 large enough because $\ell'(-\bar{x}_u) > \ell'(-\bar{x}_\ell +)$.

Therefore, $w^*(y_1, \tilde{c}(y_1)y_1)$ satisfies

$$u'(w^*(y_1, \tilde{c}(y_1)y_1)) + y_1 \ell'(-w^*(y_1, \tilde{c}(y_1)y_1)) = \tilde{c}(y_1)y_1$$

for y_1 large enough. Due to $\tilde{c}(y_1) \geq c \geq \ell'(-\bar{x}_u)$, this implies

$$u'(w^*(y_1, \tilde{c}(y_1)y_1)) \geq y_1[\ell'(-\bar{x}_u) - \ell'(-w^*(y_1, \tilde{c}(y_1)y_1))]$$

and hence $\lim_{y_1 \rightarrow \infty} w^*(y_1, \tilde{c}(y_1)y_1) = \bar{x}_u = -L(c)$ due to the Inada condition (U2) and since ℓ' is strictly increasing in $-\bar{x}_u$.

Now assume that $c < \ell'(-\bar{x}_u)$. We show that $w^*(y_1, \tilde{c}(y_1)y_1)$ is bounded from below away from \bar{x}_u for large enough y_1 . For this purpose choose e such that $\ell'(-\bar{x}_\ell +) < e < \ell'(-\bar{x}_u)$ and $e > \tilde{c}(y_1)$ for y_1 large enough. It follows from (viii) that for all such y_1 we have

$$\bar{x}_u < w^*(0, \mu) = w^*(y_1, \mu + y_1 e) \leq w^*(y_1, \mu + \tilde{c}(y_1)y_1) \leq w^*(y_1, \tilde{c}(y_1)y_1),$$

where $\mu = u'(-L(e))$. This proves boundedness from below.

For y_1 large enough, we have $\tilde{c}(y_1)y_1 \geq u'(\bar{x}_\ell)$. For any such y_1 we distinguish two cases. If

$$\tilde{c}(y_1)y_1 \leq u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +),$$

then $w^*(y_1, \tilde{c}(y_1)y_1) = \bar{x}_\ell \stackrel{(*)}{=} -L(z(y_1))$, where

$$z(y_1) := \tilde{c}(y_1) - \frac{u'(w^*(y_1, \tilde{c}(y_1)y_1))}{y_1}.$$

Equation (*) can easily be checked, since $w^*(y_1, \tilde{c}(y_1)y_1) = \bar{x}_\ell$.

If $\tilde{c}(y_1)y_1 > u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell)$, then

$$u'(w^*(y_1, \tilde{c}(y_1)y_1)) + y_1\ell'(-w^*(y_1, \tilde{c}(y_1)y_1)) = \tilde{c}(y_1)y_1,$$

thus $w^*(y_1, \tilde{c}(y_1)y_1) = -L(z(y_1))$.

Since $w^*(y_1, \tilde{c}(y_1)y_1)$ is bounded away from \bar{x}_u for y_1 large enough, we have in both cases that $u'(w^*(y_1, \tilde{c}(y_1)y_1))$ is bounded, thus $z(y_1) \rightarrow c$ as $y_1 \rightarrow \infty$. The continuity of L implies

$$\lim_{y_1 \rightarrow \infty} w^*(y_1, \tilde{c}(y_1)y_1) = -L(c) \geq w^*(0, \mu) > \bar{x}_u.$$

By definition of L we have $-L(c) \leq \bar{x}_\ell$.

Finally, observe that $w^*(y_1, cy_1)$ is decreasing in c by (vii).

(x) This follows from the definition of L and basic calculus, similar to the proof of (ii). \square

Remark A.2. By continuity of w^* and Lemma A.1(vi) we may define $w^*(y_1, \infty) := \bar{x}_u$ and $w^*(y_1, 0) := \infty$ for $y_1 \geq 0$.

References

- Ankircher, Stefan & Peter Imkeller (2005), ‘Finite utility on financial markets with asymmetric information and structure properties of the price dynamics’, *Annales de l’Institut Henri Poincaré: Probability and Statistics* **41**(3), 479–503.
- Artzner, Philippe, Freddy Delbaen, Jean-Marc Eber & David Heath (1999), ‘Coherent measures of risk’, *Mathematical Finance* **9**(3), 203–228.
- Aumann, Robert J. & Micha Perles (1965), ‘A variational problem arising in economics’, *Journal of Mathematical Analysis and Applications* **11**, 488–503.
- Bank, Peter & Frank Riedel (2000), ‘Non-time additive utility maximization - the case of certainty’, *Journal of Mathematical Economics* **33**, 271–290.
- Bank, Peter & Frank Riedel (2001), ‘Optimal consumption choice with intertemporal substitution’, *Annals of Applied Probability* **11**, 750–788.
- Basak, Suleyman & Alexander Shapiro (2001), ‘Value-at-risk based risk management: Optimal policies and asset prices’, *The Review of Financial Studies* **14**, 371–405.
- Bellini, Fabio & Marco Frittelli (2002), ‘Existence of minimax martingale measures’, *Mathematical Finance* **12**, 1–21.
- Cox, John C. & Chi F. Huang (1989), ‘Optimal consumption and portfolio policies when assets follow a diffusion process’, *Journal of Economic Theory* **49**, 33–83.
- Cox, John C. & Chi F. Huang (1991), ‘A variational problem arising in financial economies’, *Journal of Mathematical Economics* **20**, 465–487.
- Delbaen, Freddy & Walter Schachermeyer (1997), Non-arbitrage and the fundamental theorem of asset pricing: Summary of main results, in ‘Proceedings of Symposia in Applied Mathematics’.
- Dunkel, Jörn & Stefan Weber (2005), ‘Efficient Monte Carlo methods for convex risk measures’. Working Paper.
- Föllmer, Hans & Alexander Schied (2004), *Stochastic Finance - An Introduction in Discrete Time*, 2nd edition, Walter de Gruyter, Berlin.

- Föllmer, Hans & Anne Gundel (2006), ‘Robust projections in the class of martingale measures’, *Illinois Journal of Mathematics* **50**(2), 439–472.
- Gabih, Abdelali, Walter Grecksch & Ralf Wunderlich (2005), ‘Dynamic portfolio optimization with bounded shortfall risks’, *Stochastic Analysis and Applications* **3**, 579–594.
- Giesecke, Kay, Thorsten Schmidt & Stefan Weber (2005), Measuring the risk of extreme events. Working Paper.
- Gilboa, Itzhak & David Schmeidler (1989), ‘Maxmin expected utility with non-unique prior’, *Journal of Mathematical Economics* **18**, 141–153.
- Goll, Thomas & Ludger Rüschendorf (2001), ‘Minimax and minimal distance martingale measures and their relationship to portfolio optimization’, *Finance and Stochastics* **5**(4), 557–581.
- Gundel, Anne (2005), ‘Robust utility maximization for complete and incomplete market models’, *Finance and Stochastics* **9**(2), 151–176.
- Gundel, Anne & Stefan Weber (2007), ‘Robust utility maximization with limited downside risk in incomplete markets’, *Stochastic Processes and Their Applications* **117**(11), 1663–1688.
- He, Hua & Neil D. Pearson (1991), ‘Consumption and portfolio policies with incomplete markets and shortsale constraints: The infinite-dimensional case’, *Journal of Economic Theory* **54**, 259–304.
- Jacod, Jean (1975), ‘Multivariate point processes: Predictable projection, Radon-Nikodym derivatives, representation of martingales’, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **31**, 235–253.
- Karatzas, Ioannis, John P. Lehoczky & Steven E. Shreve (1987), ‘Optimal portfolio and consumption decisions for a small investor on a finite time horizon’, *SIAM Journal of Control and Optimization* **25**, 1557–1586.
- Karatzas, Ioannis, John P. Lehoczky, Steven E. Shreve & Gan-Lin Xu (1991), ‘Martingale and duality methods for utility maximization in an incomplete market’, *SIAM Journal of Control and Optimization* **29**, 702–730.

- Kramkov, Dmitry & Walter Schachermayer (1999), ‘The asymptotic elasticity of utility functions and optimal investment in incomplete markets’, *Annals of Applied Probability* **9**(3), 904–950.
- Merton, Robert C. (1969), ‘Lifetime portfolio selection under uncertainty’, *The Review of Economics and Statistics* **51**, 247–257.
- Merton, Robert C. (1971), ‘Optimum consumption and portfolio rules in a continuous-time model’, *Journal of Economic Theory* **3**, 373–413.
- Pliska, Stanley R. (1986), ‘A stochastic calculus model of continuous trading: optimal portfolios’, *Mathematics of Operations Research* **11**, 371–382.
- Protter, Philip E. (2004), *Stochastic Integration and Differential Equations*, Springer-Verlag, New York. Second edition.
- Quenez, Marie-Claire (2004), Optimal portfolios in multiple-prior model, in ‘Seminar on Stochastic Analysis, Random Fields and Applications IV’, number 58, Birkhäuser, Basel, pp. 291–321.
- Schied, Alexander (2005), ‘Optimal investments for robust utility functionals in complete market models’, *Mathematics of Operations Research* **30**(3), 750–764.
- Schied, Alexander (2007), ‘Optimal investments under aversion against risk and ambiguity: a duality approach’, *Finance and Stochastics* **11**(1), 107–129.
- Schied, Alexander & Ching-Tang Wu (2005), ‘Duality theory for optimal investments under model uncertainty’, *Statistics & Decisions* **23**(3), 199–217.
- Weber, Stefan (2006), ‘Distribution-invariant risk measures, information, and dynamic consistency’, *Mathematical Finance* **16**, 419–442.