Asymptotics of robust utility maximization

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Abstract

For a stochastic factor model we maximize the long term growth rate of robust expected power utility with parameter $\lambda \in (0,1)$. Using duality methods the problem is reformulated as an infinite time horizon, risk-sensitive control problem. Our results characterize the optimal growth rate, an optimal long term trading strategy, and an asymptotic worst-case model in terms of an ergodic Bellman equation. With these results we develop a duality approach to a "robust large deviations" criterion for optimal long term investment.

Key words: robust utility maximization, risk-sensitive control, ergodic Bellman equation, large deviations

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1 Introduction

One of the basic tasks in mathematical finance is to choose an "optimal" payoff among all available financial positions which are affordable given an initial capital endowment. Any formulation of optimality will involve the investor's individual *preferences* $\succ$ on the set of financial positions $\mathfrak{X}$. The relation $X \succ Y$ means that the investor prefers the payoff X over Y. Under mild conditions such preferences admit a *numerical representation* $\mathcal{U} : \mathfrak{X} \to \mathbb{R}$ (see, e. g., [17]), i. e., for $X, Y \in \mathfrak{X}$ it holds that

$$X \succ Y \iff \mathcal{U}(X) > \mathcal{U}(Y).$$

In this context, Savage [36] clarified the conditions which guarantee that a preference order admits the specific numerical representation

$$\mathcal{U}(X) = E_Q[u(X)] = \int u(X(\omega)) \, Q(d\omega), \quad X \in \mathfrak{X}, \tag{1}$$

in terms of an increasing continuous function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and a probability measure Q on $(\Omega, \mathcal{F})$. Here Q appears as a "subjective" probability measure which is implicit in the investor's preferences, and which may differ from a given "objective"

probability measure. The function u in (1) will be concave if the investor is assumed to be risk averse. In that case, u is called a *utility function*.

The literature on optimal investment decisions in a financial market usually involves the maximization of a utility functional (1) with respect to a given measure Q. Typically, Q is assumed to model the evolution of future stock prices and is thus viewed as the "objective" measure. But the price dynamics is not really known accurately, and so the choice of the evaluation measure Q is itself subject to *model uncertainty* or *model ambiguity*, also called *Knightian uncertainty* in the economic literature. There is another reason to depart from the standard setting of expected utility as formulated in (1): Some very plausible preferences such as the famous *Ellsberg paradox* are not consistent with (1); see, e. g., [17], Example 2.75. In order to overcome this limitation, Gilboa and Schmeidler [19] proposed a more flexible set of axioms for preference orders which leads to a "robust" extension of (1): Instead of a single measure Q the numerical representation of the preference order involves a whole class $\mathcal{Q}$ of probability measures and takes the form of a "coherent" robust utility functional

$$\mathcal{U}(X) = \inf_{Q \in \mathcal{Q}} E_Q[u(X)]. \tag{2}$$

This representation suggests the following interpretation: The investor has in mind a collection of possible probability distributions of market events and takes a worst-case approach in evaluating the expected utility of a given payoff. In recent years, there is an increasing interest in the maximization of the robust expected utility (2) of wealth $X_T^{x_0,\xi}$ attainable at time $T > 0$ by investing in a financial market using some self-financing trading strategy ξ and the initial capital x_0:

$$\text{maximize} \quad \inf_{Q \in \mathcal{Q}} E_Q[u(X_T^{x_0,\xi})] \quad \text{among all self-financing strategies } \xi. \tag{3}$$

For general semimartingale models, this optimization problem can be solved by a *duality approach* (sometimes also called martingale approach); see, e. g., Quenez [35], Schied and Wu [39], or Föllmer and Gundel [14]. Their results provide a robust extension of the seminal paper by Kramkov and Schachermayer [27] for the classical utility maximization problem in incomplete markets. The main advantage of the duality approach lies in the fact that the primal saddle point problem is reduced to a minimization problem on the dual side. In many cases, the dual problem is much simpler and can be tackled with another optimization technique (dynamic programming, BSDE).

For a finite maturity, however, the optimal investment strategies for (3) will typically be time-dependent, and they are often difficult to compute. Instead we propose an asymptotic approach: We consider a long term investment model with one riskless and one risky asset whose drift coefficients are affected by an external *stochastic factor process* of diffusion type. Our model takes into account ambiguity about the "true" drift terms of both the factor process and the risky asset. The class $\mathcal{Q}$ of possible prior models corresponds to affine perturbations of the drift terms in a given reference model and is parameterized by stochastic controls. In this paper we focus on *power utility* $u(x) = \frac{1}{\lambda} x^\lambda$ with parameter $\lambda \in (0,1)$, but other utility functions are also feasible; cf. Remark 2.1. In our model the robust expected power utility will grow exponentially as time $T \uparrow \infty$, and this suggest to

$$\text{maximize} \quad \overline{\lim_{T \uparrow \infty}} \frac{1}{T} \ln \inf_{Q \in \mathcal{Q}} E_Q[(X_T^{x_0,\xi})^\lambda] \quad \text{among all strategies } \xi. \tag{4}$$

This asymptotic formulation has the advantage of allowing for stationary optimal policies and may thus be more tractable. On the other hand, the asymptotic Ansatz provides useful insight for portfolio management with long but finite time horizon.

For the non-robust case $\mathcal{Q} = \{Q\}$, problem (4) is closely related to the maximization of the portfolio's *risk-sensitized expected growth rate*

$$\Lambda_Q(\theta, \xi) := \varliminf_{T \uparrow \infty} -\tfrac{2}{\theta T} \ln E_Q[\exp(-\tfrac{\theta}{2} \ln X_T^{x_0, \xi})], \quad \theta \neq 0. \tag{5}$$

In order to explain the nature of this criterion, let us consider the *entropic monetary utility functional* $\mathcal{U}_\theta(X) := -\tfrac{2}{\theta} \ln E_Q[\exp(-\tfrac{\theta}{2} X)]$, where θ is a positive constant. The functional $\mathcal{U}_\theta$ is also well-defined for $\theta < 0$, and it can be extended to $\theta = 0$ via $\mathcal{U}_0(X) := \lim_{\theta \to 0} \mathcal{U}_\theta(X) = E_Q[X]$. A Taylor expansion around $\theta = 0$ (cf., e. g., [41], p. 5) yields

$$\mathcal{U}_\theta(X) = E_Q[X] + \tfrac{\theta}{4} \operatorname{Var}_Q[X] + O(\theta^2). \tag{6}$$

Thus θ can be interpreted as a "risk sensitivity" parameter that weights the impact of variance. In particular, the Taylor expansion (6) suggests that

$$\Lambda_Q(\theta, \xi) = \varliminf_{T \uparrow \infty} \tfrac{1}{T} E_Q[\ln X_T^{x_0, \xi}] + \tfrac{\theta}{4} \varliminf_{T \uparrow \infty} \tfrac{1}{T} \operatorname{Var}_Q[\ln X_T^{x_0, \xi}].$$

The first term at the right-hand side is the portfolio's *risk-neutral expected growth rate*. The second term provides a risk adjustment specified by the portfolio's asymptotic variance and the risk sensitivity parameter θ, and so $\Lambda_Q(\theta, \xi)$ can indeed be seen as the risk-sensitized expected growth rate of wealth. On the other hand, the long run growth rates of expected power utility $u(x) = (\theta/2)x^{\theta/2}$ are, up to constants, of the form $\Lambda_Q(\theta, \xi)$, and the limit $\theta \to 0$ corresponds to the growth rate of expected logarithmic utility. Such risk-sensitized portfolio optimization problems on an infinite time horizon have received much attention; see, e. g., [9], [10], [3], [4], [33], [31], [28]. In those papers, the maximization of (5) among a class of trading strategies, viewed as dynamic controls, is reformulated as an infinite time horizon, risk-sensitive control problem of the kind studied in Fleming and McEneaney [8]. The rewritten problem leads to an auxiliary finite horizon "exponential of integral criterion". This is a standard problem in stochastic control theory, and its value function can be described by an appropriate Hamilton-Jacobi-Bellman (HJB) equation. As time tends to infinity, a heuristic separation of time and space variables in the HJB equation yields an *ergodic Bellman equation*. The optimal growth rate and an optimal trading strategy is characterized by a specific solution of this ergodic Bellman equation.

In contrast to (5), our robust problem (4) involves also the minimization among the class $\mathcal{Q}$, and this would lead to a *stochastic differential game* on an infinite time horizon. Our main purpose, however, is to develop an alternative approach: The main idea consists in combining the duality approach in [39] with methods from risk-sensitive control. Our main results characterize the optimal growth rate

$$\Lambda(\lambda) := \sup_{\pi \in \mathcal{A}} \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln \inf_{Q \in \mathcal{Q}} E_Q[(X_T^{x_0, \xi})^\lambda],$$

an optimal long term investment strategy, and an asymptotic worst-case model $Q^* \in \mathcal{Q}$ for robust expected power utility in terms of an appropriate ergodic Bellman equation.

Such asymptotic results on robust utility maximization are not only of intrinsic interest but also relevant in connection to "robust large deviations" criteria to optimal long term investment. Suppose that the investor takes into account a class $\mathcal{Q}$ of prior models and wants to maximize the worst-case probability that the portfolio's growth rate $L_T^{x_0, \xi} := \tfrac{1}{T} \ln X_T^{x_0, \xi}$ exceeds some threshold $c \in \mathbb{R}$. In the spirit of large deviations theory (see, e. g., [6]) the asymptotic problem then consists in

$$\text{maximizing} \quad \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln \inf_{Q \in \mathcal{Q}} Q[L_T^{x_0, \xi} \geq c] \quad \text{among all } \xi. \tag{7}$$

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The solution can be derived by a duality approach similar to the Gärtner-Ellis theorem, but here the dual problem involves the optimal growth rates $\Lambda(\lambda)$, $\lambda \in (0,1)$, of robust expected power utility.

The paper is organized as follows: The setup is introduced in Section 2. Section 3 contains a heuristic derivation of our main results that are verified in Section 4. In Section 5 we discuss the existence of a solution to our ergodic Bellman equation. Explicit case studies are given in Section 6. In Section 7 we describe the duality approach to the robust outperformance criterion (7).

2   The model and problem formulation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, Q_0)$ be the canonical path space of a two-dimensional Wiener process $W = (W_t^1, W_t^2)_{t \geq 0}$. We shall consider a long term horizon investment model with one locally riskless asset S^0 and one risky asset S^1. The performance of the market is determined by an external "economic factor" Y, driven by the Wiener process W. The spectrum of possible factors includes dividend yields, short-term interest rates, price-earning ratios, yields on various bonds, the rate of inflation, etc.. Both the price processes S^0, S^1 and the factor process Y will be subject to model ambiguity. This will be described by a class $\mathcal{Q}$ of probabilistic models, viewed as perturbations of the following reference model Q_0. Under Q_0 the dynamics of the locally riskless asset is given by

$$dS_t^0 = S_t^0 r(Y_t)\, dt, \quad S_0^0 = 1,$$

and the price process of the risky asset is governed by the SDE

$$dS_t^1 = S_t^1(m(Y_t)\, dt + \sigma\, dW_t^1). \tag{8}$$

Thus the market price of risk is defined by

$$\theta(y) := \tfrac{m(y) - r(y)}{\sigma}. \tag{9}$$

The factor process evolves according to

$$dY_t = g(Y_t)\, dt + \rho\, dW_t = g(Y_t)\, dt + \rho_1\, dW_t^1 + \rho_2\, dW_t^2. \tag{10}$$

We suppose that the economic factor can be observed but cannot be traded directly. Therefore the market model is typically incomplete. This class of market models is widely used in mathematical finance and economics; see, e.g., [18], [7], [5], and the references therein. Typically the diffusion Y is also assumed to be mean reverting and ergodic with some invariant distribution μ. A special example is the Ornstein-Uhlenbeck (OU) process with dynamics

$$dY_t = \eta_0(\overline{y} - Y_t)\, dt + \sigma\, dW_t^1, \quad \eta_0 > 0, \sigma \neq 0, \tag{11}$$

and invariant distribution $\mu = \mathrm{N}(\overline{y}, \tfrac{\sigma^2}{2\eta_0})$.

We shall use the following general assumptions on the coefficients of the diffusions, summarized as

Assumption 2.1. *The functions g, m admit derivatives $g_y, m_y \in C_b^1(\mathbb{R})$, and r belongs to $C_b^2(\mathbb{R})$, where $C_b^k(\mathbb{R})$ denotes the class of all bounded functions with bounded derivatives up to order k. Moreover, we assume that σ and $\|\rho\|$ are positive and that the short rate function r is bounded below by some constant $a_1 > 0$.*

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Here we use $\|\cdot\|$ to indicate the Euclidian norm in $\mathbb{R}^2$, and in the sequel $(\cdot, \cdot)$ will denote the corresponding inner product. In particular, our assumptions ensure that the functions g and θ satisfy the linear growth conditions

$$|g(y)| \leq a_2(1+|y|) \text{ and } |\theta(y)| \leq a_3|y| + a_4 \quad \text{for } a_2, a_3, a_4 > 0.$$

Note also that Assumption 2.1 is consistent with linear drift functions g and m. In this paper, such a choice of the reference model will be particularly useful to obtain explicit solutions; cf. Section 6.

In reality, however, the "true" price dynamics is not really known exactly. Here we focus on model uncertainty with respect to the drift terms appearing in (8) and (10). More precisely, we consider the parameterized class of possible probabilistic models

$$\mathcal{Q} := \{Q^\eta | \eta = (\eta_t)_{t \geq 0} \in \mathcal{C}\},$$

on $(\Omega, \mathcal{F})$, where $\mathcal{C}$ denotes the set of all progressively measurable processes $\eta = (\eta_t)_{t \geq 0}$ such that $\eta_t = (\eta_t^{11}, \eta_t^{12}, \eta_t^{21}, \eta_t^{22})$ belongs $dt \otimes Q_0$-a. e. to some fixed compact and convex set $\Gamma \subset \mathbb{R}^4$ which contains the origin. For $\eta \in \mathcal{C}$ and any fixed horizon T, the restriction of Q^η to the σ-field $\mathcal{F}_T$ is given by the Radon-Nikodým density

$$D_T^\eta := \frac{dQ^\eta}{dQ_0}\big|_{\mathcal{F}_T} := \mathcal{E}(\int_0^{\cdot} \eta_t^{1\cdot} Y_t + \eta_t^{2\cdot} \, dW_t)_T \tag{12}$$

with respect to the reference measure Q_0. Here $\mathcal{E}(\cdot)$ denotes the Itô exponential. To see that D_T^η is indeed the density of a probability measure on $(\Omega, \mathcal{F}_T)$, we can argue as follows: By Assumption 2.1 the diffusion process Y satisfies the regularity conditions required in Lemma 8.1, and so there exists some $\delta > 0$ such that $\sup_{0 \leq t \leq T} E_{Q_0}[\exp(\delta Y_t^2)] < \infty$. The compactness of Γ thus ensures that

$$\sup_{0 \leq t \leq T} E_{Q_0}[\exp(\epsilon \|\eta_t^{1\cdot} Y_t + \eta_t^{2\cdot}\|^2)] < \infty \tag{13}$$

as soon as $\epsilon > 0$ is chosen sufficiently small. According to [29], Example 3, Subsection 6.2, this yields $E_{Q_0}[D_T^\eta] = 1$ as desired.

In view of (12) we have $Q_0 = Q^0 \in \mathcal{Q}$, and it follows as in [23], Lemma 3.1, that $\mathcal{Q}$ is a convex set of locally equivalent measures on $(\Omega, \mathcal{F})$. By Girsanov's theorem,

$$W_t^\eta := (W_t^1 - \int_0^t \eta_s^{11} Y_s + \eta_s^{21} \, ds, W_t^2 - \int_0^t \eta_s^{12} Y_s + \eta_s^{22} \, ds), \quad t \geq 0,$$

is a two-dimensional Wiener process under the measure Q^η, and the dynamics of S^1, Y under Q^η takes the form

$$dY_t = [g(Y_t) + (\rho, \eta_t^{1\cdot} Y_t + \eta_t^{2\cdot})] \, dt + \rho \, dW_t^\eta, \tag{14a}$$

$$dS_t^1 = S_t^1([m(Y_t) + \sigma(\eta_t^{11} Y_t + \eta_t^{21})] \, dt + \sigma \, dW_t^{1,\eta}. \tag{14b}$$

Roughly speaking each element of $\mathcal{Q}$ corresponds to an *affine perturbation* of the drifts in our reference model Q_0. In particular, our "robust" market model includes the following special cases (see Section 6):

Example 2.1. Black-Scholes model with uncertain drift:

$$r(y) \equiv r, \ m(y) \equiv m, \ \Gamma = \{(0,0)\} \times [a, b] \times \{0\}$$

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Example 2.2. Geometric OU model with uncertain mean reversion:

The factor process Y is an OU process under Q_0 with rate of mean reversion $\eta_0 > 0$, mean reversion level $\bar{y} = 0$ and volatility $\sigma > 0$; cf. (11). We also assume $S_t^0 = \exp(rt)$, $r > 0$, and $S_t^1 := \exp(Y_t + \alpha t)$, $\alpha \in \mathbb{R}$. By Itô's formula this corresponds to

$$g(y) = -\eta_0 y, \ \rho_1 = \sigma, \ \rho_2 = 0, \ m(y) = -\eta_0 y + \tfrac{1}{2}\sigma^2 + \alpha.$$

Moreover, we take the set $\Gamma := [\frac{\eta_0 - b}{\sigma}, \frac{\eta_0 - a}{\sigma}] \times \{(0,0,0)\}$ for $0 < a \le b < \infty$. For any $Q^\eta \in \mathcal{Q}$ the process Y thus follows under $Q^\eta \in \mathcal{Q}$ an OU type dynamics with mean reversion process $\eta_0 - \sigma\eta_t^{11}$ taking values in $[a, b]$.

Let us now formulate our main problem. We consider an investor with initial capital $x_0 > 0$ who aims at optimizing his portfolio in the long run. A *trading strategy* will be a predictable stochastic process $\xi = (\xi^0, \xi^1)$ whose components ξ^0 and ξ^1 describe the successive amounts invested into the bond and into the risky asset. The value of such a portfolio at time t is given by $X_t^\xi = \xi_t^0 S_t^0 + \xi_t^1 S_t^1$. We also assume that ξ^1 is S^1-integrable. Such a trading strategy ξ is said to be *self-financing* for the given initial capital x_0 if its wealth process $X^\xi = (X_t^\xi)_{t \ge 0}$ takes the form

$$X_t^\xi = x_0 + \int_0^t \xi_u^0 \, dS_u^0 + \int_0^t \xi_u^1 \, dS_u^1. \tag{15}$$

Here the (stochastic) integrals can be interpreted as cumulative gains or losses, i.e., any change in the portfolio value equals the profit or loss due to changes in the asset prices. For notational convenience we omit the explicit dependence of X^ξ on the initial capital x_0, since it will be irrelevant for our purpose of long term investment.

Definition 2.1. *A self-financing trading strategy ξ is called T-admissible if $X_t^\xi \ge 0$ for all $t \in [0, T]$. A strategy ξ will be called admissible if it is T-admissible for any time horizon $T > 0$. We denote by $\mathcal{A}_T$ the class of all T-admissible strategies and by $\mathcal{A}$ the class of all admissible strategies.*

Clearly, a self-financing trading strategy ξ can also be described by the fractions

$$\pi_t := \frac{\xi_t^1 S_t^1}{X_t^\xi}, \quad t \ge 0,$$

of the current wealth which should be invested into the risky asset. Throughout this paper we identify a strategy ξ with the fractions $\pi = (\pi_t)_{t \ge 0}$. In terms of π the wealth process defined in (15) takes the form

$$X_t^\pi = x_0 + \int_0^t \frac{X_u^\pi (1 - \pi_u)}{S_u^0} \, dS_u^0 + \int_0^t \frac{X_u^\pi \pi_u}{S_u^1} \, dS_u^1$$

i.e., the investor's wealth X^π evolves according to the SDE

$$\begin{aligned} dX_t^\pi &= X_t^\pi \big((1 - \pi_t)\tfrac{dS_t^0}{S_t^0} + \pi_t \tfrac{dS_t^1}{S_t^1}\big) \\ &= X_t^\pi \big(r(Y_t) \, dt + \pi_t \sigma[(\theta(Y_t) + \eta_t^{11} Y_t + \eta_t^{21}) \, dt + dW_t^{1,\eta}]\big) \end{aligned} \tag{16}$$

with initial condition $X_0^\pi = x_0$.

In order to specify optimality, we assume that the investor's preferences in the face of model ambiguity are specified by a power utility function

$$u(x) = \tfrac{1}{\lambda} x^\lambda \quad \text{with risk aversion parameter } \lambda \in (0, 1),$$

and the set of prior probabilistic models $\mathcal{Q}$; cf. page 2. For a finite maturity T, his robust portfolio selection problem then consists in

$$\text{maximizing} \quad \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[u(X_T^\pi)] \quad \text{among all} \ \pi \in \mathcal{A}_T. \tag{17}$$

In a general semimartingale setting, this problem is well understood from a theoretical point of view, in particular due to the articles [35], [39], [14]. For robust market models of the diffusion type described above and for power utility, problem (17) has been discussed recently by Schied [38]. Applying dynamic programming methods to the dual problem, he determines the maximal robust expected utility and a worst-case model in terms of a *Hamilton-Jacobi-Bellman equation*. Here we do not limit the analysis to a fixed maturity. Instead the objective of our investor consists in maximizing the long term growth of robust expected power utility. A priori estimates, as established in Lemma 3.1, suggest that the maximal values

$$U_T^Q(x_0) := \sup_{\pi \in \mathcal{A}_T} E_Q[u(X_T^\pi)], \quad U_T(x_0) := \sup_{\pi \in \mathcal{A}_T} \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[u(X_T^\pi)] \tag{18}$$

for the classical utility maximization problem under Q and for its robust extension will grow exponentially as $T \uparrow \infty$. Thus it is natural to try to

$$\text{maximize} \quad \varlimsup_{T \uparrow \infty} \frac{1}{T} \ln \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^\pi)^\lambda] \text{ among all } \pi \in \mathcal{A}. \tag{19}$$

Our goal is to identify the optimal growth rate

$$\Lambda(\lambda) := \sup_{\pi \in \mathcal{A}} \varlimsup_{T \uparrow \infty} \frac{1}{T} \ln \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^\pi)^\lambda], \quad \lambda \in (0,1), \tag{20}$$

an optimal long term investment strategy π^*, and an asymptotic worst-case model $Q^{\eta^*} \in \mathcal{Q}$. Heuristically this means that, as $T \uparrow \infty$,

$$U_T(x_0) \approx \frac{1}{\lambda} x_0^\lambda e^{\Lambda(\lambda)T} \tag{21}$$

$$\approx \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[u(X_T^{\pi^*})] \tag{22}$$

$$\approx U_T^{Q^{\eta^*}}(x_0) = \sup_{\pi \in \mathcal{A}_T} E_{Q^{\eta^*}}[u(X_T^\pi)] \tag{23}$$

$$\approx E_{Q^{\eta^*}}[u(X_T^{\pi^*})]. \tag{24}$$

Here (22) corresponds to asymptotic optimality of the trading strategy π^*, (23) to the property of Q^{η^*} of being the asymptotic worst-case model, and (24) identifies π^* also as the asymptotically optimal strategy for the model Q^{η^*}. In particular, Q^{η^*} and π^* can be viewed as a saddle point for the problem of asymptotic robust utility maximization with control parameters $\eta \in \mathcal{C}$ and $\pi \in \mathcal{A}$. Moreover, (22) suggests that an optimal strategy π^* of the asymptotic criterion (19) should provide a good approximation of an optimal investment process $\pi^{*,T}$ for the robust power utility maximization problem with a large but finite time horizon T.

Remark 2.1. *The asymptotic approach to robust utility maximization can be extended to the following cases (see [26], Chapter 4):*

- *For power utility $u(x) = \frac{1}{\lambda} x^\lambda$ with parameter $\lambda < 0$ the distance between*

$$U_T(x_0) = \frac{1}{\lambda} \inf_{\pi \in \mathcal{A}_T} \sup_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^\pi)^\lambda]$$

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and its upper bound 0 *will typically decrease exponentially. This suggests to compute the optimal growth rate*

$$\Lambda(\lambda) := \inf_{\pi \in \mathcal{A}} \varliminf_{T \uparrow \infty} \frac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^\pi)^\lambda].$$

- *For logarithmic utility* $u(x) = \ln(x)$ *the growth of robust expected utility will be linear. Thus we want to*

$$maximize \quad \varlimsup_{T \uparrow \infty} \frac{1}{T} \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[\ln(X_T^\pi)] \quad among\ all\ \pi \in \mathcal{A}.$$

3 Heuristic outline of the dynamic programming approach

We start with a heuristic derivation of our main results. They provide a characterization of the optimal growth rate $\Lambda(\lambda)$, of an asymptotic worst-case model Q^{η^*}, and of an optimal long term investment strategy π^* in terms of an *ergodic Bellman equation* (EBE). Our method combines the *duality approach* to robust utility maximization with *dynamic programming methods* for a varying time horizon. As a byproduct of the duality approach, we also show that $U_T(x_0)$ grows exponentially at rate $\Lambda(\lambda)$ as $T \uparrow \infty$. A more direct, but not more tractable approach to the saddle-point problem (19) via stochastic differential games will be discussed in Remark 4.2.

First, we setup the duality approach based on the results of Schied and Wu [39] for a utility function u on the positive halfline. This will allow us to transform the primal saddle-point problem (19) to a simpler minimization problem on the dual side. The dual value function at time T is defined by

$$V_T(y) := \inf_{Q \in \mathcal{Q}} \inf_{Y \in \mathcal{Y}^Q} E_Q[v(yY_T/S_T^0)], \quad y > 0, \tag{25}$$

where $v(y) := \sup_{x>0}\{u(x) - xy\}$, $y > 0$, is the convex conjugate function of u. This definition also involves the class of supermartingales

$$\mathcal{Y}_T^Q := \{Y \geq 0 | Y_0 = 1 \,\&\, \forall\, \pi \in \mathcal{A}_T : (Y_t X_t^\pi/S_t^0)_{t \leq T} \text{ is a } Q\text{-supermartingale}\},$$

as introduced by Kramkov and Schachermayer [27]. Note that $\mathcal{Y}_T^Q$ contains the density processes (taken with respect to Q and the numéraire S^0) of the class $\mathcal{P}_T$ of all equivalent local martingale measures on $(\Omega, \mathcal{F}_T)$. For power utility we have $v(y) = -\beta^{-1}y^\beta$, $\beta := \frac{\lambda}{\lambda-1}$, and this yields the scaling property $V_T(y) = y^\beta V_T(1)$. Due to [39], Theorem 2.2, the primal value function (18) can then be obtained as

$$U_T(x_0) = \inf_{y>0}\{V_T(y) + x_0 y\} = \frac{1}{\lambda} x_0^\lambda (-\beta V_T(1))^{1-\lambda}. \tag{26}$$

Since power utility has asymptotic elasticity $\varlimsup_{x \uparrow \infty} \frac{xu'(x)}{u(x)} < 1$, it follows from [39], Theorem 2.5, also that

$$V_T(1) = \inf_{P \in \mathcal{P}_T} \inf_{Q \in \mathcal{Q}} E_Q[v(\tfrac{dP}{dQ}|_{\mathcal{F}_T}/S_T^0)]. \tag{27}$$

We now parameterize the sets $\mathcal{Y}_T^Q$ and $\mathcal{P}_T$. Since $Z_t := dP/dQ_0|_{\mathcal{F}_t}$, $t \leq T$, is a positive Q_0-martingale for any $P \in \mathcal{P}_T$, the martingale representation theorem yields the existence of an $\mathbb{R}^2$-valued progressively measurable process $\phi = (\phi^1, \phi^2)$ with

$\int_0^T \|\phi_s\|^2 \, ds < \infty$ Q_0-a.s. such that $Z_t = \mathcal{E}(\int_0^{\cdot} \phi_s \, dW_s)_t$. By Girsanov's theorem, the discounted wealth process X^π/S^0 is a local martingale under P if and only if $\phi_s^1 = -\theta(Y_s) \, ds \otimes Q_0$-a.e.. Thus the Q_0-density process of an martingale measure $P \in \mathcal{P}_T$ necessarily takes the form

$$Z_t^\nu := \mathcal{E}(-\int_0^{\cdot} \theta(Y_s) \, dW_s^1 - \int_0^{\cdot} \nu_s \, dW_s^2)_t \tag{28}$$

for some progressively measurable process ν such that $\int_0^T \nu_s^2 \, ds < \infty$ Q_0-a.s.. Conversely, Z_T^ν corresponds to the Q_0-density of an equivalent local martingale measure on $(\Omega, \mathcal{F}_T)$ as soon as the martingale condition $E_{Q_0}[Z_T^\nu] = 1$ holds. This can verified if, for instance, the process ν is assumed to be bounded. Thus our market model admits a variety of equivalent local martingale measures up to any finite horizon T, i.e., the restriction of our model to a finite horizon is *arbitrage-free* but *incomplete*.

More generally, we will denote by $\mathcal{M}$ the set of all progressively measurable processes $\nu = (\nu_t)_{t \geq 0}$ such that $\int_0^T \nu_t^2 \, dt < \infty$ Q_0-a.s. for all $T > 0$. Via (28) every $\nu \in \mathcal{M}$ gives rise to a positive Q_0-supermartingale Z^ν. Using Itô's formula one easily shows that $(D^\eta)^{-1} Z^\nu X^\pi/S^0$ is a positive local martingale under Q^η for any $\nu \in \mathcal{M}$ and $\pi \in \mathcal{A}_T$, and hence a Q^η-supermartingale. Thus,

$$\{(\tfrac{dP}{dQ^\eta}\big|_{\mathcal{F}_t})_{t \leq T} | P \in \mathcal{P}_T\} \subset \{((D_t^\eta)^{-1} Z_t^\nu)_{t \leq T} | \nu \in \mathcal{M}\} \subset \mathcal{Y}_T^{Q^\eta}.$$

In view of (25), (27) and (26) this inclusion and a change of measure yield

$$U_T(x_0) = \tfrac{1}{\lambda} x_0^\lambda \big(\inf_{\nu \in \mathcal{M}} \inf_{\eta \in \mathcal{C}} E_{Q_0}[(Z_T^\nu (S_T^0)^{-1})^{\frac{\lambda}{\lambda-1}} (D_T^\eta)^{\frac{1}{1-\lambda}}] \big)^{1-\lambda}. \tag{29}$$

In a second step, we derive an ergodic Bellman equation by applying dynamic programming methods to the dual minimization problem. Since Z_T^ν, D_T^η, and the bond price S_T^0 depend on the factor process Y, the expectation at the right-hand side of (29) is a function of the initial state $Y_0 = y$. For all processes $\eta \in \mathcal{C}$ and $\nu \in \mathcal{M}$ we can thus define

$$V(\eta, \nu, y, T) := E_{Q_0}[(Z_T^\nu (S_T^0)^{-1})^{\frac{\lambda}{\lambda-1}} (D_T^\eta)^{\frac{1}{1-\lambda}}]. \tag{30}$$

Inserting the definitions of Z_T^ν, D_T^η and S_T^0 we then obtain the decomposition

$$V(\eta, \nu, y, T) = E_{Q_0}[\mathcal{E}_T^{\eta,\nu} e^{\int_0^T l(\eta_t, \nu_t, Y_t) \, dt}]. \tag{31}$$

Here the function $l : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is defined by

$$l(\eta, \nu, y) := \tfrac{1}{2} \tfrac{\lambda}{(1-\lambda)^2} [(\theta(y) + \eta^{11} y + \eta^{21})^2 + (\nu + \eta^{12} y + \eta^{22})^2] + \tfrac{\lambda}{1-\lambda} r(y) \tag{32}$$

and

$$\mathcal{E}_T^{\eta,\nu} := \mathcal{E}(\tfrac{1}{1-\lambda}(\int_0^{\cdot} \lambda\theta(Y_t) + \eta_t^{11} Y_t + \eta_t^{21} \, dW_t^1 + \int_0^{\cdot} \lambda\nu_t + \eta_t^{12} Y_t + \eta_t^{22} \, dW_t^2))_T.$$

To simplify the expression for $V(\eta, \nu, y, T)$, we shall interpret the Itô exponential as the density of a probability measure $R^{\eta,\nu}$ on $(\Omega, \mathcal{F}_T)$. This requires $E_{Q_0}[\mathcal{E}_T^{\eta,\nu}] = 1$ which is satisfied, for example, if $\int_0^T \nu_t^2 \, dt$ is bounded. For arbitrary $\nu \in \mathcal{M}$ we may have $E_{Q_0}[\mathcal{E}_T^{\eta,\nu}] < 1$, but here we argue heuristically, and so we postpone this technical problem to the proof of Theorem 4.1. In terms of the measure $R^{\eta,\nu}$ we can write

$$V(\eta, \nu, y, T) = E_{R^{\eta,\nu}}[e^{\int_0^T l(\eta_t, \nu_t, Y_t) \, dt}]. \tag{33}$$

Moreover, Girsanov's theorem yields that the factor process $(Y_t)_{t \leq T}$ evolves under $R^{\eta,\nu}$ according to the SDE

$$dY_t = h(\eta_t, \nu_t, Y_t)\, dt + \rho\, dW_t^{\eta,\nu}, \tag{34}$$

where $W^{\eta,\nu}$ is a Wiener process under $R^{\eta,\nu}$, and where h is defined by

$$h(\eta, \nu, y) := g(y) + \tfrac{1}{1-\lambda}\rho_1(\lambda\theta(y) + \eta^{11}y + \eta^{21})$$
$$+ \tfrac{1}{1-\lambda}\rho_2(\lambda\nu + \eta^{12}y + \eta^{22}). \tag{35}$$

Putting (29), (30) and (33) together, we get

$$U_T(x_0) = \tfrac{1}{\lambda}x_0^\lambda v(y, T)^{1-\lambda}, \tag{36}$$

where

$$v(y, T) := \inf_{\nu \in \mathcal{M}} \inf_{\eta \in \mathcal{C}} E_{R^{\eta,\nu}}[e^{\int_0^T l(\eta_t, \nu_t, Y_t)\, dt}]$$

denotes the value function of the finite horizon optimization problem on the dual side of (29). Such an "expected exponential of integral cost criterion" with a dynamics of the form (34) is standard in stochastic control theory; see, e.g., [11], Remark IV.3.3. As a result, v can be described as the solution to the *Hamilton-Jacobi-Bellman* (HJB) equation

$$v_t = \tfrac{1}{2}\|\rho\|^2 v_{yy} + \inf_{\nu \in \mathbb{R}} \inf_{\eta \in \Gamma}\{l(\eta, \nu, \cdot)v + h(\eta, \nu, \cdot)v_y\}, \quad v(\cdot, 0) \equiv 1. \tag{37}$$

The following lemma establishes a priori bounds for the exponential growth of robust expected power utility, and this justifies the scaling in (19):

Lemma 3.1. *Suppose in addition to Assumption 2.1 that one of the following conditions is satisfied:*

(1) *The market price of risk function θ in (9) is bounded.*

(2) *There exist constants $K, M_1, M_2 > 0$ such that*

$$-Ky + M_1 \leq g(y) + \tfrac{\lambda}{1-\lambda}\rho_1\theta(y) \leq -Ky + M_2, \quad 2\tfrac{\lambda}{(1-\lambda)^2}\|\rho\|^2 a_3^2 < K^2.$$

Then there are constants $K_1, K_2 > 0$ such that for any initial capital $x_0 > 0$

$$K_1 \leq \varliminf_{T \uparrow \infty} \tfrac{1}{T} \ln U_T(x_0) \leq \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln U_T(x_0) \leq K_2. \tag{38}$$

Proof. If at any time the whole capital is put into the money market account, then the investor's utility at time T is given by $\tfrac{1}{\lambda}x_0^\lambda \exp(\lambda \int_0^T r(Y_t)\, dt)$ which, by Assumption 2.1, is bounded from below by $\tfrac{1}{\lambda}x_0^\lambda \exp(\lambda a_1 T)$. This implies the lower bound

$$0 < K_1 := \lambda a_1 \leq \varliminf_{T \uparrow \infty} \tfrac{1}{T} \ln U_T(x_0).$$

To obtain the upper bound, observe first that

$$v(y, T) \leq V(0, 0, y, T) \leq E_R[e^{\frac{1}{2}\frac{\lambda}{(1-\lambda)^2}\int_0^T \theta^2(Y_t)\, dt}]e^{\frac{\lambda}{1-\lambda}\|r\|_\infty T},$$

where $R := R^{0,0}$ is the probability measure defined by $\mathcal{E}_T^{0,0}$. In view of (36) we thus get the estimate

$$\varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln U_T(x_0) \leq (1 - \lambda) \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln E_R[e^{\frac{1}{2}\frac{\lambda}{(1-\lambda)^2}\int_0^T \theta^2(Y_t)\, dt}] + \lambda\|r\|_\infty. \tag{39}$$

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In particular, the upper bound in (38) holds with $K_2 := \frac{1}{2}\frac{\lambda}{1-\lambda}\|\theta\|_\infty^2 + \lambda\|r\|_\infty$ if the market price of risk function θ is bounded.

Case (2) requires more effort. By (39) it is sufficient to show that

$$\varlimsup_{T\uparrow\infty} \frac{1}{T} \ln E_R[\exp(\frac{1}{2}\frac{\lambda}{(1-\lambda)^2}\int_0^T \theta^2(Y_t)\,dt)] < \infty. \tag{40}$$

To this end, recall from (34) that the dynamics of Y under R is given by

$$dY_t = h(0,0,Y_t)\,dt + \rho\,dW_t^{0,0} \quad \text{with} \quad h(0,0,y) = g(y) + \frac{\lambda}{1-\lambda}\rho_1\theta(y).$$

Consider now the R-OU processes $dZ_{it} = [-KZ_{it}+M_i]\,dt + \rho\,dW_t^{0,0}$, $Z_{i0} = y$, $i = 1,2$. Then a comparison argument for the solutions of SDE's ensures that

$$R[Z_{1t} \leq Y_t \leq Z_{2t} \text{ for all } t \geq 0] = 1. \tag{41}$$

Take now $\epsilon > 0$ satisfying $2\frac{\lambda}{(1-\lambda)^2}\|\rho\|^2(a_3^2 + \epsilon) < K^2$. By Assumption 2.1 there exist constants C_1, C_2 depending on ϵ such that

$$\theta^2(y) \leq (a_3|y| + a_4)^2 \leq (a_3^2 + \tfrac{\epsilon}{2})y^2 + C_1 \leq (a_3^2 + \epsilon)(y - M_i/K)^2 + C_1 + C_2$$

for any $y \in \mathbb{R}$. Together with (41) and Hölder's inequality (applied in line 3) this leads to

$$
\begin{aligned}
E_R[e^{\frac{1}{2}\frac{\lambda}{(1-\lambda)^2}\int_0^T \theta^2(Y_t)\,dt}] &\leq E_R[e^{\frac{1}{2}\frac{\lambda}{(1-\lambda)^2}(a_3^2+\frac{\epsilon}{2})\int_0^T Y_t^2\,dt}]e^{\frac{1}{2}\frac{\lambda}{(1-\lambda)^2}C_1 T} \\
&\leq E_R[e^{\frac{1}{2}\frac{\lambda}{(1-\lambda)^2}(a_3^2+\frac{\epsilon}{2})\int_0^T Z_{1t}^2 + Z_{2t}^2\,dt}]e^{C_3 T} \\
&\leq \max_{i=1,2} E_R[e^{\frac{\lambda}{(1-\lambda)^2}(a_3^2+\frac{\epsilon}{2})\int_0^T Z_{it}^2\,dt}]e^{C_3 T} \\
&\leq \max_{i=1,2} E_R[e^{\frac{\lambda}{(1-\lambda)^2}(a_3^2+\epsilon)\int_0^T (Z_{it}-M_i/K)^2\,dt}]e^{C_4 T} \\
&= \max_{i=1,2} E_R[e^{\frac{\lambda}{(1-\lambda)^2}\|\rho\|^2(a_3^2+\epsilon)\int_0^T \widetilde{Z}_{it}^2\,dt}]e^{C_4 T} \tag{42}
\end{aligned}
$$

Here we use the processes $\widetilde{Z}_i$, $i = 1,2$, defined by $\widetilde{Z}_{it} := \|\rho\|^{-1}(Z_{it} - M_i/K)$. Note that $\widetilde{Z}_i$ is an OU process with rate of mean reversion K, equilibrium level 0, and volatility 1, since $B := \int_0(\|\rho\|)^{-1}\rho\,dW_t^{0,0}$ is a standard one-dimensional R-Brownian motion, due to P. Lévy's characterization. Applying Lemma 4.2 in [13] (here with $\lambda = 0$, $\mu = \frac{\lambda}{(1-\lambda)^2}\|\rho\|^2(a_3^2 + \epsilon)$ and $\theta_0 = -K$) for the asymptotics of the Laplace transform of the energy integral of a normalized OU process, we obtain

$$\lim_{T\uparrow\infty} \frac{1}{T}\ln E_R[e^{\frac{\lambda}{(1-\lambda)^2}\|\rho\|^2(a_3^2+\epsilon)\int_0^T \widetilde{Z}_{it}^2\,dt}] = \frac{1}{2}\left(K - \sqrt{K^2 - 2\frac{\lambda}{(1-\lambda)^2}\|\rho\|^2(a_3^2+\epsilon)}\right).$$

In view of (42) we have thus shown (40). This completes the proof. □

Combining the discussion of (21) with (36), it is natural to expect that the optimal growth rate $\Lambda(\lambda)$ in (20) satisfies

$$\Lambda(\lambda) = \varlimsup_{T\uparrow\infty} \frac{1}{T}\ln U_T(x_0) = \varlimsup_{T\uparrow\infty} \frac{1}{T}\ln(v(y,T)^{1-\lambda}).$$

As in Fleming and McEneaney [8] we now use a formal separation of time and space variables and formulate the heuristic Ansatz

$$(1-\lambda)\ln v(y,T) = \ln U_T(x_0) \approx \Lambda(\lambda)T + \varphi(y). \tag{43}$$

Here the function $\varphi : \mathbb{R} \to \mathbb{R}$ incorporates the influence of the initial state $Y_0 = y$. Inserting this Ansatz into the HJB equation (37), we obtain a steady-state dynamic programming equation for the pair $(\Lambda(\lambda), \varphi)$:

$$\Lambda(\lambda) = \tfrac{1}{2} \|\rho\|^2 [\varphi_{yy} + \tfrac{1}{1-\lambda} \varphi_y^2] + \inf_{\nu \in \mathbb{R}} \inf_{\eta \in \Gamma} \{(1-\lambda) l(\eta, \nu, \cdot) + \varphi_y h(\eta, \nu, \cdot)\} \qquad (44)$$

An equation of this type is called *ergodic Bellman equation* (EBE); see, e. g., [1], [25], [30], and the references therein. For fixed $\eta \in \Gamma$ the minimizer $\nu^*(\eta, y)$ among all $\nu \in \mathbb{R}$ can be computed explicitly as

$$\nu^*(\eta, y) = -\eta^{12} y - \eta^{22} - \rho_2 \varphi_y(y). \qquad (45)$$

Thus the EBE (44) can be rewritten in condensed form that involves only an infimum among the set Γ. Let us now assume that our EBE (44) admits a solution $\Lambda(\lambda) \in \mathbb{R}_+$, $\varphi \in C^2(\mathbb{R})$. In addition, assume that $\eta^*(y)$ is a minimizer in (44), and let $Q^{\eta^*} \in \mathcal{Q}$ be the probabilistic model corresponding to the feedback control $\eta_t^* = \eta^*(Y_t)$. We are now going to give a heuristic argument to identify a candidate for the optimal long run investment process π^*. To this end, we suppose that the measure Q^{η^*} is a worst-case model in the asymptotic sense that

$$\Lambda(\lambda) = \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln U_T(x_0) = \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln \sup_{\pi \in \mathcal{A}_T} E_{Q^{\eta^*}}[(X_T^\pi)^\lambda]. \qquad (46)$$

Later on we will show that this assumption is indeed justified. We are now going to introduce a change of measure which will allow us to interpret the finite time maximization problem at the right-hand side of (46) as an exponential of integral criterion. For this purpose, note that an optimal wealth process should stay positive, and this suggests to focus on those strategies $\pi \in \mathcal{A}$, where the unique strong solution to (16) takes the form

$$X_t^\pi = x_0 e^{\int_0^t \pi_u \sigma \, dW_u^{1,\eta^*} + \int_0^t r(Y_u) + \sigma \pi_u (\theta(Y_u) + \eta_u^{11,*} Y_u + \eta_u^{21,*}) - \frac{1}{2} \sigma^2 \pi_u^2 \, du}.$$

In this case the expectation at the right-hand side of (46) can be rewritten as

$$E_{Q^{\eta^*}}[(X_T^\pi)^\lambda] = x_0^\lambda E_{R^{\pi,\eta^*}} [e^{\int_0^T \widetilde{l}(\pi_t, \eta^*(Y_t), Y_t) \, dt}].$$

Here we use the notation

$$\widetilde{l}(\pi, \eta, y) := \tfrac{1}{2} \lambda (\lambda - 1) \sigma^2 \pi^2 + \lambda \sigma [\theta(y) + \eta^{11} y + \eta^{21}] \pi + \lambda r(y), \qquad (47)$$

and $R^{\pi,\eta}$ denotes the probability measure on $(\Omega, \mathcal{F}_T)$ defined by

$$\tfrac{dR^{\pi,\eta}}{dQ^\eta} \big|_{\mathcal{F}_T} := \mathcal{E}(\int_0^{\cdot} \lambda \pi_t \sigma \, dW_t^{1,\eta})_T. \qquad (48)$$

By Girsanov's theorem, the dynamics of $(Y_t)_{t \le T}$ under $R^{\pi,\eta}$ is described by

$$dY_t = \widetilde{h}(\pi_t, \eta_t, Y_t) \, dt + \rho \, dW_t^{\pi,\eta} \qquad (49)$$

in terms of the function $\widetilde{h}$ defined by

$$\widetilde{h}(\pi, \eta, y) := g(y) + (\rho, \eta^{1\cdot} y + \eta^{2\cdot}) + \lambda \rho_1 \sigma \pi \qquad (50)$$

and the one-dimensional Wiener process $W^{\pi,\eta}$. We have thus shown that the finite horizon maximization problem appearing in the right-hand side of (46) can be viewed as a finite horizon control problem with value function

$$\widetilde{v}(y, T) := \sup_{\pi \in \mathcal{A}} E_{Q^{\eta^*}}[(X_T^\pi)^\lambda] = x_0^\lambda \sup_{\pi \in \mathcal{A}} E_{R^{\pi,\eta^*}} [e^{\int_0^T \widetilde{l}(\pi_t, \eta^*(Y_t), Y_t) \, dt}]$$

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and with dynamics (49). In analogy to (37), we expect that $\widetilde{v}$ is the solution to the HJB equation

$$\widetilde{v}_t = \tfrac{1}{2}\|\rho\|^2 \widetilde{v}_{yy} + \sup_{\pi \in \mathbb{R}}\{\widetilde{l}(\pi, \eta^*, \cdot)\widetilde{v} + \widetilde{h}(\pi, \eta^*, \cdot)\widetilde{v}_y\}, \quad \widetilde{v}(\cdot, 0) \equiv 1. \tag{51}$$

Our Ansatz (43) combined with (46) for the worst-case measure Q^{η^*} now suggests the heuristic separation of variables $\ln \widetilde{v}(y, T) \approx \Lambda(\lambda)T + \varphi(y)$. Inserting this asymptotic identity into (51), we finally obtain an alternative version of the EBE:

$$\Lambda(\lambda) = \tfrac{1}{2}\|\rho\|^2[\varphi_{yy} + \varphi_y^2] + \sup_{\pi \in \mathbb{R}}\{\widetilde{l}(\pi, \eta^*, \cdot) + \varphi_y \widetilde{h}(\pi, \eta^*, \cdot)\}. \tag{52}$$

Note that the role played by the controls η and ν in (44) is now taken over by the "trading strategies" π. We expect that the maximizing function

$$\pi^*(y) = \tfrac{1}{1-\lambda}\tfrac{1}{\sigma}(\rho_1 \varphi_y(y) + \theta(y) + \eta^{11,*}(y)y + \eta^{12,*}(y)). \tag{53}$$

in (52) provides an optimal feedback control $\pi_t^* = \pi^*(Y_t)$, $t \geq 0$, for the asymptotic maximization of power utility with respect to the specific model Q^{η^*} and at the same time for the original robust problem (19).

4 Verification theorems

In this section we verify our heuristic results. For this purpose, we first return to the heuristic change of measure in (33) which is crucial to translate the dual problem (29) into a standard "exponential of integral criterion". From the technical point of view this requires the condition $E_{Q_0}[\mathcal{E}_T^{\eta,\nu}] = 1$ that can be violated if the supermartingale Z^ν is not a true Q_0-martingale. This fact will create some technical difficulties. To overcome this obstacle, we shall employ a localization argument.

Lemma 4.1. *Let $\eta \in \mathcal{C}$ and $\nu \in \mathcal{M}$ be arbitrary controls, and suppose that $(\tau_n)_{n \in \mathbb{N}}$ is a localizing sequence of stopping times for the local Q_0-martingale Z^ν. Then $V(\eta, \nu, y, T \wedge \tau_n) \nearrow V(\eta, \nu, y, T)$ as $n \uparrow \infty$, and the integrands in (30) even converge in $L^1(Q_0)$ if $V(\eta, \nu, y, T) < \infty$.*

Proof. The proof is given in [38], Lemma 3.2. The main idea consists in applying the concept of *extended martingale measures* introduced in [14]. $\qquad \square$

In a second step we are going to show that the value $\widetilde{\Lambda}(\lambda)$ given by a specific solution to the EBE (44) is actually the exponential growth rate of the maximal robust power utility $U_T(x_0)$. For this purpose, we need

Assumption 4.1. *Suppose that $\widetilde{\Lambda}(\lambda) \in \mathbb{R}_+$, $\varphi \in C^2(\mathbb{R})$ is a solution to*

$$\widetilde{\Lambda}(\lambda) = \tfrac{1}{2}\|\rho\|^2[\varphi_{yy} + \tfrac{1}{1-\lambda}\varphi_y^2] + \inf_{\nu \in \mathbb{R}} \inf_{\eta \in \Gamma}\{(1 - \lambda)l(\eta, \nu, \cdot) + \varphi_y h(\eta, \nu, \cdot)\} \tag{54}$$

which fulfills the following regularity conditions:

(a) *Either the first derivative φ_y is bounded or φ is bounded below and its derivative φ_y has at most linear growth, i. e.,*

$$|\varphi_y(y)| \leq C_1(1 + |y|) \quad \text{for some constant } C_1 > 0.$$

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(b) *There exist $C_2, C_3 > 0$ such that $y\kappa(\eta, y) \leq -C_2 y^2 + C_3$, where*

$$\kappa(\eta, y) := g(y) + \tfrac{\lambda}{1-\lambda}\rho_1(\theta(y) + \eta^{11}y + \eta^{21})$$
$$+ (\rho, \eta^{1\cdot}y + \eta^{2\cdot}) + [\tfrac{1}{1-\lambda}\rho_1^2 + \rho_2^2]\varphi_y(y). \quad (55)$$

In full generality, we are unfortunately not able to clarify whether the EBE (54) has such a solution $(\widetilde{\Lambda}(\lambda), \varphi)$. In Subsection 5 we are going to state sufficient (but rather restrictive) conditions under which the existence of a solution to our EBE (54) is already known. Moreover, Section 6 contains two case studies with linear drift coefficients, where the solution can be derived even explicitly. But as illustrated in Subsection 6.2 in case of the geometric OU model, there may exist multiple such pairs $(\widetilde{\Lambda}(\lambda), \varphi)$, even beyond the fact that φ is determined only except for an additive constant. However, the verification theorems will require a certain "uniform ergodicity condition" such as Assumption 4.1 (b) for the diffusion Y, and this condition selects the "good candidate" for the optimal growth rate $\Lambda(\lambda)$; cf. Remark 6.2.

Theorem 4.1. *If Assumption 4.1 is satisfied, then we get the identity*

$$\widetilde{\Lambda}(\lambda) = \lim_{T\uparrow\infty} \tfrac{1}{T}\ln(\inf_{\nu\in\mathcal{M}} \inf_{\eta\in\mathcal{C}} V(\eta, \nu, y_0, T)^{1-\lambda}) \quad \textit{for any } Y_0 = y_0. \quad (56)$$

Moreover, the infima at the right-hand side are attained for feedback controls

$$\eta_t^* := \eta^*(Y_t), \quad \nu_t^* := \nu^*(Y_t), \quad t \geq 0, \quad (57)$$

defined in terms of a measurable Γ-valued function η^* and the function

$$\nu^*(y) := \nu^*(\eta^*(y), y) = -\eta^{12,*}(y)y - \eta^{22,*}(y) - \rho_2\varphi_y(y)$$

such that the infima in (54) are realized. Thus,

$$\widetilde{\Lambda}(\lambda) = \lim_{T\uparrow\infty} \tfrac{1}{T}\ln(V(\eta^*, \nu^*, y_0, T)^{1-\lambda}). \quad (58)$$

In particular, the duality relations for robust utility maximization yield that

$$\widetilde{\Lambda}(\lambda) = \lim_{T\uparrow\infty} \tfrac{1}{T}\ln U_T(x_0) = \lim_{T\uparrow\infty} \tfrac{1}{T}\ln U_T^{Q^{\eta^*}}(x_0) \quad \textit{for any } X_0^\pi = x_0. \quad (59)$$

Remark 4.1. *In view of (59), Q^{η^*} can be seen as the asymptotic worst-case measure for robust expected power utility with parameter $\lambda \in (0,1)$. On the other hand, the probability measure P^{ν^*} on $(\Omega, \mathcal{F})$ with Radon-Nikodým density process $(Z_t^{\nu^*})_{t\geq 0}$ is a martingale measure which is equivalent to Q_0 on each σ-algebra $\mathcal{F}_t$, $t > 0$. In view of (58) and the duality relation (26) it can be interpreted as the asymptotic worst-case martingale measure.*

Proof. 1) In order to show that the constant $\widetilde{\Lambda}(\lambda)$ given by the specific solution $(\widetilde{\Lambda}(\lambda), \varphi)$ to the EBE (54) coincides with the exponential growth rate of the maximal robust power utility, we first prove that $\widetilde{\Lambda}(\lambda)$ provides a lower bound for the growth rate. To this end, we use the duality relation

$$U_T(x_0) = \tfrac{1}{\lambda}x_0^\lambda \inf_{\nu\in\mathcal{M}} \inf_{\eta\in\mathcal{C}} V(\eta, \nu, y, T)^{1-\lambda}$$

(cf. (29)) with V introduced in (30), derive suitable lower bounds for any fixed horizon T, and then pass to the limit.

Let $\eta \in \mathcal{C}$, $\nu \in \mathcal{M}$ be fixed controls, and let T be a given maturity. Then $\tau_n := \inf\{t \geq 0 | |Y_t| \geq n \text{ or } \int_0^t \nu_s^2 \, ds \geq n\} \wedge T$, $n \in \mathbb{N}$, is a localizing sequence for the local Q_0-martingale $(Z_t^\nu)_{t \leq T}$. This will allow us to apply the change of measure (33) locally and to use the localization Lemma 4.1 for $\tau_n \uparrow T$. In analogy to (31) we obtain

$$V(\eta, \nu, y_0, \tau_n) = E_{Q_0}[\mathcal{E}_{\tau_n}^{\eta,\nu} e^{\int_0^{\tau_n} l(\eta_t, \nu_t, Y_t) \, dt}], \quad n \in \mathbb{N},$$

where l is the auxiliary function defined in (32), and where

$$\mathcal{E}_{\tau_n}^{\eta,\nu} = \mathcal{E}\left(\tfrac{1}{1-\lambda}\left(\int_0^{\cdot} \lambda\theta(Y_u) + \eta_u^{11} Y_u + \eta_u^{21} \, dW_u^1 + \int_0^{\cdot} \lambda\nu_u + \eta_u^{12} Y_u + \eta_u^{22} \, dW_u^2\right)\right)_{\tau_n}.$$

To eliminate the Itô exponential $\mathcal{E}_{\tau_n}^{\eta,\nu}$, we pass to the new probability measure $R_n^{\eta,\nu}$ on $(\Omega, \mathcal{F}_T)$ with density process $dR_n^{\eta,\nu}/dQ_0|_{\mathcal{F}_t} := \mathcal{E}_{t \wedge \tau_n}^{\eta,\nu}$, $t \in [0, T]$. It remains to justify this change of measure. For this purpose, note that the process $\eta \in \mathcal{C}$ takes its values in a compact subset $\Gamma \subset \mathbb{R}^4$ and that $\theta^2(y) \leq (a_3|y| + a_4)^2 \leq 2(a_3^2 y^2 + a_4^2)$, due to Assumption 2.1. Using the definition of τ_n we can verify the Novikov condition; see, e.g., [29], Theorem 6.1 and the note after it. This allows us to write

$$V(\eta, \nu, y_0, \tau_n) = E_{R_n^{\eta,\nu}}[e^{\int_0^{\tau_n} l(\eta_t, \nu_t, Y_t) \, dt}], \quad n \in \mathbb{N}. \tag{60}$$

By Girsanov's theorem, the dynamics of Y follows under $R_n^{\eta,\nu}$ the SDE

$$dY_t = h(\eta_t, \nu_t, Y_t) \, dt + \rho \, dW_t^{\eta,\nu} \quad \text{on } \{t \leq \tau_n\} \tag{61}$$

for the drift function h given by (35) and for a two-dimensional $R_n^{\eta,\nu}$-Wiener process $W^{\eta,\nu}$. Note that (60) can be viewed as a cost functional of an "expected exponential of integral criterion" with dynamics (61); cf. p. 10.

Let us next introduce the auxiliary function $\gamma \geq 0$ by

$$\gamma(\eta, \nu, y) := (1 - \lambda)l(\eta, \nu, y) + \varphi_y(y)h(\eta, \nu, y)$$
$$- \inf_{\nu \in \mathbb{R}}\{(1 - \lambda)l(\eta, \nu, y) + \varphi_y(y)h(\eta, \nu, y)\}. \tag{62}$$

Inserting the minimizer $\nu^*(\eta, y)$ introduced in (45), we then see that γ takes the condensed form

$$\gamma(\eta, \nu, y) = \tfrac{1}{2}\tfrac{\lambda}{1-\lambda}(\nu - \nu^*(\eta, y))^2. \tag{63}$$

Later on this representation of γ will be crucial to eliminate the control ν in the dynamics of Y. In terms of γ our EBE (54) yields the inequality

$$\widetilde{\Lambda}(\lambda) \leq \tfrac{1}{2}\|\rho\|^2[\varphi_{yy} + \tfrac{1}{1-\lambda}\varphi_y^2] + (1 - \lambda)l(\eta, \nu, \cdot) + \varphi_y h(\eta, \nu, \cdot) - \gamma(\eta, \nu, \cdot). \tag{64}$$

By Itô's formula applied to $\varphi \in C^2(\mathbb{R})$ and to the dynamics of Y in (61), this estimate translates on $\{u \leq \tau_n\}$ into

$$\varphi(Y_u) - \varphi(y_0)$$
$$= \int_0^u \varphi_y(Y_t)h(\eta_t, \nu_t, Y_t) + \tfrac{1}{2}\|\rho\|^2 \varphi_{yy}(Y_t) \, dt + \int_0^u \varphi_y(Y_t)\rho \, dW_t^{\eta,\nu}$$
$$\geq \int_0^u \widetilde{\Lambda}(\lambda) - \tfrac{1}{2}\tfrac{1}{1-\lambda}\|\rho\|^2 \varphi_y^2(Y_t) - (1 - \lambda)l(\eta_t, \nu_t, Y_t) + \gamma(\eta_t, \nu_t, Y_t) \, dt$$
$$+ \int_0^u \varphi_y(Y_t)\rho \, dW_t^{\eta,\nu}. \tag{65}$$

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Dividing through $1 - \lambda$, rearranging the terms, and taking the exponential on both sides, we thus obtain from (60) that

$$V(\eta, \nu, y_0, \tau_n)$$

$$\geq E_{R_n^{\eta, \nu}}[e^{\frac{1}{1-\lambda}(\widetilde{\Lambda}(\lambda)\tau_n + \varphi(y_0) - \varphi(Y_{\tau_n}) + \int_0^{\tau_n} \gamma(\eta_t, \nu_t, Y_t) dt)} \mathcal{E}(\int_0^{\cdot} \frac{\varphi_y(Y_t)}{1-\lambda} \rho \, dW_t^{\eta, \nu})_{\tau_n}]$$

$$= E_{\overline{R}_n^{\eta, \nu}}[e^{\frac{1}{1-\lambda}(\widetilde{\Lambda}(\lambda)\tau_n + \varphi(y_0) - \varphi(Y_{\tau_n}) + \int_0^{\tau_n} \gamma(\eta_t, \nu_t, Y_t) dt)}]. \tag{66}$$

Here the last expectation is taken with respect to the probability measure $\overline{R}_n^{\eta, \nu}$ on $(\Omega, \mathcal{F}_T)$ with density process

$$\frac{d\overline{R}_n^{\eta, \nu}}{dR_n^{\eta, \nu}}\big|_{\mathcal{F}_t} := \mathcal{E}\left(\int_0^{\cdot} \frac{\rho\varphi_y(Y_u)}{1-\lambda} \, dW_u^{\eta, \nu}\right)_{t \wedge \tau_n}.$$

Indeed, since φ_y grows at most linearly according to Assumption 4.1 (a) and since ρ_1, ρ_2 are bounded, this change of measure can be justified again by Novikov's condition; cf., e.g., [29], Theorem 6.1 and the note after it. By Girsanov's theorem, the factor process Y evolves under $\overline{R}_n^{\eta, \nu}$ according to

$$dY_t = [h(\eta_t, \nu_t, Y_t) + \tfrac{1}{1-\lambda}\|\rho\|^2 \varphi_y(Y_t)] \, dt + \rho \, d\overline{W}_t^{\eta, \nu} \quad \text{on } \{t \leq \tau_n\},$$

where $\overline{W}^{\eta, \nu}$ denotes a two-dimensional $\overline{R}_n^{\eta, \nu}$-Wiener process. But this dynamics still depends on the irrepressible control ν. To eliminate this dependence, we apply once more a Girsanov transformation. Consider the probability measure R_n^{η} on $(\Omega, \mathcal{F}_T)$ with density process

$$\frac{d\widehat{R}_n^{\eta}}{d\overline{R}_n^{\eta, \nu}}\big|_{\mathcal{F}_t} := \mathcal{E}\left(\int_0^{\cdot} \frac{\lambda}{1-\lambda}(\nu^*(\eta_s, Y_s) - \nu_s) \, d\overline{W}_t^{2, \eta, \nu}\right)_{t \wedge \tau_n}.$$

Verifying once more the Novikov's condition, we see that R_n^{η} is well-defined, and the inequality (66) translates into

$$V(\eta, \nu, y_0, \tau_n) \geq E_{\widehat{R}_n^{\eta}}[e^{\frac{1}{1-\lambda}(\widetilde{\Lambda}(\lambda)\tau_n + \varphi(y_0) - \varphi(Y_{\tau_n}) + \int_0^{\tau_n} \gamma(\eta_t, \nu_t, Y_t) dt)} \frac{d\overline{R}_n^{\eta, \nu}}{d\widehat{R}_n^{\eta}}\big|_{\mathcal{F}_{\tau_n}}]. \tag{67}$$

Moreover, Girsanov's theorem yields that the dynamics of Y under $\widehat{R}_n^{\eta}$ on $\{t \leq \tau_n\}$ takes the form

$$dY_t = [h(\eta_t, \nu_t, Y_t) + \tfrac{1}{1-\lambda}\|\rho\|^2 \varphi_y(Y_t) + \tfrac{\lambda}{1-\lambda}\rho_2(\nu^*(\eta_t, Y_t) - \nu_t)] dt + \rho \, d\widehat{W}_t^{\eta}$$

in terms of the two-dimensional $\widehat{R}_n^{\eta}$-Wiener process $\widehat{W}^{\eta}$. Recalling from (35) and (45) the definitions of the drift function h and of the minimizer $\nu^*(\eta, y)$, a straightforward computation shows that this SDE is equivalent to

$$dY_t = \kappa(\eta_t, Y_t) \, dt + \rho \, d\widehat{W}_t^{\eta}, \tag{68}$$

where κ denotes the auxiliary function introduced in Assumption 4.1 (b). To eliminate the density $d\overline{R}_n^{\eta, \nu}/d\widehat{R}_n^{\eta}\big|_{\mathcal{F}_{\tau_n}}$, we define $p := \frac{\lambda - 1}{\lambda} < 0$ and apply Hölder's inequality with $1/p + 1/q = 1$ to (67) (see, e.g., [24], p. 191, for an extension of the classical result to $p < 0$, $q \in (0, 1)$). This leads to

$$V(\eta, \nu, y_0, \tau_n) \geq E_{\widehat{R}_n^{\eta}}[e^{\frac{q}{1-\lambda}(\widetilde{\Lambda}(\lambda)\tau_n + \varphi(y_0) - \varphi(Y_{\tau_n}))}]^{1/q}$$

$$\times E_{\widehat{R}_n^{\eta}}\left[\left(\frac{d\overline{R}_n^{(\eta, \nu)}}{d\widehat{R}_n^{(\eta)}}\big|_{\mathcal{F}_{\tau_n}} e^{\frac{1}{1-\lambda}\int_0^{\tau_n} \gamma(\eta_t, \nu_t, Y_t) \, dt}\right)^p\right]^{1/p}. \tag{69}$$

But in view of (63) and our choice of p we see that

$$\left(\frac{d\overline{R}_n^{\eta,\nu}}{d\widehat{R}_n^{\eta}}\Big|_{\mathcal{F}_{\tau_n}} e^{\frac{1}{1-\lambda}\int_0^{\tau_n} \gamma(\eta_t,\nu_t,Y_t)\,dt} \right)^p = \mathcal{E}\left(\int_0^{\cdot} \frac{p\lambda}{1-\lambda}(\nu^*(\eta_t,Y_t) - \nu_t)\,d\widehat{W}_t^{2,\eta} \right)_{\tau_n}.$$

Since the Itô exponential of a local martingale is always a supermartingale, it follows that the expectation in (69) is less than 1. Raised to the power of $1/p < 0$, this estimate is reversed, and we obtain

$$V(\eta,\nu,y_0,\tau_n) \geq E_{\widehat{R}_n^{\eta}}[e^{\frac{q}{1-\lambda}(\widetilde{\Lambda}(\lambda)\tau_n + \varphi(y_0) - \varphi(Y_{\tau_n}))}]^{1/q}, \quad n \in \mathbb{N}. \tag{70}$$

In our next step, we shall extend the measures $\widehat{R}_n^{\eta}|_{\mathcal{F}_{\tau_n}}$, $n \in \mathbb{N}$, to a probability measure $\widehat{R}^{\eta}$ on the σ-field $\mathcal{F}_T$ whose restrictions to $\mathcal{F}_{\tau_n}$ are equal to $\widehat{R}_n^{\eta}|_{\mathcal{F}_{\tau_n}}$ for all $n \in \mathbb{N}$. To this end, note that the sequence τ_n increases to T and that the family $(\widehat{R}_n^{\eta}|_{\mathcal{F}_{\tau_n}})_{n \in \mathbb{N}}$ is consistent in the sense that $\widehat{R}_{n+1}^{\eta}(A) = \widehat{R}_n^{\eta}(A)$ for all $A \in \mathcal{F}_{\tau_n}$ since

$$\frac{d\widehat{R}_n^{\eta}}{dQ_0}\Big|_{\mathcal{F}_{\tau_n}} = \frac{d\widehat{R}_n^{\eta}}{d\overline{R}_n^{\eta,\nu}}\Big|_{\mathcal{F}_{\tau_n}} \frac{d\overline{R}_n^{\eta,\nu}}{dR_n^{\eta,\nu}}\Big|_{\mathcal{F}_{\tau_n}} \frac{dR_n^{\eta,\nu}}{dQ_0}\Big|_{\mathcal{F}_{\tau_n}}$$

$$= \mathcal{E}\left(\int_0^{\cdot} \frac{1}{1-\lambda}(\lambda\theta(Y_u) + \eta_u^{11}Y_u + \eta_u^{21} + \varphi_y(Y_u)\rho_1)dW_u^1 \right.$$

$$\left. + \int_0^{\cdot} \eta_u^{12}Y_u + \eta_u^{22} + \varphi_y(Y_u)\rho_2\,dW_u^2 \right)_{\tau_n}, \quad n \in \mathbb{N}, \tag{71}$$

is a discrete-time Q_0-martingale. Thus the existence of a unique extension $\widehat{R}^{\eta}$ to $\sigma(\cup_{n \in \mathbb{N}} \mathcal{F}_{\tau_n}) = \mathcal{F}_T$ follows from [32], Theorem V.4.2. More directly, (71) suggests to define the probability measure $\widehat{R}^{\eta}$ on $(\Omega, \mathcal{F}_T)$ by

$$\frac{d\widehat{R}^{\eta}}{dQ_0}\Big|_{\mathcal{F}_T} := \mathcal{E}\left(\int_0^{\cdot} \frac{1}{1-\lambda}(\lambda\theta(Y_u) + \eta_u^{11}Y_u + \eta_u^{21} + \varphi_y(Y_u)\rho_1)dW_u^1 \right.$$

$$\left. + \int_0^{\cdot} \eta_u^{12}Y_u + \eta_u^{22} + \varphi_y(Y_u)\rho_2\,dW_u^2 \right)_T. \tag{72}$$

Since the functions θ, φ grow at most linearly, it follows similar to page 5 that $\widehat{R}^{\eta}$ is well-defined, i.e., $E_{Q_0}[d\widehat{R}^{\eta}/dQ_0|_{\mathcal{F}_T}] = 1$. In particular, the corresponding Itô exponential is a Q_0-martingale up to time T, and in view of (71) this yields $\widehat{R}^{\eta}|_{\mathcal{F}_{\tau_n}} = \widehat{R}_n^{\eta}|_{\mathcal{F}_{\tau_n}}$ for all $n \in \mathbb{N}$. We thus see that the estimate (70) is equivalent to

$$V(\eta,\nu,y_0,\tau_n) \geq E_{\widehat{R}^{\eta}}[e^{\frac{q}{1-\lambda}(\widetilde{\Lambda}(\lambda)\tau_n + \varphi(y_0) - \varphi(Y_{\tau_n}))}]^{1/q} \quad \text{for any } n \in \mathbb{N}.$$

Now we are ready to replace the stopping times τ_n by the deterministic time T by passing to the limit $n \uparrow \infty$. Indeed, as shown in Lemma 4.1, the left-hand side increases to $V(\eta,\nu,y_0,T)$ as $n \uparrow \infty$; cf. Lemma 4.1. Applying Fatou's lemma and afterwards Jensen's inequality to the rightmost expectation, we now obtain the lower bound

$$V(\eta,\nu,y_0,T) \geq E_{\widehat{R}^{\eta}}[e^{\frac{1}{1-\lambda}(\widetilde{\Lambda}(\lambda)T + \varphi(y_0) - \varphi(Y_T))}] \geq e^{\frac{1}{1-\lambda}(\widetilde{\Lambda}(\lambda)T + \varphi(y_0) + E_{\widehat{R}^{\eta}}[-\varphi(Y_T)])}$$

for any finite horizon T and for all controls $\eta \in \mathcal{C}$, $\nu \in \mathcal{M}$. Taking the scaling $\frac{1}{T}\ln(\cdot)^{1-\lambda}$ on both sides and passing to the limit $T \uparrow \infty$, this yields

$$\lim_{T \uparrow \infty} \frac{1}{T}\ln\left(\inf_{\nu \in \mathcal{M}} \inf_{\eta \in \mathcal{C}} V(\eta,\nu,y_0,T)^{1-\lambda} \right) \geq \widetilde{\Lambda}(\lambda) + \lim_{T \uparrow \infty} \frac{1}{T} \inf_{\eta \in \mathcal{C}} E_{\widehat{R}^{\eta}}[-\varphi(Y_T)].$$

Thus, the constant $\widetilde{\Lambda}(\lambda)$ provides a lower bound if

$$\lim_{T \uparrow \infty} \tfrac{1}{T} \inf_{\eta \in \mathcal{C}} E_{\widehat{R}^\eta}[-\varphi(Y_T)] = 0. \tag{73}$$

Indeed, Assumption 4.1 (a) ensures that φ grows at most quadratically, i.e., there exists some constant $K_1 > 0$ with $|\varphi(y)| \leq K_1(1 + y^2)$. Therefore, we have the bounds

$$-K_1(1 + \sup_{\eta \in \mathcal{C}} E_{\widehat{R}^\eta}[Y_T^2]) \leq \inf_{\eta \in \mathcal{C}} E_{\widehat{R}^\eta}[-\varphi(Y_T)] \leq K_1(1 + \sup_{\eta \in \mathcal{C}} E_{\widehat{R}^\eta}[Y_T^2]). \tag{74}$$

Recall now from (68) that Y evolves under $\widehat{R}^\eta$, $\eta \in \mathcal{C}$, according to the SDE

$$dY_t = \kappa(\eta_t, Y_t) \, dt + \rho(Y_t) \, d\widehat{W}_t^\eta.$$

Due to Assumption 4.1 (b) there exist constants $C_2, C_3 > 0$ such that the drift function κ satisfies $y\kappa(\eta, y) \leq -C_2 y^2 + C_3$ for all $\eta \in \Gamma$. Therefore, Lemma 8.2 ensures that

$$\sup_{T \geq 0} \sup_{\eta \in \mathcal{C}} E_{\widehat{R}^\eta}[Y_T^2] \leq y_0^2 + \text{const.} < \infty.$$

But in view of (74) this implies (73), and hence

$$\varliminf_{T \uparrow \infty} \tfrac{1}{T} \ln \left(\inf_{\nu \in \mathcal{M}} \inf_{\eta \in \mathcal{C}} V(\eta, \nu, y_0, T)^{1-\lambda} \right) \geq \widetilde{\Lambda}(\lambda). \tag{75}$$

2) In the second part we identify controls $\eta^* \in \mathcal{C}$ and $\nu^* \in \mathcal{M}$ such that

$$\widetilde{\Lambda}(\lambda) = \lim_{T \uparrow \infty} \tfrac{1}{T} \ln(V(\eta^*, \nu^*, y_0, T)^{1-\lambda}). \tag{76}$$

Together with (75) this implies (56). Indeed, by compactness of Γ and continuity of the functions l, h and $\nu^*(\cdot, y)$ with respect to η, there exists

$$\eta^*(y) \in \arg\min_{\eta \in \Gamma}\{(1 - \lambda)l(\eta, \nu^*(y, \eta), y) + \varphi_y(y)h(\eta, \nu^*(y, \eta), y)\}. \tag{77}$$

By a measurable selection argument $\eta^*(\cdot)$ can be chosen as a measurable function. Set $\nu^*(y) := \nu^*(\eta^*(y), y)$ (cf. (45)), and let η^*, ν^* be the feedback controls defined by $\eta_t^* := \eta^*(Y_t)$, $\nu_t^* := \nu^*(Y_t)$, $t \geq 0$. In that case, we have $\eta^* \in \mathcal{C}$, and one easily proves that the process ν^* belongs to the class $\mathcal{M}$.

In order to verify (76), we now proceed as in part 1). As in (60) we obtain

$$V(\eta^*, \nu^*, y_0, T) = E_{R^{\eta^*, \nu^*}}[e^{\int_0^T l(\eta_t^*, \nu_t^*, Y_t) \, dt}].$$

The measure R^{η^*, ν^*} is defined on $(\Omega, \mathcal{F}_T)$ in terms of the density $\mathcal{E}_T^{\eta^*, \nu^*}$. Since $\nu^*(\eta, \cdot)$ grows at most linearly, this change of measure can be justified in analogy to page 5. By Girsanov's theorem, the dynamics of Y under R^{η^*, ν^*} follows the SDE

$$dY_t = h(\eta_t^*, \nu_t^*, Y_t) \, dt + \rho \, dW_t^{\eta^*, \nu^*}, \tag{78}$$

where the drift function h is given by (35), and where $(W_t^{\eta^*, \nu^*})_{t \leq T}$ is a two-dimensional Wiener process under R^{η^*, ν^*}; cf. p. 15. Using the specific controls η^*, ν^*, the auxiliary function γ in (62) satisfies $\gamma(\eta_t^*, \nu_t^*, Y_t) = 0$, and we also obtain equality in (64). Along the lines of part 1) this implies

$$V(\eta^*, \nu^*, y_0, T) = E_{R^{\eta^*, \nu^*}}[e^{\frac{1}{1-\lambda}(\widetilde{\Lambda}(\lambda)T + \varphi(y_0) - \varphi(Y_T))} \mathcal{E}(\int_0^\cdot \tfrac{\varphi_y(Y_t)}{1-\lambda} \rho \, dW_t^{\eta^*, \nu^*})_T],$$

in analogy to (66). Once more the Itô exponential is interpreted as the density of a new probability measure $\widehat{R}^{\eta^*}$ on $(\Omega, \mathcal{F}_T)$. Since the drift function $h(\eta^*(\cdot), \nu^*(\cdot), \cdot)$ of Y under R^{η^*,ν^*} only depends on the control η^* and satisfies the linear growth condition $|h(\eta^*(y), \nu^*(y), y)| \leq K_2(1 + |y|)$, we may proceed in analogy to page 5 to justify this change of measure. Then we get

$$V(\eta^*, \nu^*, y_0, T) = e^{\frac{1}{1-\lambda}(\widetilde{\Lambda}(\lambda)T + \varphi(y_0))} E_{\widehat{R}^{\eta^*}}[e^{-\frac{1}{1-\lambda}\varphi(Y_T)}]. \tag{79}$$

Moreover, by Girsanov's theorem, the dynamics of Y with respect to $\widehat{R}^{\eta^*}$ is given by

$$dY_t = \kappa(\eta_t^*, Y_t)\, dt + \rho\, d\widehat{W}_t^{\eta^*}, \tag{80}$$

where $(\widehat{W}_t^{\eta^*})_{t \leq T}$ is a two-dimensional Wiener process, and where the drift function κ satisfies Assumption 4.1 (b). In analogy to part 1), we now take the scaling $\frac{1}{T}\ln(\cdot)^{1-\lambda}$ on both sides of (79) and then pass to the limit $T \uparrow \infty$. For this purpose, note that

$$\sup_{T \geq 0} E_{\widehat{R}^{\eta^*}}[Y_T^2] < \infty \text{ and that } \sup_{T \geq 0} E_{\widehat{R}^{\eta^*}}[\exp(k|Y_T|)] < \infty \quad \text{for any } k \in \mathbb{R},$$

due to Assumption 4.1 (b) and Lemma 8.2 applied to the SDE (80). If φ_y is bounded and consequently $|\varphi(y)| \leq K_3(1 + |y|)$, then this implies the uniform upper bound

$$\sup_{T \geq 0} E_{\widehat{R}^{\eta^*}}[\exp(-\tfrac{1}{1-\lambda}\varphi(Y_T))] \leq \sup_{T \geq 0} E_{\widehat{R}^{\eta^*}}[\exp(\tfrac{1}{1-\lambda}K_3(1 + |Y_T|))] < \infty.$$

This uniform boundedness among all T clearly also holds, if φ is bounded below. In particular, the identity (79) translates into

$$\lim_{T \uparrow \infty} \tfrac{1}{T}\ln(V(\eta^*, \nu^*, y_0, T)^{1-\lambda}) = \widetilde{\Lambda}(\lambda).$$

Thus we have shown (76). This ends the proof of (56).

3) In our last step we return to the initial problem of robust utility maximization. The finite horizon duality relation (26) holds for any (regular) convex class of measures, and in particular for the one-point set $\{Q^{\eta^*}\}$. In analogy to (29) it thus follows that the maximal value for expected power utility in the specific model Q^{η^*} satisfies the duality formula

$$U_T^{Q^{\eta^*}}(x_0) = \tfrac{1}{\lambda}x_0^\lambda(\inf_{\nu \in \mathcal{M}} V(\eta^*, \nu, y_0, T))^{1-\lambda}.$$

Using this representation and the duality relation (29) for the whole set $\mathcal{Q}$, we obtain (59) immediately from (56) and (58). $\qquad\square$

Theorem 4.1 shows that the solution $(\widetilde{\Lambda}(\lambda), \varphi)$ to the EBE (54) specified in Assumption 4.1 describes the exponential growth of the maximal robust power utility $U_T(x_0)$ as $T \uparrow \infty$. We have also seen that the maximal utility in the specific model Q^{η^*} grows at the same rate as $U_T(x_0)$. In the next step we shall use these facts in order to identify an optimal long term investment strategy $\pi^* \in \mathcal{A}$. For this purpose, we introduce the additional regularity

Assumption 4.2. *Let $(\widetilde{\Lambda}(\lambda), \varphi)$ be the solution to the EBE (54) introduced in Assumption 4.1, and let η^* denote the corresponding minimizing function. Then the function $\widetilde{\kappa}$ defined by*

$$\widetilde{\kappa}(\eta, y) := g(y) + \tfrac{\lambda}{1-\lambda}\rho_1(\theta(y) + \eta^{11,*}(y)y + \eta^{21,*}(y))$$
$$+ (\rho, \eta^{1\cdot}y + \eta^{2\cdot}) + [\tfrac{1}{1-\lambda}\rho_1^2 + \rho_2^2]\varphi_y(y). \tag{81}$$

satisfies $y\widetilde{\kappa}(\eta, y) \leq -C_4 y^2 + C_5$ for all $\eta \in \Gamma$ with constants $C_4, C_5 > 0$.

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Theorem 4.2. *Under the regularity Assumptions 4.1 and 4.2 we have:*

i) *The value $\widetilde{\Lambda}(\lambda)$ given by the solution to the EBE (54) can be identified as the optimal exponential growth rate*

$$\Lambda(\lambda) = \sup_{\pi \in \mathcal{A}} \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^\pi)^\lambda]$$

for robust expected power utility. In particular, (59) implies

$$\Lambda(\lambda) = \lim_{T \uparrow \infty} \tfrac{1}{T} \ln U_T(x_0) = \lim_{T \uparrow \infty} \tfrac{1}{T} \ln U_T^{Q^{\eta^*}}(x_0),$$

where $Q^{\eta^*} \in \mathcal{Q}$ is defined in terms of the control η^* in (57).

ii) *In the specific model Q^{η^*}, the maximal growth rate of power utility*

$$\Lambda_{Q^{\eta^*}}(\lambda) := \sup_{\pi \in \mathcal{A}} \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln E_{Q^{\eta^*}}[(X_T^\pi)^\lambda]$$

coincides with $\Lambda(\lambda)$.

iii) *Let $\pi_t^* = \pi^*(Y_t)$, $t \geq 0$, be the trading strategy defined in terms of the function (53). Then π^* belongs to class $\mathcal{A}$, and it satisfies the optimality condition*

$$\Lambda(\lambda) = \lim_{T \uparrow \infty} \tfrac{1}{T} \ln \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^{\pi^*})^\lambda] = \lim_{T \uparrow \infty} \tfrac{1}{T} \ln E_{Q^{\eta^*}}[(X_T^{\pi^*})^\lambda]. \qquad (82)$$

In other words, the strategy π^* and the measure $Q^{\eta^*} \in \mathcal{Q}$ form a saddle point for the robust optimization problem (19).

Proof. 1) Theorem 4.1 shows that the maximal power utility $U_T^{Q^{\eta^*}}(x_0)$ in the specific model Q^{η^*} grows exponentially with rate $\widetilde{\Lambda}(\lambda)$, i.e.,

$$\widetilde{\Lambda}(\lambda) = \lim_{T \uparrow \infty} \tfrac{1}{T} \ln U_T^{Q^{\eta^*}}(x_0) = \lim_{T \uparrow \infty} \tfrac{1}{T} \ln \sup_{\pi \in \mathcal{A}_T} E_{Q^{\eta^*}}[(X_T^\pi)^\lambda].$$

Since $\mathcal{A} \subseteq \mathcal{A}_T$, this implies

$$\widetilde{\Lambda}(\lambda) \geq \sup_{\pi \in \mathcal{A}} \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln E_{Q^{\eta^*}}[(X_T^\pi)^\lambda] \geq \sup_{\pi \in \mathcal{A}} \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^\pi)^\lambda] = \Lambda(\lambda).$$

In order to verify that this chain of inequalities is indeed a series of equalities, it suffices to show that π^* belongs to $\mathcal{A}$, and that

$$\widetilde{\Lambda}(\lambda) \leq \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^{\pi^*})^\lambda]. \qquad (83)$$

This yields the converse inequality $\widetilde{\Lambda}(\lambda) \leq \Lambda(\lambda)$, and hence the identity $\widetilde{\Lambda}(\lambda) = \Lambda(\lambda) = \Lambda_{Q^{\eta^*}}(\lambda)$. In particular, the strategy π^* satisfies (82).

Let us first show that π^* is admissible in the sense of Definition 2.1. For this purpose, note that the adapted process $\pi_t^* = \pi^*(Y_t)$, $t \geq 0$, admits continuous paths and that the unique strong solution to (16) takes the form

$$X_t^{\pi^*} = x_0 e^{\int_0^t \pi_u^* \sigma \, dW_u^{1,\eta} + \int_0^T r(Y_u) + \sigma \pi_u^*[\theta(Y_u) + \eta_u^{11} Y_u + \eta_u^{21}] - \frac{1}{2}\sigma^2 (\pi_u^*)^2 \, du} > 0 \qquad (84)$$

for any $t \geq 0$. Thus the processes defined by the number of shares

$$\xi_t^{*,0} = \tfrac{X_t^{\pi^*}(1-\pi_t^*)}{S_t^0} \quad \text{and} \quad \xi_t^{*,1} = \tfrac{X_t^{\pi^*}\pi_t^*}{S_t^1}, \quad t \geq 0,$$

are continuous and adapted to the Brownian filtration, hence predictable. Moreover, the integrals in (15) are well-defined for $\xi^* = (\xi^{*,0}, \xi^{*,1})$. In other words, π^* associated with ξ^* is an admissible long term investment process.

To verify (83), we derive suitable lower bounds for $\inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^{\pi^*})^\lambda]$ for any finite horizon T and then pass to the limit. We first argue for a fixed control $\eta \in \mathcal{C}$ and the corresponding model $Q^\eta \in \mathcal{Q}$. The representation (84) yields the decomposition

$$E_{Q^\eta}[(X_T^{\pi^*})^\lambda] = x_0^\lambda E_{Q^\eta}[\mathcal{E}(\int_0^\cdot \lambda \sigma \pi_t^* \, dW_t^{1,\eta})_T e^{\int_0^T \widetilde{l}(\pi_t^*, \eta_t, Y_t) \, dt}], \tag{85}$$

where we use, as in (47), the function $\widetilde{l}$. In order to eliminate the Itô exponential, we introduce a new probability measure $\overline{Q}^\eta$ on $(\Omega, \mathcal{F}_T)$ with density

$$\frac{d\overline{Q}^\eta}{dQ^\eta}\big|_{\mathcal{F}_T} := \mathcal{E}(\int_0^\cdot \lambda \sigma \pi_t^* \, dW_t^{1,\eta})_T = \mathcal{E}(\int_0^\cdot \lambda \sigma \pi^*(Y_t) \, dW_t^{1,\eta})_T. \tag{86}$$

This requires to verify $E_{Q^\eta}[d\overline{Q}^\eta/dQ^\eta|_{\mathcal{F}_T}] = 1$. Indeed, the factor process Y evolves under Q^η according to the SDE (14a), and the drift function satisfies

$$|g(y) + (\rho, \eta^{1\cdot}y + \eta^{2\cdot})|^2 \le K_1(1 + y^2),$$

due to Assumption 2.1 and compactness of $\Gamma \subset \mathbb{R}^4$. Thus, by Lemma 8.1, there exists some constant $K_2 > 0$ such that $\sup_{0 \le t \le T} E_{Q^\eta}[\exp(K_2 Y_t^2)] < \infty$. Since $|\pi^*(y)| \le K_3(1 + |y|)$, this implies $\sup_{t \le T} E_{Q^\eta}[\exp(\delta(\lambda \sigma \pi^*(Y_t))^2] < \infty$ as soon as $\delta > 0$ is chosen sufficiently small. Therefore, [29], Example 3 of Subsection 6.2, guarantees that (86) defines a probability measure on $(\Omega, \mathcal{F}_T)$. In particular, equation (85) becomes equivalent to

$$E_{Q^\eta}[(X_T^{\pi^*})^\lambda] = x_0^\lambda E_{\overline{Q}^\eta}[e^{\int_0^T \widetilde{l}(\pi_t^*, \eta_t, Y_t) \, dt}]. \tag{87}$$

By Girsanov's theorem, the factor process Y follows under $\overline{Q}^\eta$ the SDE

$$dY_t = \widetilde{h}(\pi_t^*, \eta_t, Y_t) \, dt + \rho \, d\overline{W}_t^\eta, \quad t \le T, Y_0 = y_0. \tag{88}$$

Here $(\overline{W}_t^\eta)_{t \le T}$ is a two-dimensional $\overline{Q}^\eta$-Wiener process and the drift function $\widetilde{h}$ is defined by (50). Note that the right-hand side of (87) can be viewed as a cost functional of an "exponential of integral criterion" with dynamics (88).

In terms of the functions $\widetilde{l}$ and $\widetilde{h}$ the EBE (54) for the pair $(\widetilde{\Lambda}(\lambda), \varphi)$ can be rewritten as

$$\widetilde{\Lambda}(\lambda) = \tfrac{1}{2}\|\rho\|^2[\varphi_{yy} + \varphi_y^2] + \inf_{\eta \in \Gamma}\{\widetilde{l}(\pi^*, \eta, \cdot) + \varphi_y \widetilde{h}(\pi^*, \eta, \cdot)\}. \tag{89}$$

For clarity of exposition the precise arguments are postponed to part 2) of this proof. We now proceed in analogy to the proof of Theorem 4.1. Note that the roles played by l, h are taken over by $\widetilde{l}, \widetilde{h}$.

Applying Itô's formula to $\varphi \in C^2(\mathbb{R})$ and to the dynamics (88) we obtain

$$\varphi(Y_T) = \varphi(y_0) + \int_0^T \varphi_y(Y_t)\widetilde{h}(\pi_t^*, \eta_t, Y_t) + \tfrac{1}{2}\|\rho\|^2 \varphi_{yy}(Y_t)dt + \int_0^T \varphi_y(Y_t)\rho \, d\overline{W}_t^\eta.$$

The alternative version (89) of our EBE thus yields the inequality

$$\varphi(Y_T) \ge \varphi(y_0) + \int_0^T \widetilde{\Lambda}(\lambda) - \widetilde{l}(\pi_t^*, \eta_t, Y_t) \, dt + \ln \mathcal{E}(\int_0^\cdot \varphi_y(Y_t)\rho \, d\overline{W}_t^\eta)_T.$$

Rearranging the terms and taking the exponential on both sides, (87) allows us to deduce that

$$E_{Q^\eta}[(X_T^{\pi^*})^\lambda] = x_0^\lambda E_{\overline{Q}^\eta}[e^{\int_0^T \widetilde{l}(\pi_t^*, \eta_t, Y_t)\, dt}]$$

$$\geq x_0^\lambda e^{\widetilde{\Lambda}(\lambda)T + \varphi(y_0)} E_{\overline{Q}^\eta}[e^{-\varphi(Y_T)} \, \mathcal{E}(\int_0^\cdot \varphi_y(Y_t)\rho \, d\overline{W}_t^\eta)_T].$$

Applying once more a Girsanov transformation to eliminate the Itô exponential, we obtain

$$E_{Q^\eta}[(X_T^{\pi^*})^\lambda] \geq x_0^\lambda e^{\widetilde{\Lambda}(\lambda)T + \varphi(y_0)} E_{\widehat{Q}^\eta}[e^{-\varphi(Y_T)}], \tag{90}$$

where the expectation is taken with respect to the probability measure $\widehat{Q}^\eta$ on $(\Omega, \mathcal{F}_T)$ defined by

$$\frac{d\widehat{Q}^\eta}{d\overline{Q}^\eta}\big|_{\mathcal{F}_T} := \mathcal{E}(\int_0^\cdot \varphi_y(Y_t)\rho \, d\overline{W}_t^\eta)_T.$$

In particular, (90) means that

$$\varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^{\pi^*})^\lambda] \geq \widetilde{\Lambda}(\lambda) + \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln \inf_{Q^\eta \in \mathcal{Q}} E_{\widehat{Q}^\eta}[e^{-\varphi(Y_T)}]. \tag{91}$$

Since $|\widetilde{h}(\pi^*(y), \eta, y)|^2 \leq K_4(1 + y^2)$, this second change of measure can be justified again by Lemma 8.1 combined with [29], Example 3 of Subsection 6.2. By Girsanov's theorem, the dynamics of Y under the new probability measure $\widehat{Q}^\eta$ is given by

$$dY_t = (\widetilde{h}(\pi_t^*, \eta_t, Y_t) + \|\rho\|^2 \varphi_y(Y_t))\, dt + \rho \, d\widehat{W}_t^\eta, \quad t \leq T,$$

where $\widehat{W}_t^\eta$ is a two-dimensional $\widehat{Q}^\eta$-Wiener process. Moreover, inserting the definition (53) of $\pi^*(y)$, a straightforward computation yields the identity

$$\widetilde{h}(\pi^*(y), \eta, y) + \|\rho\|^2 \varphi_y(y) = \widetilde{\kappa}(\eta, y).$$

Here the function $\widetilde{\kappa}$ introduced in Assumption 4.2 satisfies the inequality $y\widetilde{\kappa}(\eta, y) \leq -C_4 y^2 + C_5$ for all $\eta \in \Gamma$ with appropriate constants $C_4, C_5 > 0$. Thus, by Lemma 8.2, the quadratic moments $E_{\widehat{Q}^\eta}[Y_T^2]$ are bounded above uniformly with respect to all processes $\eta \in \mathcal{C}$ and $T \geq 0$, i.e.,

$$\sup_{T\geq 0} \sup_{\eta\in\mathcal{C}} E_{\widehat{Q}^\eta}[Y_T^2] \leq K_5(1 + y_0^2).$$

Note now that $|\varphi(y)| \leq K_6(1 + y^2)$ for some constant $K_6 > 0$, since the first derivative φ_y grows at most linearly; cf. Assumption 4.1 (a). Using Jensen's inequality, we obtain the lower bound

$$\ln \inf_{Q^\eta \in \mathcal{Q}} E_{\widehat{Q}^\eta}[e^{-\varphi(Y_T)}] \geq \inf_{\eta\in\mathcal{C}} E_{\widehat{Q}^\eta}[-\varphi(Y_T)] \geq -K_6(1 + \sup_{\eta\in\mathcal{C}} E_{\widehat{Q}^\eta}[Y_T^2])$$

$$\geq -K_6(1 + K_5(1 + y_0^2))$$

for any finite horizon T. Thus the last term in (91) non-negative, and so the desired estimate (83) follows from (91).

2) It remains to verify that the solution $(\widetilde{\Lambda}(\lambda), \varphi)$ to our EBE (54) also satisfies (89) and vice versa. In other words, the EBE (89) is an alternative version of the original equation (54). For this purpose, we use the minimizing functions η^* and ν^* defined

in Theorem 4.1 and write η^*, ν^* and π^* instead of $\eta^*(y)$, $\nu^*(y)$ and $\pi^*(y)$ to simplify the notation. Then an easy but tedious computation yields the identity

$$\widetilde{\Lambda}(\lambda) = \tfrac{1}{2}\|\rho\|^2[\varphi_{yy}(y) + \varphi_y^2(y)] + \widetilde{l}(\pi^*, \eta^*, y) + \varphi_y(y)\widetilde{h}(\pi^*, \eta^*, y).$$

Thus the pair $(\widetilde{\Lambda}(\lambda), \varphi)$ also solves the EBE (89) if and only if for all $\eta \in \Gamma$

$$0 \leq \widetilde{l}(\pi^*, \eta, y) + \varphi_y(y)\widetilde{h}(\pi^*, \eta, y) - [\widetilde{l}(\pi^*, \eta^*, y) + \varphi_y(y)\widetilde{h}(\pi^*, \eta^*, y)]. \qquad (92)$$

Inserting formula (53) for π^*, this inequality takes the explicit form

$$0 \leq \tfrac{\lambda}{1-\lambda}[(\eta^{11} - \eta^{11,*})y + (\eta^{21} - \eta^{21,*})][\theta(y) + \eta^{11,*}y + \eta^{21,*}]$$
$$+ \tfrac{1}{1-\lambda}\rho_1\varphi_y(y)[(\eta^{11} - \eta^{11,*})y + (\eta^{21} - \eta^{21,*})]$$
$$+ \rho_2\varphi_y(y)[(\eta^{21} - \eta^{21,*})y + (\eta^{22} - \eta^{22,*})] \quad (93)$$

for all $\eta \in \Gamma$. To derive (93), we fix $\eta \in \Gamma$ and define the convex combination $\widetilde{\eta}_\alpha := \eta^* + \alpha(\eta - \eta^*)$, $\alpha \in (0,1)$. Then $\widetilde{\eta}_\alpha$ belongs to Γ, due to convexity of this set. Moreover, using the minimizers η^*, ν^* and the specific choice $\nu_\alpha^*(y) := \nu^*(\widetilde{\eta}_\alpha, y) = -\widetilde{\eta}_\alpha^{12}y - \widetilde{\eta}_\alpha^{22} - \rho_2(y)\varphi_y(y)$, we easily derive the inequality

$$0 \leq (1-\lambda)l(\widetilde{\eta}_\alpha, \nu_\alpha^*(y), y) + \varphi_y(y)h(\widetilde{\eta}_\alpha, \nu_\alpha^*(y), y) - [(1-\lambda)l(\eta^*, \nu^*, y) + \varphi_y(y)h(\eta^*, \nu^*, y)]$$
$$= \alpha[\text{terms in (88)}] + \tfrac{1}{2}\tfrac{\lambda}{1-\lambda}\alpha^2[(\eta^{11} - \eta^{11,*})y + (\eta^{21} - \eta^{21,*})]^2].$$

Dividing finally by α and letting afterwards α tend to zero yields the desired estimate (93) and equivalently (92). Thus we have shown that the solution $(\widetilde{\Lambda}(\lambda), \varphi)$ to the EBE (54) also satisfies (89). This ends the proof. $\qquad\square$

Remark 4.2. *The duality approach used above requires two verification theorems. The first one characterizes the growth rate of $U_T(x_0)$ in terms of the EBE (54), and the second one identifies an optimal long term investment strategy and the associated optimal growth rate $\Lambda(\lambda)$. In this remark, we discuss heuristically a more direct approach to (19) via stochastic differential game techniques; see, e.g., [12] for an introduction. To this end, note that (if $X_t^\pi > 0$ for all t)*

$$E_{Q^\eta}[(X_T^\pi)^\lambda] = x_0^\lambda E_{R^{\eta,\pi}}[e^{\int_0^T \widetilde{l}(\pi_t, \eta_t, Y_t)\, dt}],$$

where $\widetilde{l}$ is defined in (47), the measure $R^{\pi,\eta}$ is introduced in (48), and the dynamics of Y under $R^{\eta,\pi}$ is specified in (49). This suggests that

$$U_T(x_0) = x_0^\lambda v^u(y, T) := x_0^\lambda \sup_{\pi \in \mathcal{A}_T} \inf_{\eta \in \mathcal{C}} E_{R^{\eta,\pi}}[e^{\int_0^T \widetilde{l}(\pi_t, \eta_t, Y_t)\, dt}], \quad Y_0 = y,$$

where v^u can be seen as the upper value function of a stochastic differential game with maximizing "player" π and minimizing "player" η. The function v^u should be determined by the HJB-Isaacs equation

$$v_t^u = \tfrac{1}{2}\|\rho\|^2 v_{yy}^u + \sup_{\pi \in \mathbb{R}} \inf_{\eta \in \Gamma}\{\widetilde{l}(\pi, \eta, \cdot)v^u + \widetilde{h}(\pi, \eta, \cdot)v_y^u\}, \quad v^u(\cdot, 0) \equiv 1.$$

Using the heuristic transform $\ln v^u(y, T) \approx \ln U_T(x_0) \approx \Lambda(\lambda)T + \varphi(y)$, this translates into the following EBE of Isaacs type:

$$\Lambda(\lambda) = \tfrac{1}{2}\|\rho\|^2[\varphi_{yy} + \varphi_y^2] + \sup_{\pi \in \mathbb{R}} \inf_{\eta \in \Gamma}\{\widetilde{l}(\pi, \eta, \cdot) + \varphi_y\widetilde{h}(\pi, \eta, \cdot)\}.$$

If this equation has a solution $(\Lambda(\lambda), \varphi)$, *then it is easy to show that* sup *and* inf *can be interchanged and that the saddle-point is attained by* $\pi^*(y)$ *in (53) and* $\eta^*(y)$ *defined in (57), i. e., the EBE of Isaacs type is actually a version of (54). We conjecture that the alternative approach via differential games is also feasible. However, the detailed derivation would be a lenghty and technical exercise that is beyond the scope of the present paper.*

5 Existence of a solution to the ergodic Bellman equation

Our results rely on the existence of a specific solution $(\widetilde{\Lambda}(\lambda), \varphi) \in \mathbb{R}_+ \times C^2(\mathbb{R})$ to the EBE (54). More generally, an EBE is given by

$$\widetilde{\Lambda} = D\varphi(x) + H(x, \nabla\varphi) + q(x), \quad x \in \mathbb{R}^d, \tag{94}$$

where q maps from $\mathbb{R}^d$ to $\mathbb{R}$, D is a second order differential operator, and where H is a real-valued non-linear function of the gradient $\nabla\varphi$, called the Hamiltonian. A solution to (94) is a pair $(\widetilde{\Lambda}, \varphi)$ of a constant $\widetilde{\Lambda}$ and a function $\varphi : \mathbb{R}^d \to \mathbb{R}$. Such equations have been analyzed by various authors; see, e. g., [8], [25], [30] for a discussion related to risk-sensitive control problems, or [1], [2]. Unfortunately their existence results do in general not apply to our EBE (54). The main difficulty relies on three facts: We consider a model with non-linear coefficients r, g and m appearing in the functions l and h, the cost function l may grow quadratically in y, and (54) exhibits a non-linearity with respect to the first derivative φ_y. If the discussion is limited to linear coefficients, then a quadratic Ansatz may yield an explicit solution to (54); see, e. g., [9], [33], and also Subsection 6.2 for a case study.

Let us now turn to the existence problem for non-linear coefficients. The EBE (54) can be rewritten in the condensed form

$$\widetilde{\Lambda}(\lambda) = \tfrac{1}{2}\|\rho\|^2 \varphi_{yy}(y) + \tfrac{1}{2}(\widehat{\rho}\varphi_y(y))^2 + \inf_{\eta \in \Gamma} \{n(\eta, y) + \varphi_y(y)m(\eta, y)\}, \tag{95}$$

where we use the notation $\widehat{\rho} := \sqrt{\tfrac{1}{1-\lambda}\rho_1^2 + \rho_2^2}$,

$$n(\eta, y) := \tfrac{1}{2}\tfrac{\lambda}{1-\lambda}[\theta(y) + \eta^{11}y + \eta^{21}]^2 + \lambda r(y),$$
$$m(\eta, y) := g(y) + \tfrac{1}{1-\lambda}\rho_1(\lambda\theta(y) + \eta^{11}y + \eta^{21}) + \rho_2(\eta^{12}y + \eta^{22}).$$

The following existence result is deduced from Fleming and McEneaney [8]. Their construction of a solution involves a parameterized family of finite time horizon *stochastic differential games*; see, e. g., Fleming and Souganidis [12]. The associated value function is characterized in terms of a parabolic PDE, called *Isaac's equation*, and the existence of a solution $(\widetilde{\Lambda}(\lambda), \varphi)$ follows by taking appropriate limits of the Isaac's PDE when both "time" tends to infinity and the underlying parameter converges to zero.

Lemma 5.1. *In addition to Assumption 2.1 let us assume that* θ *is bounded, that* $\Gamma \subset \{(0,0) \times \mathbb{R}^2\}$, *and that*

$$\exists\, K > 0 : \; g_y(y) + \tfrac{\lambda}{1-\lambda}\rho_1\theta_y(y) \leq -K \quad \text{for all } y \in \mathbb{R}. \tag{97}$$

Then there exist a pair $\widetilde{\Lambda}(\lambda) \in \mathbb{R}_+$, $\varphi \in C^2(\mathbb{R})$ *that solves the EBE (54). Moreover, we have* $|\varphi_y| \leq \max_{\eta \in \Gamma} \|n_y(\cdot, y)\|_\infty / K$, *and so this solution also satisfies the regularity Assumptions 4.1 and 4.2.*

Proof. Our assumptions ensure boundedness of $n \geq 0$, η_y and m_y on $\Gamma \times \mathbb{R}$. Moreover, the mean value theorem combined with (97) gives

$$(x - y)(m(\eta, x) - m(\eta, y)) \leq -K|x - y|^2 \quad \text{for all } x, y \in \mathbb{R}, \eta \in \Gamma.$$

The functions n, m thus satisfy condition (7.2) in Fleming and McEneaney [8], and applying [8], Theorem 7.1, for $\gamma := (\sqrt{2}\widehat{\rho})^{-1}$ and $\epsilon := \|\rho\|^2/\widehat{\rho}^2$ the desired existence result follows. $\qquad\square$

6 Explicit results

6.1 Black-Scholes model with uncertain drift

For constant coefficients $r(y) \equiv r$ and $m(y) \equiv m$, the reference model Q_0 in Section 2 becomes the Black-Scholes model with price dynamics

$$dS_t^0 = S_t^0 r \, dt, \quad dS_t^1 = S_t^1(m \, dt + \sigma \, dW_t^1).$$

In particular, the market price of risk function $\theta(y) = \frac{m-r}{\sigma}$ is constant. Taking the specific set $\Gamma = \{(0,0)\} \times [a, b] \times \{0\}$, $a \leq 0 \leq b$, each measure $Q^\eta \in \mathcal{Q}$ corresponds to a drift perturbation of the following type:

$$dS_t^1 = S_t^1([m + \sigma\eta_t^{21}] \, dt + \sigma \, dW_t^{1,\eta}).$$

In this example the factor process Y plays no role. In particular, the maximal expected utility for a finite horizon does not depend on the initial state of the factor process. Hence the function φ appearing in the heuristic separation of time and space variables (43) is constant, and its derivatives φ_y, φ_{yy} vanish. The EBE (54) thus reduces to

$$\widetilde{\Lambda}(\lambda) = \inf_{\nu \in \mathbb{R}} \inf_{\eta \in \Gamma} \{ \tfrac{1}{2} \tfrac{\lambda}{1-\lambda} [(\theta + \eta^{21})^2 + \nu^2] + \lambda r \} = \tfrac{1}{2} \tfrac{\lambda}{1-\lambda} \inf_{\eta \in \Gamma} \{(\theta + \eta^{21})^2\} + \lambda r.$$

The number $\widetilde{\Lambda}(\lambda)$ can be expressed in terms of the element $\eta^{21,*} \in [a, b]$ which minimizes the absolute value $|\theta + \eta^{21}|$ among all $\eta^{21} \in [a, b]$. Defining the constant controls $\eta_t^* := (0, 0, \eta^{21,*}, 0)$ and $\nu_t^* := 0$, $t \geq 0$, the verification theorems can be transferred to our present example in a simplified form which does not not require any additional conditions as in Assumptions 4.1 and 4.2. As a result we get the following description of the aymptotics of robust expected power utility:

- The maximal robust utility $U_T(x_0)$ grows exponentially with rate

$$\widetilde{\Lambda}(\lambda) = \tfrac{1}{2} \tfrac{\lambda}{1-\lambda} (\theta + \eta^{21,*})^2 + \lambda r > 0.$$

- $\Lambda(\lambda) = \sup_{\pi \in \mathcal{A}} \varlimsup_{T \uparrow \infty} \tfrac{1}{T} \ln \inf_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^\pi)^\lambda] = \widetilde{\Lambda}(\lambda)$

- The optimal long term strategy takes the form

$$\pi_t^* := \tfrac{1}{1-\lambda} \tfrac{1}{\sigma} (\theta + \eta^{21,*}), \quad t \geq 0.$$

- The asymptotic worst-case model Q^{η^*} is given by the constant control $\eta_t^* = (0, 0, \eta^{21,*}, 0)$, and it does not depend on the parameter λ.

Remark 6.1. *Using methods from robust statistics, Schied [37] shows that the measure Q^{η^*} is actually* least favorable *in the following sense: For any finite maturity, the robust utility maximization problem (17) is equivalent to the classical problem for Q^{η^*}, regardless of the choice of the underlying utility function u.*

6.2 Geometric Ornstein-Uhlenbeck model with uncertain mean reversion

As our second case study, we consider the case where the economic factor Y is of OU type, and where there interest rate r is constant. In our reference model Q_0, the factor Y is assumed to be a classical OU process with constant rate of mean reversion $\eta_0 > 0$ and volatility $\sigma > 0$, i.e.,

$$dY_t = -\eta_0 Y_t\, dt + \sigma\, dW_t^1, \quad Y_0 = y_0. \tag{98}$$

We assume that $S_t^1 := \exp(Y_t + \alpha t)$, $\alpha \in \mathbb{R}$, describes the price process of the risky asset. By Itô's formula, the dynamics of S^1 is governed by the SDE

$$dS_t^1 = S_t^1(\alpha\, dt + dY_t + \tfrac{1}{2}\, d\langle Y \rangle_t) = S_t^1((-\eta_0 Y_t + \tfrac{1}{2}\sigma^2 + \alpha)dt + \sigma\, dW_t^1).$$

Hence this example corresponds to the general model of Section 2 for the choice $g(y) = -\eta_0 y$, $\rho_1 = \sigma$, $\rho_2 = 0$, $m(y) = -\eta_0 y + \tfrac{1}{2}\sigma^2 + \alpha$, and for the affine market price of risk function

$$\theta(y) = \tfrac{1}{\sigma}(-\eta_0 y + \tfrac{1}{2}\sigma^2 + \alpha - r).$$

Let us suppose that the investor is uncertain about the "true" future rate of mean reversion: Instead of a constant rate we admit any rate process that is progressively measurable and that takes its values in some interval $[a, b]$, $0 < a \le \eta_0 \le b < \infty$. This uncertainty about the true rate of mean reversion can be embedded into our general model by choosing the set

$$\Gamma = \left[\tfrac{\eta_0 - b}{\sigma}, \tfrac{\eta_0 - a}{\sigma}\right] \times \{(0, 0, 0)\}.$$

Indeed, let $Q^\eta \in \mathcal{Q}$ denote the probabilistic model generated by a Γ-valued, progressively measurable process $\eta = (\eta_t)_{t \ge 0}$; cf. (12). In view of (14a), the factor process Y then evolves under Q^η according to

$$dY_t = -(\eta_0 - \sigma\eta_t^{11})Y_t\, dt + \sigma\, dW_t^{1,\eta},$$

and the resulting mean reversion process $(\eta_0 - \sigma\eta_t^{11})_{t \ge 0}$ takes values in $[a, b]$.

To prepare the analysis of the asymptotic robust utility maximization problem (19), we first solve its non-robust version

$$\text{maximize} \quad \overline{\lim_{T \uparrow \infty}}\, \tfrac{1}{T} \ln E_{Q_0}[(X_T^\pi)^\lambda] \quad \text{among all } \pi \in \mathcal{A} \tag{99}$$

for the specific model Q_0. This problem has been studied, amongst others, by Fleming and Sheu [9] and Pham [33]. By the following proposition we recover their results as a special case of our general robust duality approach. To indicate the non-robust case, we denote the optimal growth rate for (99) by $\Lambda_{Q_0}(\lambda)$. Note that $\mathcal{Q} = \{Q_0\}$ if we take the one-point set $\Gamma = \{(0, 0, 0, 0)\}$. Thus our general EBE (54) takes the simplified form

$$\widetilde{\Lambda}_{Q_0}(\lambda) = \tfrac{1}{2}\sigma^2[\varphi_{yy}(y) + \tfrac{1}{1-\lambda}\varphi_y^2(y)] + \lambda r + \tfrac{1}{2}\tfrac{\lambda}{1-\lambda}\left(\tfrac{-\eta_0 y + \tfrac{1}{2}\sigma^2 + \alpha - r}{\sigma}\right)^2$$
$$+ \varphi_y(y)\left[-\tfrac{1}{1-\lambda}\eta_0 y + \tfrac{\lambda}{1-\lambda}(\tfrac{1}{2}\sigma^2 + \alpha - r)\right], \tag{100}$$

where the infimum among all $\nu \in \mathbb{R}$ is attained for $\nu^*(y) \equiv 0$.

Proposition 6.1. *The EBE (100) has the solution*

$$\widetilde{\Lambda}_{Q_0}(\lambda) = \tfrac{1}{2}(1 - \sqrt{1 - \lambda})\eta_0 + \lambda(r + \tfrac{1}{2\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r)^2), \qquad (101a)$$

$$\varphi(y) = \tfrac{1}{2}(1 - \sqrt{1 - \lambda})\tfrac{\eta_0}{\sigma^2}y^2 - \tfrac{\lambda}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r)y \qquad (101b)$$

which satisfies our regularity Assumptions 4.1 and 4.2. Thus it holds that

$$\widetilde{\Lambda}_{Q_0}(\lambda) = \lim_{T\uparrow\infty} \tfrac{1}{T}\ln U_T^{Q_0}(x_0) = \Lambda_{Q_0}(\lambda) = \sup_{\pi\in\mathcal{A}} \overline{\lim_{T\uparrow\infty}} \tfrac{1}{T}\ln E_{Q_0}[(X_T^\pi)^\lambda].$$

Moreover, an optimal feedback strategy $\pi_t^* = \pi^*(Y_t)$, $t \geq 0$, for our investment problem (99) is given by the affine function

$$\pi^*(y) = -\frac{1}{\sqrt{1-\lambda}}\frac{\eta_0}{\sigma^2}y + \frac{1}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r). \qquad (102)$$

Proof. Following [9] and [33] we are looking for a quadratic solution $\varphi(y) = \tfrac{1}{2}Ay^2 + By$. Inserting the derivatives in (100) and comparing the coefficients of the terms in y^2, in y, and the constants yields that the EBE (100) holds for every triple $(A, B, \widetilde{\Lambda}_{Q_0}(\lambda))$ satisfying the system of equations

$$0 = \tfrac{1}{2}\sigma^2 A^2 - \eta_0 A + \tfrac{\lambda}{2\sigma^2}\eta_0^2$$

$$0 = \sigma^2 AB + \lambda(\tfrac{1}{2}\sigma^2 + \alpha - r)A - B\eta_0 - \tfrac{\lambda}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r)\eta_0$$

$$\widetilde{\Lambda}_{Q_0}(\lambda) = \tfrac{1}{2}\sigma^2(A + \tfrac{1}{1-\lambda}B^2) + \tfrac{\lambda}{1-\lambda}(\tfrac{1}{2}\sigma^2 + \alpha - r)B + \lambda r + \tfrac{1}{2}\tfrac{\lambda}{1-\lambda}(\tfrac{\tfrac{1}{2}\sigma^2 + \alpha - r}{\sigma})^2.$$

The quadratic equation for A has the solutions $A_\pm = (1 \pm \sqrt{1-\lambda})\tfrac{\eta_0}{\sigma^2}$. We choose $A = A_-$, and we shall explain in Remark 6.2 why the other solution is irrelevant. A straightforward calculation gives $B = -\tfrac{\lambda}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r)$ and finally the expressions for $\widetilde{\Lambda}_{Q_0}(\lambda)$ and φ in (101). The parabola φ is bounded below, φ_y grows linearly and the functions $\kappa, \widetilde{\kappa}$ defined in Assumption 4.1 (b) and 4.2 satisfy the regularity condition

$$y\kappa(0, y) = y\widetilde{\kappa}(0, y) = -\tfrac{1}{\sqrt{1-\lambda}}\eta_0 y^2.$$

Applying Theorem 4.1 and Theorem 4.2 ends the proof. $\qquad\square$

Remark 6.2. *Using the other root A_+ yields $\varphi(y) = \tfrac{1}{2}A_+ y^2 + By$ and*

$$\widetilde{\Lambda}_{Q_0}(\lambda) = \tfrac{1}{2}(1 + \sqrt{1-\lambda})\eta_0 + \lambda(r + \tfrac{1}{2\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r)^2).$$

In particular, this example illustrates that the solution to an EBE is not necessarily unique. On the other hand, the "ergodicity" Assumption 4.1 (b) selects the good candidate. Indeed, the proof of Theorem 4.1 requires that $\lim_{T\uparrow\infty} \tfrac{1}{T}E_{\widehat{R}^\eta}[Y_T^2] = 0$, where Y follows the SDE (68). Given the geometric OU model and the solutions $\varphi(y) = \tfrac{1}{2}A_\pm y^2 + By$ this SDE takes the form

$$dY_t = \pm\frac{\eta_0}{\sqrt{1-\lambda}}Y_t\, dt + \sigma\, d\widehat{W}_t^{1,\eta}, \quad Y_0 = y_0.$$

The factor process Y is an "explosive" Gaussian process for the root A_+ in the sense that $\lim_{T\uparrow\infty} \tfrac{1}{T}E_{\widehat{R}^\eta}[Y_T^2] = \infty$. Thus the arguments used in the proof of Theorem 4.1 fail and so the solution associated with A_+ is irrelevant. Conversely, taking A_-, Theorem 8.2 applies to the ergodic process Y.

As a complement to Proposition 6.1 we look at the maximal robust utility $U_T^{Q_0}(x_0)$ attainable at time T and the asymptotics of the optimal investment strategy $\pi^{*,T}$ as $T \uparrow \infty$. The following proposition extends Proposition 5.6 and 5.7 in Föllmer and Schachermayer [16] by including an interest rate $r > 0$ and the additional drift component α for the price process S^1.

Proposition 6.2. *For any initial condition $Y_0 = y_0$, the maximal robust expected utility $U_T^{Q_0}(x_0)$ takes the form*

$$U_T^{Q_0}(x_0) = \tfrac{1}{\lambda} x_0^\lambda [(A_T^-)^{-1/2} e^{B_T(y_0) + \frac{\lambda}{1-\lambda} rT + (A_T^-)^{-1} C_T(y_0)}]^{1-\lambda}, \tag{103}$$

where we use the notation

$$A_T^\pm := 1 - \tfrac{1}{2}(1 - (1-\lambda)^{-1/2})(1 \pm \exp(-2\eta_0(1-\lambda)^{-1/2}T)),$$
$$B_T(y) := -\tfrac{\eta_0}{2\sigma^2}[(1-\lambda)^{-1/2} - (1-\lambda)^{-1}]y^2 + \tfrac{1}{\sigma^2}\tfrac{\lambda}{\lambda-1}(\tfrac{1}{2}\sigma^2 + \alpha - r)y$$
$$\qquad - \tfrac{1}{2}[\eta_0((1-\lambda)^{-1/2} - (1-\lambda)^{-1}) + \tfrac{1}{\sigma^2}\tfrac{\lambda}{\lambda-1}(\tfrac{1}{2}\sigma^2 + \alpha - r)^2]T,$$
$$C_T(y) := \tfrac{\eta_0}{2\sigma^2}((1-\lambda)^{-1/2} - (1-\lambda)^{-1}) \exp(-2\eta_0(1-\lambda)^{-1/2}T)y^2$$
$$\qquad - \tfrac{1}{\sigma^2}\tfrac{\lambda}{\lambda-1}(\tfrac{1}{2}\sigma^2 + \alpha - r) \exp(-\eta_0(1-\lambda)^{-1/2}T)y$$
$$\qquad + \tfrac{1}{4\sigma^2}\tfrac{\lambda^2}{(1-\lambda)^{3/2}}(\tfrac{1}{2}\sigma^2 + \alpha - r)^2(1 - \exp(-2\eta_0(1-\lambda)^{-1/2}T)).$$

The optimal proportion $\pi_t^{*,T}$ is an affine function of the current state Y_t of the factor process given by $\pi_t^{*,T} = a[T-t]Y_t + b[T-t]$, where

$$a[T-t] := -\tfrac{\eta_0}{\sigma^2}(1-\lambda)^{-1/2}A_{T-t}^+(A_{T-t}^-)^{-1},$$
$$b[T-t] := \tfrac{1}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r)[1 + (A_{T-t}^-)^{-1}\tfrac{\lambda}{1-\lambda}e^{-\eta_0(1-\lambda)^{-1/2}(T-t)}].$$

Proof. Detailed computations can be found in [26], Chapter 4. $\qquad\square$

Since $A_T^\pm$ and $C_T(y_0)$ converge to a finite limit as $T \uparrow \infty$, we thus obtain

$$\lim_{T\uparrow\infty} \tfrac{1}{T} \ln U_T^{Q_0}(x_0) = (1-\lambda) \lim_{T\uparrow\infty} \tfrac{1}{T}(B_T(y_0) + \tfrac{\lambda}{1-\lambda} rT) = \widetilde{\Lambda}_{Q_0}(\lambda),$$

in accordance with Proposition 6.1. Moreover, we have

$$\lim_{T\uparrow\infty} a[T-t] = -\tfrac{\eta_0}{\sigma^2\sqrt{1-\lambda}} \quad \text{and} \quad \lim_{T\uparrow\infty} b[T-t] = \tfrac{1}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r),$$

due to $\lim_{T\uparrow\infty} A_T^\pm = \tfrac{1}{2}(1 + (1-\lambda)^{-1/2})$. Thus the asymptotic form of the optimal strategy $\pi^{*,T}$ as $T \uparrow \infty$ is given by

$$\lim_{T\uparrow\infty} \pi_t^{*,T} = -\tfrac{1}{\sigma^2\sqrt{1-\lambda}}\eta_0 Y_t + \tfrac{1}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r), \tag{104}$$

and so it coincides with the optimal long term strategy π^* in (102). On the other hand, Fleming and Sheu [9] observed that $\lim_{T\uparrow\infty} \pi_t^{*,T}$ does not provide an optimal long term strategy for power utility with parameter $\lambda \le -3$.

Let us now analyze the *robust* case. Since $\Lambda_{Q_0}(\lambda)$ is increasing in η_0, it is natural to expect that the asymptotic worst-case measure Q^{η^*} corresponds to the probabilistic model, under which Y has the minimal rate of mean reversion a. The following proposition confirms this conjecture.

Proposition 6.3. *For the geometric OU model with uncertain rate of mean reversion, the optimal growth rate of robust power utility is given by*

$$\Lambda(\lambda) = \tfrac{1}{2}(1 - \sqrt{1-\lambda})a + \lambda(r + \tfrac{1}{2\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r)^2) > 0,$$

and the maximal robust utility $U_T(x_0)$ grows exponentially at this rate. The asymptotic worst-case model Q^{η^*} is determined by $\eta_t^* = (\tfrac{\eta_0 - a}{\sigma}, 0, 0, 0)$, and the optimal long term strategy $\pi_t^* = \pi^*(Y_t)$ is specified by the affine function

$$\pi^*(y) = -\frac{1}{\sqrt{1-\lambda}}\frac{a}{\sigma^2}y + \frac{1}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r).$$

Proof. Replacing η_0 in (101) by the minimal mean reversion a provides

$$\widetilde{\Lambda}(\lambda) = \tfrac{1}{2}(1 - \sqrt{1-\lambda})a + \lambda(r + \tfrac{1}{2\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r)^2) > 0,$$
$$\varphi(y) = \tfrac{1}{2}(1 - \sqrt{1-\lambda})\frac{a}{\sigma^2}y^2 - \frac{\lambda}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r)y$$

as a candidate for the solution to the EBE (54), and it is easy to verify that $(\widetilde{\Lambda}(\lambda), \varphi)$ is indeed a solution. The corresponding minimizers are $\nu^*(y) \equiv 0$, $\eta^*(y) \equiv (\tfrac{\eta_0 - a}{\sigma}, 0, 0, 0) \in \Gamma$. It remains to verify that $(\widetilde{\Lambda}(\lambda), \varphi)$ satisfies our Assumptions 4.1 and 4.2: Since $\varphi \in C^2(\mathbb{R})$ is a parabola, it is bounded below and its first derivative φ_y grows linearly. Moreover, the auxiliary functions κ, $\widetilde{\kappa}$ appearing in our Assumptions 4.1 and 4.2 satisfy for all $\eta \in \Gamma$

$$y\kappa(\eta, y) = [-\tfrac{1}{1-\lambda}(\eta_0 - a - \sigma\eta^{11}) - \tfrac{1}{\sqrt{1-\lambda}}a]y^2 \leq -\tfrac{1}{\sqrt{1-\lambda}}ay^2$$

and

$$y\widetilde{\kappa}(\eta, y) = [-(\eta_0 - a) + \sigma\eta^{11} - \tfrac{1}{\sqrt{1-\lambda}}a]y^2 \leq -\tfrac{1}{\sqrt{1-\lambda}}ay^2$$

due to $\eta^{11} \leq \tfrac{\eta_0 - a}{\sigma}$ and $\eta^{11,*} = \tfrac{\eta_0 - a}{\sigma}$. We thus derive Proposition 6.3 as a special case of Theorem 4.1 and Theorem 4.2. $\qquad\square$

7 Application to a robust outperformance criterion

Utility maximization is conceptually related to specific numerical representations of the investor's preferences. The application requires to know the utility function u which is by nature subjective. For institutional managers utility maximization thus creates severe difficulties. On the one hand, the preferences of their customers and the corresponding numerical representations are not really known exactly. On the other hand, the individual preferences of the managers and of the various customers with shares in the same investment fund will typically be different. This suggests to look for an "intersubjective" criterion for optimal portfolio management which is acceptable for a large variety of investors. Such an alternative consists in evaluating the performance of the portfolio relative to a given *benchmark* such as a stock index. The investor aims at outperforming the benchmark with maximal probability. If the benchmark is a contingent claim H at a terminal time T, then the outperformance problem reduces to maximizing the probability $Q[X_T^\pi \geq H]$ of a *successful hedge*. This criterion, known as *quantile hedging*, has been developed as a substitute for investors who are not willing or not able to raise the initial costs required by a *perfect hedging* or *superhedging* strategy of H; see, e.g., Föllmer and Leukert [15] and the references therein.

Pham [33] proposed an asymptotic benchmark criterion for optimal long term investment. Here the investor has in mind a level of return c and aims at maximizing the probability that the portfolio's growth rate

$$L_T^\pi := \tfrac{1}{T} \ln X_T^\pi$$

(or more generally $\tfrac{1}{T} \ln(X_T^\pi / I_T)$ for an index process I) exceeds this threshold. For finite T this corresponds to quantile hedging for $H = \exp(cT)$. But what happens in the long run? If the growth rates L_T^π converge Q-a.s. as $T \uparrow \infty$ and satisfy under Q a *large deviations principle* with rate function I^π, then $Q[L_T^\pi \geq c] \approx \exp(-I^\pi(c)T)$ as $T \uparrow \infty$, i.e., the probability that L_T^π departs from its limiting value decays to zero exponentially fast. Thus the long term view amounts to minimizing the rates $I^\pi(c)$, or equivalently to

$$\text{maximizing} \quad \varlimsup_{T\uparrow\infty} \tfrac{1}{T} \ln Q[L_T^\pi \geq c] \quad \text{among all } \pi. \tag{105}$$

An asymptotic benchmark criterion of this form may be of particular interest for institutional managers with long term horizon, such as mutual fund managers. Note, however, that this Ansatz does not take into account the size of the shortfall if it does occur. From a mathematical point of view, it leads to a non-standard *large deviations control problem*. On the other hand, standard results from the large deviations theory (such as the Gärtner-Ellis theorem; see, e.g., [6], Theorem 2.3.6) suggest that the rate function I^π is a Fenchel-Legendre transform of the logarithmic moment generating function

$$\Lambda_Q(\lambda, \pi) := \varlimsup_{T\uparrow\infty} \tfrac{1}{T} \ln E_Q[\exp(\lambda T L_T^\pi)] = \varlimsup_{T\uparrow\infty} \tfrac{1}{T} \ln E_Q[(X_T^\pi)^\lambda].$$

In this spirit, Pham developed a duality approach to (105). His Theorem 3.1, relying on large deviations arguments, but not on the specific structure of the underlying market model, states that

$$\sup_\pi \varlimsup_{T\uparrow\infty} \tfrac{1}{T} \ln Q[L_T^\pi \geq c] = - \sup_{\lambda\in(0,\lambda')} \{\lambda c - \Lambda_Q(\lambda)\}, \tag{106}$$

where $\Lambda_Q(\lambda) := \sup_\pi \Lambda_Q(\lambda, \pi)$ is the optimal growth rate of expected power utility with respect to Q. Applications of Pham's theorem to specific market models can be found in [33], [34], [40], [22], and [20].

However, the benchmark criterion (105) does not account for model ambiguity. To overcome this limitation, it is natural to study its robust version:

$$\text{maximize} \quad \varlimsup_{T\uparrow\infty} \tfrac{1}{T} \ln \inf_{Q\in\mathcal{Q}} Q[L_T^\pi \geq c] \quad \text{among all } \pi. \tag{107}$$

The solution is derived in [26], Chapter 6, for the *robust* stochastic factor model of Section 2, and it is closely related to the asymptotics of robust utility maximization. Under suitable regularity assumptions (e.g., $\Lambda \in C^1((0,1))$ and $\lim_{\lambda\uparrow\lambda'} \Lambda'(\lambda) = \infty$ for some $\lambda' \leq 1$) we obtain the duality formula

$$\sup_{\pi\in\mathcal{A}} \varlimsup_{T\uparrow\infty} \tfrac{1}{T} \ln \inf_{Q^\eta\in\mathcal{Q}} Q^\eta[L_T^\pi \geq c] = - \sup_{\lambda\in(0,\lambda')} \{\lambda c - \Lambda(\lambda)\}. \tag{108}$$

This can be seen as a robust extension of (106), but here the duality formula involves the optimal growth rates $\Lambda(\lambda)$, $\lambda \in (0,1)$, of robust power utility. Moreover, the sequence of investment processes $\widehat{\pi}^{c,n}$, $n \in \mathbb{N}$, defined by

$$\widehat{\pi}_t^{c,n} = \begin{cases} \pi_t^*(\lambda[c+1/n]) & \text{for } c > \Lambda'(0) \\ \pi_t^*(\lambda[\Lambda'(0)+1/n]) & \text{for } c \leq \Lambda'(0) \end{cases}$$

in terms of the optimal long term strategies $\pi^*(\lambda)$ for robust power utility and in terms of parameters $\lambda[c] \in \operatorname{argmax}_{\lambda \in (0,\lambda')}\{\lambda c - \Lambda(\lambda)\}$ is nearly optimal for (107). The proof is beyond the scope of this paper and therefore omitted.

Example 7.1. *For the geometric OU model with uncertain mean reversion (see Subsection 6.2) Proposition 6.3 shows that:*

- $\Lambda(\lambda) = \frac{1}{2}(1 - \sqrt{1-\lambda})a + \lambda\gamma$ *with* $\gamma := r + \frac{1}{2\sigma^2}(\frac{1}{2}\sigma^2 + \alpha - r)^2$,

- $\pi_t^*(\lambda) = -\frac{1}{\sqrt{1-\lambda}}\frac{a}{\sigma^2}Y_t + \frac{1}{\sigma^2}(\frac{1}{2}\sigma^2 + \alpha - r)$, $t \geq 0$.

We thus obtain from (108) the optimal rate of exponential decay

$$\sup_{\pi \in \mathcal{A}} \overline{\lim_{T \uparrow \infty}} \frac{1}{T} \ln \inf_{Q \in \mathcal{Q}} Q[L_T^\pi \geq c] = \begin{cases} -\frac{(\frac{a}{4} - c + \gamma)^2}{c - \gamma} & \text{for } c > \frac{a}{4} + \gamma \\ 0 & \text{for } c \leq \frac{a}{4} + \gamma \end{cases}.$$

Since $\lambda[c] = 1 - (\frac{a}{4(c-\gamma)})^2$, *the nearly optimal strategies are given by*

$$\widehat{\pi}_t^{c,n} = \begin{cases} -\frac{4}{\sigma^2}(c + \frac{1}{n} - \gamma)Y_t + \frac{1}{\sigma^2}(\frac{1}{2}\sigma^2 + \alpha - r) & \text{for } c > \frac{a}{4} + \gamma \\ -\frac{4}{\sigma^2}(\frac{a}{4} + \frac{1}{n})Y_t + \frac{1}{\sigma^2}(\frac{1}{2}\sigma^2 + \alpha - r) & \text{for } c \leq \frac{a}{4} + \gamma \end{cases}.$$

Remark 7.1. *Another natural problem is to minimize the robust large deviations probability of downside risk*

$$\lim_{T \uparrow \infty} \frac{1}{T} \ln \sup_{Q \in \mathcal{Q}} Q[L_T^\pi \leq c]. \tag{109}$$

Here the investor is interested in minimizing, in the long run, the worst-case probability that his portfolio underperforms a bond with interest rate c. In the non-robust case, this large deviation criterion has been proposed by Pham [33], but a rigorous solution was given first by Hata, Nagai and Sheu [21] for the special case of a linear Gaussian factor model. The solution can be derived by a duality approach which, in contrast to (106) and (108), involves the optimal growth rates $\Lambda(\lambda)$ of power utility with negative parameter λ; cf. Remark 2.1. For a detailed discussion of problem (109) see [26].

8 Appendix

Let us finally summarize some technicalities.

Lemma 8.1. *Let W be a two-dimensional Brownian motion on the stochastic base $(\Omega, \mathcal{G}, \mathbb{G}, Q)$, and let η be a $\mathbb{G}$-progressively measurable process taking its values in a compact subset $\Gamma \subset \mathbb{R}^d$. Moreover, let $(Y_t)_{t \leq T}$ be a continuous process that is a strong solution of the SDE*

$$dY_t = h(\eta_t, Y_t)\, dt + \sigma\, dW_t, \quad Y_0 = y_0, \|\sigma\| > 0, \tag{110}$$

where the drift function $h : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfies for all $\eta \in \Gamma$, $y \in \mathbb{R}$

$$h^2(\eta, y) \leq K^2(1 + y^2) \quad \text{for some constant } K.$$

Then there exists $\delta = \delta(T) > 0$ such that $\sup_{t \leq T} E_Q[\exp(\delta Y_t^2)] < \infty$.

Proof. The local martingale $B_t := \|\sigma\|^{-1}\sigma W_t$, $t \in [0,T]$, satisfies $\langle B \rangle_t = t$. Thus, B is a one-dimensional Brownian motion, due to P. Lévy's characterization. In particular, the SDE (110) can be rewritten as

$$dY_t = h(\eta_t, Y_t)\,dt + \|\sigma\|\,dB_t. \tag{111}$$

The proof now follows in two steps: First we argue for a constant function $h(y) \equiv h$. In that case, the solution to (111) is given by the Gaussian OU process $Y_t = e^{ht}(y_0 + \int_0^t e^{-hs}\|\sigma\|\,dB_s)$, $t \in [0,T]$, and the claim follows easily. In a second step, we extend this result to the general case by a comparison argument. The details are given in [29], Theorem 4.7, restricted to the special case $h(\eta, y) = h(y)$. $\qquad\square$

Lemma 8.2. *Let $(\Omega, \mathcal{G}, \mathbb{G}, Q)$ be a reference probability system supporting a two-dimensional Brownian motion $W = (W^1, W^2)$, and let η be a $\mathbb{G}$-progressively measurable process with values in a compact subset $\Gamma \subset \mathbb{R}^d$. Furthermore, we suppose that Y is a strong solution to the SDE (110), where h is real-valued function such that*

$$\exists\, K, M > 0 \,\forall\, \eta \in \Gamma: \quad yh(\eta, y) \leq -Ky^2 + M,$$

and where the volatility vector satisfies $\|\sigma\| > 0$. Then it holds that:

i) *There exist constants $C, C_n > 0$, $n \in \mathbb{N}$, such that*

$$\sup_{t \geq 0} E_Q[Y_t^{2n}] \leq y_0^{2n} + C_n \quad and \quad \sup_{t \geq 0} E_Q[|Y_t|] \leq C(1 + |y_0|).$$

ii) *For all $k \in \mathbb{R}$, $\sup_{t \geq 0} E_Q[\exp(k|Y_t|)] < \infty$.*

In particular, these bounds are uniform among the class of all progressively measurable Γ-valued processes η.

Proof. The proof is rather standard in ergodic control theory and appears in single components under slight different assumptions in various papers; see, e. g., [21] or [8]. For a unifying version see [26], Lemma A.2. $\qquad\square$

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