Robust Utility Maximization with Limited Downside Risk in Incomplete Markets

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Abstract

In this article we consider the portfolio selection problem of an agent with robust preferences in the sense of Gilboa & Schmeidler (1989) in an incomplete market. Downside risk is constrained by a robust version of utility-based shortfall risk. We derive an explicit representation of the optimal terminal wealth in terms of certain worst case measures which can be characterized as minimizers of a dual problem. This dual problem involves a three-dimensional analogue of f-divergences which generalize the notion of relative entropy.

Key words: Robust utility maximization, optimal portfolio choice, utility-based shortfall risk, convex risk measures, semimartingales

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1 Introduction

The measurement and management of the downside risk of portfolios is a key issue for financial institutions. The industry standard Value at Risk (VaR) shows serious deficiencies as a measure of the downside risk. It penalizes diversification in many situations and does not take into account the size of very large losses exceeding the value at risk. These problems motivated intense research on alternative risk measures whose foundation was provided by Artzner, Delbaen, Eber & Heath (1999). An excellent summary of recent results can be found in the book by Föllmer & Schied (2004).

While axiomatic results are an important first step towards better risk management, an analysis of the economic implications of different approaches to risk measurement is indispensable. In the current article we investigate the agent's optimal payoff profile under

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a joint budget and risk measure constraint. A first step in this direction has already been made by Gundel & Weber (2005) where the utility maximization problem is analyzed for fixed probabilistic models. In contrast, the current paper considers the situation of model uncertainty and extends the results of Gundel & Weber (2005).

Here model uncertainty has three dimensions. The first dimension concerns the preferences of the maximizing agent. In most articles on optimal portfolio selection, preferences are represented by von Neumann-Morgenstern utility functionals. These utility functionals can be expressed in terms of a Bernoulli utility function and a single subjective probability measure. A more general class of preferences can be constructed if the single representing probability measure is replaced by a set of subjective measures. Robust utility functionals of this type have been analyzed by Gilboa & Schmeidler (1989). We will study the portfolio selection problem on this level of generality. Here, we will always assume that the essential domain of the Bernoulli utility function is bounded from below.

The second dimension of model uncertainty is related to the budget constraint. In a complete market, this constraint can be formalized in terms of an expectation under the single pricing measure. In an incomplete market the set of equivalent martingale measures is infinite, and the analysis of the budget constraint requires more care. We consider the case of a financial market that is not necessarily complete.

Finally, the measurement of the downside risk can also be a source of model uncertainty. We define the risk constraint in terms of *utility-based shortfall risk* (UBSR). This risk measure does not share the deficiencies of Value at Risk. For a detailed description of its properties, we refer to Föllmer & Schied (2004), Weber (2006), Dunkel & Weber (2005), and Giesecke, Schmidt & Weber (2005). The definition of shortfall risk involves a subjective probability measure. The choice of this measure can be a third source of model uncertainty.

In this article we consider the portfolio selection problem of an agent with robust preferences in the sense of Gilboa & Schmeidler (1989) in an incomplete market. Downside risk is constrained by a robust version of UBSR. We derive an explicit representation of the optimal terminal wealth in terms of certain worst case measures which can be characterized as minimizers of a dual problem. This dual problem involves a three-dimensional analogue of f-divergences which generalize the notion of relative entropy.

The paper is organized as follows. Section 2 describes the agent's preferences, budget and risk constraint in detail. The portfolio selection problem is stated in Section 2.4. The interpretation of the budget constraint in an incomplete market is further analyzed in Section 2.5. Section 3 explains the notion of extended martingale measures which will be used in our characterization of optimal wealth. Extended martingale measures have been introduced by Föllmer & Gundel (2006) and correspond exactly to the class of supermartingales which appear in the duality approach of Kramkov & Schachermayer (1999). Section 4 describes the solution in the absence of model uncertainty and summarizes the findings of Gundel & Weber (2005). In addition, Section 4.2 presents a dual characterization which provides the basis for the solution of the robust problem. The robust problem in an incomplete market is solved in Section 5. To improve readability, some of the proofs are postponed to Section 6.

2 The Constrained Maximization Problem

We consider a market over a finite time horizon $[0, T]$ for $T > 0$ which consists of $d + 1$ assets, one bond and d stocks. W.l.o.g. we suppose that prices are discounted by the bond, i.e., that the bond price is constant and equal to 1. The price processes of the stocks are given by an $\mathbb{R}^d$-valued semimartingale S on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, R)$ satisfying the usual conditions, where $\mathcal{F} = \mathcal{F}_T$; see Protter (2004), page 3.

An $\mathcal{F}$-measurable random variable will be interpreted as the value of a *financial position* or *contingent claim* at maturity T. Positions which are R-almost surely equal can be identified. The set of all terminal financial positions is denoted by L^0.

2.1 Utility functionals

The classical problem of expected utility maximization consists in maximizing the utility functional

$$U(X) = E_{Q_0}[u(X)]$$

over all *feasible* financial positions X, where Q_0 is some subjective probability measure which is equivalent to the reference measure R and $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is a Bernoulli utility function.

Expected utility is a numerical representation of certain preferences which have been characterized by von Neumann & Morgenstern (1944) and Savage (1954). The utility functional is defined in terms of the single probability measure Q_0. A more general class of preferences admits a robust representation as suggested by Gilboa & Schmeidler (1989). Instead of a single measure Q_0, a set $\mathcal{Q}_0$ of *subjective* or *model measures* provides a numerical representation of these preference orders via a *robust utility functional*

$$U(X) := \inf_{Q_0 \in \mathcal{Q}_0} E_Q[u(X)]. \tag{1}$$

These more general preferences resolve several well-known paradoxa which arise in the classical framework; see, for instance, Gilboa & Schmeidler (1989) or Föllmer & Schied (2004).

The representation (1) suggests also another interpretation. An agent with Bernoulli utility functional u is evaluating her expected utility, but is uncertain about the correct subjective probability measure. Instead the agent is faced with a whole set of conceivable probabilities. In this situation of model uncertainty, she considers the infimum of all possible expectations in order to be on the safe side.

In the current article we consider the problem of maximizing robust utility under a joint budget and downside risk contraint. We impose some standard assumptions on the Bernoulli utility function u. We suppose that the utility function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is strictly increasing, strictly concave, continuously differentiable with existing second derivative in the interior of dom $u := \{x \in \mathbb{R} : u(x) > -\infty\}$. $\bar{x}_u := \inf\{x \in \mathbb{R} : u(x) > -\infty\}$ is assumed to be finite, i.e., $\bar{x}_u > -\infty$. It follows that the interior of the essential domain of u is given by the open interval dom $u = (\bar{x}_u, \infty)$. We suppose that u satisfies the Inada conditions

$$u'(\infty) := \lim_{x\to\infty} u'(x) = 0, \tag{U1}$$

$$\text{(U2)} \qquad u'(\bar{x}_u) := \lim_{x \searrow \bar{x}_u} u'(x) = \infty.$$

Moreover, we assume that u has *regular asymptotic elasticity* (RAE) in the sense of Kramkov & Schachermayer (1999), Frittelli & Gianin (2004), i.e.,

$$\text{(2)} \qquad \limsup_{x \to \infty} \frac{xu'(x)}{u(x)} < 1.$$

The last assumptions allows us to simplify the analysis considerably. We will later emphasize where we use the notion of RAE.

By shifting the utility function along the x-axis it is no loss of generality to suppose that $\bar{x}_u = 0$, and we will make this assumption in the following. The inverse of the derivative of u will be denoted by $I := (u')^{-1}$.

We also impose some restrictions on the set $\mathcal{Q}_0$. We assume that the set $\mathcal{Q}_0$ is convex and that all measures $Q_0 \in \mathcal{Q}_0$ are equivalent to the reference measure. In addition, we suppose that

$$\text{(3)} \qquad \forall \epsilon > 0 \; \exists \delta > 0 \text{ such that } R(A) < \delta \Longrightarrow Q_0(A) < \epsilon \; \forall Q_0 \in \mathcal{Q}_0.$$

Since the set of densities

$$\mathcal{K}_{\mathcal{Q}_0} := \left\{ \frac{dQ_0}{dR} : Q_0 \in \mathcal{Q}_0 \right\}$$

is bounded in $L^1(R)$, (3) is equivalent to the uniform integrability of the densities. Intuitively, the assumption corresponds to a generalized uniform moment condition on the densities. Namely, by the de la Vallée-Poussin criterion, (3) is equivalent to the existence of a function $g : [0, \infty) \to [0, \infty)$ with $\lim_{x \to \infty} g(x)/x = \infty$ such that

$$\sup_{\psi \in \mathcal{K}_{\mathcal{Q}_0}} E_R[g(|\psi|)] < \infty.$$

W.l.o.g we will also assume that $\mathcal{K}_{\mathcal{Q}_0}$ is closed in $L^1(R)$. Indeed, if $\mathcal{K}_{\mathcal{Q}_0}$ is not closed, then its $L^1(R)$-closure $\bar{\mathcal{K}}_{\mathcal{Q}_0}$ defines yet another set $\bar{\mathcal{Q}}_0$ of subjective measures by setting

$$Q(A) := E_R[\psi; A]$$

for $\psi \in \bar{\mathcal{K}}_{\mathcal{Q}_0}$. For any claim X with $u(X)^- \in L^1(Q_0)$ for all $Q_0 \in \mathcal{Q}_0$ we obtain

$$\inf_{Q_0 \in \mathcal{Q}_0} E_{\mathcal{Q}_0}[u(X)] = \inf_{Q_0 \in \bar{\mathcal{Q}}_0} E_{\mathcal{Q}_0}[u(X)].$$

In summary, we suppose that $\mathcal{K}_{\mathcal{Q}_0}$ is $L^1(R)$-closed and uniformly integrable. By the Dunford-Pettis Theorem our hypothesis can therefore be rephrased in the following way:

Assumption 2.1. We assume that all measures in $\mathcal{Q}_0$ are equivalent to R and that the set $\mathcal{K}_{\mathcal{Q}_0}$ is weakly compact, i.e., $\mathcal{K}_{\mathcal{Q}_0}$ is $\sigma(L^1(R), L^\infty(R))$-compact.

2.2 Budget Constraint

We are interested in maximizing the terminal robust utility over all *feasible* financial positions. *Feasibility* is, of course, a term which needs to be defined in detail, and we will do so in the following three sections. We will solve the optimization problem in two steps. Using convex duality, we solve a portfolio optimization problem which is essentially static. We investigate in Section 2.5 how this solution is linked to the problem of finding an optimal self-financing trading strategy.

Definition 2.2. A self-financing portfolio with initial value x is a d-dimensional predictable, S-integrable process $(\xi_t)_{0\leq t\leq T}$ which specifies the amount of each asset in the portfolio. The corresponding value process of the portfolio is given by

$$V_t := x + \int_0^t \xi_s dS_s \qquad (0 \leq t \leq T). \tag{4}$$

The family $\mathcal{V}(x)$ denotes all non-negative value processes of self-financing portfolios with initial value equal to x.

Let us fix an initial wealth $x_2 > 0$. We are interested in finding a self-financing portfolio in $\mathcal{V}(x_2)$ with bounded downside risk that maximizes terminal robust utility. The budget constraint can be expressed in terms of martingale measures.

Definition 2.3. A probability measure P which is absolutely continuous with respect to R is called an *absolutely continuous martingale measure* if S is a local martingale under P. The family of these measures is denoted by $\mathcal{P}$. Any $P \in \mathcal{P}$ which is equivalent to R is called an *equivalent local martingale measure*. The family of these measures will be denoted by $\mathcal{P}_e$.

We interpret measures in the set $\mathcal{P}$ as pricing measures and assume throughout that

$$\mathcal{P}_e \neq \emptyset. \tag{5}$$

The financial market which we consider will thus have the *no free lunch with vanishing risk* (NFLVR) property, see Delbaen & Schachermayer (1994).

Fixing initial wealth of $x_2 > 0$, a contingent claim $X \geq 0$ is *affordable* if there is a self-financing portfolio $V \in \mathcal{V}(x_2)$ such that

$$V_T \geq X \quad R - a.s. \tag{6}$$

The optional decomposition theorem by Kramkov (1996) and Föllmer & Kabanov (1998) states that this notion of affordability is equivalent to

$$\sup_{P\in\mathcal{P}} E_P[X] \leq x_2. \tag{7}$$

We will choose (7) as the budget constraint of our robust utility maximization problem. A simple argument in Section 2.5 will later show that the optimal claim can actually be replicated. This connects the static optimization result to the dynamic optimization problem.

2.3 The Risk Constraint

Besides the budget constraint, we will also require feasible financial positions to satisfy a downside risk constraint. Downside risk of financial positions can be quantified by risk measures. We let $\mathcal{D}$ be some vector space of random variables.

Definition 2.4. A mapping $\rho : \mathcal{D} \to \mathbb{R}$ is called a *risk measure* (on $\mathcal{D}$) if it satisfies the following conditions for all $X_1, X_2 \in \mathcal{D}$:

- *Inverse Monotonicity:* If $X_1 \leq X_2$, then $\rho(X_1) \geq \rho(X_2)$.
- *Translation Invariance*: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

Monotonicity refers to the property that risk decreases if the payoff profile is increased. Translation invariance formalizes that risk is measured on a monetary scale: if a monetary amount $m \in \mathbb{R}$ is added to a position X, then the risk of X is reduced by m.

Value at risk (VaR in the following) is a risk measure according to the above definition, but it does in general not encourage diversification of positions – it is not a *convex* risk measure, if $L^\infty \subseteq \mathcal{D}$. A risk measure ρ is *convex* (on $\mathcal{D}$), if it satisfies the following conditions for all $X_1, X_2 \in \mathcal{D}$:

- *Convexity:* $\rho(\alpha X_1 + (1-\alpha)X_2) \leq \alpha\rho(X_1) + (1-\alpha)\rho(X_2)$ for all $\alpha \in (0, 1)$.

In this article, we focus on a particular example of a convex risk measure for measuring the downside risk, namely *utility-based shortfall risk*. Utility-based shortfall risk is most easily defined as a *capital requirement*, i.e., the smallest monetary amount that has to be added to a position to make it acceptable.[1] We will now give the definition of utility-based shortfall risk.

Let $\ell : \mathbb{R} \to [0, \infty]$ be a loss function, i.e., an increasing function that is not constant. The level x_1 shall be a point in the interior of the range of ℓ. Let Q_1 be a fixed subjective probability measure equivalent to R, which we will use for the purpose of risk management. The space of financial positions $\mathcal{D}$ is chosen in such a way that for $X \in \mathcal{D}$ the integral $\int \ell(-X)dQ_1$ is well defined.

Define an acceptance set

$$\mathcal{A}_{Q_1} = \{X \in \mathcal{D} : \ E_{Q_1}[\ell(-X)] \leq x_1\}. \tag{8}$$

A financial position is thus acceptable if the expected value of $\ell(-X)$ under the subjective probability measure Q_1, i.e., the expected loss $E_{Q_1}[\ell(-X)]$, is not more than x_1.

The acceptance set $\mathcal{A}_{Q_1}$ induces the risk measure *utility-based shortfall risk* (*UBSR* in the following) ρ_{Q_1} as the associated capital requirement

$$\rho_{Q_1}(X) = \inf\{m \in \mathbb{R} : \ X + m \in \mathcal{A}_{Q_1}\}. \tag{9}$$

[1] Note that every static risk measure can be defined as a capital requirement. To be more precise, if ρ is a risk measure, then $\mathcal{A} = \{X \in \mathcal{D} : \rho(X) \leq 0\}$ defines its acceptance set, i.e., the set of positions with non-positive risk. ρ is then recovered as $\rho(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}$, see e.g. Föllmer & Schied (2004), Chapter 4.

Utility-based shortfall risk is convex and does therefore encourage diversification. Examples of loss functions ℓ include exponentials $\exp(\alpha x)$, $\alpha > 0$, which lead to the so-called *entropic risk measures*, for which a simple explicit formula is available; see Föllmer & Schied (2004), Example 4.105. Alternatively, one-sided loss functions can be used to measure downside risk only. These risk measures look at losses only and do not consider tradeoffs between gains and losses. Examples include $(x + \bar{x}_\ell)^\alpha \cdot 1_{(-\bar{x}_\ell,\infty)}(x)$, $\alpha > 1$, $\bar{x}_\ell \in \mathbb{R}$, or exponentials $(\exp\{\alpha(x + \bar{x}_\ell)\} - 1) \cdot 1_{(-\bar{x}_\ell,\infty)}(x)$, $\alpha > 0$, $\bar{x}_\ell \in \mathbb{R}$.

Our aim is to solve the utility maximization problem under a joint budget and risk measure constraint. If there is no model uncertainty, the shortfall risk constraint (*UBSR constraint* in the following) shall be given by

$$\rho_{Q_1}(X) \leq 0. \tag{10}$$

A financial position X which satisfies (10) is acceptable from the point of view of the risk measure ρ. This is equivalent to

$$E_{Q_1}[\ell(-X)] \leq x_1. \tag{11}$$

In the case where the agent faces model uncertainty, we consider a second set $\mathcal{Q}_1$ of subjective measures which are equivalent to the reference measure R. The *robust UBSR constraint* is given by

$$\sup_{Q_1 \in \mathcal{Q}_1} \rho_{Q_1}(X) \leq 0. \tag{12}$$

That is, any financial position must be acceptable from the point of view of *all* risk measures ρ_{Q_1} $(Q_1 \in \mathcal{Q}_1)$. This is equivalent to

$$\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X)] \leq x_1. \tag{13}$$

As for the set $\mathcal{Q}_0$ we impose also convexity and weak compactness on the set $\mathcal{Q}_1$.

Assumption 2.5. We assume that all measures in the convex set $\mathcal{Q}_1$ are equivalent to the reference measure R, and that the set of densities

$$\mathcal{K}_{\mathcal{Q}_1} := \left\{ \frac{dQ_1}{dR} : \; Q_1 \in \mathcal{Q}_1 \right\} \tag{14}$$

is weakly compact in $L^1(R)$, i.e., that $\mathcal{K}_{\mathcal{Q}_1}$ is $\sigma(L^1(R), L^\infty(R))$-compact.

Weak compactness, of course, means that $\mathcal{K}_{\mathcal{Q}_1}$ is weakly closed (or equivalently $L^1(R)$-closed) and uniformly integrable by the Dunford-Pettis. The uniform integrability can be rephrased as a generalized moment condition by the de la Vallée-Poussin criterion.

We require the loss function ℓ to satisfy the following technical conditions. We assume that ℓ is strictly convex, strictly increasing, and continuous. We suppose in addition that ℓ is continuously differentiable on the interval[2] $(-\bar{x}_\ell, \infty)$ for some $\bar{x}_\ell \in (0, \infty]$, and that $\ell(x) = 0$ for $x \leq -\bar{x}_\ell$. We assume that $\lim_{x \to -\infty} \ell(x) = 0$ and $\lim_{x \to -\infty} \ell'(x) = 0$ if $\bar{x}_\ell = \infty$. As for the utility function, we suppose that ℓ has regular asymptotic elasticity (RAE) if $\bar{x}_\ell = \infty$, i.e., $\liminf_{x \to -\infty} \frac{x\ell'(x)}{\ell(x)} < 1$. The last assumption implies that the associated Bernoulli utility function $x \mapsto -\ell(-x)$ has RAE for $x \to \infty$.

[2] If $\bar{x}_\ell \leq 0$, the risk constraint will trivially be satisfied for all claims with utility larger than $-\infty$.

2.4 The Robust Problem in an Incomplete Market Model

We can now pose the robust utility maximization problem under a joint budget and downside risk constraint which we will solve in the current paper. It can be seen as a auxiliary static problem. Its relationship with the solution to the dynamic portfolio selection problem is discussed in Section 2.5.

Let us denote the set of terminal financial positions with well defined utility and prices by

$$\mathcal{I} = \left\{ X \geq 0 : \quad X \in L^1(P) \text{ for all } P \in \mathcal{P} \text{ and } u(X)^- \in L^1(Q_0) \text{ for all } Q_0 \in \mathcal{Q}_0 \right\}. \tag{15}$$

For x_0, $x_1 > 0$, we will solve the following optimization problem under a joint budget and UBSR constraint:

$$\begin{aligned} &\text{Maximize } \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X)] \text{ over all } X \in \mathcal{I} \\ &\text{that satisfy } \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X)] \leq x_1 \text{ and } \sup_{P \in \mathcal{P}} E_P[X] \leq x_0. \end{aligned} \tag{16}$$

The set of all financial positions in $\mathcal{I}$ that satisfy the two constraints is denoted by $\mathcal{X}(x_0, x_1)$, i.e.,

$$\mathcal{X}(x_0, x_1) := \{X \in \mathcal{I} : \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X)] \leq x_1 \text{ and } \sup_{P \in \mathcal{P}} E_P[X] \leq x_0\}. \tag{17}$$

We will first solve an auxiliary problem (20) without model uncertainty and then use this result to tackle problem (16).

2.5 Replication

If S is locally bounded, then the solution to the static problem above is equivalent to the following dynamic problem under a joint budget and UBSR constraint:

$$\begin{aligned} &\text{Maximize } \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(V_T)] \text{ over all } V \in \mathcal{V}(x_0) \\ &\text{that satisfy } \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-V_T)] \leq x_1. \end{aligned} \tag{18}$$

Although the proof of the following theorem requires some results which will be proven in later sections, we state it already at this point. This allows us to to motivate our analysis of (16) more clearly.

Theorem 2.6. *The optimization problem* (16) *admits a solution, if and only if the optimization problem* (18) *admits a solution.*

If $X^* \in \mathcal{X}(x_0, x_1)$ is a solution to problem (16)*, then there exists a solution $V^* \in \mathcal{V}(x_0)$ to* (18) *with $V_T^* \geq X^*$ R-almost surely. In this case, $V_T^* = X^*$ R-almost surely, if the solution to* (16) *is R-almost surely unique. If, conversely, $V^* \in \mathcal{V}(x_0)$ is a solution to* (18)*, then $V_T^* \in \mathcal{X}(x_0, x_1)$ is a solution to* (16)*.*

Proof. Assume first that (16) admits a solution. Let Z be a right-continuous version of

$$Z_t = \text{ess sup}_{P\in\mathcal{P}_e} E_P[X^*|\mathcal{F}_t].$$

By Proposition 4.2 in Kramkov (1996) Z is a supermartingale for every $P \in \mathcal{P}_e$. By Theorem 2.1 in Kramkov (1996) there exists a predictable, S-integrable process ξ such that

$$V_T = x_0 + \int_0^T \xi_s dS_s \geq X^* \geq 0.$$

Under all $P \in \mathcal{P}_e$, V is a σ-martingale which is bounded from below, thus a supermartingale. Thus, $\sup_{P\in\mathcal{P}_e} E_P[V_T] \leq x_0$ which implies by Lemma 3.3 that $\sup_{P\in\mathcal{P}} E_P[V_T] \leq x_0$. Since $V_T \geq X^*$, we obtain $\inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(V_T)] \geq \inf_{Q_0\in\mathcal{Q}_0} E[u(X^*)]$. We also get $V_T \in \mathcal{X}(x_0, x_1)$ which implies that $\inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(V_T)] = \inf_{Q_0\in\mathcal{Q}_0} E[u(X^*)]$.

It remains to be shown that V is a solution to (18). Letting $V^* \in \mathcal{V}(x_0)$ such that

$$\sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-V_T^*)] \leq x_1 \quad \text{and} \quad \inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(V_T^*)] \geq \inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(V_T)],$$

similar arguments as above show that $V_T^* \in \mathcal{X}(x_0, x_1)$. Thus,

$$\inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(V_T^*)] \leq \inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(X^*)] = E_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(V_T)].$$

This implies that V is a solution to (18). If X^* is R-almost surely unique, then $V_T = X^*$ R-almost surely, since V_T is a solution to (16).

Conversely, if a $V^* \in \mathcal{V}(x_0)$ is a solution to (18), then V^* is a σ-martingale which is bounded from below. With similar arguments as above, it follows that $\sup_{P\in\mathcal{P}} E_P[V_T^*] \leq x_0$. This implies $V^* \in \mathcal{X}(x_0, x_1)$. If there was $X \in \mathcal{X}(x_0, x_1)$ with $\inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(V_T^*)] < \inf_{Q_0\in\mathcal{Q}_0} E[u(X)]$, arguments as above would imply the existence of $V \in \mathcal{V}(x_0)$ such that

$$\inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(V_T^*)] < \inf_{Q_0\in\mathcal{Q}_0} E[u(X)] = \inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(V_T)],$$

contradicting the optimality of V^*. It follows that V_T^* is a solution to problem (16). □

Remark 2.7. In both the static and the dynamic problem (16) and (18) the risk constraint is imposed at initial time 0 and not updated later. Optimal strategies are contingent on future information, but have to respect the risk constraint at 0. They can be interpreted as commitment solutions.

3 Extended Martingale Measures

Our characterization of a solution to the robust utility maximization problem (16) requires an enlarged set of martingale measures. For this purpose, consider an additional *default time* ζ, defined as the second coordinate $\zeta(\omega, s) := s$ on the product space $\bar{\Omega} := \Omega \times (0, \infty]$. Set $\mathcal{F}_t := \mathcal{F}_T$ for $t > T$ and let

$$\bar{\mathcal{F}} := \sigma(\{A \times (t, \infty] : A \in \mathcal{F}_t, t \geq 0\})$$

denote the predictable σ-field on $\bar{\Omega}$; the predictable filtration $(\bar{\mathcal{F}}_t)_{t\geq 0}$ is defined in the same manner.

An adapted process $Y = (Y_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ will be identified with the adapted process $\bar{Y} = (\bar{Y}_t)_{t\geq 0}$ on $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\geq 0})$ defined by $\bar{Y}_t := Y_t I_{\{\zeta > t\}}$, i.e.,

$$\bar{Y}_t(\omega, s) := Y_t(\omega) 1_{(t,\infty]}(s) \qquad (t \geq 0).$$

To any probability measure Q on $(\Omega, \mathcal{F})$ corresponds the probability measure $\bar{Q} := Q \times \delta_\infty$ on $(\bar{\Omega}, \bar{\mathcal{F}})$. Conversely, for any probability measure $\bar{Q}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ we define its projections Q^t on $(\Omega, \mathcal{F}_t)$ by

$$Q^t(A) := \bar{Q}(A \times (t, \infty]) \qquad (A \in \mathcal{F}_t).$$

Note that Q^t is a finite measure, but not necessarily a probability measure.

In order to introduce the class $\bar{\mathcal{P}}$ of extended martingale measures, let us denote by $\bar{\mathcal{V}}(x)$ the class of value processes $\bar{V} = (\bar{V}_t)_{t\geq 0}$ on $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\geq 0})$ with $\bar{V}_t = V_t 1_{\{\zeta > t\}}$ for $V \in \mathcal{V}(x)$.

Definition 3.1. A probability measure $\bar{P}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ will be called an *extended martingale measure* if

(i) $P^t \ll R$ on $\mathcal{F}_t$ $(t \geq 0)$,

(ii) Any $\bar{V} \in \bar{\mathcal{V}}(1)$ is a supermartingale under $\bar{P}$.

We denote by $\bar{\mathcal{P}}$ the class of all extended martingale measure on $(\bar{\Omega}, \bar{\mathcal{F}})$, and by $\mathcal{P}^T := \{P^T : \bar{P} \in \bar{\mathcal{P}}\}$ the class of projections of $\bar{\mathcal{P}}$ on $(\Omega, \mathcal{F})$.

Remark 3.2. (i) $P \in \mathcal{P}^T$ is not necessarily a probability measure, but a measure with $P(\Omega) \leq 1$.

(ii) For any martingale measure $P \in \mathcal{P}$ the corresponding measure $\bar{P} := P \times \delta_\infty$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ belongs to $\bar{\mathcal{P}}$. This implies that $\mathcal{P} \subseteq \mathcal{P}^T$. In particular, for any financial position X we have $\sup_{P\in\mathcal{P}} E_P[X] \leq \sup_{P\in\mathcal{P}^T} E_P[X]$.

(iii) The class $\bar{\mathcal{P}}$ of extended martingale measures corresponds exactly to the class of supermartingales which appear in the duality approach of Kramkov & Schachermayer (1999) to the problem of maximizing expected utility in incomplete financial markets, see Föllmer & Gundel (2006).

Lemma 3.3. *For a contingent claim $X \geq 0$ the following conditions are equivalent:*

(i) $\sup_{P\in\mathcal{P}} E_P[X] \leq x_2$.

(ii) $\sup_{P\in\mathcal{P}_e} E_P[X] \leq x_2$.

(iii) There exists a value process $V \in \mathcal{V}(x_2)$ such that $V_T \geq X$ R-almost surely.

(iv) The corresponding claim $\bar{X} := X 1_{\{\zeta > T\}}$ satisfies the constraint

$$\sup_{\bar{P}\in\bar{\mathcal{P}}} E_{\bar{P}}[\bar{X}] \leq x_2.$$

(v) $\sup_{P\in\mathcal{P}^T} E_P[X] \leq x_2$.

Proof. See Föllmer & Gundel (2006). □

4 An Auxiliary Non-Robust Problem in a "Complete Market"

4.1 The Non-Robust Problem in a "Complete Market" Setting

We fix a projection $P := P^T$ of an extended martingale measure $\bar{P} \in \bar{\mathcal{P}}$, a subjective measure $Q_0 \in \mathcal{Q}_0$ for the utility evaluation, and a subjective measure $Q_1 \in \mathcal{Q}_1$ for the risk constraint. Since $\mathcal{P} \subseteq \mathcal{P}^T$, our analysis includes all martingale measures, but it covers also cases in which P is not necessarily a probability measure and has total mass less than one.

We denote the set of terminal financial positions with well defined utility by

$$\mathcal{I}_{P,Q_0} = \left\{ X \geq 0 : \quad X \in L^1(P) \text{ and } u(X)^- \in L^1(Q_0) \right\}. \tag{19}$$

Let $x_0 > 0$ be an initial endowment and $x_1 > 0$ be a risk limit. We consider an auxiliary optimization problem under a joint budget and UBSR constraint:

$$\begin{gathered} \text{Maximize } E_{Q_0}[u(X)] \text{ over all } X \in \mathcal{I}_{P,Q_0} \\ \text{that satisfy } E_{Q_1}[\ell(-X)] \leq x_1 \text{ and } E_P[X] \leq x_0. \end{gathered} \tag{20}$$

The set of all financial positions in $\mathcal{I}_{P,Q_0}$ that satisfy the two constraints is denoted by $\mathcal{X}_{P,Q_1,Q_0}(x_0, x_1)$, i.e.,

$$\mathcal{X}_{P,Q_1,Q_0}(x_0, x_1) := \{ X \in \mathcal{I}_{P,Q_0} : \ E_{Q_1}[\ell(-X)] \leq x_1 \text{ and } E_P[X] \leq x_0 \}. \tag{21}$$

Recall that $\bar{x}_\ell \in (0, \infty]$.

It has been shown in Gundel & Weber (2005) that the unique solution to the constrained maximization problem (20) can be written in the form

$$X_{P,Q_1,Q_0} = x^* \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right),$$

where $x^* : [0, \infty) \times (0, \infty) \to (0, \infty)$ is a continuous deterministic function, and λ_1^*, λ_2^* are suitable real parameters. x^* is obtained as the solution of a family of deterministic maximization problems.

To be more specific, let us define a family of functions g_{y_1,y_2} with $y_1, y_2 \geq 0$ by

$$g_{y_1,y_2}(x) := u(x) - y_1 \ell(-x) - y_2 x.$$

For each pair $y_1 \geq 0, y_2 > 0$, the maximizer of g_{y_1,y_2} is unique and equals

$$x^*(y_1, y_2) \quad := \quad \begin{cases} J(y_1, y_2) & \text{if } y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +), \\ \bar{x}_\ell & \text{if } u'(\bar{x}_\ell) \leq y_2 \leq u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +), \\ I(y_2) & \text{if } y_2 < u'(\bar{x}_\ell). \end{cases} \tag{22}$$

Here, $J(y_1, y_2)$ denotes the unique solution to the equation $u'(x) + y_1 \ell'(-x) = y_2$ for the case that $y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +)$, and $I := (u')^{-1}$. Note that $x^*(0, y_2) = I(y_2) = J(0, y_2)$.

In order to characterize the solution to the utility maximization problem, we will also need to determine a financial position $Y_{P,Q_1} \geq 0$ that minimizes the expected loss under the budget constraint:

$$\text{Minimize } E_{Q_1}[\ell(-Y)] \text{ over all financial positions } Y \geq 0 \tag{23}$$
$$\text{with } Y \in L^1(P) \text{ and } E_P[Y] \leq x_0.$$

The solution to this problem is of the form

$$Y_{P,Q_1} = -L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right).$$

Here $L : \mathbb{R} \to [-\bar{x}_\ell, 0]$ is defined as the generalized inverse of the derivative of the loss function ℓ, i.e.,

$$L(y) := \begin{cases} 0 & \text{if } y \geq \ell'(0), \\ (\ell')^{-1}(y) & \text{if } \ell'(-\bar{x}_\ell+) < y < \ell'(0), \\ -\bar{x}_\ell & \text{if } y \leq \ell'(-\bar{x}_\ell+). \end{cases} \tag{24}$$

L is a continuous function which is strictly increasing on $[\ell'(-\bar{x}_\ell+), \ell'(0)]$. Properties of the functions x^* and L are collected in Section A.

We make the following technical assumption.

Assumption 4.1. Let the function x^* be defined as in (22). We impose the following integrability assumptions for all $\lambda_1 \geq 0, \lambda_2 > 0$:

(a) $x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right) \in L^1(P)$,

(b) $\ell\left(-x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right) \in L^1(Q_1)$,

(c) $u\left(x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right) \in L^1(Q_0)$.

Assumption 4.1 imposes the standard integrability conditions which guarantee that the price, the expected loss and the utility of the solution are well defined.

Let us now state the solution to the loss minimization problem (23).

Proposition 4.2. *Let $x_0 \in (0, \bar{x}_\ell)$. Then the equation*

$$x_0 = -E_P\left[L\left(c\frac{dP}{dQ_1}\right)\right] \tag{25}$$

has a solution $c_{P,Q_1} > 0$. A solution to Problem (23) is given by

$$Y_{P,Q_1} := -L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right). \tag{26}$$

On the set $\{dP/dR > 0\}$, the loss minimizing contingent claim is R-almost surely unique, i.e., $Y_{P,Q_1} \cdot 1_{\{dP/dR>0\}} = \tilde{Y} \cdot 1_{\{dP/dR>0\}}$ R-almost surely for any other solution $\tilde{Y}$ to (23).

If Assumption 4.1(a) holds for $\lambda_1 = 0$ and all $\lambda_2 > 0$, then there exists a unique constant $\tilde{\lambda}_2 > 0$ that solves the equation

$$x_0 \;=\; E_P\left[I\left(\tilde{\lambda}_2 \frac{dP}{dQ_0}\right)\right]. \tag{27}$$

$I(\tilde{\lambda}_2 dP/dQ_0)$ is the unique solution to the utility maximization problem without risk constraint.

The following theorem provides a solution to the utility maximization problem (20).

Theorem 4.3. *Suppose that Assumption 4.1 holds. Let $x_1 > 0$, $x_0 > \bar{x}_u$, let c_{P,Q_1} and $\tilde{\lambda}_2$ be defined as in (25) and (27), and let Y_{P,Q_1} be the solution to the loss minimization problem (23) defined in (26). There are four cases:*

(i) *We have $x_0 < \bar{x}_\ell$ and $x_1 < E_{Q_1}\left[\ell\left(-Y_{P,Q_1}\right)\right]$.*

Then there is no financial position which satisfies both constraints.

(ii) *We have $x_0 < \bar{x}_\ell$ and $x_1 = E_{Q_1}\left[\ell\left(-Y_{P,Q_1}\right)\right]$.*

If $u\left(Y_{P,Q_1}\right)^- \in L^1(Q_0)$, then

$$\begin{aligned} X_{P,Q_1,Q_0} &:= Y_{P,Q_1} \cdot 1_{\left\{\frac{dP}{dR}>0\right\}} + \infty \cdot 1_{\left\{\frac{dP}{dR}=0\right\}} \\ &= -L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right) \cdot 1_{\left\{\frac{dP}{dR}>0\right\}} + \infty \cdot 1_{\left\{\frac{dP}{dR}=0\right\}} \end{aligned}$$

is a solution to the maximization problem (20), and both constraints are binding. Otherwise the maximization problem has no solution. X_{P,Q_1,Q_0} is the unique solution if $u\left(X_{P,Q_1,Q_0}\right) \in L^1(Q_0)$.

(iii) *We have $E_{Q_1}[\ell(-I(\tilde{\lambda}_2 dP/dQ_0))] < x_1$. This implies that either $x_0 \geq \bar{x}_\ell$ or, if $x_0 < \bar{x}_\ell$, $x_1 > E_{Q_1}[\ell(-Y_{P,Q_1})]$.*

Then

$$X_{P,Q_1,Q_0} := I\left(\tilde{\lambda}_2 \frac{dP}{dQ_0}\right)$$

is the unique solution to the maximization problem (20), and the UBSR constraint is not binding.

(iv) *We have either $x_0 \geq \bar{x}_\ell$ or, if $x_0 < \bar{x}_\ell$, $x_1 > E_{Q_1}[\ell(-Y_{P,Q_1})]$, and in both cases $E_{Q_1}[\ell(-I(\tilde{\lambda}_2 dP/dQ_0))] \geq x_1$.*

Then a solution to the maximization problem (20) exists and both constraints are binding. The unique solution is given by

$$X_{P,Q_1,Q_0} := x^*\left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0}\right)$$

$$= \begin{cases} J\left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0}\right) & on \; \left\{\lambda_2^* \frac{dP}{dQ_0} > u'(\bar{x}_\ell) + \lambda_1^* \frac{dQ_1}{dQ_0} \ell'(-\bar{x}_\ell +)\right\}, \\ \bar{x}_\ell & on \; \left\{u'(\bar{x}_\ell) \leq \lambda_2^* \frac{dP}{dQ_0} \leq u'(\bar{x}_\ell) + \lambda_1^* \frac{dQ_1}{dQ_0} \ell'(-\bar{x}_\ell +)\right\}, \\ I\left(\lambda_2^* \frac{dP}{dQ_0}\right) & on \; \left\{\lambda_2^* \frac{dP}{dQ_0} < u'(\bar{x}_\ell)\right\}, \end{cases}$$

where x^* and J are defined as in (22), and $\lambda_1^* \geq 0$, $\lambda_2^* > 0$ satisfy

$$x_1 \;=\; E_{Q_1}\left[\ell\left(-X_{P,Q_1,Q_0}\right)\right] \tag{28}$$

and

$$x_0 \;=\; E_P\left[X_{P,Q_1,Q_0}\right]. \tag{29}$$

4.2 Dual Characterization

The solution of the utility maximization problem (20) can alternatively be characterized by dual functionals. These results provide the basis for the solution of the general robust problem in an incomplete market.

Define the convex function

$$v(y_2, y_1, y_0) := \sup_{x>0}\{y_0 u(x) - y_1 \ell(-x) - y_2 x\}$$

for $(y_2, y_1, y_0) \in [0,\infty) \times [0,\infty) \times (0,\infty)$. Then, for $\lambda_1 \geq 0$ and $\lambda_2 > 0$, a convex functional on $\mathcal{P}^T \times \mathcal{Q}_1 \times \mathcal{Q}_0$ is given by

(30)
$$\begin{aligned}
v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) &= E_R\left[v\left(\lambda_2\frac{dP}{dR}, \lambda_1\frac{dQ_1}{dR}, \frac{dQ_0}{dR}\right)\right] \\
&= E_{Q_0}\left[u\left(x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right)\right] \\
&\quad - \lambda_1 E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right)\right] - \lambda_2 E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right].
\end{aligned}$$

Define the convex function

$$\tilde{v}(y_2, y_1) := \sup_{x>0}\{-y_1\ell(-x) - y_2 x\}$$

for $(y_2, y_1) \in (0,\infty) \times [0,\infty)$. Then, for $c \geq 0$, a convex funtional on $\mathcal{P}^T \times \mathcal{Q}_1$ is given by

$$\begin{aligned}
\tilde{v}_c(P|Q_1) &= E_R\left[\tilde{v}\left(c\frac{dP}{dR}, \frac{dQ_1}{dR}\right)\right] \\
&= -E_{Q_1}\left[\ell\left(L\left(c\frac{dP}{dQ_1}\right)\right)\right] + cE_P\left[L\left(c\frac{dP}{dQ_1}\right)\right].
\end{aligned} \tag{31}$$

Proposition 4.4. *For all $\lambda_1 \geq 0$, $\lambda_2 > 0$, and $c \geq 0$ the functions v_{λ_1,λ_2} and $\tilde{v}_c$ are well-defined, and $v_{\lambda_1,\lambda_2} : \mathcal{P}^T \times \mathcal{Q}_1 \times \mathcal{Q}_0 \to \mathbb{R} \cup \{\infty\}$ and $\tilde{v}_c : \mathcal{P}^T \times \mathcal{Q}_1 \to (-\infty, 0]$.*

Proof. For any $x > 0$,

$$v\left(\lambda_2\frac{dP}{dR}, \lambda_1\frac{dQ_1}{dR}, \frac{dQ_0}{dR}\right) \geq \frac{dQ}{dR}u(x) - \lambda_1\frac{dQ_1}{dR}\ell(-x) - \lambda_2\frac{dP}{dR}x =: Z.$$

But $E_R[Z] = u(x) - \lambda_1 \ell(-x) - \lambda_2 x \in \mathbb{R}$. Thus,

$$E_R\left[v\left(\lambda_2 \frac{dP}{dR}, \lambda_1 \frac{dQ_1}{dR}, \frac{dQ_0}{dR}\right)^- \right] \in \mathbb{R},$$

which implies that v_{λ_1,λ_2} is well-defined. Equality with the right hand side of (30) follows from Lemma A.1(x). The proof for $\tilde{v}_c$ is analogous using Lemma A.1(xi). Moreover, $\tilde{v}(y_2, y_1) \leq 0$ for all $y_1 \geq 0$ and $y_2 > 0$ and hence $\tilde{v}_c(P|Q_1) \leq 0$. $\square$

The following assumption replaces the integrability conditions from the last section.

Assumption 4.5. We suppose that

$$v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) < \infty \quad \text{for all } \lambda_1 \geq 0\ \lambda_2 > 0. \tag{32}$$

In order to verify Assumption 4.5, it is sufficient to consider specific pairs (λ_1, λ_2). This is a consequence of the assumption of the RAE of the utility function.

Proposition 4.6. *The following statement are equivalent:*

(i) $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) < \infty$ *for all* $\lambda_1 \geq 0$, $\lambda_2 > 0$.

(ii) $v_{0,1}(P|Q_1|Q_0) < \infty$.

(iii) $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) < \infty$ *for some* $\lambda_1 \geq 0$, $\lambda_2 > 0$.

Proof. (ii) $\Rightarrow$ (i): There exist functions $a > 0$ and $b \geq 0$ such that for $\lambda_2 > 0$ and $y_2, y_0 > 0$

$$v(\lambda_2 y_2, 0, y_0) \leq a(\lambda_2) v(y_2, 0, y_0) + b(\lambda_2)(y_2 + 1),$$

see e.g. Lemma 2.1.6(iv) in Gundel (2006). Since v is decreasing in y_1, (i) follows from (ii).
(iii) $\Rightarrow$ *(i)*: Assume that $v_{\tilde{\lambda}_1,\tilde{\lambda}_2}(P|Q_1|Q_0) < \infty$. Then

$$\begin{aligned} v(\lambda_2 y_2, \lambda_1 y_1, y_0) &\leq v(\lambda_2 y_2, 0, y_0) \\ &\leq a\left(\frac{\lambda_2}{\tilde{\lambda}_2}\right) v\left(\tilde{\lambda}_2 y_2, 0, y_0\right) + b\left(\frac{\lambda_2}{\tilde{\lambda}_2}\right)(\tilde{\lambda}_2 y_2 + 1) \\ &\leq a\left(\frac{\lambda_2}{\tilde{\lambda}_2}\right)(v(\tilde{\lambda}_2 y_2, \tilde{\lambda}_1 y_1, y_0) + \tilde{\lambda}_1 y_1 \ell(0)) + b\left(\frac{\lambda_2}{\tilde{\lambda}_2}\right)(\tilde{\lambda}_2 y_2 + 1) \end{aligned}$$

for $\lambda_1 \geq 0$ and $\lambda_2 > 0$. Thus, (i) follows from (iii). $\square$

Assumption 4.5 is equivalent to the integrability assumptions that were needed for the solution of the primal utility maximization problem (20) without model uncertainty, i.e., Assumption 4.1.

Lemma 4.7. *Assumptions 4.1 and 4.5 are equivalent.*

Proof. By Lemma A.1(x) v is continuously differentiable in $y_1 \geq 0$ and $y_2 > 0$. We will first show that Assumption 4.5 implies Assumption 4.1.

• Assumption 4.5 $\Rightarrow$ Assumption 4.1:

(a) In order to simplify the notation, we define the convex function

$$f(y_2) := v(y_2, y_1, y_0).$$

Letting $\lambda_1 \geq 0$ be fixed, we set $y_0 := dQ_0/dR > 0$, $y_1 := \lambda_1 dQ_1/dR \geq 0$, $\phi := dP/dR$, $y_2 := \lambda_2\phi$ for $\lambda_2 > 0$. Since f is convex, we obtain for $0 < \mu < \nu$ and $\phi > 0$

$$f(\nu\phi) - f((\nu-\mu)\phi) \leq \mu\phi f'(\nu\phi) \leq f((\nu+\mu)\phi) - f(\nu\phi).$$

For $\phi = 0$ we have to argue more carefully. If $f(0) < \infty$, the above inequality is trivially satisfied. If $f(0) = \infty$ and $R[\phi = 0] > 0$, then $E_R[f(\phi)] = \infty$, contradicting Assumption 4.5. In summary, we obtain that

$$\begin{aligned} E_R\left[f\left(\nu\phi\right)\right] - E_R\left[f\left((\nu-\mu)\phi\right)\right] &\leq \mu E_P\left[f'\left(\nu\phi\right)\right] \\ &\leq E_R\left[f\left((\nu+\mu)\phi\right)\right] - E_R\left[f\left(\nu\phi\right)\right]. \end{aligned}$$

By Lemma A.1(x), $f'(y_2) = -x^*\left(y_1/y_0, y_2/y_0\right)$. Multiplying all parts by -1 thus leads to

$$\begin{aligned} v_{\lambda_1,\nu}(P|Q_1|Q_0) - v_{\lambda_1,\nu+\mu}(P|Q_1|Q_0) &\leq \mu E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \nu\frac{dP}{dQ_0}\right)\right] \\ &\leq v_{\lambda_1,\nu-\mu}(P|Q_1|Q_0) - v_{\lambda_1,\nu}(P|Q_1|Q_0). \end{aligned}$$

Both the upper and the lower bound are finite due to Proposition 4.4 and Assumption 4.5. This implies Assumption 4.1(a).

(b) follows analogously with $\partial v(y_2, y_1, y_0)/\partial y_1 = -\ell\left(-x^*\left(y_1/y_0, y_2/y_0\right)\right)$.

(c) Finally, we obtain from Lemma A.1(x)

$$\begin{aligned} \frac{dQ_0}{dR}u\left(x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right) \;&=\; v\left(\lambda_2\frac{dP}{dR}, \lambda_1\frac{dQ_1}{dR}, \frac{dQ_0}{dR}\right) \\ &+ \lambda_1\frac{dQ_1}{dR}\ell\left(-x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right) \;+\; \lambda_2\frac{dP}{dR}x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right). \end{aligned} \tag{33}$$

Since we just showed that the right-hand side is in $L^1(R)$, 4.5(c) is also proven.

• Assumption 4.1 $\Rightarrow$ Assumption 4.5: This direction is immediate from (33). $\square$

The following theorem gives an alternative solution of the robust utility maximization problem in the absence of model uncertainty using the dual functionals v_{λ_1,λ_2} and $\tilde{v}_c$.

Theorem 4.8. *Suppose that Assumption 4.5 holds.*

(i) Let Y_{P,Q_1} be the solution to the the loss minimization problem (23) *defined in Proposition 4.2. Assume that either $x_0 \geq \bar{x}_\ell$ or, if $x_0 < \bar{x}_\ell$, $x_1 > E_{Q_1}\left[\ell\left(-Y_{P,Q_1}\right)\right]$. The following conditions are equivalent:*

(a) $x_1 = E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0}, \lambda_2^*\frac{dP}{dQ_0}\right)\right)\right]$, $x_0 = E_P\left[x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0}, \lambda_2^*\frac{dP}{dQ_0}\right)\right]$ *and* $\lambda_1^* > 0$

(b) $(\lambda_1^*, \lambda_2^*) = \operatorname{argmin}_{\lambda_1\geq 0, \lambda_2 > 0}\ (v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0)$ *and* $\lambda_1^* > 0$

For the case $\lambda_1^* = 0$ *the following conditions are equivalent:*

(c) $x_1 \geq E_{Q_1}\left[\ell\left(-x^*\left(0, \lambda_2^*\frac{dP}{dQ_0}\right)\right)\right]$, $x_0 = E_P\left[x^*\left(0, \lambda_2^*\frac{dP}{dQ_0}\right)\right]$

(d) $(0, \lambda_2^*) = \operatorname{argmin}_{\lambda_1\geq 0, \lambda_2 > 0}\ (v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0)$

If any of these conditions is satisfied, $X_{P,Q_1,Q_0} = x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0}, \lambda_2^*\frac{dP}{dQ_0}\right)$ *is a solution to the utility maximization problem* (20) *and*

$$E_{Q_0}[u(X_{P,Q_1,Q_0})] = v_{\lambda_1^*,\lambda_2^*}(P|Q_1|Q_0) + \lambda_1^* x_1 + \lambda_2^* x_0. \tag{34}$$

(ii) *Let* $x_0 \in (0, \bar{x}_\ell)$. *The following conditions are equivalent:*

(a) $x_0 = E_P\left[-L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right)\right]$

(b) $c_{P,Q_1} = \operatorname{argmin}_{c>0}\ (\tilde{v}_c(P|Q_1) + c x_0)$

In this case, $Y_{P,Q_1} = -L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right)$ *is a solution to the the loss minimization problem* (23) *defined in Proposition 4.2 and* $E_{Q_1}\left[-\ell\left(-Y_{P,Q_1}\right)\right] = \tilde{v}_{c_{P,Q_1}}(P|Q_1) + c_{P,Q_1} x_0$.

The proof of the last theorem is based on the following lemma.

Lemma 4.9. *Let Assumption 4.5 hold. Then* $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0)$ *is continuously differentiable in* $\lambda_1 \geq 0$ *and* $\lambda_2 > 0$ *with*

$$\frac{\partial}{\partial \lambda_1} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) = -E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right)\right] \tag{35}$$

and

$$\frac{\partial}{\partial \lambda_2} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) = -E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right]. \tag{36}$$

Furthermore, $\tilde{v}_c(P|Q_1)$ *is continuously differentiable in* $c > 0$ *with*

$$\frac{\partial}{\partial c}\tilde{v}_c(P|Q_1) = E_P\left[L\left(c\frac{dP}{dQ_1}\right)\right]. \tag{37}$$

Proof. By Lemma A.1(x)&(xi) v and $\tilde{v}$ are continuously differentiable with

$$\frac{\partial v}{\partial y_1}(y_2, y_1, y_0) = -\ell\left(-x^*\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right)\right), \quad \frac{\partial v}{\partial y_2}(y_2, y_1, y_0) = -x^*\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right), \quad \frac{\partial \tilde{v}}{\partial y_2}(y_2, y_1) = L\left(\frac{y_2}{y_1}\right).$$

By Lemma 4.7,

$$x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right) \in L^1(P), \quad \ell\left(-x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right) \in L^1(Q_1), \quad L\left(c\frac{dP}{dQ_1}\right) \in L^1(P)$$

for any $\lambda_1 \geq 0$, $\lambda_2 > 0$, and $c > 0$. Furthermore, x^* is decreasing in y_2, $\ell \circ (-x^*)$ is decreasing in y_1, and L is increasing. Thus, the continuity of the right hand sides of (35), (36), and (37) follows from the dominated convergence theorem. Moreover we may use Fubini's theorem to obtain for $0 < \lambda_2^1 < \lambda_2^2$

$$
\begin{aligned}
v_{\lambda_1,\lambda_2^2}(P|Q_1|Q_0) &= v_{\lambda_1,\lambda_2^1}(P|Q_1|Q_0) - E_R\left[\int_{\lambda_2^1}^{\lambda_2^2} x^*\left(\lambda_1\frac{dQ_1}{dQ_0},\nu\frac{dP}{dQ_0}\right)\frac{dP}{dR}d\nu\right] \\
&= v_{\lambda_1,\lambda_2^1}(P|Q_1|Q_0) - \int_{\lambda_2^1}^{\lambda_2^2} E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0},\nu\frac{dP}{dQ_0}\right)\right]d\nu,
\end{aligned}
$$

and for $0 \le \lambda_1^1 < \lambda_1^2$

$$
\begin{aligned}
v_{\lambda_1^2,\lambda_2}(P|Q_1|Q_0) &= v_{\lambda_1^1,\lambda_2}(P|Q_1|Q_0) - E_R\left[\int_{\lambda_1^1}^{\lambda_1^2} \ell\left(-x^*\left(\nu\frac{dQ_1}{dQ_0},\lambda_2\frac{dP}{dQ_0}\right)\right)\frac{dQ_1}{dR}d\nu\right] \\
&= v_{\lambda_1^1,\lambda_2}(P|Q_1|Q_0) - \int_{\lambda_1^1}^{\lambda_1^2} E_{Q_1}\left[\ell\left(-x^*\left(\nu\frac{dQ_1}{dQ_0},\lambda_2\frac{dP}{dQ_0}\right)\right)\right]d\nu,
\end{aligned}
$$

and for $0 < c^1 < c^2$

$$
\begin{aligned}
\tilde{v}_{c^2}(P|Q_1) &= \tilde{v}_{c^1}(P|Q_1) + E_R\left[\int_{c^1}^{c^2} L\left(\nu\frac{dP}{dQ_1}\right)\frac{dP}{dR}d\nu\right] \\
&= \tilde{v}_{c^1}(P|Q_1) + \int_{c^1}^{c^2} E_P\left[L\left(\nu\frac{dP}{dQ_1}\right)\right]d\nu.
\end{aligned}
$$

This completes the proof. □

Proof of Theorem 4.8. (i) Note that $(\lambda_1,\lambda_2) \mapsto v_{\lambda_1,\lambda_2}(P|Q_1|Q_2) + \lambda_1 x_1 + \lambda_2 x_0 =: g(\lambda_1,\lambda_2)$ is convex and continuously differentiable.
(a) $\Rightarrow$ (b): By Lemma 4.9, $\frac{\partial g}{\partial\lambda_1}(\lambda_1^*,\lambda_2^*) = 0$ and $\frac{\partial g}{\partial\lambda_2}g(\lambda_1^*,\lambda_2^*) = 0$. Thus, $(\lambda_1^*,\lambda_2^*)$ is a global minimum of g.
(b) $\Rightarrow$ (a): Since $\lambda_1^* > 0$, we have

$$
0 = \frac{\partial g}{\partial\lambda_1}(\lambda_1^*,\lambda_2^*) = -E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0},\lambda_2^*\frac{dP}{dQ_0}\right)\right)\right] + x_1.
$$

Moreover, $\frac{\partial g}{\partial\lambda_2}(\lambda_1^*,0) = -\infty$ by Lemma A.1(vi), thus $\lambda_2^* > 0$ and

$$
0 = \frac{\partial g}{\partial\lambda_2}(\lambda_1^*,\lambda_2^*) = -E_P\left[x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0},\lambda_2^*\frac{dP}{dQ_0}\right)\right] + x_0.
$$

(c) $\Rightarrow$ (d): By Lemma 4.9, $\frac{\partial g}{\partial\lambda_1}(0,\lambda_2^*) \ge 0$ and $\frac{\partial g}{\partial\lambda_2}g(0,\lambda_2^*) = 0$. Thus, $(0,\lambda_2^*)$ is a global minimum of g.
(d) $\Rightarrow$ (c): We have

$$
0 \le \frac{\partial g}{\partial\lambda_1}(\lambda_1^*,\lambda_2^*) = -E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0},\lambda_2^*\frac{dP}{dQ_0}\right)\right)\right] + x_1.
$$

The second claim follows as in the part "(b) $\Rightarrow$ (a)."

It remains to prove (34). By Theorem 4.3, X_{P,Q_1,Q_0} is a solution to the maximization problem (20) and

$$\begin{aligned} E_{Q_0}\left[u(X_{P,Q_1,Q_0})\right] &= E_{Q_0}\left[u\left(x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0}, \lambda_2^*\frac{dP}{dQ_0}\right)\right)\right] \\ &\quad -\lambda_1^* E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0}, \lambda_2^*\frac{dP}{dQ_0}\right)\right)\right] - \lambda_2^* E_P\left[x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0}, \lambda_2^*\frac{dP}{dQ_0}\right)\right] \qquad (38) \\ &= v_{\lambda_1^*,\lambda_2^*}(P|Q_1|Q_0). \end{aligned}$$

The first equality follows, since the last two terms in (38) are 0. The last equality follows from (30).

(ii) Note that $c \mapsto \tilde{v}_c(P, Q_1) + cx_0 =: k(c)$ is convex and continuously differentiable.

(a) $\Rightarrow$ (b): By Lemma 4.9, $\frac{\partial k}{\partial c}(c_{P,Q_1}) = 0$. Thus, c_{P,Q_1} is a global minimum.

(b) $\Rightarrow$ (a): $\frac{\partial k}{\partial c}(0) = E_P[L(0)] = -\bar{x}_\ell < 0$ by Lemma A.1(viii). Thus, $c_{P,Q_1} > 0$ and

$$0 = \frac{\partial k}{\partial c}(c_{P,Q_1}) = E_P\left[L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right)\right] + x_0.$$

□

5 The Robust Problem in an Incomplete Market

In this section we finally solve the robust utility maximization problem (16) under a joint budget and risk constraint. In order to keep the presentation clear, we postpone all proofs to Section 6. The relationship of the solution to (16) with the dynamic portfolio optimization problem (18) was already investigated in Section 2.5.

It turns out that the robust solution can be constructed from the non robust solution with the help of certain worst-case measures. In the robust case, we replace Assumption (4.5) by the following robust version:

Assumption 5.1.

$$\inf_{P\in\mathcal{P}^T} \inf_{Q_1\in\mathcal{Q}_1} \inf_{Q_0\in\mathcal{Q}_0} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) < \infty \quad \text{for all } \lambda_1 \geq 0,\ \lambda_2 > 0. \qquad (39)$$

In order to verify Assumption 5.1, it is again sufficient to consider specific pairs (λ_1, λ_2). This is a consequence of the assumption of RAE of the utility function.

Proposition 5.2. *Assumption 5.1 is equivalent to*

$$\inf_{P\in\mathcal{P}^T} \inf_{Q_1\in\mathcal{Q}_1} \inf_{Q_0\in\mathcal{Q}_0} v_{0,1}(P|Q_1|Q_0) < \infty.$$

Proof. The proposition follows from Proposition 4.6. □

5.1 Loss Minimization

As in the non-robust case, a first step consists in solving the problem of minimizing the expected loss over all contingent claims $Y \geq 0$ under the budget constraint in an incomplete market, i.e.,

$$\text{Minimize} \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-Y)] \text{ over all } Y \geq 0$$

(40)

$$\text{with } Y \in L^1(P) \text{ for all } P \in \mathcal{P}^T \text{ and } \sup_{P \in \mathcal{P}^T} E_P[Y] \leq x_0.$$

Proposition 5.3. *Let Assumption 5.1 hold and let $x_0 \in (0, \bar{x}_\ell)$.*

(i) There exists $c^* \in (0, \infty)$ that minimizes the convex function

$$\tilde{G}(c) = \inf_{P \in \mathcal{P}^T} \inf_{Q_1 \in \mathcal{Q}_1} \tilde{v}_c(P|Q_1) + cx_0.$$

(ii) There exist $\tilde{P} \in \mathcal{P}^T$ and $\tilde{Q}_1 \in \mathcal{Q}_1$ that achieve the infimum of $\tilde{v}_{c^*}(P|Q_1)$ over the sets $\mathcal{P}^T$ and $\mathcal{Q}_1$.

(iii) The solution to Problem (40) is R-almost surely unique on the set $\{d\tilde{P}/dR > 0\}$ and given by

$$Y^* = -L\left(c^* \frac{d\tilde{P}}{d\tilde{Q}_1}\right).$$

Furthermore, Problem (40) is equivalent to the classical problem (23) under the measures $\tilde{P}$ and $\tilde{Q}_1$, $\sup_{P \in \mathcal{P}^T} E_P[Y^*] = E_{\tilde{P}}[Y^*]$, and

$$-\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-Y^*)] = -E_{\tilde{Q}_1}[\ell(-Y^*)] = \tilde{v}_{c^*}(\tilde{P}|\tilde{Q}_1) + c^* x_0. \tag{41}$$

5.2 Utility Maximization

We will now solve the robust utility maximization problem (16) under a joint budget and risk constraint.

Assumption 5.4. There exists a minimizer $(\lambda_1^*, \lambda_2^*) \in [0, \infty) \times (0, \infty)$ of the convex function

$$\inf_{P \in \mathcal{P}^T} \inf_{Q_1 \in \mathcal{Q}_1} \inf_{Q_0 \in \mathcal{Q}_0} \left\{ v_{\lambda_1, \lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0 \right\}.$$

Proposition 5.5. *Suppose that Assumption 5.4 holds and that the sets $\mathcal{Q}_0$ and $\mathcal{Q}_1$ satisfy Assumptions 2.1 & 2.5. The convex functional*

$$(P, Q_1, Q_0) \mapsto v_{\lambda_1^*, \lambda_2^*}(P|Q_1|Q_0)$$

attains its infimum on $\mathcal{P}^T \times \mathcal{Q}_1 \times \mathcal{Q}_0$. We denote the minimizing measures by $P^* \in \mathcal{P}^T$, $Q_1^* \in \mathcal{Q}_1$, and $Q_0^* \in \mathcal{Q}_0$.

We impose the following additional hypothesis:

Assumption 5.6. For any $Q_0 \in \mathcal{Q}_0$, there exists $\alpha \in (0,1]$ such that

$$v_{\lambda_1^*,\lambda_2^*}(P^*|Q_1^*|\alpha Q_0 + (1-\alpha)Q_0^*) < \infty.$$

Lemma 5.7. *If $u(\infty) < \infty$, then Assumption 5.6 is automatically satisfied.*

The minimizers P^*, Q_1^*, and Q_0^* in Proposition 5.5 can be characterized as worst case measures.

Proposition 5.8. *Suppose that Assumptions 5.1, 5.4 & 5.6 hold, and define*

$$X^* := x^* \left(\lambda_1^* \frac{dQ_1^*}{dQ_0^*}, \lambda_2^* \frac{dP^*}{dQ_0^*} \right).$$

Then

(i) $X^* \in L^1(P)$ *for all* $P \in \mathcal{P}^T$, *and*

$$E_{P^*}[X^*] = \sup_{P \in \mathcal{P}^T} E_P[X^*], \tag{42}$$

(ii) $\ell(-X^*) \in L^1(Q_1)$ *for all* $Q_1 \in \mathcal{Q}_1$, *and*

$$E_{Q_1^*}[\ell(-X^*)] = \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X^*)], \tag{43}$$

(iii) $u(X^*) \in L^1(Q_0)$ *for all* $Q_0 \in \mathcal{Q}_0$, *and*

$$E_{Q_0^*}[u(X^*)] = \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X^*)]. \tag{44}$$

Finally, we state the solution to the robust utility maximization problem (16) under both a budget and a risk constraint. Recall that $v_{0,\lambda_2}(P|Q_1|Q_0)$ does not depend on Q_1. Uniqueness in the following is meant in the R-almost sure sense.

Theorem 5.9. *Let the sets $\mathcal{Q}_0$ and $\mathcal{Q}_1$ satisfy the Assumptions 2.1 & 2.5, let the integrability assumptions 5.1 and 5.6 hold, and let x_1, $x_0 > 0$. Define Y^* as the loss-minimizing claim from Proposition 5.3. Furthermore, let $\tilde{\lambda}_2$ be a minimizer of the convex function*

$$\inf_{P \in \mathcal{P}^T} \inf_{Q_0 \in \mathcal{Q}_0} v_{0,\lambda_2}(P|Q_1|Q_0) + \lambda_2 x_0,$$

and $\hat{P}$ and $\hat{Q}_0$ minimizer of $v_{0,\tilde{\lambda}_2}(P|Q_1|Q_0)$ over $\mathcal{P}^T$ and $\mathcal{Q}_0$.

(i) If $x_0 < \bar{x}_\ell$ and $x_1 < \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-Y^*)]$, then there is no contingent claim which satisfies both constraints.

(ii) Assume that $x_0 < \bar{x}_\ell$ and $x_1 = \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-Y^*)]$.
If $u(Y^*)^- \in L^1(Q_0)$ for all $Q_0 \in \mathcal{Q}_0$, then

$$X^* : = Y^* \cdot 1_{\left\{\frac{d\tilde{P}}{dR} > 0\right\}} + \infty \cdot 1_{\left\{\frac{d\tilde{P}}{dR} = 0\right\}}$$

is a solution to the maximization problem (16), and both constraints are binding. Otherwise the maximization problem has no solution. X^* is the unique solution on the set $\{d\tilde{P}/dR > 0\}$.

(iii) Assume that $\sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0))] < x_1$. *Then*

$$X^* := I\left(\tilde{\lambda}_2 \frac{d\hat{P}}{d\hat{Q}_0}\right)$$

is the unique solution to the maximization problem (16)*, and the UBSR constraint is not binding.*

(iv) Assume that $x_1 \geq \sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-Y^*)]$ *and* $\sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0))] \geq x_1$. *Then a solution to the maximization problem* (16) *exists and both constraints are binding.*

Assume in addition that Assumption 5.4 holds. Then the unique solution is given by

$$X^* := x^*\left(\lambda_1^* \frac{dQ_1^*}{dQ_0^*}, \lambda_2^* \frac{dP^*}{dQ_0^*}\right),$$

where x^* *is defined as in* (22)*. Furthermore,* P^**,* Q_1^**, and* Q_0^* *are worst case measures, i.e., they satisfy* (42)*,* (43)*, and* (44)*, and the utility of the optimal claim is given by*

$$\inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(X^*)] = v_{\lambda_1^*,\lambda_2^*}(P^*|Q_1^*|Q_0^*) + \lambda_1^* x_1 + \lambda_2^* x_0. \tag{45}$$

The preceding theorem provides a solution to the robust utility maximization problem (16) under both a budget and a risk constraint. The solution is of the same form as the solution to Problem (20) without model uncertainty.

Note that in case (ii), the robust problem (16) has the same solution as the classical problem (20) under $\tilde{Q}_1$ and $\tilde{P}$, and these two measures may be interpreted as worst case measures for the utility maximization problem. In case (iii), the robust problem (16) can be reduced to a utility maximization problem with utility functional $E_{\hat{Q}_0}[u(X)]$ and budget constraint $E_{\hat{P}}[X]$. The risk constraint is automatically satisfied in this case, and $\hat{P}$ and $\hat{Q}_0$ are worst case measures for the optimal claim. In the last case (iv), X^* is the solution to the utility maximization problem (20) with a joint budget and risk constraint under the measures Q_0^*, Q_1^*, and P^*.

6 Proofs

6.1 Loss Minimization

Proof of Proposition 5.3. (i) The function $\tilde{G}(c) := \inf_{P\in\mathcal{P}^T} \inf_{Q_1\in\mathcal{Q}_1} \tilde{v}_c(P|Q_1) + cx_0$ is convex. Lemma A.1 implies that $\lim_{c\to\infty} \tilde{v}(c,1)/c = 0$. Observe that for $x > 0$ we have

$$\tilde{v}\left(c\frac{dP}{dR}, \frac{dQ_1}{dR}\right) \geq -\frac{dQ_1}{dR}\ell(-x) - c\frac{dP}{dR}x.$$

We obtain $\tilde{v}_c(P|Q_1) \geq \tilde{v}(c,1)$ by taking expectations with respect to the reference measure R and then the supremum over $x > 0$. Thus,

$$\lim_{c\to\infty} \tilde{G}(c) \geq \lim_{c\to\infty} (\tilde{v}(c,1) + cx_0) = \lim_{c\to\infty} c\left(\frac{\tilde{v}(c,1)}{c} + x_0\right) = \infty$$

because $x_0 > 0$.

Assume now that the infimum is achieved in $c^* = 0$. Observe that

$$\tilde{G}(c) \geq \tilde{v}(c, 1) + cx_0.$$

With $c \to 0$ we obtain by Lemma A.1(xi) that $\tilde{G}(0) \geq 0$. Thus, for any $c > 0$,

$$0 \leq \tilde{G}(0) \leq \inf_{P \in \mathcal{P}^T} \inf_{Q_1 \in \mathcal{Q}_1} \tilde{v}_c(P|Q_1) + cx_0 \leq \tilde{v}_c(P|Q_1) + cx_0 \leq c \left(E_P \left[L \left(c \frac{dP}{dQ_1} \right) \right] + x_0 \right)$$

for any $Q_1 \in \mathcal{Q}$ and $P \in \mathcal{P}^T$. Noting that $\tilde{v}_c(P|Q_1) + cx_0$ is zero for $c = 0$, the last inequality follows from the convexity of $c \mapsto \tilde{v}_c(P|Q_1) + cx_0$ and Lemma 4.9.

$L\left(c\frac{dP}{dQ_1}\right)$ converges to $-\bar{x}_\ell$ as $c \to 0$ and is bounded. Since $\bar{x}_\ell > x_0$, the bounded convergence theorem implies that there exists $c > 0$ such that the last term in the brackets is strictly negative, a contradiction. Hence, the convex function $\tilde{G}$ achieves its infimum in some $c^* \in (0, \infty)$.

(ii) For the properties of the function $\tilde{v}$ the reader is referred to Lemma A.1. Let $f(x) = \tilde{v}(c^* x, 1)$. f is continuous, convex and

$$\lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{\tilde{v}(c^* x, 1)}{x} = 0.$$

Moreover,

$$\tilde{v}(c^* x, y) = \begin{cases} 0 & \text{if } y = 0, \\ yf\left(\frac{x}{y}\right) & \text{if } y > 0. \end{cases}$$

Since $\mathcal{Q}_1$ is weakly compact by Assumption 2.1, we can apply Theorem 1.2.8 of Gundel (2006).

(iii) $\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}(\ell(-Y^*)) = E_{\tilde{Q}_1}(\ell(-Y^*))$ and $\sup_{P \in \mathcal{P}^T} E_P(Y^*) = E_{\tilde{P}}(Y^*)$ follow from Proposition 2.3.8 in Gundel (2006). Here, $-\ell(-\cdot)$ replaces the Bernoulli utility function. Gundel's Assumption 2.3.2 is automatically satisfied, since $\tilde{v}_c(P|Q_1) \leq 0$ for all $c \geq 0$.

By Theorem 4.8(ii), Y^* is a solution to the classical loss minimization problem (23) under $\tilde{P}$ and $\tilde{Q}_1$, and $x_0 = E_{\tilde{P}}(Y^*)$. Thus, Y^* satisfies also the robust budget constraint in (40). Then,

$$-\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}(\ell(-Y)) \leq -E_{\tilde{Q}_1}(\ell(-Y)) \leq -E_{\tilde{Q}_1}(\ell(-Y^*)) = -\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}(\ell(-Y^*)).$$

This implies that Y^* is a solution to (40). Moreover, by Theorem 4.8(ii),

$$-E_{\tilde{Q}_1}(\ell(-Y^*)) = \tilde{v}_{c^*}(\tilde{P}|\tilde{Q}_1) + c^* x_0.$$

In order to show uniqueness, assume that $\tilde{Y}$ solves Problem (40). Then we have $E_{\tilde{P}}[\tilde{Y}] \leq x_0$ and hence

$$\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-\tilde{Y})] \geq E_{\tilde{Q}_1}[\ell(-\tilde{Y})] \geq E_{\tilde{Q}_1}[\ell(-Y^*)].$$

The second inequality holds strictly unless $\tilde{Y} = Y^*$ R-almost surely on $\{d\tilde{P}/dR > 0\}$. This follows from the fact that Y^* is the solution to Problem (23) under $\tilde{P}$ and $\tilde{Q}_1$ and from the uniqueness result in Proposition 4.2. But the strict inequality is a contradiction to $E_{\tilde{Q}_1}[\ell(-Y^*)] = \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-Y^*)] = \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-\tilde{Y})]$. Thus $\tilde{Y} = Y^*$ R-almost surely on $\{d\tilde{P}/dR > 0\}$. $\square$

6.2 Utility Maximization

For the proof of Proposition 5.5, we need the following auxiliary result. In order to simplify the notations, we define $f\left(\phi,\psi_1,\psi_0\right) := v\left(\lambda_2^*\phi,\lambda_1^*\psi_1,\psi_0\right)$ and $f(P|Q_1|Q_0) := v_{\lambda_1^*,\lambda_2^*}(P|Q_1|Q_0)$.

Lemma 6.1. *The set*

$$\left\{ f\left(\frac{dP}{dR}+\epsilon,\frac{dQ_1}{dR},\frac{dQ_0}{dR}\right)^- : P\in\mathcal{P}^T, Q_1\in\mathcal{Q}_1, Q_0\in\mathcal{Q}_0 \right\}$$

is uniformly integrable with respect to R.

Proof. We obtain from the proof of Theorem 4.5 in Föllmer & Gundel (2006) that

$$\left\{ \left[\sup_{x>0}\left(\frac{dQ_0}{dR}u(x)-\lambda_2 x\left(\frac{dP}{dR}+\epsilon\right)\right)\right]^- : P\in\mathcal{P}^T, Q_0\in\mathcal{Q}_0 \right\}$$

is uniformly integrable. $\sup_{x>0}\left(\frac{dQ_0}{dR}u(x)-\lambda_2 x\left(\frac{dP}{dR}+\epsilon\right)\right)$ takes the role of the term "$f(\psi_0+\epsilon,\psi_0)$" in the proof in Föllmer & Gundel (2006). The details are left to the reader. Now the result follows from

$$\begin{aligned} f\left(\frac{dP}{dR}+\epsilon,\frac{dQ_1}{dR},\frac{dQ_0}{dR}\right) &= \sup_{x>0}\left\{\frac{dQ_0}{dR}u(x)-\lambda_1\ell(-x)\frac{dQ_1}{dR}-\lambda_2 x\left(\frac{dP}{dR}+\epsilon\right)\right\} \\ &\geq \sup_{x>0}\left\{\frac{dQ_0}{dR}u(x)-\lambda_2 x\left(\frac{dP}{dR}+\epsilon\right)\right\}-\lambda_1\ell(0)\frac{dQ_1}{dR}, \end{aligned}$$

the uniform integrability of $\mathcal{K}_{\mathcal{Q}_1}$ due to Assumption 2.5, and the fact that the sum of two uniformly integrable sets is again uniformly integrable. ☐

Proof of Proposition 5.5. W.l.o.g assume that $\inf_{P\in\mathcal{P}}\inf_{Q_1\in\mathcal{Q}_1}\inf_{Q_0\in\mathcal{Q}_0} f(P|Q_1|Q_0) < \infty$; otherwise, any $(P,Q_1,Q_0)\in\mathcal{P}\times\mathcal{Q}_1\times\mathcal{Q}_0$ is a minimizer of the generalized divergence. $f(\phi,\psi_1,\psi_0)$ is continuous on $[0,\infty)\times[0,\infty)\times(0,\infty)$, since the functions g and x^* defined in Lemma A.1 are continuous and $f(\phi,\psi_1,\psi_0)=\psi_0 g(x^*(\lambda_1\psi_1/\psi_0,\lambda_2\phi/\psi_0))$.

Let $(Q_0^n)_{n\geq 1}\subseteq\mathcal{Q}_0$, $(Q_1^n)_{n\geq 1}\subseteq\mathcal{Q}_1$, and $(P_n)_{n\geq 1}\subseteq\mathcal{P}^T$ be such that $f(P^n|Q_1^n|Q_0^n)$ converges to the infimum of the values $f(P|Q_1|Q_0)$ over $P\in\mathcal{P}^T$, $Q_1\in\mathcal{Q}_1$ and $Q_0\in\mathcal{Q}_0$, and define

$$\psi_i^n := \frac{dQ_i^n}{dR}$$

for $i=0,1$. By Delbaen & Schachermayer (1994), Lemma A1.1, we can choose

$$\psi_i^{n,0}\in\operatorname{conv}(\psi_i^n,\psi_i^{n+1},...) \quad (n=1,2,...)$$

and functions ψ_i^* such that

$$\psi_i^{n,0}\longrightarrow\psi_i^* \qquad R-\text{almost surely.}$$

Since the sets $\mathcal{K}_{\mathcal{Q}_i}$ are weakly compact we have $\psi_i^*\in\mathcal{K}_{\mathcal{Q}_i}$, i.e., ψ_i^* are the densities of some measures $Q_i^*\in\mathcal{Q}_i$. Due to Lemma 4.4 in Föllmer & Gundel (2006), we can also choose

$$P^{n,0}\in\ \operatorname{conv}(P^n,P^{n+1},...) \quad (n=1,2,...)$$

and $P^* \in \mathcal{P}^T$ such that

$$\frac{dP^{n,0}}{dR} \longrightarrow \frac{dP^*}{dR} \qquad R-\text{almost surely.} \tag{46}$$

Define $\phi^{n,0} := dP^{n,0}/dR$ and $\phi^* := dP^*/dR$. Note first that

$$f(P^*|Q_1^*|Q_0^*) = E_R\left[f\left(\phi^*, \psi_1^*, \psi_0^*\right)\right] = E_R\left[\lim_{\epsilon\to 0} f\left(\phi^* + \epsilon, \psi_1^*, \psi_0^*\right)\right] = \lim_{\epsilon\to 0} E_R\left[f\left(\phi^* + \epsilon, \psi_1^*, \psi_0^*\right)\right]$$

by monotone convergence, since $f(\cdot, \psi_1, \psi_0)$ is continuous and decreasing on $[0, \infty)$, and

$$E_R\left[f\left(\phi^* + \epsilon, \psi_1^*, \psi_0^*\right)\right] \geq f(E_R[\phi^*] + \epsilon, 1, 1) > -\infty$$

by definition of f as a supremum. Lemma 6.1 implies

$$\begin{aligned}
E_R\left[f\left(\phi^* + \epsilon, \psi_1^*, \psi_0^*\right)\right] \quad &= \quad E_R\left[\lim_{n\to\infty} f(\phi^{n,0} + \epsilon, \psi_1^{n,0}, \psi_0^{n,0})\right] \\
= \quad E_R\left[\lim_{n\to\infty} f^+(\phi^{n,0} + \epsilon, \psi_1^{n,0}, \psi_0^{n,0})\right] &- E_R\left[\lim_{n\to\infty} f^-(\phi^{n,0} + \epsilon, \psi_1^{n,0}, \psi_0^{n,0})\right] \\
\leq \quad \liminf_{n\to\infty} E_R[f(\phi^{n,0} + \epsilon, \psi_1^{n,0}, \psi_0^{n,0})] \quad &\leq \quad \liminf_{n\to\infty} E_R[f(\phi^{n,0}, \psi_1^{n,0}, \psi_0^{n,0})] \\
\leq \quad \liminf_{n\to\infty} E_R[f(\phi^{n}, \psi_1^{n}, \psi_0^{n})] \quad &= \quad \inf_{P\in\mathcal{P}^T} \inf_{Q_1\in\mathcal{Q}_1} \inf_{Q_0\in\mathcal{Q}_0} f(P|Q_1|Q_0).
\end{aligned}$$

The first equality follows from the continuity of $f(\cdot + \epsilon, \cdot, \cdot)$ on $[0, \infty)^2 \times (0, \infty)$, the first inequality follows from Fatou's lemma (applied to the first term) and Lebesgue's theorem (applied to the second term) due to Lemma 6.1, and the last one from the convexity of $f(\cdot, \cdot, \cdot)$. This shows that $f(\cdot|\cdot|\cdot)$ attains its minimum in (P^*, Q_1^*, Q_0^*). □

Proof of Lemma 5.7. Let $Q_0 \in \mathcal{Q}_0$, $\alpha \in (0, 1)$, and define $\psi_0^* := dQ_0^*/dR$, $\psi_0 := dQ_0/dR$, $\psi_0^\alpha := \alpha\psi_0 + (1-\alpha)\psi_0^*$, $\psi_1^* := dQ_1^*/dR$, and $\phi^* := dP^*/dR$. The convex function $f(\psi_0) := v(\lambda_2^*\phi^*, \lambda_1^*\psi_1^*, \psi_0)$ has increasing derivative $f'(\psi_0) = u(x^*(\lambda_1^*\psi_1^*/\psi_0, \lambda_2^*\phi^*/\psi_0)) \leq u(\infty)$ due to Lemma A.1(vii)&(x). Hence

$$\begin{aligned}
f(\psi_0^\alpha) &\leq f(\psi_0^*) - f'(\psi_0^\alpha)(\psi_0^* - \psi_0^\alpha) \\
&\leq f(\psi_0^*) - f'((1-\alpha)\psi_0^*)\psi_0^* + u(\infty)\psi_0^\alpha \\
&= f(\psi_0^*) - u\left(x^*\left(\frac{\lambda_1^*}{1-\alpha}\frac{\psi_1^*}{\psi_0^*}, \frac{\lambda_2^*}{1-\alpha}\frac{\phi^*}{\psi_0^*}\right)\right)\psi_0^* + u(\infty)\psi_0^\alpha,
\end{aligned}$$

which is in $L^1(R)$ due to Assumption 5.1 and Lemma 4.7(i). □

Proof of Proposition 5.8. This can be shown in exactly the same way as Proposition 3.12 in Föllmer & Gundel (2006) or Proposition 2.3.8 in Gundel (2006) by setting
(i) $f(\phi) := v(\lambda_2^*\phi, \lambda_1^* dQ_1^*/dR, dQ_0^*/dR)$ for $P \in \mathcal{P}^T$ and $\phi := dP/dR$,
(ii) $f(\psi_1) := v(\lambda_2^* dP^*/dR, \lambda_1^*\psi_1, dQ_0^*/dR)$ for $Q_1 \in \mathcal{Q}_1$ and $\psi_1 := dQ_1/dR$,
(iii) $f(\psi_0) := v(\lambda_2^* dP^*/dR, \lambda_1^* dQ_1^*/dR, \psi_0)$ for $Q_0 \in \mathcal{Q}_0$ and $\psi_0 := dQ_0/dR$,
and using Lemma A.1(x).

Note that in (i) for any $P \in \mathcal{P}^T$ there is $\alpha \in (0,1]$ such that $v_{\lambda_1^*,\lambda_2^*}(\alpha P + (1-\alpha)P^*|Q_1^*|Q_0^*) < \infty$. Indeed, let $P \in \mathcal{P}^T$, $\alpha \in (0,1)$, and define $\phi^* := dP^*/dR$, $\phi := dP/dR$, $\phi^\alpha := \alpha\phi + (1-\alpha)\phi^*$, $\psi_1^* := dQ_1^*/dR$, and $\psi_0^* := dQ_0^*/dR$. The convex function $f(\phi) := v(\lambda_2^*\phi, \lambda_1^*\psi_1^*, \psi_0^*)$ has increasing derivative $f'(\phi) = -\lambda_2^* x^*(\lambda_1^*\psi_1^*/\psi_0^*, \lambda_2^*\phi/\psi_0^*) \leq 0$ on $\{\phi > 0\}$. Hence we obtain on $\{\phi^\alpha > 0\}$,

$$
\begin{aligned}
f(\phi^\alpha) &\leq f(\phi^*) - f'(\phi^\alpha)(\phi^* - \phi^\alpha) \\
&\leq f(\phi^*) - \lambda_2^* f'((1-\alpha)\phi^*)\phi^* \\
&= f(\phi^*) + \lambda_2^* x^* \left(\lambda_1^* \frac{\psi_1^*}{\psi_0^*}, (1-\alpha)\lambda_2^* \frac{\phi^*}{\psi_0^*}\right)\phi^*,
\end{aligned}
$$

which is in $L^1(R)$ due to Assumption 5.1 and Lemma 4.7(i). If $f(0) = u(\infty) - \lambda_1^*\ell(-\infty) = \infty$, then $R(\phi^\alpha > 0) = 1$ since $E_R[f(\phi^*)] < \infty$. Otherwise $v_{\lambda_1^*,\lambda_2^*}(\alpha P + (1-\alpha)P^*|Q_1^*|Q_0^*) = E_R[f(\phi^\alpha); \phi^\alpha > 0] + (u(\infty) - \lambda_1^*\ell(-\infty)) \cdot R(\phi^\alpha = 0)$, and the second term is bounded for any $P \in \mathcal{P}^T$.

Similarly, note for the proof of (ii) that the set $\mathcal{Q}_1$ satisfies an assumption like Assumption 2.3.2 in Gundel (2006). That is, for any $Q_1 \in \mathcal{Q}_1$ and $\alpha \in (0,1)$ we have $v_{\lambda_1^*,\lambda_2^*}(P^*|\alpha Q_1^* + (1-\alpha)Q_1|Q_0^*) < \infty$. Indeed, let $Q_1 \in \mathcal{Q}_1$ and define $\psi_1 := dQ_1/dR$ and $\psi_1^\alpha := \alpha\psi_1 + (1-\alpha)\psi_1^*$. For the convex function $f(\psi_1) := v(\lambda_2^*\phi^*, \lambda_1^*\psi_1, \psi_0^*)$ with increasing derivative $f'(\psi_1) = -\lambda_1^*\ell(-x^*(\lambda_1^*\psi_1/\psi_0^*, \lambda_2^*\phi^*/\psi_0^*)) \leq 0$, we obtain

$$
\begin{aligned}
f(\psi_1^\alpha) &\leq f(\psi_1^*) - f'(\psi_1^\alpha)(\psi_1^* - \psi_1^\alpha) \\
&\leq f(\psi_1^*) + \lambda_1^*\ell\left(-x^*\left((1-\alpha)\lambda_1^*\frac{\psi_1^*}{\psi_0^*}, \lambda_2^*\frac{\phi^*}{\psi_0^*}\right)\right)\psi_1^*,
\end{aligned}
$$

which is in $L^1(R)$ due to Assumption 5.1 and Lemma 4.7(i). □

Proof of Theorem 5.9. (i) follows from Proposition 5.3.

(ii) X^* solves the loss minimization problem (40) by Proposition 5.3. Hence it satisfies both constraints, and by Proposition 5.3, any other contingent claim satisfying both constraints equals X^* on the set $\{d\tilde{P}/dR > 0\}$. On $\{d\tilde{P}/dR = 0\}$ we cannot do any better than setting X^* equal to ∞. Hence, X^* solves the utility maximization problem (16), and it is the unique solution on the set $\{d\tilde{P}/dR > 0\}$.

In order to show (iii) and (iv), take a contingent claim $X \in \mathcal{X}(x_0, x_1)$ that satisfies the constraints, and $\lambda_1 \geq 0$, $\lambda_2 > 0$. Letting $P' \in \mathcal{P}^T$, $Q_1' \in \mathcal{Q}_1$, and $Q_0' \in \mathcal{Q}_0$ with

$v_{1,1}(P'|Q'_1|Q'_0) < \infty$, we obtain

$$
\begin{aligned}
\inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(X)] &\leq E_{Q'_0}[u(X)] \\
&\leq v_{\lambda_1,\lambda_2}(P'|Q'_1|Q'_0) + \lambda_1 x_1 + \lambda_2 x_0 \\
&= E_{Q'_0}\left[u\left(x^*\left(\lambda_1\frac{dQ'_1}{dQ'_0}, \lambda_2\frac{dP'}{dQ'_0}\right)\right)\right] \\
&\quad + \quad \lambda_1\left(x_1 - E_{Q'_1}\left[\ell\left(-x^*\left(\lambda_1\frac{dQ'_1}{dQ'_0}, \lambda_2\frac{dP'}{dQ'_0}\right)\right)\right]\right) \\
&\quad + \quad \lambda_2\left(x_0 - E_{P'}\left[x^*\left(\lambda_1\frac{dQ'_1}{dQ'_0}, \lambda_2\frac{dP'}{dQ'_0}\right)\right]\right).
\end{aligned} \tag{47}
$$

(iii) Let $P' = \hat{P}$ and $Q'_0 = \hat{Q}_0$ in (47). If $\sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0))] < x_1$, then the last two summands in (47) are equal to zero for $\lambda_1 = 0$, $\lambda_2 = \tilde{\lambda}_2$. Since $x^*(0, y_2) = I(y_2)$, this implies

$$
\sup_{X\in\mathcal{X}(x_0,x_1)} \inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(X)] \leq E_{\hat{Q}_0}\left[u\left(I\left(\tilde{\lambda}_2\frac{d\hat{P}}{d\hat{Q}_0}\right)\right)\right].
$$

By Proposition 2.3.8 in Gundel (2006) the last term equals $\inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0))]$, and $I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0)$ satisfies the budget constraint. Thus, $I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0)$ is a solution to Problem (16), and the UBSR constraint is not binding. In order to prove uniqueness we proceed as follows: Assume that $\tilde{X} \in \mathcal{X}(x_0, x_1)$ solves Problem (16). Then we have $E_{\hat{P}}[\tilde{X}] \leq x_0$ and hence

$$
\inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(\tilde{X})] \leq E_{\hat{Q}_0}[u(\tilde{X})] \leq E_{\hat{Q}_0}[u(X^*)].
$$

The second inequality holds strictly unless $\tilde{X} = X^*$ $\hat{Q}_0$- and hence R-almost surely. This follows from the fact that X^* is the solution to Problem (20) under $\hat{P}$ and $\hat{Q}_0$ and from the uniqueness result in Theorem 4.3. But the strict inequality is a contradiction to $E_{\hat{Q}_0}[u(X^*)] = \inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(X^*)] = \inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(\tilde{X})]$. Thus $\tilde{X} = X^*$ R-almost surely.

(iv) Let $P' = P^*$, $Q'_1 = Q^*_1$, and $Q'_0 = Q^*_0$. Since $(\lambda^*_1, \lambda^*_2)$ minimizes $v_{\lambda_1,\lambda_2}(P^*|Q^*_1|Q^*_0) + \lambda_1 x_1 + \lambda_2 x_0$ it follows from Corrolary 4.8 that the two terms in the brackets on the right-hand side of (47) equal zero for $\lambda_1 = \lambda^*_1$ and $\lambda_2 = \lambda^*_2$. Proposition 5.8 implies that X^* satisfies the constraints and that $E_{Q^*_0}[u(X^*)] = \inf_{Q_0\in\mathcal{Q}_0} E_{Q_0}[u(X^*)]$. This concludes the proof of (45) and of the optimality of X^*. Both constraints are binding due to the assumption $\sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0))] \geq x_1$. Furthermore, in this case, the robust utility maximization problem is equivalent to the classical problem with $\mathcal{Q}_0 = \{Q^*_0\}$. Now the uniqueness follows in the same way as in (iii). □

A Auxiliary Results

In this section we collect properties of the deterministic functions x^* and L. Remember that $\bar{x}_u = 0$.

We consider a family of functions g_{y_1,y_2} with $y_1, y_2 \geq 0$, defined by

$$g_{y_1,y_2}(x) := u(x) - y_1 \ell(-x) - y_2 x.$$

In the following we will sometimes drop the indices y_1, y_2 if there is no danger of confusion.

Lemma A.1. *(i) g_{y_1,y_2} is strictly concave and thus continuous on its essential domain*

$$dom(g_{y_1,y_2}) = dom(u).$$

(ii) g_{y_1,y_2} attains its supremum on $\mathbb{R}$ if and only if $y_2 > 0$. In this case, the maximizer is unique and equals

$$x^*(y_1,y_2) := \begin{cases} J(y_1,y_2) & \text{if } y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell+), \\ \bar{x}_\ell & \text{if } u'(\bar{x}_\ell) \leq y_2 \leq u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell+), \\ I(y_2) & \text{if } y_2 < u'(\bar{x}_\ell). \end{cases} \tag{48}$$

Here $J(y_1,y_2)$ denotes the unique solution to the equation $u'(x) + y_1 \ell'(-x) = y_2$ for the case that $y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell+)$, and $I := (u')^{-1}$.

(iii) If $\bar{x}_\ell = \infty$, (48) simplifies to

$$x^*(y_1,y_2) = J(y_1,y_2).$$

(iv) The function $x^* : [0,\infty) \times (0,\infty) \to (0,\infty)$, defined in (48), is continuous.

(v) $x^*(y_1,y_2)$ is decreasing in y_2 for $y_1 \geq 0$ fixed, and increasing in y_1 for $y_2 > 0$ fixed.

(vi) For fixed $y_1 \geq 0$, we have $x^*(y_1,\infty) := \lim_{y_2\to\infty} x^*(y_1,y_2) = 0$, $x^*(y_1,0) := \lim_{y_2\to 0} x^*(y_1,y_2) = \infty$.

(vii) If $\alpha \geq 1$, then $x^*(\alpha y_1, \alpha y_2) \leq x^*(y_1,y_2)$.

(viii) Let $L : \mathbb{R} \to [-\bar{x}_\ell, 0]$ be the generalized inverse of the derivative of the loss function ℓ, i.e.,

$$L(y) := \begin{cases} 0 & \text{if } y \geq \ell'(0), \\ (\ell')^{-1}(y) & \text{if } \ell'(-\bar{x}_\ell+) < y < \ell'(0), \\ -\bar{x}_\ell & \text{if } y \leq \ell'(-\bar{x}_\ell+). \end{cases} \tag{49}$$

L is a continuous function which is strictly increasing on $[\ell'(-\bar{x}_\ell+), \ell'(0)]$.
If $e > 0$ is such that $\ell'(-\bar{x}_\ell+) < e < \ell'(0)$, and $\mu := u'(-L(e))$, then we have for all $y_1 \geq 0$,

$$x^*(0,\mu) = x^*(y_1, \mu + y_1 e).$$

(ix) Let $\tilde{c} : \mathbb{R}_+ \to \mathbb{R}_+$ be decreasing with $\lim_{y_1\to\infty} \tilde{c}(y_1) = c > 0$. Then

$$\lim_{y_1\to\infty} x^*(y_1, \tilde{c}(y_1)\cdot y_1) \;=\; -L(c) \;\in\; [0,\bar{x}_\ell].$$

Moreover, $x^*(y_1, cy_1)$ converges for $y_1 \to \infty$ to $-L(c)$ monotonously from above.

(x) Define

$$\begin{aligned} v(y_2,y_1,y_0) :&= \sup_{x>0}\{y_0 u(x) - y_1 \ell(-x) - y_2 x\} \\ &= y_0 u\left(x^*\left(\frac{y_1}{y_0},\frac{y_2}{y_0}\right)\right) - y_1 \ell\left(-x^*\left(\frac{y_1}{y_0},\frac{y_2}{y_0}\right)\right) - y_2 x^*\left(\frac{y_1}{y_0},\frac{y_2}{y_0}\right) \end{aligned} \tag{50}$$

for $y_2 > 0$, $y_1 \geq 0$, and $y_0 > 0$. v is convex and continuously differentiable with derivatives

$$\frac{\partial}{\partial y_0} v(y_2,y_1,y_0) = u\left(x^*\left(\frac{y_1}{y_0},\frac{y_1}{y_0}\right)\right), \tag{51}$$

(52) $$\frac{\partial}{\partial y_1} v(y_2, y_1, y_0) = -\ell\left(-x^*\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right)\right),$$

and

(53) $$\frac{\partial}{\partial y_2} v(y_2, y_1, y_0) = -x^*\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right).$$

Hence v is decreasing in y_1, and it is decreasing in y_2 if $0 = 0$.

Furthermore,

$$v(0, y_1, y_0) := \lim_{y_2 \to 0} v(y_2, y_1, y_0) = y_0 u(\infty) - y_1 \ell(-\infty) := \lim_{x \to \infty} (y_0 u(x) - y_1 \ell(-x))$$

for $y_1 \geq 0$, $y_0 > 0$.

(xi) Define

(54) $$\tilde{v}(y_2, y_1) := \sup_{x>0}\{-y_1 \ell(-x) - y_2 x\} = -y_1 \ell\left(L\left(\frac{y_2}{y_1}\right)\right) + y_2 L\left(\frac{y_2}{y_1}\right)$$

for $y_2 > 0$ and $y_1 > 0$. $\tilde{v}$ is convex and continuously differentiable with derivatives

$$\frac{\partial}{\partial y_1} \tilde{v}(y_2, y_1) = -\ell\left(L\left(\frac{y_2}{y_1}\right)\right),$$

and

$$\frac{\partial}{\partial y_2} \tilde{v}(y_2, y_1) = L\left(\frac{y_2}{y_1}\right).$$

Furthermore, $\lim_{y_2 \to 0} \tilde{v}(y_2, y_1) = 0$ *and* $\lim_{c \to \infty} \frac{\tilde{v}(cx,1)}{c} = 0$.

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