From the equivalence principle to market consistent valuation

Thomas Knispel[a] Gerhard Stahl[b] Stefan Weber[c]

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Abstract

Insurance companies are exposed to many different types of risk, in particular actuarial as well as financial risks. As a consequence, the classical actuarial principle of pooling does not provide a sufficient basis for the valuation and risk management of the total portfolio of an insurance company. Instead, the methodology needs to be complemented by modern financial mathematics that enables a market consistent valuation. The current article provides an introduction to the fundamental principles of financial mathematics that were originally developed by Fischer Black, Robert Merton and Myron Scholes in the beginning of the 1970s. We will discuss the relevance of these concepts for insurance firms in the context of internal models and the computation of the *market consistent embedded value* (MCEV).

Key words:
Actuarial equivalence principle, fundamental theorems of asset pricing, market consistent valuation, risk measures

AMS Subject Classification (2010): 62P05; 91B24; 91B25; 91B30

1 Background

Modern financial mathematics – also known as *financial engineering* – is today an indispensible tool for the risk management of banks and insurance companies. A thorough quantitative analysis is a necessary prerequisite for the valuation of financial instruments and portfolios, the construction of optimal investment strategies and the design of products. Insurance companies and banks deal with uncertain future events and cash flows. Actuarial and financial mathematics, thus, provide mathematical models for structures and patterns on insurance and financial markets that are driven by randomness.

An important link between economic questions and mathematical techniques was developed at MIT by Fischer Black (*1938 - †1995), Myron S. Scholes (*1941) and Robert C. Merton (*1944) in the beginning of the 1970s. Their universal ideas how financial options and derivatives must be priced are not limited to the classical *Black-Scholes model*, but can also be applied to more general models of financial markets. In 1997 Merton and Scholes received the Nobel prize for their work. The seminal papers triggered new quantitative techniques and products, and provided the foundation for modern

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[a]Leibniz Universität Hannover, Institut für Mathematische Stochastik, Welfengarten 1, 30167 Hannover, Germany, email: knispel@stochastik.uni-hannover.de

[b]Talanx AG, Quantitative Risk Management, Riethorst 2, 30659 Hannover, Germany, email: gerhard.stahl@talanx.com

[c]Leibniz Universität Hannover, Institut für Mathematische Stochastik, Welfengarten 1, 30167 Hannover, Germany, email: sweber@stochastik.uni-hannover.de

financial mathematics, an applied field that combines methods from stochastic analysis, differential equations, functional analysis and numerics.

In contrast to financial mathematics, traditional actuarial mathematics does not have the reputation of being the most exciting field of applied mathematics. This dichotomy is not only apparent in academia, but also in industry practice. While *high-tech quants* – also known as *rocket scientists* – have been playing a key role in investment banking since the middle of the 1990s, insurance companies relied on a more conservative and traditional methodology. However, the insurance business has been experiencing significant changes for the last ten years.

Insurance companies are not only exposed to traditional actuarial risks that can be handled quite easily, but struggle at the same time with systematic risks that need to be priced and hedged. In particular, insurance companies are confronted with financial risk that cannot be understood on the basis of actuarial techniques. Modern insurance mathematics needs to integrate financial mathematics. A prime example provides the notion of *market consistent embedded value* that has been discussed in the context of *Solvency II*[1] and that can only be understood within the framework of 'Black-Scholes'.

High-tech quants have been joining the insurance industry employing techniques from financial engineering. These enable two main applications. Firstly, the total portfolios of insurance companies can be priced consistently with market data. Secondly, complex insurance products with exposure to financial risk and embedded options can be designed, valued and hedged. Like financial derivatives, these structures provide investment tools that can be used for hedging or speculation. Examples include catastrophe bonds ('CAT-bonds') and variable annuities. New products offer new opportunities for companies and investors: insurance firms can mitigate risks to financial market participants which contributes to their risk management strategies; investors, on the other hand, profit from a broader spectrum of instruments which can be used to better diversify and further optimize their portfolios. – In view of the recent financial crisis, the conditions under which these positive effects do indeed materialize must, of course, be characterized very carefully. Complex products did indeed contribute to the crisis, but are they primarily responsible?

This question has frequently been discussed recently, but the answer is all but simple. Among the key words that were mentioned in this context are, for example, insufficient due diligence, high leverage, inadequate bonus systems without liability, Ponzi schemes a la Madoff, ignorance, fraud, short-sighted interventions from politics and central banks, etc. Math was sometimes also blamed: Markets + Math = Mayhem?[2] However, the truth is that a complex world cannot be understood without adequate models. The random nature of market movements forces us to employ probabilistic models, if we want to partially understand the reality of the banking and insurance business. Without math, there will be no understanding: Markets - Math = Mayhem!

The current article reviews the key concepts of financial mathematics. We will explain how economic questions can be translated into proper mathematics and how financial engineering and insurance mathematics are linked to each other.

2 Principles of financial engineering

2.1 Classical actuarial mathematics vs. financial mathematics

Classical actuarial mathematics relies on the key principle of the insurance business – the principle of pooling. The exposure of individuals to idiosyncratic risks is transferred to a community of insured members that bear all risks together; as a consequence, individuals can be covered under insurance policies for fixed premiums that are small compared to possible individual claim sizes. The total claim should, however, be covered by the sum of the premia.

[1] Solvency II is the updated regulatory framework for the insurance industry in the European Union. Its key aim is to ensure the liquidity and solvency of insurance companies.

[2] Financial Times, March 21, 2009

A particularly simple situation occurs if insurance claims are independent and no systematic risk components are involved. In this case, classical valuation methods can be applied very easily. Consider, for example, a random payment stream $C_0, C_1, \ldots, C_n$ at dates $0, 1, \ldots, n$ associated with a pension or a pure life insurance. With deterministic interest rate $r \geq 0$, its value at time 0 is given as the expected present value PV_0, defined as the expectation of the sum of the discounted payoffs $(1+r)^{-t} C_t$, $t = 0, 1, \ldots, n$, i. e.,

$$\mathrm{PV}_0(C_0, C_1, \ldots, C_n) = E\left[\sum_{t=0}^{n} \tfrac{1}{(1+r)^t} C_t\right].$$

This valuation principle provides the conceptual basis for the calculation of premiums in classical life insurance, as postulated by the *equivalence principle*:

expected present value of premiums = expected present value of benefits

A key feature of the classical insurance valuation principle consists in the computation of all expectations under the *statistical probability measure* that models the relative frequency with which events occur. In the context of life insurance mathematics, quantities like death and survival probabilities are, for example, estimated by the fraction of the dead and the survivors in the total population of each age group. In case of perfect *pooling of risks*, a valuation based on statistical means is mathematically justified by the *law of large numbers*.

The following example outlines the computation of single net premiums for pure life insurance policies. For a comprehensive introduction to life insurance mathematics we refer to the textbooks by Gerber [21], Milbrodt & Helbig [34] and Koller [30].

Example 2.1 *Consider a pure life insurance contract of a male insured of age x with a term of ten years that promises the death benefit of 100,000 € to be paid retrospectively at the end of the year of the insuree's death. Denoting by T_x the random remaining lifetime of the man, the insurance benefits are described by the payment stream*

$$C_0 = 0 \quad \text{and} \quad C_t = \begin{cases} 100,000 & \text{if } t-1 < T_x \leq t \\ 0 & \text{otherwise} \end{cases}, \quad t = 1, \ldots, 10.$$

The expected present value of the benefits is thus given by

$$\mathrm{PV}_0(C_0, C_1, \ldots, C_{10}) = 100,000 \cdot \sum_{t=1}^{10} \tfrac{1}{(1+r)^t} P[t-1 < T_x \leq t], \tag{1}$$

where $P[t-1 < T_x \leq t]$ signifies the probability of dying between $t-1$ and t. These expressions can be rewritten in terms of the one-year death probabilities q_z and survival probabilities p_z of male policy holders aged z:

$$P[t-1 < T_x \leq t] = q_{x+t-1} \prod_{j=0}^{t-2} p_{x+j}$$

The one-year statistical probabilities are listed in mortality tables, and for a given technical interest rate r the expected present value of the benefits PV_0 can easily be computed from (1). The insurance premium is implied by the equivalence principle which states that the present value of the benefits equals the single net premium.

The principle of pooling is applicable to the valuation of products, if loss events occur independently. Systematic risks, however, alter the picture and require a more complex analysis that cannot solely be based on the equivalence principle. Both modern insurance products like variable annuities and the total portfolio of insurance firms are both exposed to financial as well as *systematic* actuarial risk components.

In the case of systematic risks, the classical actuarial equivalence principle needs to be replaced by the *principle of risk-neutral valuation.* Risk-neutral valuation is not merely an ad-hoc method, but a mathematical consequence of specific axiomatic assumptions in the context of market models. The key hypothesis states that efficient markets do not admit any risk-free profits ('no free lunch') excluding arbitrage strategies, i. e., trading strategies that yield profits without any downside risk. Conceptually, a rationale to the assumption of the absence of arbitrage is provided by the market forces of supply and demand. A '*free lunch*' would trigger a high demand for the profit opportunity on markets with profit-orientated participants and affect prices adversely. As a consequence, the arbitrage would vanish very quickly.

Formally, *risk-neutral valuation* resembles the equivalence principle. Prices are computed as expectation of future payments, i. e., *as if* market participants were risk-neutral. This <u>formal</u> observation is the reason why the valuation procedure is called the principle of *risk-neutral* valuation. However, risk-neutral and actuarial pricing are substantially different concepts. Risk-neutral valuation weights scenarios differently when expectations are computed. The statistical measure (of actuarial mathematics) must be replaced by a <u>technical</u> probability measure which is called *risk-neutral measure*, *martingale measure*, or *pricing measure.*

Risk-neutrality refers to the form of the valuation procedure only, but does not imply that economic agents are assumed to be risk-neutral. In fact, the risk-neutral measure encodes the risk-aversion of the market which is reflected by the modified weights that are assigned to scenarios. Market consistent valuation of modern insurance products and total portfolios of firms requires an integrated approach that combines techniques from financial engineering as well as actuarial mathematics.

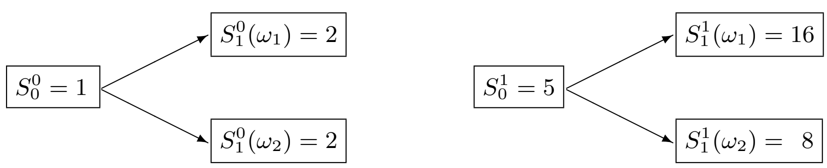
2.2 A simple one-period model with two scenarios

What are the basic concepts of financial mathematics? And how can the valuation principle be derived from basic axioms? A simple exposition to the answers can be provided in the context of elementary one-period models. The key ideas, however, generalize to advanced multi-period models.

In a one-period model, time is discrete with two dates, say $t = 0, 1$. Time 0 is interpreted as today, time $t = 1$ as the future. Today's market prices are observable, but typically not future prices which are modeled as random. In the simplest case, the world is described by only two scenarios labeled ω_1 and ω_2 with strictly positive probabilities $P[\{\omega_1\}] > 0$ and $P[\{\omega_2\}] > 0$ under the statistical measure; scenarios with probability 0 would be irrelevant. The set of all scenarios is typically denoted by $\Omega = \{\omega_1, \omega_2\}$. A probability space with two scenarios can be interpreted as a model of a random experiment in which a (not necessarily fair) coin is tossed: scenario ω_1 can e. g. be identified with the event 'head', scenario ω_2 with the event 'tail'.

In the simplest case, the market consists of two primary products, a savings account and a stock, whose initial prices are observable in the market. The asset prices at time 1 may depend on the realized scenario which is not revealed before time 1. Formally, we label the savings account with the security identification number '0', the stock with number '1'. With this convention, S_t^0, S_t^1 correspond to the prices of the savings account and the stock at times $t = 0, 1$, respectively.

A specific one-period market model with two scenarios is provided by the following example:



In this case, the performance of the savings account does not depend on the future state of the world. However, its is not difficult to incorporate stochastic interest rates in the model.

Agents can buy or sell shares of the primary products. Positive amounts are called *long positions*, negative amounts *short positions*. A short position in the savings account corresponds to debt.

Definition 2.2 (Trading Strategy) *In a one-period market model, a trading strategy is a vector $\vartheta = (\vartheta^0, \vartheta^1) \in \mathbb{R}^2$ whose components correspond to the number of shares invested into the savings account and the stock, respectively. The initial value V_0^ϑ of the strategy is given by*

$$V_0^\vartheta := \vartheta^0 S_0^0 + \vartheta^1 S_0^1, \tag{2}$$

and its terminal value V_1^ϑ by

$$V_1^\vartheta := \vartheta^0 S_1^0 + \vartheta^1 S_1^1. \tag{3}$$

The analysis of the model is particularly simple under the following standard assumptions: Products are liquidly tradable, all assets can be sold and bought for the same price ('no bid-ask spread'), agents can hold short and non integer positions, taxes and transaction costs are neglected. In practice, these aspects are, of course, highly relevant, and can adequately be taken into account in more complex models.

The key assumption of financial engineering is the **absence of arbitrage opportunities**:

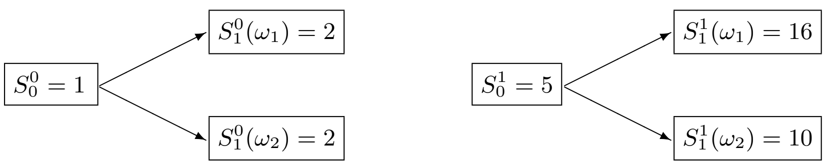
Definition 2.3 (Arbitrage) *An* arbitrage opportunity *is a trading strategy ϑ with the following properties:*

1. *The cost of implementing ϑ is zero, i. e., $V_0^\vartheta = 0$.*
2. *Losses at time 1 are impossible, i. e., $V_1^\vartheta(\omega) \geq 0$ for all $\omega \in \Omega$.*
3. *The strategy ϑ generates a profit with positive probability, i. e., $P[V_1^\vartheta > 0] > 0$.*

A financial market model is said to be free of arbitrage*, if it does not admit arbitrage opportunities.*

Financial market models are not always arbitrage-free. Whether or not a model is arbitrage-free, needs to be investigated. This is illustrated by the following example of a one-period model that permits arbitrage opportunities.

Example 2.4 *Suppose that the price dynamics of the primary products is specified as follows:*



The stock systematically outperforms the savings account, since the terminal stock price is always at least twice its initial price. An agent with trading strategy $\vartheta = (-5, 1)$ finances the purchase of one share of stock by debt. The strategy does not cost anything at time $t = 0$, but yields terminal wealth $V_1^\vartheta(\omega_1) = 6 > 0$ resp. $V_1^\vartheta(\omega_2) = 0$. This is an arbitrage opportunity! The strategy – not involving any downside risk – is scalable and thus offers the chance of a profit of arbitrary size.

Financial engineering provides appropriate methods for the valuation and (partial) hedging of financial products, commonly called *contingent claims*. Contingent claims are contracts between two or more parties which specify the size of future payments to be exchanged between the parties conditional or contingent on future events. These describe both financial derivatives as well as insurance payments. In the first case, the size of payments may e. g. depend on the performance of reference products like stocks, bonds, currencies, commodities, and indexes. In the second case, payments are determined by the occurrence of loss events and the extent of insurance liabilities.

Definition 2.5 (Contingent Claim) *A European contingent claim with maturity* 1 *is a contract specifying the payments* $C_1(\omega)$ *due at time* 1 *for all scenarios* $\omega \in \Omega$. *Mathematically, a contingent claim is described by a measurable mapping* $C_1 : \Omega \to [0, \infty)$.

The basic examples are call and put options.

- A *European call option* on the stock with *maturity* 1 and *strike price* K is a financial contract that gives the contract holder the right to buy one share of the stock at time 1 for the fixed price K (without any obligation). This right will be exercised by a rational investor if and only if the stock price S_1^1 exceeds K; otherwise the strike price is higher than the spot price, and it is cheaper to buy the stock on the spot market. The random payoff of the call option at time 1 is given by

  $$C_1 = \max\{S_1^1 - K, 0\}.$$

  In practice, option contracts are frequently not settled physically, but in cash.

- A *European put option* on the stock with *maturity* 1 and *strike price* K entitles its owner to sell the stock at time 1 at the price of K, corresponding to a payoff of the form

  $$C_1 = \max\{K - S_1^1, 0\}.$$

Two questions are immediate. What is the fair price C_0 at time 0 of a contingent claim C_1? How can the issuer of a claim manage and hedge the risks associated with the random payment C_1? The answers to both questions are closely related and can be derived from the 'no arbitrage' hypothesis.

Assumption 2.6 *The price* C_0 *at time* $t = 0$ *of a contingent claim* C_1 *maturing at time* $t = 1$ *is* fair *if the extended market model specified by the savings account, the stock, and the contingent claim with price process* (C_0, C_1) *is free of arbitrage.*

The notion of *fair* price is closely related to the *cost of perfect replication.*

Definition 2.7 (Replicable Claim) *A contingent claim* C_1 *is replicable if there exists a trading strategy* $\vartheta = (\vartheta^0, \vartheta^1)$ *such that its terminal value* V_1^ϑ *coincides with the payoff* C_1 *for all scenarios. In this case,* ϑ *is called a* replicating trading strategy*, and the initial investment* V_0^ϑ *is the cost of replication.*

Example 2.8 *Let us consider a contingent claim with payoffs* $C_1(\omega_1) = 12$, $C_1(\omega_2) = 4$. *A replicating strategy* ϑ *has to satisfy* $V_1^\vartheta(\omega) = C_1(\omega)$ *for each scenario* $\omega \in \Omega$, *thus (3) yields the following system of linear equations for* ϑ:

$$\vartheta^0 S_1^0 + \vartheta^1 S_1^1(\omega_1) = C_1(\omega_1), \tag{4a}$$

$$\vartheta^0 S_1^0 + \vartheta^1 S_1^1(\omega_2) = C_1(\omega_2). \tag{4b}$$

Inserting the specific asset prices of the model specified on page 4, the system takes the form

$$\vartheta^0 2 + \vartheta^1 16 = 12,$$

$$\vartheta^0 2 + \vartheta^1 8 = 4,$$

admitting the unique solution $\vartheta^0 = -2$, $\vartheta^1 = 1$. *The replicating strategy thus consists in buying one share of the stock and borrowing* 2. *The strategy* ϑ *requires the initial capital*

$$V_0^\vartheta = -2S_0^0 + 1S_0^1 = -2 \cdot 1 + 1 \cdot 5 = \underline{3},$$

and this amount is equal to the unique fair (i. e. arbitrage-free) price C_0 *at time* $t = 0$ *of the contingent claim* C_1.

How can this result be derived from the absence of arbitrage? Suppose that $C_0 > V_0^\vartheta$. In this case, an agent could sell at time 0 one share of the contingent claim C_1 at a price of C_0, implement the replicating strategy ϑ with cost V_0^ϑ, and invest the remaining capital $C_0 - V_0^\vartheta > 0$ into the savings account. This trading strategy does not cost anything at time 0. At maturity 1, the agent must pay C_1, the replicating strategy ϑ generates the terminal wealth $V_1^\vartheta = C_1$, and the investment into the savings account yields $S_1^0(C_0 - V_0^\vartheta)$. The strategy thus provides the payoff $S_1^0(C_0 - V_0^\vartheta) > 0$ at no cost. A price $C_0 > V_0^\vartheta$ is, thus, not consistent with the absence of arbitrage. Analogously, an arbitrage opportunity can be constructed if the price C_0 is strictly less than V_0^ϑ.

The example illustrates a general valuation principle that applies to any *replicable* contingent claim C_1:

arbitrage-free price C_0 = cost of perfect replication!

In Example 2.8, a replicating strategy is characterized by a system (4) of two linear equations and two unknowns. This system of equations admits a unique solution for any contingent claim $C_1(\omega_1) = c_1$, $C_1(\omega_2) = c_2$, i. e., any contingent claim is replicable in the simple one-period model with two possible scenarios.

Definition 2.9 (Complete Market Model) *A financial market model is called* complete*, if all contingent claims can be replicated.*

In more complex multi-period models the computation of a replicating strategy becomes more involved and requires, for example, the recursive solution of various systems of equations. The computations can be simplified by the application of martingale methods that are particularly powerful in the context of financial market models in continuous time, see, e. g., Musiela & Rutkowski [36].

The pricing of contingent claims can be separated from the computation of a replicating strategy via the principle of *risk-neutral valuation.* In one-period models this alternative approach does, of course, not provide any advantages; but it provides substantial simplifications in more complex market models. The key ideas can easily be illustrated in the context of one-period models.

Risk-neutral pricing resembles *formally* the classical actuarial pricing principle: the fair price is a (conditional) expectation of all future discounted payments. However, there are important differences. In contrast to classical actuarial mathematics, expectations have to be computed with respect to a *risk-neutral measure* rather than the statistical measure. At the same time, payments are discounted with respect to a reference product, the *numéraire.* This encodes that from an economic point of view only relative prices are meaningful, but not prices measured in absolute monetary units (e. g. € or DM). In the one-period model above, the savings account is taken as numéraire.[3]

Definition 2.10 (Risk-Neutral Measure) *A* risk-neutral measure *with respect to the numéraire S^0 is a technical probability measure Q such that*

1. *$Q[\{\omega\}] > 0$ for all $\omega \in \Omega$, and*

2. *the discounted price processes of all primary assets satisfy*

$$\frac{S_0^i}{S_0^0} = E_Q\left[\frac{S_1^i}{S_1^0}\right] := Q[\{\omega_1\}]\frac{S_1^i}{S_1^0}(\omega_1) + Q[\{\omega_2\}]\frac{S_1^i}{S_1^0}(\omega_2), \quad i = 0, 1. \tag{5}$$

[3]The choice of the numéraire is flexible. The savings account (or, more generally, money market account) constitutes only one possible alternative. The choice of the price process of zero coupon bonds as numéraire and a corresponding martingale measure, a *forward measure*, facilitates the efficient valuation of interest rate derivatives ; cf., e. g., Filipović [14], Chapter 7.

Property 1 formalizes that each scenario $\omega \in \Omega$ that possibly occurs according to the real-world measure P is also incorporated into the pricing (mathematically, this property is the equivalence of the probability measures P and Q). Property 2 states that for all primary products – in the present setting these are the savings account and the stock – the discounted price at time 0 is given by the discounted prices at time 1 weighted by the Q-probabilities.[4] In the language of probability theory the discounted price processes are *martingales*[5] with respect to the measure Q, and for this reason Q is also called *equivalent martingale measure*.

In the one-period model above, equation (5) takes an explicit form

$$5 = Q[\{\omega_1\}]8 + Q[\{\omega_2\}]4.$$

Since $Q[\{\omega_1\}] + Q[\{\omega_2\}] = 1$, it follows that

$$Q[\{\omega_1\}] = \tfrac{1}{4} \quad \text{and} \quad Q[\{\omega_2\}] = \tfrac{3}{4}, \tag{6}$$

i. e., the arbitrage-free and complete simple market model admits a unique risk-neutral measure. This observation holds, in fact, more generally, and also the converse implication is true (see Section 2.4 and Section 3). Arbitrage-free prices of contingent claim can be computed using the risk-neutral measure Q. A replication problem does not need to be solved.

Theorem 2.11 (Risk-neutral Valuation Formula, cf., e. g., [18], Corollary 1.35 and Theorem 1.32) *In the context of the complete financial market model above, the unique arbitrage-free price C_0 at time 0 of a contingent claim with terminal payoff C_1 is given by risk-neutral valuation with respect to the risk-neutral measure Q, i. e.,*

$$C_0 = S_0^0 E_Q \left[\tfrac{C_1}{S_1^0} \right].$$

In particular, the price computed with this formula is equal to the cost of perfect replication.

The following example illustrates that the risk-neutral valuation formula is indeed consistent with the valuation via perfect replication.

Example 2.12 *Example 2.8 shows that the contingent claim $C(\omega_1) = 12$, $C(\omega_2) = 4$ can be replicated with initial investment $V_0^\vartheta = 3$ and that the cost of replication corresponds to the unique arbitrage-free price C_0. Alternatively risk-neutral valuation with the technical probabilities (6) yields*

$$\begin{aligned} C_0 = S_0^0 \cdot E_Q \left[\tfrac{C}{S_1^0} \right] &= S_0^0 \cdot Q[\{\omega_1\}] \tfrac{C_1(\omega_1)}{S_1^0} + Q[\{\omega_2\}] \tfrac{C_1(\omega_2)}{S_1^0} \\ &= 1 \cdot \left(\tfrac{1}{4} \cdot \tfrac{12}{2} + \tfrac{3}{4} \cdot \tfrac{4}{2}\right) = \underline{3}. \end{aligned}$$

2.3 A simple one-period model with three scenarios

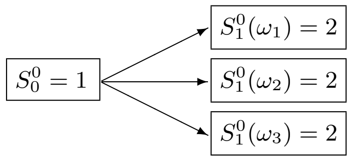
Reality is, however, more complex than the simple model above. Markets are typically incomplete, i. e., not every contingent claim can be replicated in terms of liquidly traded primary products. This fact is captured by incomplete market models. Pricing in incomplete markets is very different than in complete market models. The absence of arbitrage does not imply a unique price for contingent claims that cannot be replicated. Instead, 'no free lunch' is consistent with an interval of prices.

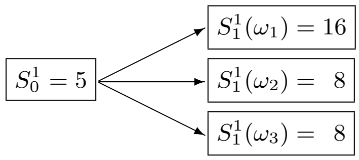
The main features of incomplete markets can easily be illustrated in the context of one-period models. For this purpose, we consider again a one-period model with the primary products 'savings account' S^0 and 'stock' S^1. However, now we include three possible scenarios $\omega_1, \omega_2, \omega_3$ that occur with strictly positive probability under the statistical measure P. Formally, this corresponds to the set of scenarios $\Omega = \{\omega_1, \omega_2, \omega_3\}$.

[4] In view of $S_0^0/S_0^0 = S_1^0/S_1^0 = 1$ condition (5) is automatically satisfied for the numéraire product, the savings account. It thus remains to compute the weights $Q[\{\omega_1\}]$ and $Q[\{\omega_2\}]$ based on the discounted stock prices.

[5] Martingales are a synonym for 'fair games'. Here 'fair' is used in the sense that the expected future value of the game equals the present value; in average there are no gains, but also no losses.

The price dynamics of the primary products is modeled as follows:





The performance of the savings account in this example does not depend on the scenario ω that is actually realized. The future stock price S_1^1 is affected by the state of the world, but takes the same value if either scenario ω_2 or scenario ω_3 occurs. At first glance, the incorporation of a third scenario might seem like a minor modification only; however, it has serious consequences for the properties of the market model. The model with three scenarios is free of arbitrage, but not all contingent claims can be replicated.

Example 2.13 *The contingent claim with payoffs $C_1(\omega_1) = 24$, $C_1(\omega_2) = 16$, and $C_1(\omega_3) = 8$ is not replicable, i. e., there exists no trading strategy $\vartheta = (\vartheta^0, \vartheta^1)$ with terminal wealth $V_1^\vartheta = C_1$. Mathematically, this simply means that the system*

$$\begin{aligned} \vartheta^0 S_1^0 + \vartheta^1 S_1^1(\omega_1) =& C_1(\omega_1) \\ \vartheta^0 S_1^0 + \vartheta^1 S_1^1(\omega_2) =& C_1(\omega_2) \\ \vartheta^0 S_1^0 + \vartheta^1 S_1^1(\omega_3) =& C_1(\omega_3) \end{aligned}$$

of three linear equations with two unknowns does not have a solution $(\vartheta^0, \vartheta^1)$.

Definition 2.14 (Incomplete Market Model)*A financial market model that contains at least one non-replicable contingent claim is said to be incomplete.*

A distinction between replicable and non-replicable contingent claims emerges immediately from Example 2.13. Replicable contingent claims possess terminal payoffs of the form

$$C_1(\omega) = \vartheta^0 S_1^0 + \vartheta^1 S_1^1(\omega), \quad \omega \in \Omega,$$

and their unique fair price C_0 coincides with the cost of perfect replication

$$V_0^\vartheta = \vartheta^0 S_0^0 + \vartheta^1 S_0^1. \tag{7}$$

Any other contingent claim is not replicable and cannot be priced on the basis of perfect replication. The principle of risk-neutral valuation, however, is still applicable, but is interpretation needs to be modified.

Let us first determine all risk-neutral measures in the sense of Definition 2.10: these are the technical probability measures Q such that

$$\tfrac{S_0^1}{S_0^0} = E_Q\left[\tfrac{S_1^1}{S_1^0}\right] := Q[\{\omega_1\}]\tfrac{S_1^1}{S_1^0}(\omega_1) + Q[\{\omega_2\}]\tfrac{S_1^1}{S_1^0}(\omega_2) + Q[\{\omega_3\}]\tfrac{S_1^1}{S_1^0}(\omega_3). \tag{8}$$

Since $Q[\{\omega_1\}] + Q[\{\omega_2\}] + Q[\{\omega_3\}] = 1$, we obtain in the specific model that

$$\begin{aligned} 5 =& Q[\{\omega_1\}] \cdot \tfrac{16}{2} + Q[\{\omega_2\}] \cdot \tfrac{8}{2} + Q[\{\omega_3\}] \cdot \tfrac{8}{2} \\ =& Q[\{\omega_1\}] \cdot \tfrac{16}{2} + (1 - Q[\{\omega_1\}]) \cdot \tfrac{8}{2}. \end{aligned}$$

This implies $Q[\{\omega_1\}] = 1/4$ and $Q[\{\omega_2\}] + Q[\{\omega_3\}] = 3/4$. Any choice $Q[\{\omega_2\}] = \frac{3}{4}\alpha$, $Q[\{\omega_3\}] = \frac{3}{4}(1-\alpha)$ with $\alpha \in (0,1)$ defines a risk-neutral measure. The class of risk-neutral measures $\mathcal{Q}$ consists of a continuum of elements:

$$\mathcal{Q} \quad = \quad \left\{Q_\alpha : \quad Q_\alpha[\{\omega_1\}] = \tfrac{1}{4},\ Q_\alpha[\{\omega_2\}] = \tfrac{3}{4}\alpha,\ Q_\alpha[\{\omega_3\}] = \tfrac{3}{4}(1-\alpha) \quad \text{with } \alpha \in (0,1)\right\}.$$

The existence of infinitely many risk-neutral measures characterizes incomplete market models.

It can be proven that for any given contingent claim with payoff C_1 any risk-neutral measure Q_α produces a price

$$C_{0,\alpha} := S_0^0 E_{Q_\alpha}\left[\tfrac{C_1}{S_1^0}\right] \tag{9}$$

that is consistent with the absence of arbitrage. Conversely, any arbitrage-free price for the contingent claim can be written as in equation (9) for some $Q_\alpha \in \mathcal{Q}$. For a contingent claim the model thus admits a set of fair (i. e. arbitrage-free) prices, namely

$$\mathcal{C}_0 := \left\{ S_0^0 E_{Q_\alpha}\left[\tfrac{C_1}{S_1^0}\right] \middle| Q_\alpha \in \mathcal{Q} \right\}.$$

The lower and upper value

$$C_0^{\inf} := \inf_{Q\in\mathcal{Q}} S_0^0 E_Q\left[\tfrac{C_1}{S_1^0}\right] \quad \text{and} \quad C_0^{\sup} := \sup_{Q\in\mathcal{Q}} S_0^0 E_Q\left[\tfrac{C_1}{S_1^0}\right]$$

are arbitrage bounds for the contingent claim with payoff C_1. The set $\mathcal{C}_0$ is always either a singleton $C_0^{\inf} = C_0^{\sup}$ or the open interval $(C_0^{\inf}, C_0^{\sup})$.

Example 2.15 (Replicable Contingent Claim) *The contingent claim with terminal payoffs $C_1(\omega_1) = 12$, $C_1(\omega_2) = 4$, and $C_1(\omega_3) = 4$ can be replicated by buying one share of the stock and borrowing the amount of 2, corresponding to trading strategy $\vartheta = (-2, 1)$. According to (7), the cost of replication is $V_0^\vartheta = 3$. Alternatively, risk-neutral valuation with respect to an arbitrary risk-neutral measure yields*

$$\begin{aligned} C_{0,\alpha} =& S_0^0\left(Q_\alpha[\{\omega_1\}]\tfrac{C_1(\omega_1)}{S_1^0} + Q_\alpha[\{\omega_2\}]\tfrac{C_1(\omega_2)}{S_1^0} + Q_\alpha[\{\omega_3\}]\tfrac{C_1(\omega_3)}{S_1^0}\right) \\ =& 1\left(\tfrac{1}{4}\cdot\tfrac{12}{2} + \tfrac{3}{4}\alpha\cdot\tfrac{4}{2} + \tfrac{3}{4}(1-\alpha)\cdot\tfrac{4}{2}\right) = \underline{3}, \end{aligned}$$

and this price does not depend on the specific choice of a risk-neutral measure Q_α, $\alpha \in (0, 1)$. In other words, the class of arbitrage-free prices consists of exactly one element, namely the cost of perfect replication.

Example 2.16 (Non-replicable Contingent Claim) *For the non-replicable contingent claim $C_1(\omega_1) = 24$, $C_1(\omega_2) = 16$, $C_1(\omega_3) = 8$ introduced in Example 2.13 the arbitrage-free price computed in terms of the risk-neutral measure Q_α, $\alpha \in (0, 1)$, equals*

$$\begin{aligned} C_{0,\alpha} =& S_0^0\left(Q_\alpha[\{\omega_1\}]\tfrac{C_1(\omega_1)}{S_1^0} + Q_\alpha[\{\omega_2\}]\tfrac{C_1(\omega_2)}{S_1^0} + Q_\alpha[\{\omega_3\}]\tfrac{C_1(\omega_3)}{S_1^0}\right) \\ =& 1\left(\tfrac{1}{4}\cdot\tfrac{24}{2} + \tfrac{3}{4}\alpha\cdot\tfrac{16}{2} + \tfrac{3}{4}(1-\alpha)\cdot\tfrac{8}{2}\right) \\ =& \underline{6 + 3\alpha}. \end{aligned}$$

This price depends on the parameter α resp. the measure Q_α explicitly. In particular, the non-replicable claim C_1 admits a whole class of arbitrage-free prices, namely the open interval $\mathcal{C}_0 = (6, 9)$.

Examples 2.15 and 2.16 illustrate the following dichotomy which generalizes to more complex incomplete financial market models.

Theorem 2.17 (Arbitrage-free Prices, cf., e. g., [18], Corollary 1.35) *Let C_1 be a contingent claim.*

1. *The contingent claim C_1 can be replicated if and only if it admits a unique arbitrage-free price. In particular, this price is given by the cost of perfect replication.*
2. *If C_1 is not replicable, then $C_0^{\inf} < C_0^{\sup}$, and the set of arbitrage-free price is the open interval*

$$\mathcal{C}_0 = (C_0^{\inf}, C_0^{\sup}) = \left(\inf_{Q\in\mathcal{Q}} S_0^0 E_Q\left[\tfrac{C_1}{S_1^0}\right], \sup_{Q\in\mathcal{Q}} S_0^0 E_Q\left[\tfrac{C_1}{S_1^0}\right]\right).$$

Incomplete market models are characterized by the existence of contingent claims that cannot perfectly be replicated. While replicable claims can, by definition, be hedged, risk management of non replicable claims is more sophisticated.

An extreme approach is *superhedging.*[6] A superhedge is a trading strategy ϑ with minimal initial capital whose terminal value V_1^ϑ dominates the payoff of the contingent claim for any possible scenario, i. e.,

$$V_1^\vartheta(\omega) = \vartheta^0 S_1^0 + \vartheta^1 S_1^1(\omega) \geq C_1(\omega) \quad \text{for all } \omega \in \Omega. \tag{10}$$

Superhedging excludes any downside risk at maturity. Superhedging strategies do indeed always exist. The minimal capital to set up the superhedging portfolio is given by the upper arbitrage bound $C_0^{\sup}$, commonly labeled the *superhedging price.*

Example 2.18 *The contingent claim $C_1(\omega_1) = 24$, $C_1(\omega_2) = 16$, $C_1(\omega_3) = 8$ in Example 2.13 is not replicable. According to Example 2.16, its superhedging price is $C_0^{\sup} = 9$, i. e., the initial cost of a superhedging strategy ϑ is given by*

$$9 = V_0^\vartheta = \vartheta^0 S_0^0 + \vartheta^1 S_0^1.$$

This yields the relation $\vartheta^0 = 9 - 5\vartheta^1$, and the superhedging condition (10) provides a system of inequalities for the number of shares ϑ_1:

$$\begin{aligned} (9 - 5\vartheta^1) \cdot 2 + \vartheta^1 \cdot 16 &\geq 24, \\ (9 - 5\vartheta^1) \cdot 2 + \vartheta^1 \cdot 8 &\geq 16, \\ (9 - 5\vartheta^1) \cdot 2 + \vartheta^1 \cdot 8 &\geq 8. \end{aligned}$$

The first and the second inequality imply $\vartheta^1 \geq 1$ and $\vartheta^1 \leq 1$, and the third inequality is equivalent to $\vartheta^1 \leq 5$. Thus, the superhedging strategy consists in buying $\vartheta^1 = 1$ shares of the stock and investing the amount $\vartheta^0 = 9 - 5\vartheta^1 = 4$ into the savings account.

In practice, superhedging strategies are usually not acceptable, since they are typically very expensive. Even in complete markets, agents might not follow replication strategies, but accept a certain amount of risk in return for potential profits. Partial hedges that require less capital and reduce risk at the same time provide sensible alternatives. One possible approach, the *quantile hedging* method, has been proposed by several authors in the 90s. Quantile hedging tolerates a positive probability (with respect to the statistical measure P) that the payoff of the contingent claim is not covered at maturity, i. e., a probability of failure. Quantile hedging strategies can be constructed either by maximizing the probability of a successful hedge among all trading strategies, given a constraint on the required cost; or by minimizing the cost of the strategy, given a constraint on the probability of failure. A detailed discussion of quantile hedging for both complete and incomplete market models in continuous time can be found in Föllmer & Leukert [15].

Quantile hedging controls the probability of the hedging error, but it does not take into account the economically relevant size of the shortfall $(C_1 - V_1^\vartheta)^+$ if it occurs. This deficiency motivates the construction of alternative methods like *efficient hedging* suggested by Föllmer & Leukert [16]. *Efficient hedging* measures the hedging error in terms of a loss function l that weights losses of different size. An economic agent can either minimize his expectation of the weighted shortfall

$$E_P[l((C_1 - V_1^\vartheta)^+)]$$

among all trading strategies ϑ, given a constraint on his available capital; or, he might seek the trading strategy with minimal cost that does not exceed a fixed upper bound for the expected shortfall.

Extensions of quantile and efficient hedging incorporating the aspect of model uncertainty are discussed in Kirch [27]. Further references are, among others, Nakano [37], Cvitanic & Karatzas [8],

[6] For dynamic market models, seminal papers on superhedging are El Karoui & Quenez [11] and Kramkov [31].

Kirch & Runggaldier [28], Favero [12], Favero & Runggaldier [13], Schied [44, 45], Rudloff [42, 41], Sekine [46], and Klöppel & Schweizer [29]. The solution of the partial hedging problem relies in continuous time models, for example, on duality methods, stochastic control techniques, or backward stochastic differential equations (BSDE). A survey on this field and the related topic of robust portfolio optimization can be found in Föllmer, Schied & Weber [19].

2.4 Risk-neutral valuation – the general principles

The toy models above are, of course, too simple to capture the complex reality of financial markets. In practice, more realistic models are needed that include various risk factors, multiple primary products, and many trading times at which portfolio proportions can be readjusted. Such models are available both in discrete and continuous time. The classical Black-Scholes model in continuous time will be discussed in Section 3.

In discrete time models, the key results of asset pricing theory are summarized in the current section. We denote the random prices of $d+1$ primary financial products at times $t = 0, 1, \ldots, T$ by $S_t^0, S_t^1, \ldots, S_t^d$. Product '0' is typically an asset with strictly positive prices (e. g., a savings or money market account), playing the role of the numéraire.

Trading decisions at different times must be based on facts that are known at the specific date. Symbolically, the information that is available at time t is often denoted by $\mathcal{F}_t$, $t = 0, 1, \ldots, T$. The information sets $\mathcal{F}_t$ are sigma-algebras that are interpreted as the classes of events (subsets of Ω) for which it is known at time t whether they have occurred or not. The increasing family of information sets $\mathcal{F}_t$, $t = 0, 1, \ldots, T$, is called information filtration.

In the multi-period case, a risk-neutral measure Q is determined (as extension of (5)) by the condition

$$\frac{S_t^i}{S_t^0} = E_Q\left[\frac{S_{t+1}^i}{S_{t+1}^0}\Big|\mathcal{F}_t\right], \quad t = 0, 1, \ldots, T-1,\ i = 0, 1, \ldots, d,$$

where $E_Q[\cdot|\mathcal{F}_t]$ denotes the expectation under Q conditioned on the available information $\mathcal{F}_t$. This condition specifies that the discounted price processes of all primary products have to be martingales with respect to the risk-neutral resp. martingale measure.

Absence of arbitrage and market completeness can be described in terms of risk-neutral measures. These characterizations are provided by the *fundamental theorems of asset pricing.*

Theorem 2.19 (1. Fundamental Theorem, cf., e. g., [18], Theorem 5.16)[7] *A market model is arbitrage-free if and only if there exists a risk-neutral measure.*

In the simple one-period models above, we verified that arbitrage free markets admit indeed risk-neutral measures. Conversely, the following example illustrates that for market models with arbitrage opportunities risk-neutral measures do not exist.

Example 2.20 *The market model in Example 2.4 admits arbitrage. Equation (5) has a unique solution $Q[\{\omega_1\}] = 0$, $Q[\{\omega_2\}] = 1$, i. e., the measure Q assigns probability* 0 *to scenario ω_1. This contradicts property* 1 *in Definition 2.10, thus Q does not define a risk-neutral measure.*

As illustrated by the toy models above, there exist always either one or infinitely many risk-neutral measures. These two cases are in a one-to-one correspondence with the completeness and incompleteness of the underlying market model.

[7] In continuous time market models the formulation of the fundamental theorems involves the general notion of equivalent local martingale measures (resp. equivalent sigma-martingale measures in the case of price processes that are not locally bounded). The existence of such a measure immediately implies the absence of arbitrage opportunities in the market model. In general, however, the converse does not hold. The existence of an equivalent local martingale measure (resp. sigma-martingale measures) requires a stronger condition, namely 'no free lunch with vanishing risk', cf. Delbaen & Schachermayer [9].

Theorem 2.21 (2. Fundamental Theorem, cf., e. g., [18], Theorem 5.37) *The completeness of an arbitrage-free market model is equivalent to the uniqueness of the risk-neutral measure.*

An arbitrage-free valuation of contingent claims can always be provided by risk-neutral measures.

Theorem 2.22 (Risk-neutral Valuation, cf., e. g., [18], Theorem 5.32) *Let C_{T^*} be the payoff of a contingent claim with maturity $T^* \leq T$. Any risk-neutral measure Q defines an arbitrage-free price process for the contingent claim by*

$$C_t := S_t^0 E_Q \left[\frac{C_{T^*}}{S_{T^*}^0} \middle| \mathcal{F}_t \right], \quad t \leq T^*.$$

In analogy to Theorem 2.17 the contingent claim C_{T^*} will be replicable if and only if there is exactly one arbitrage-free price C_0 at time 0. This price corresponds to the cost of setting up a perfect hedge at time 0. Conversely, the contingent claim C_{T^*} admits no perfect hedge if the set of arbitrage-free prices forms a non-empty open interval.

An excellent survey of the *asset pricing theory* in discrete time is provided by the monograph of Föllmer & Schied [18]. A general discussion of the *fundamental theorems of asset pricing theory* that covers both the discrete time and the continuous time case with general semimartingale price processes can be found in the monograph of Delbaen & Schachermayer [9]. Applications in the context of life-insurance mathematics are presented in Møller & Steffensen [35].

3 The Black-Scholes model

The fundamental concepts of asset pricing and hedging were originally discovered by Fischer Black, Robert C. Merton and Myron S. Scholes in the context of the famous Black-Scholes model in the beginning of the 1970s. Black-Scholes is a model in continuous time, but the same basic ideas that provide solutions in discrete time models can successfully be applied in the Black-Scholes model and other continuous time models. Black, Merton and Scholes developed their techniques within an asset model that was suggested by the MIT-economist Paul A. Samuelson[8] (*1915 - †2009) in 1965 as an extension of the Bachelier model [43]. The Bachelier model dates back to the Ph.D. dissertation [3] of Louis Bachelier from 1900.

Black-Scholes is a financial market model with two primary financial products: agents can either invest at a fixed interest rate into a money market account or into a stock. Time is continuous and corresponds to an interval $[0, T]$ for some $T > 0$. As discussed on page 5 in a different context, the Black-Scholes model relies on certain simplifying assumptions that are also standard for many more complex models.

The constant interest rate of the money market account is $r > 0$. Thus, the dynamics of an investment of 1 is described by

$$S_t^0 = \exp(rt), \quad 0 \leq t \leq T.$$

Here, the upper index signifies the 'security identification number' which is equal to '0' in the case of the money market account. The stock is the investment product with security identification number '1' and price process $(S_t^1)_{0 \leq t \leq T}$, modeled by a *geometric Brownian motion*, a strictly positive stochastic process defined in terms of a standard Brownian motion, see equation (11) below.

Brownian motion is a key building block of the theory of stochastic processes in continuous time. It constitutes the driver of all random fluctuations in the model. The model itself specifies how the random innovations influence the movement of the objects in the model, e. g., the price processes. As illustrated in Figure 1, Brownian motion is a specific stochastic process with rough paths that resemble the local behavior of price movements of stock indices.

[8]Nobel prize in economics, 1970

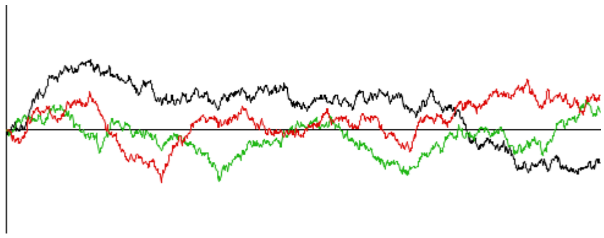


Figure 1: Paths $t \mapsto W_t(\omega)$ of a Brownian motion for different scenarios ω

Brownian motion has a very simple structure and is mathematically easily tractable, but it does not already provide a good model for asset price processes. Paths will, for example, take negative values with positive probability (= negative prices). In contrast to Samuelson in 1965, about sixty years earlier Bachelier used Brownian motion without any proper modification as a model for prices. In the Black-Scholes model Brownian motion is only a building block.

Brownian motion can be interpreted as a time-continuous random walk. A random walk $(S_n)_{n=0,1,\ldots}$ is a simple process in discrete time that is generated by the summation of independent, identically distributed random variables X_n, $n \in \mathbb{N}$:

$$S_n := X_1 + \ldots + X_n, \quad n = 0, 1, \ldots.$$

Like Brownian motion in continuous time, a random walk can be used as a basic building block for the construction of realistic financial market models in discrete time. Paths of a random walk are displayed in Figure 2. For example, a symmetric random walk $(S_n)_{n=0,1,\ldots}$ on a probability space $(\Omega, \mathcal{F}, P)$ with $P[X_n = 1] = P[X_n = -1] = \frac{1}{2}$ can be generated by repeated coin tossing. At each point in time, the result of a coin toss determines whether the path of the process increases or decreases by 1. These two options correspond to the events 'head' and 'tail'.

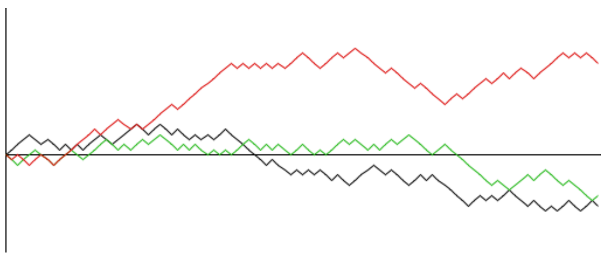


Figure 2: Paths $n \mapsto S_n(\omega)$ of a random walk for different scenarios ω

The relation between random walks and Brownian motion is described by the invariance principle [10] of Monroe David Donsker (*1925 - †1991). It provides a rationale why Brownian motion can be interpreted as a symmetric random walk in continuous time.

Theorem 3.1 (Donsker's invariance principle, functional central limit theorem, cf., e. g., [26], Theorem 14.9) *Suppose that $(X_n)_{n\in\mathbb{N}}$ is a sequence of independent, identically distributed random variables on a probability space $(\Omega, \mathcal{F}, P)$ with mean $E[X_k] = 0$ and variance $Var[X_k] = 1$. Letting $(S_n)_{n=0,1,\ldots}$ be the corresponding random walk,*

$$B^n_t := \tfrac{1}{\sqrt{n}}(S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)X_{\lfloor nt \rfloor + 1}), \quad t \in [0, 1],$$

defines a sequence of stochastic processes in continuous time with continuous paths. The sequence B^n of random elements of $C([0, 1])$ converges in law to a Brownian motion $(W_t)_{t\in[0,1]}$.

The value of the random walk in Figure 2 changes by 1 on each unit time interval. Randomness with a frequency of 1 determines the movement of the process. The frequency of the innovations can be increased by decreasing the length of the time intervals. At the same time, the variance for fixed time horizons can be held constant, if simultaneously the size of the jumps is reduced. This procedure will generate a random walk that fluctuates up and down more quickly. At very high frequencies the constructed object looks very similar to a Brownian motion. Donsker's Theorem states this relationship precisely: in the limiting case, when coins are tossed with infinite frequency, the rescaled random walk becomes a Brownian motion. Mathematically, Brownian motion can be defined as follows.

Definition 3.2 *A Brownian motion is a stochastic process $W = (W_t)_{t\geq 0}$ with the following properties:*

1. *For P-almost all scenarios $\omega \in \Omega$ the Brownian motion starts at 0, i. e., $W_0(\omega) = 0$.*
2. *For all $t > 0$ we have $W_t \sim \mathcal{N}(0, t)$, i. e.,*

$$P[W_t \leq x] = \Phi\left(\frac{x}{\sqrt{t}}\right),$$

   where Φ denotes the cumulative distribution function of the standard normal distribution.

3. *The increments are independent and stationary:*
   - *For all $t < s$ the distribution of the increments $W_s - W_t$ is equal to the distribution of W_{s-t}.*
   - *For all $t_1 < t_2 < \ldots < t_n$ the family $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ is independent.*
4. *The paths $t \to W_t(\omega)$ are continuous for almost all $\omega \in \Omega$.*

The stock price process in the Black-Scholes model, a *geometric Brownian motion*, is a stochastic process with continuous paths that is driven by a Brownian motion W:

$$S_t^1(\omega) = S_0 \exp\left(\sigma W_t(\omega) + (\mu - \tfrac{1}{2}\sigma^2)t\right), \quad \omega \in \Omega. \tag{11}$$

This equation determines also the distribution of the stock price process S^1 under the statistical measure P. Returns are normally distributed, prices log-normal. The parameter σ is called *volatility*; the parameter μ *drift*. The volatility determines the impact of the random fluctuations of Brownian motion on the stock price dynamics; the drift specifies the size of the trend in the exponent.[9]

The information filtration $\mathbb{F} = (\mathcal{F}_t)$ is generated by the observations of the stock price process and, thus, given by the Brownian filtration. As in the case of discrete time models, investments are described by self-financing strategies. In contrast to the discrete time case, in continuous time models investors can adjust the composition of their portfolio at any point in time. The analysis of the Black-Scholes model relies on Itô calculus that was invented by the Japanese mathematician Kiyoshi Itô [24] in the early 1940s.[10]

On the basis of these techniques, it can be shown that any derivative can be replicated by a self-financing strategy in the Black-Scholes model – the model is *complete*. Any derivative possesses

[9] The infinitesimal dynamics of geometric Brownian motion is described by the *stochastic differential equation* (SDE)

$$dS_t^1 = S_t^1(\sigma\, dW_t + \mu\, dt)$$

that can be derived from equation (11) by the Itô-Döblin-formula. In contrast to classical calculus the deterministic drift equals μ and not $\mu - 1/2 \cdot \sigma^2$ – which might naively be conjectured from equation (11).

[10] Wolfgang Döblin (*1915 - †1940) obtained similar results in the late 1930s. His notes on the topic were, however, not discovered before the year 2000 – sixty years after his death. Wolfgang Döblin, known in France as Vincent Doblin, escaped with his family from Nazi Germany to Paris. In World War II he became a French soldier and took his own life in the village Housseras, when his company surrendered, facing immediate capture. Wolfgang Döblin was the son of the writer Alfred Döblin (*1878 - †1957), best known for his novel 'Berlin Alexanderplatz' (1929). A literary portrait of Wolfgang and Alfred Döblin is provided in the a book by Marc Petit [39].

a unique price that is solely implied by the absence of arbitrage: this price equals the cost of perfect replication. On the technical side, the concept of martingale measures generalizes to continuous time models like the Black-Scholes model.

The typical choice for the numéraire in the Black-Scholes model is the money market account. The construction of the unique equivalent martingale measure relies on techniques from *Stochastic Analysis*. Itô's formula implies a stochastic differential equation (SDE) for the dynamics of the discounted stock price process $\widetilde{S}^1_t := S^1_t / S^0_t = e^{-rt} S^1_t$, i. e.,

$$d\widetilde{S}^1_t = \widetilde{S}^1_t (\sigma \, dW_t + (\mu - r) \, dt) \tag{12}$$

resp. in integral form

$$\widetilde{S}^1_t = \widetilde{S}^1_0 + \int_0^t \widetilde{S}^1_u \sigma \, dW_u + \int_0^t \widetilde{S}^1_u (\mu - r) \, du.$$

The integral with respect to the Brownian motion is not a classical integral a la Stieltjes, since the paths of Brownian motion are 'rough and wild' and are, in particular, not of finite variation. Instead a stochastic integral, also called Itô-integral, is used which admits in its general form semimartingale-integrators. Itô-integrals with respect to Brownian motion are (local) martingales, and Itô's Theorem of martingale representation states that, conversely, any continuous local martingale can be represented as an Itô-integral with respect to Brownian motion. Due to the SDE (12) of the discounted price process, an equivalent measure Q is a martingale measure if the stochastic process $W^*_t := W_t + \frac{\mu - r}{\sigma} t$, $t \in [0, T]$, is a Brownian motion with respect to Q. The construction of this martingale measure relies on Girsanov's Theorem.[11]

Corollary 3.3 (Corollary of Girsanov's Theorem for Brownian Motion, cf., e. g., [36], Proposition A.15.1) *For $b \in \mathbb{R}$, let P^* be a probability measure on $(\Omega, \mathcal{F}_T)$ defined by the Radon-Nikodym density*

$$\tfrac{dP^*}{dP} = \exp(bW_T - \tfrac{1}{2} b^2 T),$$

i. e., for any event $A \in \mathcal{F}_T$,

$$P^*[A] = \int_A \tfrac{dP^*}{dP} \, dP.$$

Then the process $W^*_t := W_t - bt$, $t \in [0, T]$, is a Brownian motion with respect to the measure P^*.

Thus, a martingale measure in the Black-Scholes model is defined by the Radon-Nikodym density

$$\tfrac{dQ}{dP} = \exp(-\tfrac{\mu - r}{\sigma} W_T - \tfrac{1}{2} \left(\tfrac{\mu - r}{\sigma}\right)^2 T).$$

A more general version of Girsanov's Theorem implies that the equivalent martingale measure is indeed unique. As a consequence of versions of the fundamental theorems of asset pricing for continuous-time financial market models, the Black-Scholes model is free of arbitrage and the financial market is complete. Any derivative possesses a unique price – equal to the cost of perfect replication – that can be computed by risk-neutral valuation.

Theorem 3.4 (Risk-neutral Valuation, cf., e. g., [36], Corollary 3.3.1) *In the Black-Scholes model, the price of a financial derivative C_T with maturity T is given by*

$$C_0 = S^0_0 E_Q \left[\tfrac{C_T}{S^0_T} \right] = e^{-rT} \cdot E_P \left[\exp(-\tfrac{\mu - r}{\sigma} W_T - \tfrac{1}{2} \left(\tfrac{\mu - r}{\sigma}\right)^2 T) \cdot C_T \right]$$

which equals the cost at time 0 of setting up a perfect replication strategy of the claim C_T.

[11] Igor V. Girsanov (*1934 - †1965) was a Russian mathematician. Only four years after completing his Ph.D. in Moscow he died in an avalanche in the Sayan Mountains in southern Siberia. Versions of Girsanov's Theorem for general semimartingales – as e. g. discussed in Protter [40] – allow the analysis of general financial market models.

The Black-Scholes model is the simplest model for equity derivatives (besides the Bachelier model which admits negative prices). A pricing formula like the one in Theorem 3.4 can also be evaluated in more complex models by the application of Monte Carlo techniques. In the Black-Scholes model, prices of products like Vanilla European call and put options can explicitly be computed by integration with respect to a Gaussian density. Another approach for European, non path-dependent options is provided by the theory of partial differential equations. Particularly tractable is also the perfect replication of derivatives: the number of shares in the replicating portfolio is the delta of the option, the sensitivity of the option price with respect to the price of the stock (delta hedging).

The classical example of option valuation that was already discussed in the seminal papers by Black, Scholes, and Merton is the pricing of European call options. As discussed before, a call option on a stock with maturity T and strike K is the right (but not the obligation) to buy one share of the stock at time T for the strike price K. A rational economic agent will execute this right, if and only if the stock price dominates the strike price at maturity. In this case, his immediate payoff at maturity is the difference of the spot price of the stock and the strike price at maturity T. The European call option is replicable (as are all derivatives in the Black-Scholes model), and its price can be determined by risk-neutral valuation according to Theorem 3.4.

Corollary 3.5 (Black-Scholes Formula, cf., e. g., [36], Theorem 3.1.1) *In the Black-Scholes model, the price of a European call option* $C_T = \max\{S_T^1 - K, 0\}$ *on a stock with maturity* T *and strike* K *is given by*

$$E_Q\left[e^{-rT}\max\{S_T^1 - K, 0\}\right] = S_0^1\Phi(d_+(S_0^1, T)) - e^{-rT}K\Phi(d_-(S_0^1, T))$$

with constants

$$d_\pm(S_0^1, T) = \frac{\log(S_0^1/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Here Φ *denotes the distribution function of the standard normal distribution.*

The Black-Scholes model is a very simple model that illustrates fundamental principles in the context of continuous time models. However, it does not provide a realistic description of market data.

Times series methods allow a detailed analysis of the statistical properties of financial market data. Returns are neither normally distributed nor independent as assumed by the Black-Scholes model. At the same time the volatility of the stock is not constant, but fluctuates stochastically. The deficiencies of the Black-Scholes model are also apparent under the risk-neutral measure. Prices of Vanilla European call and put options are easily observable for many stocks, since these products are very liquidly traded. These options can easily priced in the Black-Scholes model, if the five input parameters stock price, strike price, maturity, interest rate and stock volatility are known. If all input parameters but stock volatility are given, Black-Scholes option prices are described as a function of volatility. The *implied volatility* σ_{imp} of an option is defined as the volatility that produces a Black-Scholes price equal to the observed market price. Since various different options are traded, implied volatility can be computed for a collection of different strikes and maturities. As illustrated in Figure 2, real market data show a dependence of implied volatility on maturity (*term structure*) and strike (*skew*). These observations contradict the assumptions of the Black-Scholes model with a constant stock volatility that determines the price of options of arbitrary strike and maturity. The implied volatility in a Black-Scholes world is a constant.

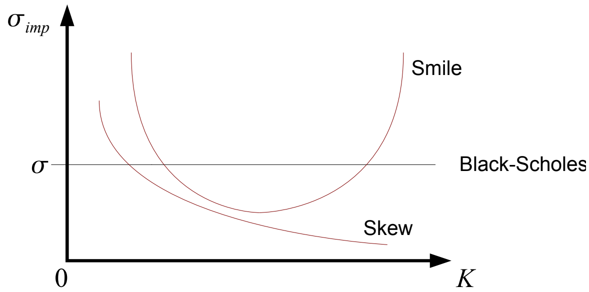


Figure 3: Volatily Smile and Skew

Over the last three decades various models for equity markets and many other asset classes have been developed in financial mathematics that provide a much better picture of reality than the Black-Scholes model. Extensions are, of course, often more complex and more difficult to handle. From a more abstract point of view, financial mathematics can be studied in the context of general semimartingale prices processes. Since markets are usually incomplete, many products cannot be perfectly replicated and partial hedging strategies need to be constructed. For practical applications, specific models that can be estimated and calibrated have been designed: local and stochastic volatility models (Heston, Hull-White, SABR,...) for equity options; structural and intensity-based models for credit products; multi-asset-models with multiple correlated products; hybrid models for hybrid products, etc. (Musiela & Rutkowski [36], Brigo & Mercurio [6], Filipović [14], Bielecki & Rutkowski [4], Overhaus et al. [38]).

The more complex and more realistic models are still not a perfect image of the world, but at most a good approximation. At the same time, practitioners are faced with model risk. The choice of a deficient model might lead to misleading results and provide inadequate guidance in reality. A thoughtful application of financial mathematics in practice should always include a thorough analysis of model risk. A robust model approach that works well for specific applications must be the goal of any quant in practice [5]:

'Essentially, all models are wrong, but some of them are useful.'

4 Market consistent valuation and embedded value

Financial mathematics a la Black & Scholes is not only highly relevant for banks and option traders, but also for insurance companies. New insurance products with exposure to financial risks have been developed that need to be priced and hedged. At the same time, on the level of the insurance companies balance sheet the valuation of assets and liabilities requires an integration of techniques from insurance and financial mathematics. The implementation of internal models under *Solvency II* and the computation of *market consistent embedded value* (MCEV) relies on the techniques that have been discussed in the previous sections.

How should a large portfolio be priced in a market consistent way? Letting $(\Omega, \mathcal{F})$ be a measurable space that describes all relevant scenarios, all insurance and financial payments at a fixed time horizon (say $t = 1$) are modeled as measurable functions on this space. The total value of the company's portfolio of assets and liabilities is a measurable function $X : \Omega \to \mathbb{R}$. The portfolio X aggregates the company's positions and provides an integrated view of assets and liabilities, capturing all financial and actuarial risks including embedded options and other derivative payments. In practice, scenarios and the corresponding portfolio values are usually produced by an *economic scenario generator*, a Monte Carlo simulation engine, that provides the back bone of market consistent valuation. The scenario generator together with suitable evaluation techniques is called *internal model*.

Valuation would be simple, if the real world was correctly described by Black-Scholes. The market of all systematic risks (equity, interest rates, credit, insurance risks) would be complete, a perfect model could easily be calibrated to market data. Unsystematic actuarial risks could be measured on the basis of the principle of risk pooling. The MCEV of the portfolio would be unique, and hedging as well as risk-management very simple.

Reality is, however, more complicated. Insurance companies are confronted with a complex world of incomplete markets, model uncertainty, and non-robust calibration algorithms. Perfect hedges are rarely available, and risk management of large portfolios is quite difficult. Since replicating strategies for portfolios are typically unavailable, the MCEV can usually not uniquely be determined. How can these problems be resolved by an insurance company that needs to value its portfolio and manage its risks?

A solution can be provided on the basis of risk measures which quantify unsystematic risks that cannot be perfectly hedged in the market. A detailed discussion of the concept of risk measures will be provided in Section 5. In incomplete markets, however, prices are not uniquely determined. Derivatives or portfolios possess a unique price in complete markets that equals the costs of perfect replication. The fair value of unsystematic insurance risks can approximately be obtained by the principle of pooling. A risk measure can be used to quantify the residual part.

We denote by $\mathcal{X}$ the set of conceivable portfolios or subportfolios whose risk needs to be assessed. Mathematically, the family $\mathcal{X}$ is a vector space of real-valued mappings X on $(\Omega, \mathcal{F})$ that contains the constants. Negative values of X correspond to debt or losses. A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ quantifies the risk of the subportfolio that cannot be priced by the principle of pooling or perfect replication. In order to achieve consistency with the pricing principles implied by the absence of arbitrage, we impose the following requirement on the risk measure. If a position H can be perfectly replicated, its fair value H should be larger than the value $-\rho(H)$ that is implied by its cost of capital. This insures that companies will compute the value of replicable positions by risk-free valuation.

An **algorithm for the computation of MCEV**[12] can be described as follows:

(a) Decompose X as $X = H + Z$ where

- H is replicable, and
- Z denotes the residual.

(b) The MCEV is then computed by

$$\mathrm{Value(X) = H_0 - \rho(Z)}$$

where

- H_0 signifies the cost of perfect replication of H, and
- $\rho(Z)$ is the cost of the risk capital for Z.

In the context of insurance portfolios, H is often called *valuation portfolio* and contains only risks that are liquidly traded. The position H must be constructed on the basis of the principle of pooling. A detailed introduction to valuation portfolios can be found in Bühlmann [7] and Wüthrich et al. [47].

From the point of view of the firm a large MCEV Value(X) is desirable. Equivalently, the costs of a hedge of X should be as small as possible, i. e., the sum of the cost $-H_0$ of a perfect hedge of H and the capital costs $\rho(Z)$ of Z should be minimized.

The **goal** of the decomposition of X into the summands (H, Z) is the maximization of the MCEV Value(X), or, equivalently, the minimization of the cost of hedging. This optimization problem must be solved by the insurance company in cooperation with regulatory authorities and rating agencies. In practice, it might be reasonable to require that the solution to the problem satisfies additional constraints:

[12] We would like to thank Stefan Jaschke for his inspiring talk in Hannover on the concept of MCEV (Common core in Swiss Solvency Test, Solvency II, IFRS phase II, CRO-Forum) [25].

(a) The value of the replicable part H might be required to be at least α, i. e. $H_0 \geq \alpha$.

(b) Alternative, one might require that the the capital cost of the residual are at most $\rho_{\max}$, i. e. $\rho(Z) \leq \rho_{\max}$.

The optimization problem that determines the MCEV of a portfolio resembles the problem of robust *efficient hedging* ([16], [27], [37], [41], [8], [28], [12], [13], [44], [45], [42], [41], [46], [29]) that was briefly discussed in Section 2.3. Implementing MCEV for large portfolios in practice is a highly complex problem.

The value of positions that can be replicated by simple strategies can directly be inferred from market data (*marking-to-market*). Complex structures, however, do frequently not admit a model-free construction of replication strategies. If perfect hedging strategies exist in reliable models, the risk-neutral value of the position can be computed from the model which needs to be calibrated to market data (*combination of marking-to-market and marking-to-model*). Non replicable risks that can quantitatively be reasonably modeled do not possess a unique risk-neutral value, but can be quantified using risk measures. A last category of non measurable risks requires a thorough qualitative analysis that must be conducted jointly by firms, regulatory authorities and rating agencies. The amount of such risk must already be constrained when portfolios are build up. At the same time, strategies must be devised how deal with extreme scenarios and how to contain potential damage. Examples of the various risk categories are provided in the following table.

Replicable risks	**Non replicable, quantifiable risk**	**Non quantifiable risks**
Equity, Interest rates, FX Futures	Long-term interest rate risks	Structural breaks
Vanilla Equity Options	Long-term volatility risks	Business development
Swaps, Swaptions	Large insurance losses	Management
Variance swaps	Systematic insurance risks	Political developments
Predictable surrender		Longevity, long-term trends

5 Risk measures

The market consistent valuation of individual positions and portfolios whose value is not uniquely determined by replication or pooling is governed by the choice of a risk measure. The risk measure defines the cost of capital of non replicable and non diversifiable positions. Different choices will typically lead to different MCEVs and thus influence the size of the required equity capital of companies, an important determinant for solvency and stability. Regulatory authorities and rating agencies must therefore carefully evaluate the measures of the downside risk that are used in internal models.

The most commonly used risk measure in industry practice is *Value at Risk* (VaR):

Definition 5.1 *The* Value at Risk *at level $\lambda \in (0,1)$ of a financial position X is the smallest monetary amount that needs to be added to X such that the probability of a loss becomes smaller than λ:*

$$\mathrm{VaR}_\lambda(X) := \inf\{m \in \mathbb{R} : P[X + m < 0] \leq \lambda\}.$$

The parameter $\lambda \in (0,1)$ is typically a number close to 0.[13]

Value at Risk allows for a very simple interpretation and can at the same time be easily implemented in practice. Although popular among practitioners due to these reasons, Value at Risk has been heavily criticized in the academic literature as a risk measurement and management tool since the middle of the 1990s. VaR has two serious deficiencies: Firstly, VaR neglects extreme events that occur with small probability. Companies that measure downside risk by VaR thus fail to adequately

[13] Alternative notation is sometimes used for VaR_λ: instead of level λ the number $1-\lambda$ can be provided as the level; X is often replaced by $-X$ in the formula. For example, *Value at Risk at level* 99% refers in some papers to the same quantity as *Value at Risk at level* 1% in the current paper.

prepare for extreme loss scenarios. In times of financial crisis, insufficient capital buffers of a large number of companies endanger the stability of the whole economy. Secondly, risk management tools should provide incentives for managers to spread risk and diversify the companies' portfolios. VaR, however, does not generally reward diversification, but charges a larger risk amount for a diversified position in many cases.

The discussion initially involved mainly academics, but spread to the financial service industry and the general public in the meantime. An article in the New York Times on January 2, 2009 entitled 'Risk Mismanagement' quotes the opinion of two practitioners:

> "David Einhorn, who founded Greenlight Capital, a prominent hedge fund, wrote not long ago that VaR was '*relatively useless as a risk-management tool and potentially catastrophic when its use creates a false sense of security among senior managers and watchdogs. This is like an air bag that works all the time, except when you have a car accident.*' [...] Nicholas Taleb, the best-selling author of 'The Black Swan', has crusaded against VaR for more than a decade. He calls it, flatly, '*a fraud.*' "

VaR was also seriously criticized in the report of the British *Financial Service Authority* 2009, 'The Turner Review – A regulatory response to the global banking crisis' [1], named after their chairman Lord Adair Turner.

The academic literature investigates risk measures since the middle of the 1990s. An axiomatic theory of risk measures was initiated by a seminal paper of Artzner, Delbaen, Eber & Heath [2] that suggests replacing the *ad hoc* risk measure VaR by alternatives that are systematically constructed on the basis of sensible properties that should be incorporated in risk management procedures. An excellent overview can be found in a monograph by Föllmer & Schied [18].

A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is a mapping that quantifies the risk of portfolios $X : (\Omega, \mathcal{F}) \to \mathbb{R}$ in a family of portfolios $\mathcal{X}$. Mathematically, $\mathcal{X}$ is a vector space of measurable functions that contains the constants.

Definition 5.2 *A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is a monetary risk measure if it satisfies the following properties:*

1. Monotonicity: *If $X(\omega) \leq Y(\omega)$ for all scenarios $\omega \in \Omega$, then $\rho(X) \geq \rho(Y)$.*
2. Cash Invariance: *For any $m \in \mathbb{R}$, $\rho(X + m) = \rho(X) - m$.*

Property 1 states that the risk of a position X is smaller than the risk of a position Y, if the future value of X is at least Y for any scenario. Property 2 states that risk is measured on a monetary scale: if m € are added to X, then the risk of X is exactly reduced by this amount.

VaR is a risk measure according to the definition above. Other examples are provided by *Average Value at Risk* – also known as *Tail Value at Risk*, *Conditional Value at Risk* or *Expected Shortfall* – and *Utility-Based Shortfall Risk.*

Example 5.3 *Suppose that $\mathcal{X}$ is the space $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ of all essentially bounded random variables on the probability space $(\Omega, \mathcal{F}, P)$.*

(a) Average Value at Risk (AVaR):

For a financial position X, its AVaR_λ *at level $\lambda \in (0, 1)$ is defined as*

$$\mathrm{AVaR}_\lambda(X) := \tfrac{1}{\lambda} \int_0^\lambda \mathrm{VaR}_\alpha(X)\, d\alpha.$$

The risk of a financial position X measured by AVaR *is always at least as large as risk measured by* VaR*, i. e.,*

$$\mathrm{AVaR}_\lambda(X) \geq \mathrm{VaR}_\lambda(X).$$

More precisely, AVaR_λ *is the smallest distribution-based*[14] *convex*[15] *risk measure that is continuous from above and which dominates* VaR_λ*.*

(b) Utility-based Shortfall Risk (UBSR):
UBSR with loss function l and loss level z is defined as

$$\mathrm{UBSR}_z(X) := \inf\{m \in \mathbb{R} : E[l(-(X+m))] \le z\}.$$

UBSR of a financial position X is the smallest monetary amount that needs to be added to X such that the expected value of $l(-X)$ does not exceed the bound z.

All risk measures can be characterized by their acceptance sets: every risk measure is uniquely determined by the 'acceptable positions' with non-negative risk. The risk measure that corresponds to a given family of acceptable positions, the acceptance set, is equal to the smallest monetary amount that needs to be added to a given position to make it acceptable. In the case of VaR with level λ, a financial position is acceptable, if the probability of a loss is at most λ. The size of the losses and their distribution is neglected by VaR. These quantities are, however, highly relevant when the size of the loss scenario to the economy is appraised. The serious deficiencies of VaR are not shared by AVaR and UBSR. For example, UBSR defines those positions X as acceptable whose rescaled losses $l(-X)$ do not exceed a level z in expectation. The size of the losses is thus explicitly measured.

Another important aspect of risk management relates to incentives to properly diversify portfolios that should be provided by risk measurement procedures. Mathematically, this aspect can be formulated in terms of the (semi-)convexity of the risk measures. Suppose that $X, Y \in \mathcal{X}$ are financial positions, then for any $\alpha \in (0,1)$ the convex combination $\alpha X + (1-\alpha)Y$ models a diversified position. A risk measure assesses diversification as positive, if the risk of the diversified position is at most as large as the maximum of the individual risks. Mathematically, this is described by the semiconvexity of ρ:

$$\rho(\alpha X + (1-\alpha)Y) \le \max\{\rho(X), \rho(Y)\}, \quad \alpha \in [0,1].$$

The semiconvexity is equivalent to the convexity of the risk measure (Föllmer & Schied, 2002):

$$\rho(\alpha X + (1-\alpha)Y) \le \alpha\rho(X) + (1-\alpha)\rho(Y), \quad \alpha \in [0,1]. \tag{13}$$

Convex risk measures can be characterized with the help of techniques from convex analysis via robust representation theorems that provide an interesting link to the aspect of model uncertainty (Föllmer & Schied [17], Föllmer, Schied & Weber [19]).

Theorem 5.4 (Robust Representation, cf., e. g.,[18], Theorem 4.33) *Letting $(\Omega, \mathcal{F}, P)$ be a probability space and $\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ be a risk measure, the following conditions are equivalent:*

(a) ρ is convex and continuous from above.

(b) ρ admits a robust representation in terms of all probability measures $\mathcal{M}_1(P)$ on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to P:

$$\rho(X) = \sup_{Q \in \mathcal{M}_1(P)} \{E_Q[-X] - \alpha(Q)\}, \quad X \in L^\infty. \tag{14}$$

Here, $\alpha(Q) = \sup_{\{X \in L^\infty :\, \rho(X) \le 0\}} E_Q[-X]$ $(Q \in \mathcal{M}_1(P))$ denotes the minimal penalty function.

Condition (b) shows that the risk ρ of a position X is the *worst case* of the expected (and penalized) losses of X computed for all absolutely continuous probability measures; the function α penalizes the probability measures according to their relevance (that is implied by the risk measure).

VaR is not a convex risk measure. Examples for convex risk measures that are also sensitive to extreme loss scenarios are *Average Value at Risk* and *Utility-based Shortfall Risk*.

[14] A risk measure ρ is *distribution-based*, if $\rho(X)$ depends only on the distribution of X.
[15] *Convexity of risk measures* is defined in (13).

Example 5.5

(a) Average Value at Risk (AVaR):

The convex risk measure AVaR *admits a simple robust representation:*

$$\mathrm{AVaR}_\lambda(X) = \max_{Q \in \mathcal{Q}_\lambda} E_Q[-X], \quad X \in L^\infty,$$

where the family $\mathcal{Q}_\lambda$ *contains all probability measures* Q *that are absolutely continuous with respect to* P *with Radon-Nikodym-densities that are* P*-a.s. bounded by* $1/\lambda$*, cf., e. g., [18], Theorem 4.52. The penalty function in equation (14) has the values* $\alpha(Q) = 0$ *for* $Q \in \mathcal{Q}_\lambda$ *and* $\alpha(Q) = \infty$ *else, and the supremum is attained.*

(b) Utility-based Shortfall Risk (UBSR):

The risk measure $\mathrm{UBSR}_z(X) = \inf\{m \in \mathbb{R} : E[l(-(X+m))] \leq z\}$ *admits a robust representation (14) where the supremum is attained. The minimal penalty function can be computed as the solution of a one-dimensional minimization problem:*

$$\alpha(Q) = \inf_{\lambda > 0} \frac{1}{\lambda}\left(z + E_P\left[l^*\left(\lambda \frac{dQ}{dP}\right)\right]\right), \quad Q \in \mathcal{M}_1(P),$$

where $l^*(u) = \sup_{x \in \mathbb{R}}(ux - l(x))$ *denotes the Fenchel-Legendre-transform of* l*, cf., e. g., [18], Theorem 4.115. In the case of the entropic risk measure with exponential loss function* $l(x) = e^{\alpha x}$ *the minimal penalty function can explicitly be computed and can be expressed in terms of the relative entropy with respect to* P*.*

The risk measure AVaR is an important building stone in the family of distribution-based risk measures. The following representation theorem was discovered by Kusuoka [33] restricted to coherent convex risk measures and by Kunze [32] and Frittelli & Rosazza Gianin [20] in the general convex case.

Theorem 5.6 (Robust Representation of Distribution-based Risk Measures, cf., e. g.,[18], Theorem 4.62) *A convex risk measure* $\rho : L^\infty \to \mathbb{R}$ *is distribution-based and continuous from above, if and only if*

$$\rho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \left(\int_{(0,1]} \mathrm{AVaR}_\lambda(X)\mu(d\lambda) - \beta(\mu)\right), \quad X \in L^\infty,$$

where $\mathcal{M}_1((0,1])$ *denotes the set of probability measures on* $(0,1]$ *and*

$$\beta(\mu) = \sup_{\{X \in L^\infty :\, \rho(X) \leq 0\}} \int_{(0,1]} \mathrm{AVaR}_\lambda(X)\mu(d\lambda) \quad (\mu \in \mathcal{M}_1((0,1])).$$

Apart from convexity, the sensitivity in detecting extreme loss scenarios is an important property of risk measures. Figure 4 illustrates the different sensitivities of VaR, AVaR and UBSR with polynomial loss functions $l(x) = x^p$ with exponent p. A detailed analysis is provided by Giesecke, Schmidt & Weber [22]. Increasing the parameter μ_{peak} increases the likelihood of extreme losses. The figure illustrates the low sensitivity of VaR and demonstrates that both AVaR as well as UBSR detect extreme risks very well.

From a theoretical point of view, AVaR and UBSR are superior to the classical risk measure VaR. The sensitivity of AVaR and UBSR to extreme losses entails, however, problems from a statistical point of view. AVaR and UBSR are distribution-based risk measures that depend on the distribution of X under the statistical measure. A robust estimation of the tail of this distribution is usually not feasible.[16] This problem is inherited by any tail-sensitive risk measure. In practice, a purely data-driven analysis of extremal risks is impossible. Additional economic assumptions, a detailed analysis

[16] An interesting discussion of this problem and a suggestion for data-driven risk measures can be found in [23].

of the structure of portfolios and reasonable case studies can provide insights into the extremes. A thorough discussion with regulator authorities, transparency and a procedure that treats all market participants fairly is advisable. The choice of the risk measure should be subject to such a process. The MCEV can then be computed on a solid basis.

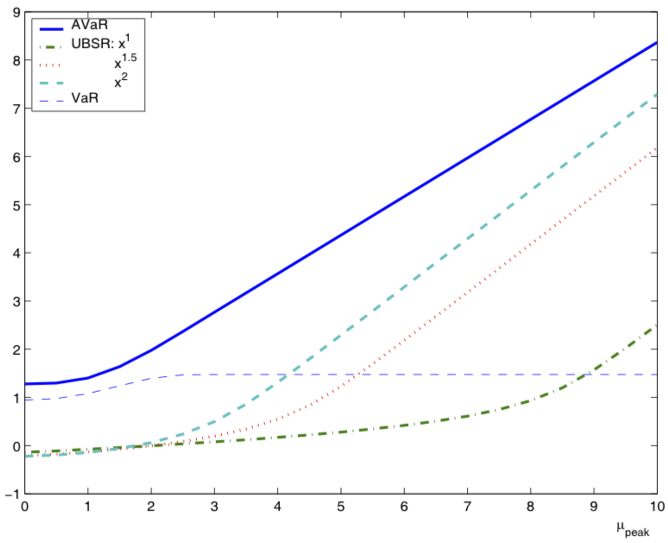


Figure 4: $\mathrm{VaR}_{0.05}$, $\mathrm{AVaR}_{0.05}$ und UBSR with $p \in \left\{1, \frac{3}{2}, 2\right\}$ and $z = 0.3$ as a function of μ for a mixture of a Student-t-distribution (weight 0.96) and a normal distribution with mean μ_{peak} (weight 0.04).

The modern world of financial and insurance companies is extremely complex. Interdisciplinary efforts are necessary to understand multiple aspects of the economy. Randomness is a key driver of the dynamics of prices and markets. A sensible analysis is nowadays impossible without sophisticated mathematical models. These provide in particular risk management strategies to minimize the likelihood of future financial crises.

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