Convex Capital Requirements for Large Portfolios

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Abstract

For a large homogeneous portfolio of financial positions, we study the asymptotic behavior of the capital requirement per position defined in terms of a convex monetary risk measure. In an actuarial context, this capital requirement can be seen as a premium per contract. We show that the premia converge to the fair premium as the portfolio becomes large, and we give a precise description of the decay of the risk premia. The analysis is carried out first for a law-invariant convex risk measure and then in a situation of model ambiguity.

Key words: Capital requirements, convex risk measures, Value at Risk, Average Value at Risk, entropic risk measures, law-invariant risk measures, comonotonic risk measures, concave distortions, asymptotics of risk measures, model ambiguity

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1 Introduction

Consider a large portfolio consisting of n financial positions whose monetary outcomes are described as random variables $X_1, \ldots, X_n$ on some probability space $(\Omega, \mathcal{F}, P)$. Given a convex risk measure ρ, the capital which is required in order to make the aggregate position $S_n = X_1 + \cdots + X_n$ acceptable is specified as $\rho(S_n)$, and we denote by

$$\pi_n := \tfrac{1}{n}\rho(S_n)$$

the resulting capital requirement per position.

From an actuarial point of view, $\rho(S_n)$ can be seen as the aggregate premium which is needed to secure a portfolio of n insurance contracts, and π_n is then the premium per contract. In the classical i. i. d. case, one expects that the premium π_n should be higher than the "fair premium" $E_P[-X_1]$, and that the "risk premium" $\pi_n - E_P[-X_1]$ should decrease as the portfolio becomes large. For the *coherent* entropic risk measures ρ_c, defined by

$$\rho_c(X) := \sup_{Q: H(Q|P) \leq c} E_Q[-X] \tag{1}$$

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in terms of the relative entropy $H(Q|P)$, this is indeed the case. More precisely, we have $\pi_n \geq E_P[-X_1]$, and

$$\lim_{n\uparrow\infty} \sqrt{n}(\pi_n - E_P[-X_1]) = \sigma_P\sqrt{2c}, \tag{2}$$

where σ_P^2 denotes the variance of X_1; cf. [8], Proposition 4.1. On the other hand, the pooling of risks does *not* have the desired effect if we take the *convex* entropic risk measure e_γ defined by

$$e_\gamma(X) := \sup_Q\{E_Q[-X] - \tfrac{1}{\gamma}H(Q|P)\} = \tfrac{1}{\gamma}\log E_P[e^{-\gamma X}] \tag{3}$$

for parameters $\gamma > 0$. Indeed, since e_γ is additive on independent positions, we have $e_\gamma(S_n) = ne_\gamma(X_1)$, and so π_n does *not* decrease as the portfolio becomes large.

In this paper, our aim is to understand the preceding two examples from a more general point of view. For the ease of exposition we restrict the discussion to portfolios consisting of positions which are i. i. d. with finite exponential moments, but our arguments have a wider scope; cf. Remarks 3.1 and 5.1. In Section 3 we study the asymptotics of the capital requirements specified by a law-invariant convex risk measure ρ. Risk measures are often considered as functionals on L^∞. But in the law-invariant case they admit a canonical extension to L^1; cf. Filipović & Svindland [7]. Under our exponential moment assumption, we will actually consider them as functionals on a suitable Orlicz space, in accordance with the general discussion in Cheridito & Li [2][3], and we will make extensive use of the estimates available in this context.

As shown by Kusuoka [15] in the coherent case and by Kunze [14], Dana [4] and Frittelli & Rosazza Gianin [12] in the general convex case, any law-invariant convex risk measure ρ can be constructed by using as building blocks the coherent risk measures *Average Value at Risk* (AVaR), defined by

$$\mathrm{AVaR}_\lambda(X) := \tfrac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\alpha(X)\,d\alpha$$

for any level $\lambda \in (0, 1]$; cf., e. g., [10], Theorem 4.62. It is therefore natural to begin by looking at Average Value at Risk and at the behavior of the corresponding premia $\pi_n^\lambda := \frac{1}{n}\mathrm{AVaR}_\lambda(S_n)$ as the portfolio becomes large. In Proposition 3.1 we show that, as a straightforward consequence of the central limit theorem, the premia π_n^λ converge to the fair premium, and that

$$\lim_{n\uparrow\infty} \sqrt{n}(\pi_n^\lambda - E_P[-X_1]) = \sigma_P\tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda)), \tag{4}$$

where σ_P^2 denotes the variance of X_1 under P, and where φ and Φ denote the density and the distribution function of the standard normal distribution.

In a second step we focus on the comonotonic case. Here the risk measure takes the form

$$\rho_\mu(X) = \int_0^1 \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda) \tag{5}$$

for some probability measure μ on $(0, 1]$. Theorem 3.1 shows that the asymptotic behavior of the corresponding premia $\pi_{\mu,n} := \frac{1}{n}\rho_\mu(S_n)$ is analogous to (4) if $\log(1/\lambda)$ is integrable under the mixing measure μ. We then pass to a general law-invariant coherent risk measure, given by

$$\rho_{\mathcal{M}} := \sup_{\mu\in\mathcal{M}} \rho_\mu \tag{6}$$

for some class $\mathcal{M}$ of probability measures μ on $(0, 1]$. Under the condition

$$\sup_{\mu\in\mathcal{M}} \int_0^1 \log\left(\tfrac{1}{\lambda}\right)\,\mu(d\lambda) < \infty \tag{7}$$

we show that the asymptotic behavior of the corresponding premia $\pi_{\mathcal{M},n} := \frac{1}{n}\rho_{\mathcal{M}}(S_n)$ is described by

$$\lim_{n\uparrow\infty} \sqrt{n}(\pi_{\mathcal{M},n} - E_P[-X_1]) = \sigma_P \sup_{\mu\in\mathcal{M}} \int_0^1 \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\,\mu(d\lambda); \tag{8}$$

cf. Theorem 3.2.

In the final part of Section 3 we consider the general case of a law-invariant convex risk measure ρ. Here the risk measure is of the form

$$\rho(X) = \sup_{\mu}\{\rho_\mu(X) - \beta(\mu)\}, \tag{9}$$

where the penalty function β for probability measures μ on $(0,1]$ is given by

$$\beta(\mu) = \sup_{X\in\mathcal{A}_\rho} \rho_\mu(X) = \sup_{X\in\mathcal{A}_\rho} \int_{(0,1]} \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda),$$

and where $\mathcal{A}_\rho := \{X|\rho(X) \leq 0\}$ denotes the class of positions X which are acceptable for ρ. In Theorem 3.3 we show that the asymptotic behavior of the risk premia is the same for ρ as for the coherent risk measure associated to $\mathcal{M}_\rho := \{\mu|\beta(\mu) < \infty\}$ via (6). Here we need two conditions: The class $\mathcal{M}_\rho$ should satisfy (7), and the penalty function should remain bounded on $\mathcal{M}_\rho$. In the coherent case we have $\beta \equiv 0$ on $\mathcal{M}_\rho$, and so the second condition is satisfied trivially. In the convex case, the second condition follows from the first if the exponential moments of the amount X^- of the loss stay bounded for positions X which are acceptable for ρ; cf. Lemma 3.2. The convex entropic risk measures e_γ satisfy this last condition. But they do not satisfy the first, and this explains why they do not have the desired convergence property.

In Section 4 we illustrate our result for law-invariant comonotonic risk measures by viewing the risk measures ρ_μ as Choquet integrals with respect to some concave distortion of the underlying probability measure P, and by considering the special distortions proposed by S. S. Wang in [19]. We also show how the coherent entropic risk measure ρ_c and the corresponding results in [8] fit into our general framework. As a further example, we introduce a truncated version $e_{\gamma,c}$ of the convex entropic risk measure e_γ, and we show that the asymptotic behavior of the induced risk premia is the same as for the coherent entropic risk measure ρ_c.

In the final Section 5 we discuss an extension of our results beyond the law-invariant case. Here we limit the discussion to coherent risk measures of the form

$$\rho_{\mathcal{P},\mathcal{M}} := \sup_{P\in\mathcal{P}} \rho_{P,\mathcal{M}},$$

where $\mathcal{P}$ is some class of reference measures, and where $\rho_{P,\mathcal{M}}$ is defined via (6) for a given $P \in \mathcal{P}$. These risk measures can be viewed as a robustification of the law-invariant coherent risk measures in (6). We formulate conditions which guarantee that the corresponding premia converge to the robustified fair premium

$$\sup_{P\in\mathcal{P}} E_P[-X_1],$$

and that they do so at the rate $n^{-1/2}$, in analogy to (8); cf. Theorem 5.1.

2 Preliminaries

First we recall some basic definitions and facts from the theory of risk measures, first developed by Artzner, Delbaen, Eber & Heath [1] and Delbaen [5] in the coherent case and then extended to the general convex case by Föllmer & Schied [9] and Frittelli & Rosazza Gianin [11]; cf. also Deprez & Gerber [6] for an earlier development in the context of actuarial premium principles. We refer to [10] for further details and a more extensive list of references, and also to Song & Yan [17].

A functional ρ on the space $\mathfrak{X}$ of bounded measurable functions on some measurable space $(\Omega, \mathcal{F})$ is called a *monetary risk measure* if it is

i) *monotone*, i. e., $\rho(X) \leq \rho(Y)$ if $X \geq Y$,

and

ii) *cash-invariant*, i. e., $\rho(X + m) = \rho(X) - m$ for $X \in \mathfrak{X}$ and $m \in \mathbb{R}$.

Such a monetary risk measure is called a *convex* risk measure if it is *quasi-convex*, i. e., if

$$\rho(\alpha X + (1-\alpha)Y) \leq \max\{\rho(X), \rho(Y)\}$$

for all positions $X, Y \in \mathfrak{X}$ and $\alpha \in (0,1)$, and in that case ρ is indeed a convex functional on $\mathfrak{X}$. A convex risk measure is called *coherent* if it is positively homogeneous, i. e.,

$$\rho(\lambda X) = \lambda \rho(X)$$

for all $X \in \mathfrak{X}$ and $\lambda \geq 0$, and in this case ρ is *normalized*, i. e., $\rho(0) = 0$.

Now let P be a probability measure on $(\Omega, \mathcal{F})$ such that the probability space $(\Omega, \mathcal{F}, P)$ is atomless. We assume that $\rho(X) = \rho(Y)$ as soon as $X = Y$ P-a. s.. Then ρ can be viewed as a functional on $L^\infty(P)$.

Definition 2.1. *A monetary risk measure ρ on $L^\infty(P)$ is called* law-invariant *if $\rho(X)$ only depends on the distribution of X under P, i. e., if $\rho(X) = \rho(Y)$ whenever X and Y have the same distribution under P.*

Any law-invariant and normalized convex risk measure ρ satisfies

$$\rho(X) \geq E_P[-X]; \tag{10}$$

cf., e. g., [10], Corollary 4.65. In particular, the corresponding premia $\pi_n := \frac{1}{n}\rho(S_n)$ satisfy

$$\pi_n \geq E_P[-X_1].$$

Moreover, law-invariance of a convex risk measure ρ on $L^\infty(P)$ implies continuity from above, as shown by Jouini, Schachermayer & Touzi [13], and so ρ admits the robust representation

$$\rho(X) = \sup_{Q \ll P} \{E_Q[-X] - \alpha(Q)\} \tag{11}$$

with some penalty function α on the class of probability measures Q on $(\Omega, \mathcal{F})$; cf., e. g., [10], Theorem 4.33. A remarkable characterization of law-invariant coherent or convex risk measures in terms of comonotonic subadditivity or convexity and of monotonicity with respect to stochastic orders is given in Song & Yan [18].

Now consider the special case of Average Value at Risk (AVaR), defined for $\lambda \in (0,1]$ by

$$\begin{aligned} \mathrm{AVaR}_\lambda(X) :=& \tfrac{1}{\lambda} \int_0^\lambda \mathrm{VaR}_\alpha(X)\, d\alpha \\ =& \tfrac{1}{\lambda} E_P[(q(\lambda) - X)^+] - q(\lambda) \end{aligned} \tag{12}$$

for any λ-quantile $q(\lambda)$ of X; cf., e. g., [10], Lemma 4.51. This definition can be extended to $\lambda = 0$ via

$$\mathrm{AVaR}_0(X) := \mathrm{VaR}_0(X) := \lim_{\lambda \downarrow 0} \mathrm{VaR}_\lambda(X) = \operatorname{ess\,sup}(-X). \tag{13}$$

AVaR is coherent, and it admits the robust representation

$$\mathrm{AVaR}_\lambda(X) = \max_{Q \in \mathcal{Q}_\lambda} E_Q[-X] \tag{14}$$

with $\mathcal{Q}_\lambda := \{Q \ll P | \frac{dQ}{dP} \leq \frac{1}{\lambda}\}$; cf., e. g., [10], Theorem 4.52. For any $Q \in \mathcal{Q}_\lambda$ we have $H(Q|P) \leq -\log \lambda$. In view of the definition (1) of the coherent risk measures ρ_c, the representation (14) thus implies the estimate

$$\mathrm{AVaR}_\lambda(X) \leq \rho_{-\log \lambda}(X); \tag{15}$$

cf. [8], Proposition 3.2.

As mentioned already in the introduction, law-invariance of a convex risk measure ρ implies that the representation (11) reduces to a representation of ρ in terms of mixtures (5) of Average Value at Risk. In the coherent case this takes the form (6); in the general convex case it involves a penalization of the mixing measures as in (9). Since $\mathrm{AVaR}_\lambda(X) \geq E_P[-X]$ due to (10) or, more directly, (14), the representations in (6) and (9) are well defined for any $X \in L^1(P)$, and they yield a natural extension of ρ from $L^\infty(P)$ to a law-invariant convex functional $\rho : L^1(P) \to (-\infty, \infty]$; cf. Filipović & Svindland [7].

In the sequel we will make repeated use of Young's inequality

$$E_P[|XY|] \leq 2\|X\|_h \|Y\|_{h^*} \tag{16}$$

with respect to the convex functions h and h^* defined by

$$h(x) = e^{|x|} - 1 \tag{17}$$

and

$$h^*(y) = (|y| \log |y| - |y| + 1) 1_{[1,\infty)}(y); \tag{18}$$

cf., e. g., Neveu [16], Appendix A.2. Recall that the Orlicz norm $\|\cdot\|_h$ is defined by

$$\|X\|_h := \inf\{a > 0 | E_P[h(\tfrac{|X|}{a})] \leq 1\}.$$

We denote by $L^h(P)$ the corresponding Orlicz space of all random variables X such that $\|X\|_h < \infty$. The Orlicz norm $\|\cdot\|_{h^*}$ and the Orlicz space $L^{h^*}(P)$ are defined in the same way.

Remark 2.1. *Clearly, $\|X\|_h < \infty$ iff $E_P[e^{\alpha|X|}] < \infty$ for some $\alpha > 0$, and in that case we have*

$$\|X\|_h \leq \tfrac{1}{\alpha} \max\{\tfrac{1}{\log 2} \log E_P[e^{\alpha|X|}], 1\}. \tag{19}$$

Moreover,

$$\|Y\|_{h^*} < \infty \quad \Leftrightarrow \quad E_P[|Y| \log |Y|] < \infty,$$

and in this case we have

$$\|Y\|_{h^*} \leq \max\{1, E_P[|Y| \log |Y|]\}. \tag{20}$$

Note also that for an indicator function $Y = 1_A$ we get

$$E_P[h^*(\tfrac{1}{a} 1_A)] = h^*(\tfrac{1}{a}) P[A],$$

hence

$$\|1_A\|_{h^*} = \frac{1}{(h^*)^{-1}\left(\frac{1}{P[A]}\right)}, \tag{21}$$

where $(h^*)^{-1}$ denotes the inverse function of h^*.

3 Asymptotics of convex risk measures for large portfolios: the law-invariant case

Consider a portfolio of n financial positions whose outcomes are described as random variables $X_1, \ldots, X_n$ on our atomless probability space $(\Omega, \mathcal{F}, P)$.

Assumption 3.1. *We assume that the random variables $X_1, X_2, \ldots$ are independent and identically distributed under P, and that X_1 has exponential moments of any order, i. e.,*

$$E_P[e^{\alpha|X_1|}] < \infty \quad \text{for any } \alpha > 0. \tag{22}$$

We also assume that the distribution of X_1 under P is non-degenerate and denote by $\sigma_P^2 > 0$ the variance of X_1 with respect to P.

Remark 3.1. *We restrict the discussion to the classical i. i. d. case, but only for the ease of exposition. We do need bounded exponential moments. But the proofs remain valid under much weaker conditions of homogeneity and weak dependence for the underlying sequence $X_1, X_2, \ldots$, as long as the standardized sums satisfy the central limit theorem and we retain control over their exponential moments.*

Let ρ be a convex risk measure which is law-invariant and normalized. In view of (10), the capital requirements $\rho(S_n)$ for the aggregate positions

$$S_n := X_1 + \ldots + X_n$$

are well defined, and the corresponding premia $\pi_n = \frac{1}{n}\rho(S_n)$ are bounded from below by the fair premium $E_P[-X_1]$. Moreover, our Assumption 3.1 together with condition (7) will imply $\pi_n < \infty$, as shown in Lemma 3.1 below.

Our aim is to clarify the behavior of the premia π_n when the portfolio becomes large. We will proceed in several steps, guided by the representation (9) of the risk measure ρ.

3.1 The building blocks: Average Value at Risk

In a first step, we focus on the coherent risk measures AVaR_λ, $\lambda \in [0, 1]$, and on the associated capital requirements per position, or insurance premia per contract, defined by

$$\pi_n^\lambda := \tfrac{1}{n} \mathrm{AVaR}_\lambda(S_n).$$

For $\lambda = 0$ we have

$$\mathrm{AVaR}_0(S_n) = \operatorname{ess\,sup}(-S_n) = n \operatorname{ess\,sup}(-X_1),$$

hence

$$\pi_n^0 = \operatorname{ess\,sup}(-X_1), \quad n \in \mathbb{N},$$

and so the pooling of risks does not reduce the capital requirement per position.

For $\lambda \in (0, 1]$, however, we have

$$E_P[-X_1] = \tfrac{1}{n} E_P[-S_n] \leq \tfrac{1}{n} \mathrm{AVaR}_\lambda(S_n) \leq \tfrac{1}{n} \rho_{-\log \lambda}(S_n),$$

due to (14) and (15), and the right-hand side decreases to $E_P[-X_1]$ as shown in [8], Corollary 4.1.

We are now going to analyze the decay of the "risk premium" $\pi_n^\lambda - E_P[-X_1]$ more precisely.

Proposition 3.1. *For any $\lambda \in (0, 1]$, the rate of decay is given by*

$$\lim_{n \uparrow \infty} \sqrt{n}(\pi_n^\lambda - E_P[-X_1]) = \sigma_P \,\mathrm{AVaR}_\lambda(Z) \tag{23}$$

$$= \sigma_P \tfrac{1}{\lambda} \varphi(\Phi^{-1}(\lambda)), \tag{24}$$

where Z is standard normally distributed, and where φ and Φ denote the density and the distribution function of the standard normal distribution.

Proof. Since AVaR_λ is cash-invariant and positively homogeneous, we can write

$$\begin{aligned}\sqrt{n}(\pi_n^\lambda - E_P[-X_1]) =& \tfrac{1}{\sqrt{n}}(\mathrm{AVaR}_\lambda(S_n) - nE_P[-X_1]) \\ =& \sigma_P \,\mathrm{AVaR}_\lambda(S_n^*) \end{aligned} \tag{25}$$

in terms of the standardized random variables

$$S_n^* := \tfrac{S_n - nE_P[X_1]}{\sqrt{n}\sigma_P}.$$

We are going to show that

$$\lim_{n\uparrow\infty} \mathrm{AVaR}_\lambda(S_n^*) = \mathrm{AVaR}_\lambda(Z). \tag{26}$$

Indeed, the central limit theorem yields weak convergence of the distributions of S_n^* to the standard normal distribution. For any choice of the quantile functions q_n of S_n^*, this is equivalent to the pointwise convergence

$$\lim_{n\uparrow\infty} q_n(\alpha) = \Phi^{-1}(\alpha), \quad \alpha \in (0,1), \tag{27}$$

to the quantile function Φ^{-1} of Z; cf., e. g., [10], Remark A.40. Moreover,

$$\sup_{n\in\mathbb{N}} \int_0^1 (q_n(\alpha))^2 \, d\alpha = \sup_{n\in\mathbb{N}} E_P[(S_n^*)^2] = 1,$$

and this ensures uniform integrability of the sequence q_n, $n \in \mathbb{N}$, with respect to Lebesque measure on $(0,1)$. Applying Lebesgue's convergence theorem, we obtain

$$\lim_{n\uparrow\infty} \int_0^\lambda q_n(\alpha)\, d\alpha = \int_0^\lambda \Phi^{-1}(\alpha)\, d\alpha,$$

and this translates into (26), since $\mathrm{VaR}_\alpha(S_n) = -q_n(\alpha)$ a. e. on $(0,1)$ and $\mathrm{VaR}_\alpha(Z) = -\Phi^{-1}(\alpha)$. Using the substitution $x = \Phi^{-1}(\alpha)$ and $\varphi'(x) = -x\varphi(x)$, we also see that

$$\begin{aligned}\mathrm{AVaR}_\lambda(Z) =& \tfrac{1}{\lambda} \int_0^\lambda -\Phi^{-1}(\alpha)\, d\alpha = \tfrac{1}{\lambda} \int_{-\infty}^{\Phi^{-1}(\lambda)} -x\varphi(x)\, dx \\ =& \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda)). \end{aligned} \tag{28}$$

□

3.2 The comonotonic case

Let us next analyze the case where the coherent risk measure is of the form

$$\rho_\mu(X) = \int_0^1 \mathrm{AVaR}_\lambda(X)\, \mu(d\lambda) \tag{29}$$

for some probability measure μ on $[0,1]$. Such a risk measure ρ_μ is comonotonic, and any law-invariant comonotonic risk measure is of this form; cf., e. g., [10], Theorem 4.93. In this case we have

$$\pi_{\mu,n} := \tfrac{1}{n}\rho_\mu(S_n) = \int_0^1 \pi_n^\lambda \,\mu(d\lambda), \quad n \in \mathbb{N}. \tag{30}$$

If $\mu[\{0\}] > 0$, then

$$\rho_\mu(S_n) \geq \mu[\{0\}] n \operatorname{ess\,sup}(-X_1) + (1 - \mu[\{0\}]) nE_P[-X_1]$$

and

$$\varliminf_{n\uparrow\infty} (\pi_{\mu,n} - E_P[-X_1]) \geq \mu[\{0\}](\operatorname{ess\,sup}(-X_1) - E_P[-X_1]) > 0,$$

i. e., the desired convergence of $\pi_{\mu,n}$ to the "fair premium" $E_P[-X_1]$ does not take place.

From now on we only consider the case where μ is concentrated on $(0,1]$. In addition we impose the integrability condition

$$\int_0^1 \log\left(\tfrac{1}{\lambda}\right)\, \mu(d\lambda) < \infty. \tag{31}$$

Condition (31) guarantees that ρ_μ is finite on the Orlicz space $L^h(P)$, where h is the convex function defined in (17):

Lemma 3.1. *Condition (31) holds if and only if*

$$\int_{(0,1]} \frac{1}{\lambda (h^*)^{-1}\left(\frac{1}{\lambda}\right)}\, \mu(d\lambda) < \infty, \tag{32}$$

where $(h^*)^{-1}$ denotes the inverse function of h^* in (18), and

$$\int_{(0,1]} \frac{1}{\lambda (h^*)^{-1}\left(\frac{1}{\lambda}\right)}\, \mu(d\lambda) \leq \int_{(0,1]} \log\left(\tfrac{1}{\lambda}\right)\, \mu(d\lambda) + 1. \tag{33}$$

In this case $\rho_\mu(X)$ is finite for any $X \in L^h(P)$ and satisfies

$$|\rho_\mu(X)| \leq 2\|X\|_h \int_{(0,1]} \frac{1}{\lambda (h^*)^{-1}\left(\frac{1}{\lambda}\right)}\, \mu(d\lambda). \tag{34}$$

Proof. 1. For any choice of a quantile function q_X for X we have

$$\begin{aligned} |\rho_\mu(X)| &\leq \int_0^1 \tfrac{1}{\lambda} \int_0^\lambda |\operatorname{VaR}_\alpha(X)|\, d\alpha\, \mu(d\lambda) \\ &= \int_0^1 \tfrac{1}{\lambda} \int_0^1 |q_X(\alpha)| 1_{[0,\lambda]}(\alpha)\, d\alpha\, \mu(d\lambda). \end{aligned}$$

Young's inequality (16) applied to the inner integral together with formula (21) yields

$$\begin{aligned} |\rho_\mu(X)| &\leq \int_0^1 \tfrac{1}{\lambda} \cdot 2\|q_X\|_h \frac{1}{(h^*)^{-1}\left(\frac{1}{\lambda}\right)}\, \mu(d\lambda) \\ &= 2\|X\|_h \int_0^1 \frac{1}{\lambda (h^*)^{-1}\left(\frac{1}{\lambda}\right)}\, \mu(d\lambda), \end{aligned}$$

since the Orlicz norm $\|q_X\|_h$ with respect to Lebesgue measure on $(0,1)$ coincides with the Orlicz norm $\|X\|_h$ with respect to P.

2. It remains to show the equivalence of conditions (31) and (32). Let us denote by $g := (h^*)^{-1}$ the inverse function of h^*. Both integrands are bounded on $(\epsilon, 1]$ for any $\epsilon > 0$. Since

$$h^*\left(\frac{x}{\log x}\right) = x + 1 - \tfrac{x}{\log x}(\log\log x + 1) \leq x$$

for $x \geq e$, we get

$$\tfrac{g(x)}{x} \geq \tfrac{1}{\log x}$$

for $x \geq e$, hence

$$\log\left(\tfrac{1}{\lambda}\right) \geq \frac{1}{\lambda g\left(\frac{1}{\lambda}\right)} \tag{35}$$

for $\lambda \leq e^{-1}$. This implies (33), since

$$\int_{(0,1]} \frac{1}{\lambda g\left(\frac{1}{\lambda}\right)}\, \mu(d\lambda) \leq \int_0^{e^{-1}} \log\left(\tfrac{1}{\lambda}\right)\, \mu(d\lambda) + \tfrac{e}{g(e)} \leq \int_0^1 \log\left(\tfrac{1}{\lambda}\right)\, \mu(d\lambda) + 1.$$

In particular, (32) follows from (31).

Conversely, for any $\delta > 0$ we have

$$h^*\left(\tfrac{x}{\log x}\right) \geq x(1-\delta),$$

and hence $g(x(1-\delta)) \leq x(\log x)^{-1}$ for large enough x. This amounts to

$$\frac{1}{\lambda g\left(\frac{1}{\lambda}\right)} \geq (1-\delta)\log\left(\tfrac{1}{\lambda}\right)$$

for $\lambda \leq \lambda(\delta)$, and so (32) implies (31). $\square$

Remark 3.2. *Assumption 3.1 ensures that the aggregate positions S_n belong to the Orlicz heart*

$$M^h := \{X \in L^h(P) | E_P[h(\alpha|X|)] < \infty \quad \text{for all } \alpha > 0\} \subset L^h(P)$$

with respect to the Young function h in (17), and so the capital requirements $\rho_\mu(S_n)$ are well defined and finite due to (34). For a systematic discussion of risk measures on Orlicz hearts and of their dual representations we refer to Cheridito & Li [2][3].

Remark 3.3. *For a standard normal random variable Z we have*

$$\rho_\mu(Z) = \int_{(0,1]} \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\, \mu(d\lambda),$$

due to (28). Thus the estimates (33) and (34) show that condition (31) implies

$$\int_{(0,1]} \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\, \mu(d\lambda) < \infty.$$

We are now ready to identify the rate of decay of the risk premium $\pi_{\mu,n} - E_P[-X_1]$.

Theorem 3.1. *Under condition (31), the premia $\pi_{\mu,n}$ converge to the fair premium $E_P[-X_1]$, and the decay of the risk premium $\pi_{\mu,n} - E_P[X_1]$ is described by*

$$\lim_{n\uparrow\infty} \sqrt{n}(\pi_{\mu,n} - E_P[-X_1]) = \sigma_P \int_{(0,1]} \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\, \mu(d\lambda). \tag{36}$$

Proof. 1. In view of (30) and (25) we have

$$\sqrt{n}(\pi_{\mu,n} - E_P[-X_1]) = \sigma_P \int_{(0,1]} \mathrm{AVaR}_\lambda(S_n^*)\, \mu(d\lambda). \tag{37}$$

It is thus enough to show that $\int_{(0,1]} \mathrm{AVaR}_\lambda(S_n^*)\, \mu(d\lambda)$ converges to $\int_{(0,1]} \mathrm{AVaR}_\lambda(Z)\, \mu(d\lambda)$, where Z is standard normally distributed. Denoting by q_n any quantile function of S_n^*, we obtain the estimate

$$\begin{aligned}
|\int_0^1 \mathrm{AVaR}_\lambda(S_n^*)\, \mu(d\lambda) - \int_0^1 \mathrm{AVaR}_\lambda(Z)\, \mu(d\lambda)| &\leq \int_0^1 |\,\mathrm{AVaR}_\lambda(S_n^*) - \mathrm{AVaR}_\lambda(Z)|\, \mu(d\lambda) \\
&= \int_0^1 |\tfrac{1}{\lambda} \int_0^\lambda -q_n(\alpha) + \Phi^{-1}(\alpha)\, d\alpha|\, \mu(d\lambda) \\
&\leq \int_0^1 \tfrac{1}{\lambda} \int_0^1 |q_n(\alpha) - \Phi^{-1}(\alpha)| 1_{(0,\lambda]}(\alpha)\, d\alpha\, \mu(d\lambda).
\end{aligned}$$

Applying Young's inequality (16) to the interior integral, we see that

$$\begin{aligned}\int_0^1 |q_n(\alpha)-\Phi^{-1}(\alpha)|1_{(0,\lambda]}(\alpha)\,d\alpha \leq & 2\|q_n-\Phi^{-1}\|_h\|1_{(0,\lambda]}\|_{h^*}\\ =&2\|q_n-\Phi^{-1}\|_h\frac{1}{(h^*)^{-1}\left(\frac{1}{\lambda}\right)},\end{aligned}$$

where $(h^*)^{-1}$ denotes the inverse function of h^*. But this translates into the estimate

$$|\int_0^1 \mathrm{AVaR}_\lambda(S_n^*)\,\mu(d\lambda)-\int_0^1 \mathrm{AVaR}_\lambda(Z)\,\mu(d\lambda)|\leq 2\|q_n-\Phi^{-1}\|_h\int_{(0,1]}\frac{1}{\lambda(h^*)^{-1}\left(\frac{1}{\lambda}\right)}\,\mu(d\lambda). \tag{38}$$

Step 2 of this proof will show that $\lim_{n\uparrow\infty}\|q_n-\Phi^{-1}\|_h=0$. Combined with Lemma 3.1, this yields

$$\lim_{n\uparrow\infty}|\int_0^1 \mathrm{AVaR}_\lambda(S_n^*)\,\mu(d\lambda)-\int_0^1 \mathrm{AVaR}_\lambda(Z)\,\mu(d\lambda)|=0,$$

hence

$$\begin{aligned}\lim_{n\uparrow\infty}\sqrt{n}(\pi_{\mu,n}-E_P[-X_1]) =&\sigma_P\lim_{n\uparrow\infty}\int_{(0,1]}\mathrm{AVaR}_\lambda(S_n^*)\,\mu(d\lambda)\\ =&\sigma_P\int_{(0,1]}\mathrm{AVaR}_\lambda(Z)\,\mu(d\lambda)\\ =&\sigma_P\int_{(0,1]}\tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\,\mu(d\lambda).\end{aligned}$$

2. Let us check that $\lim_{n\uparrow\infty}\|q_n-\Phi^{-1}\|_h=0$. For this purpose, it is enough to show that

$$\lim_{n\uparrow\infty}\int_0^1 e^{\beta|q_n(\alpha)-\Phi^{-1}(\alpha)|}\,d\alpha=1 \tag{39}$$

for any $\beta>0$. Indeed, this implies

$$\int_0^1 (e^{\beta|q_n(\alpha)-\Phi^{-1}(\alpha)|}-1)\,d\alpha\leq 1$$

for all $n\geq n_0(\beta)$, hence $\|q_n-\Phi^{-1}\|_h\leq\frac{1}{\beta}$ for $n\geq n_0(\beta)$.

To verify (39) for given $\beta>0$, note that $f_n(\alpha):=e^{\beta|q_n(\alpha)-\Phi^{-1}(\alpha)|}$ converges to 1 for all $\alpha\in(0,1)$, due to (27). Step 3 shows that the sequence f_n, $n\in\mathbb{N}$, also satisfies

$$\sup_{n\in\mathbb{N}}\int_0^1 (f_n(\alpha))^p\,d\alpha<\infty\quad\text{for } p>1, \tag{40}$$

and is thus uniformly integrable with respect to Lebesgue measure on $(0,1)$. Using Lebesgue's convergence theorem, we obtain (39).

3. It remains to verify (40). Indeed, applying Hölder's inequality we get

$$\begin{aligned}\int_0^1 (f_n(\alpha))^p\,d\alpha\leq&\int_0^1 e^{p\beta|q_n(\alpha)|}e^{p\beta|\Phi^{-1}(\alpha)|}\,d\alpha\\ \leq&(\int_0^1 e^{2p\beta|q_n(\alpha)|}\,d\alpha)^{\frac{1}{2}}(\int_0^1 e^{2p\beta|\Phi^{-1}(\alpha)|}\,d\alpha)^{\frac{1}{2}}\\ =&(E_P[e^{\gamma|S_n^*|}])^{\frac{1}{2}}(E_P[e^{\gamma|Z|}])^{\frac{1}{2}}\end{aligned}$$

for $\gamma := 2p\beta > 0$ and for a standard normally distributed random variable Z. In order to verify that $E_P[e^{\gamma|S_n^*|}]$ stays bounded for all $n \in \mathbb{N}$, we may assume $E_P[X_1] = 0$ so that $S_n^* = \frac{1}{\sqrt{n}\sigma_P} S_n$. Assumption 3.1 ensures that both $E_P[e^{\gamma X_1}]$ and $E_P[e^{-\gamma X_1}]$ are finite. The function

$$Z(\lambda) := \log E_P[e^{\lambda X_1}]$$

is smooth and satisfies

$$Z(\lambda) = \tfrac{1}{2}\lambda^2 \sigma_P^2 + o(\lambda^2).$$

Thus

$$\log E_P[e^{\gamma S_n^*}] = \log E_P[e^{\frac{\gamma}{\sqrt{n}\sigma_P} S_n}] = nZ(\tfrac{\gamma}{\sqrt{n}\sigma_P})$$

converges to $\frac{1}{2}\gamma^2$ and hence stays bounded. Applying the same argument to $-X_1$ instead of X_1, we see that

$$E_P[e^{\gamma|S_n^*|}] \leq E_P[e^{\gamma S_n^*}] + E_P[e^{-\gamma S_n^*}]$$

is bounded. We have thus shown (40), and this completes the proof. □

3.3 The coherent case

We are now going to consider the case of a general law-invariant coherent risk measure. Such a risk measure is of the form

$$\rho_{\mathcal{M}}(X) = \sup_{\mu \in \mathcal{M}} \int_0^1 \mathrm{AVaR}_\lambda(X)\, \mu(d\lambda) \tag{41}$$

for some subclass $\mathcal{M}$ of the class $\mathcal{M}_1((0,1])$ of probability measures on $(0,1]$; cf., e. g., [10], Theorem 4.62 and Remark 4.64. In this case the premium $\pi_{\mathcal{M},n} := \frac{1}{n}\rho_{\mathcal{M}}(S_n)$ computed in terms of $\rho_{\mathcal{M}}$ takes the form

$$\pi_{\mathcal{M},n} = \sup_{\mu \in \mathcal{M}} \pi_{\mu,n} = \sup_{\mu \in \mathcal{M}} \int_0^1 \pi_n^\lambda\, \mu(d\lambda), \quad n \in \mathbb{N}.$$

The following theorem describes the decay of the risk premium $\pi_{\mathcal{M},n} - E_P[-X_1]$ for a class $\mathcal{M}$ of mixing measures such that

$$\sup_{\mu \in \mathcal{M}} \int_0^1 \log\left(\tfrac{1}{\lambda}\right)\, \mu(d\lambda) < \infty. \tag{42}$$

Remark 3.4. *The estimate (33) implies*

$$\sup_{\mu \in \mathcal{M}} \int_{(0,1]} \frac{1}{\lambda (h^*)^{-1}\left(\frac{1}{\lambda}\right)}\, \mu(d\lambda) \leq \sup_{\mu \in \mathcal{M}} \int_0^1 \log\left(\tfrac{1}{\lambda}\right)\, \mu(d\lambda) + 1.$$

Thus condition (42) guarantees that the left-hand side is finite, and it follows as in Remark 3.3 that

$$\sup_{\mu \in \mathcal{M}} \int_{(0,1]} \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\, \mu(d\lambda) < \infty.$$

Theorem 3.2. *Under condition (42) the premia $\pi_{\mathcal{M},n}$ converge to the fair premium $E_P[-X_1]$, and the decay of the risk premia $\pi_{\mathcal{M},n} - E_P[-X_1]$ is described by*

$$\lim_{n\uparrow\infty} \sqrt{n}(\pi_{\mathcal{M},n} - E_P[-X_1]) = \sigma_P \sup_{\mu \in \mathcal{M}} \int_{(0,1]} \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\, \mu(d\lambda). \tag{43}$$

Proof. Since $\pi_{\mathcal{M},n} = \sup_{\mu\in\mathcal{M}} \pi_{\mu,n}$, the identity (37) yields

$$\sqrt{n}(\pi_{\mathcal{M},n} - E_P[-X_1]) = \sigma_P \sup_{\mu\in\mathcal{M}} \int_{(0,1]} \mathrm{AVaR}_\lambda(S_n^*)\,\mu(d\lambda),$$

and so the claim follows if we can prove the uniform convergence

$$\lim_{n\uparrow\infty} \sup_{\mu\in\mathcal{M}} \Big| \int_0^1 \mathrm{AVaR}_\lambda(S_n^*)\,\mu(d\lambda) - \int_0^1 \mathrm{AVaR}_\lambda(Z)\,\mu(d\lambda)\Big| = 0, \tag{44}$$

where Z is a standard normally distributed random variable. Indeed, the estimate (38) yields

$$\sup_{\mu\in\mathcal{M}} \Big| \int_0^1 \mathrm{AVaR}_\lambda(S_n^*)\,\mu(d\lambda) - \int_0^1 \mathrm{AVaR}_\lambda(Z)\,\mu(d\lambda)\Big| \le 2\|q_n - \Phi^{-1}\|_h \sup_{\mu\in\mathcal{M}} \int_{(0,1]} \frac{1}{\lambda(h^*)^{-1}\left(\frac{1}{\lambda}\right)}\,\mu(d\lambda).$$

By Remark 3.4 the last term is bounded, and

$$\lim_{n\uparrow\infty} \|q_n - \Phi^{-1}\|_h = 0,$$

as shown in step 2 in the proof of Theorem 3.1. This implies (44), and in view of $\mathrm{AVaR}_\lambda(Z) = \frac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))$ we have verified (43). □

We denote by q_μ the function on $(0,1)$ associated to $\mu \in \mathcal{M}_1((0,1])$ via

$$q_\mu(t) := \int_{(1-t,1]} \tfrac{1}{s}\,\mu(ds).$$

Corollary 3.1. *For a law-invariant coherent risk measure $\rho_{\mathcal{M}}$, defined by (41) for a class $\mathcal{M} \subseteq \mathcal{M}_1((0,1])$, the convergence in (43) holds if*

$$\sup_{\mu\in\mathcal{M}} \int_{(0,1]} q_\mu(t)\log q_\mu(t)\,dt < \infty. \tag{45}$$

Proof. We verify that (45) implies condition (42). Indeed, in analogy to the proof of Lemma 4.1 we obtain

$$\int_0^1 \log\left(\tfrac{1}{\lambda}\right)\,\mu(d\lambda) = \int_0^1 \log\left(\tfrac{1}{t}\right) q_\mu(1-t)\,dt - 1,$$

and so condition (42) is equivalent to

$$\sup_{\mu\in\mathcal{M}} \int_0^1 \log\left(\tfrac{1}{t}\right) q_\mu(1-t)\,dt < \infty. \tag{46}$$

Applying Young's inequality (16) for the uniform distribution on $[0,1]$ and for the functions h and h^* in (17) and (18), we see that

$$\sup_{\mu\in\mathcal{M}} \int_0^1 \log\left(\tfrac{1}{t}\right) q_\mu(1-t)\,dt \le 2\|\log\left(\tfrac{1}{\cdot}\right)\|_h \sup_{\mu\in\mathcal{M}} \|q_\mu\|_{h^*}.$$

Here $\|\log\left(\frac{1}{\cdot}\right)\|_h$ is finite due to Remark 2.1, since

$$\int_0^1 e^{\alpha\log\left(\frac{1}{t}\right)}\,dt = \int_0^1 t^{-\alpha}\,dt < \infty$$

for any $\alpha \in (0,1)$. Finiteness of $\sup_{\mu\in\mathcal{M}} \|q_\mu\|_{h^*}$ follows from condition (45) and inequality (20). □

3.4 The convex case

Recall from the Introduction that any law-invariant convex risk measure ρ has the form

$$\rho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \left\{ \int_{(0,1]} \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda) - \beta(\mu) \right\}, \tag{47}$$

where the penalty function β on $\mathcal{M}_1((0,1])$ is given by

$$\beta(\mu) = \sup_{X \in \mathcal{A}_\rho} \int_{(0,1]} \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda);$$

cf., e. g., [10], Theorem 4.62. Let us define

$$\mathcal{M}_\rho := \{\mu \in \mathcal{M}_1((0,1]) | \beta(\mu) < \infty\}.$$

Under the condition

$$\sup_{\mu \in \mathcal{M}_\rho} \int_{(0,1]} \log\left(\tfrac{1}{\lambda}\right) \mu(d\lambda) < \infty, \tag{48}$$

the risk measure ρ has a natural extension from $L^\infty(P)$ to the Orlicz space $L^h(P)$, with

$$|\rho(X)| \leq 2\|X\|_h \sup_{\mu \in \mathcal{M}_\rho} \int_{(0,1]} \frac{1}{\lambda (h^*)^{-1}\left(\frac{1}{\lambda}\right)}\,\mu(d\lambda);$$

cf. Lemma 3.1 and Remark 3.4.

In addition we are going to assume the condition

$$\sup_{\mu \in \mathcal{M}_\rho} \beta(\mu) < \infty. \tag{49}$$

This condition is clearly satisfied in the coherent case, since then we have $\beta \equiv 0$ on $\mathcal{M}_\rho$. In the convex case it holds if for any acceptable position $X \in \mathcal{A}_\rho$ the Orlicz norm of its negative part $X^- := \max\{-X, 0\}$ does not exceed a given threshold:

Lemma 3.2. *Condition (49) is satisfied if, in addition to (48),*

$$\sup_{X \in \mathcal{A}_\rho} \|X^-\|_h < \infty \tag{50}$$

Proof. We have

$$\begin{aligned}\int_{(0,1]} \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda) \leq& \int_{(0,1]} \mathrm{AVaR}_\lambda(-X^-)\,\mu(d\lambda)\\ \leq& 2\|X^-\|_h \sup_{\mu \in \mathcal{M}_\rho} \int_{(0,1]} \frac{1}{\lambda (h^*)^{-1}\left(\frac{1}{\lambda}\right)}\,\mu(d\lambda)\end{aligned}$$

by (34), and so we get

$$\sup_{\mu \in \mathcal{M}_\rho} \beta(\mu) \leq 2 \sup_{X \in \mathcal{A}_\rho} \|X^-\|_h \sup_{\mu \in \mathcal{M}_\rho} \int_{(0,1]} \frac{1}{\lambda (h^*)^{-1}\left(\frac{1}{\lambda}\right)}\,\mu(d\lambda) < \infty,$$

using our assumptions (48) and (50) and Remark 3.4. □

Under conditions (48) and (49) we are now going to show that the asymptotics for ρ coincides with the asymptotics for the coherent risk measure corresponding to the class $\mathcal{M}_\rho$ via (6).

Theorem 3.3. *Consider a convex risk measure ρ of the form (47) which satisfies conditions (48) and (49). Then the premia $\pi_n := \frac{1}{n}\rho(S_n)$ converge to the fair premium $E_P[-X_1]$, and*

$$\lim_{n\uparrow\infty} \sqrt{n}(\pi_n - E_P[-X_1]) = \sigma_P \sup_{\mu\in\mathcal{M}_\rho} \int_{(0,1]} \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\,\mu(d\lambda). \tag{51}$$

Proof. In order to verify (51), consider the coherent risk measure

$$\tilde{\rho}(X) := \sup_{\mu\in\mathcal{M}_\rho} \int_{(0,1]} \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda) \tag{52}$$

and denote by $\tilde{\pi}_n := \frac{1}{n}\tilde{\rho}(S_n)$, $n \in \mathbb{N}$, the corresponding capital requirements per position. Note that $\tilde{\rho} \geq \rho$, hence $\tilde{\pi}_n \geq \pi_n$. Thus we have

$$\begin{aligned}
\sqrt{n}(\tilde{\pi}_n - E_P[-X_1]) \geq& \sqrt{n}(\pi_n - E_P[-X_1]) \\
=& \sup_{\mu\in\mathcal{M}_\rho} \left(\sigma_P \int_{(0,1]} \mathrm{AVaR}_\lambda(S_n^*)\,\mu(d\lambda) - \tfrac{\beta(\mu)}{\sqrt{n}}\right) \\
\geq& \sqrt{n}(\tilde{\pi}_n - E_P[-X_1]) - \tfrac{1}{\sqrt{n}} \sup_{\mu\in\mathcal{M}_\rho} \beta(\mu).
\end{aligned}$$

This shows that the asymptotic behavior of the premia π_n defined by ρ is the same as for the premia $\tilde{\pi}_n$ defined by the coherent risk measure $\tilde{\rho}$, and so the result follows from Theorem 3.2. □

Remark 3.5. *As pointed out in the Introduction, the premia π_n induced by the convex entropic risk measure e_γ in (3) do* not *decrease to the fair premium. In fact, e_γ does not satisfy conditions (48) and (49). But it does satisfy condition (50). Indeed, if X is acceptable for e_γ, then $e_\gamma(X) \leq 0$, hence $E_P[e^{-\gamma X}] \leq 1$ and*

$$E_P[e^{\gamma X^-}] \leq E_P[e^{-\gamma X}; X < 0] + 1 \leq 2,$$

and this implies $\|X^-\|_h \leq \frac{1}{\gamma}$.

4 Examples

4.1 Concave distortions and Wang's example

Let us now check how condition (31) translates into the alternative characterization of the risk measures

$$\rho_\mu(X) = \int_{(0,1]} \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda), \quad \mu \in \mathcal{M}_1((0,1]), \tag{53}$$

in terms of concave distortions. More precisely, let ρ be defined as the Choquet integral

$$\begin{aligned}
\rho(X) &= \int (-X)\,dc_\psi \qquad (54) \\
&= \int_{-\infty}^0 (c_\psi(-X > x) - 1)\,dx + \int_0^\infty c_\psi(-X > x)\,dx
\end{aligned}$$

with respect to the submodular set function

$$c_\psi[A] := \psi(P[A]),$$

where ψ is an increasing and concave function on $[0,1]$ with $\psi(0) = 0$ and $\psi(1) = 1$, cf. [10], Section 4.6. A coherent risk measure is of the form (53) with some probability measure μ on $(0,1]$ if and only if it is of the form (54), and the corresponding concave distortion ψ is determined by

$$\psi'_+(t) = \int_{(t,1]} s^{-1}\,\mu(ds), \quad 0 < t < 1; \tag{55}$$

cf., e. g., [10], Theorem 4.70 and Corollary 4.77.

Lemma 4.1. *The probability measure μ in (53) satisfies our integrability condition (31) if and only if the corresponding distortion function ψ in (55) satisfies the condition*

$$\int_0^1 \log\left(\tfrac{1}{t}\right) \psi'_+(t)\, dt < \infty. \tag{56}$$

Proof. Since

$$\lambda \log\left(\tfrac{1}{\lambda}\right) = \int_0^\lambda \log\left(\tfrac{1}{t}\right)\, dt - \lambda,$$

the equivalence of the two conditions follows immediately by applying Fubini's theorem:

$$\begin{aligned}\int_0^1 \log\left(\tfrac{1}{\lambda}\right)\, \mu(d\lambda) &= \int_0^1 \tfrac{1}{\lambda} \int_0^\lambda \log\left(\tfrac{1}{t}\right)\, dt\, \mu(d\lambda) - 1 \\ &= \int_0^1 \log\left(\tfrac{1}{t}\right) \int_t^1 \tfrac{1}{\lambda}\, \mu(d\lambda)\, dt - 1 \\ &= \int_0^1 \log\left(\tfrac{1}{t}\right) \psi'_+(t)\, dt - 1.\end{aligned}$$

□

Let us now consider the class of concave distortion functions $\{\psi_\lambda | \lambda \geq 0\}$ defined by $\psi_\lambda(0) = 0$ and

$$\psi'_\lambda(t) := \tfrac{\varphi(\Phi^{-1}(t)+\lambda)}{\varphi(\Phi^{-1}(t))};$$

cf. Wang [19]. As before we denote by φ and Φ the density and the distribution function of the standard normal distribution.

Proposition 4.1. *For any $\lambda \geq 0$, condition (56) is satisfied for the concave distortion functions ψ_λ, and so the convergence in (36) holds for the mixing measure μ corresponding to ψ_λ via (55).*

Proof. For $\lambda = 0$ we get $\psi_0(x) = x$ and $\mu = \delta_1$, hence

$$\rho(X) = \mathrm{AVaR}_1(X) = E_P[-X].$$

In particular, we have $\psi'_0(0+) = 1$, and condition (56) is clearly satisfied.

For $\lambda > 0$ we have $\psi'_\lambda(0+) = \infty$. Using the change of variables $t = \Phi(x)$ we get

$$\int_0^1 \log\left(\tfrac{1}{t}\right) \psi'_+(t)\, dt \leq \int_0^1 t^{-\frac{1}{2}} \psi'_+(t)\, dt = \int_{-\infty}^\infty \Phi(x)^{-\frac{1}{2}} \varphi(x+\lambda)\, dx,$$

and applying the standard estimate $\Phi(x) \geq (|x| + \frac{1}{|x|})^{-1} \varphi(x)$ on $(-\infty, c]$ for any $c < 0$, we see that the right-hand side is finite. □

4.2 Coherent and truncated versions of the entropic risk measure

As we have seen in the Introduction, the capital requirements specified by a convex entropic risk measure e_γ, defined by (3) for $\gamma > 0$, do not have the desired behavior as the portfolio becomes large. We have also seen in (2) that the situation is different for the coherent entropic risk measures ρ_c defined by

$$\rho_c(X) := \sup_{Q: H(Q|P) \leq c} E_Q[-X] \tag{57}$$

for $c > 0$; cf. [8], Corollary 4.1. We are now going to explain how our results in [8] fit into the framework of Section 3. To this end, we first derive the representation of ρ_c in terms of mixtures of AVaR. As before, we denote by q_μ the function on $(0,1)$ associated to $\mu \in \mathcal{M}_1((0,1])$ via

$$q_\mu(t) := \int_{(1-t,1]} \tfrac{1}{s}\, \mu(ds). \tag{58}$$

Proposition 4.2. *The entropic risk measure ρ_c satisfies*

$$\rho_c(X) = \sup_{\mu \in \mathcal{M}_c} \int_0^1 \mathrm{AVaR}_\lambda(X)\, \mu(d\lambda), \tag{59}$$

where the class of mixing measures $\mathcal{M}_c$ is given by

$$\mathcal{M}_c := \{\mu \in \mathcal{M}_1((0,1]) | \int_0^1 q_\mu(t) \log q_\mu(t)\, dt \le c\}.$$

Proof. By [10], Lemma 4.60, we obtain

$$\begin{aligned}\rho_c(X) &= \sup_{Q: H(Q|P) \le c} E_P[-X \varphi_Q] \\ &= \sup_{Q: H(Q|P) \le c} \int_0^1 q_{-X}(t) q_{\varphi_Q}(t)\, dt,\end{aligned} \tag{60}$$

where φ_Q is the density of Q with respect to P, and where q_{-X}, q_{φ_Q} denote quantile functions of $-X$ resp. φ_Q under P. As in the proof of Theorem 4.62 in [10] we can write

$$\int_0^1 q_{-X}(t) q_{\varphi_Q}(t)\, dt = \int_0^1 \mathrm{AVaR}_\lambda(X)\, \mu(d\lambda), \tag{61}$$

where μ is the probability measure on $(0,1]$ such that the function q_μ in (58) coincides a. e. on $(0,1)$ with q_{φ_Q}. Moreover, the condition $H(Q|P) \le c$ translates into

$$\int_0^1 q_\mu(t) \log q_\mu(t)\, dt = \int_0^1 q_{\varphi_Q}(t) \log q_{\varphi_Q}(t)\, dt = E_P[\varphi_Q \log \varphi_Q] = H(Q|P) \le c.$$

Thus we have $\mu \in \mathcal{M}_c$, and this yields "$\le$" in equation (59).

Conversely, let $\mu \in \mathcal{M}_c$ be given. In that case, the function q_μ can be seen as a quantile function of the density $\varphi := q_\mu(U)$ of a measure $Q \in \mathcal{M}_1(P)$ satisfying $H(Q|P) \le c$, where U has a uniform distribution on $(0,1)$. In view of (60) and (61) this completes the proof of (59). □

Proposition 4.1 in [8] shows that the premia $\pi_{\mathcal{M}_c,n}$ computed in terms of the coherent entropic risk measure ρ_c satisfy

$$\lim_{n \uparrow \infty} \sqrt{n}(\pi_{\mathcal{M}_c,n} - E_P[-X_1]) = \sigma_P \sqrt{2c},$$

using explicit computations for exponential families. The following proposition derives the same result as a special case of Theorem 3.2, and it gives an alternative description of the factor $\sqrt{2c}$.

Proposition 4.3. *The class $\mathcal{M}_c$ satisfies our integrability condition (42). Thus*

$$\lim_{n \uparrow \infty} \sqrt{n}(\pi_{\mathcal{M}_c,n} - E_P[-X_1]) = \sigma_P \sup_{\mu \in \mathcal{M}_c} \int_0^1 \tfrac{1}{\lambda} \varphi(\Phi^{-1}(\lambda))\, \mu(d\lambda), \tag{62}$$

and the right-hand side coincides with $\sigma_P \sqrt{2c}$.

Proof. The proof of Corollary 3.1 shows that $\mathcal{M}_c$ satisfies condition (42), and so Theorem 3.2 implies (62). It remains to show that

$$\sup_{\mu\in\mathcal{M}_c}\int_0^1 \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\,\mu(d\lambda) = \sqrt{2c}. \tag{63}$$

For this purpose, note that

$$\begin{aligned}\int_0^1 \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\,\mu(d\lambda) &= \int_0^1 \tfrac{1}{\lambda}\int_0^\lambda \Phi^{-1}(1-\alpha)\,d\alpha\,\mu(d\lambda)\\ &= \int_0^1 \Phi^{-1}(1-\alpha)\int_\alpha^1 \tfrac{1}{\lambda}\,\mu(d\lambda)\,d\alpha\\ &= \int_0^1 \Phi^{-1}(\alpha)q_\mu(\alpha)\,d\alpha\\ &= \int_0^1 \Phi^{-1}(\alpha)\,\nu(d\alpha),\end{aligned}$$

where the first identity follows from (28) combined with $-\Phi^{-1}(\alpha) = \Phi^{-1}(1-\alpha)$, and where ν denotes the probability measure on $[0,1]$ defined by the Radon-Nikodym density q_μ with respect to the Lebesgue measure on $[0,1]$, denoted by $\lambda_{[0,1]}$. The condition

$$\int_0^1 q_\mu(\alpha)\log q_\mu(\alpha)\,d\alpha \le c$$

translates into the constraint

$$H(\nu|\lambda_{[0,1]}) \le c$$

for the relative entropy of ν with respect to $\lambda_{[0,1]}$. The computation of the left-hand side in (63) thus amounts to maximizing the expectation of Φ^{-1} with respect to all ν such that $H(\nu|\lambda_{[0,1]}) \le c$. This is a standard problem, and the solution ν^* is given by the Radon-Nikodym density

$$e^{\beta\Phi^{-1}}\Big(\int_0^1 e^{\beta\Phi^{-1}(\alpha)}\,d\alpha\Big)^{-1}$$

with respect to $\lambda_{[0,1]}$, with $\beta > 0$ such that $H(\nu^*|\lambda_{[0,1]}) = c$; cf., e. g., [8], Proposition 3.1. Thus we have

$$\begin{aligned}\sup_{\mu\in\mathcal{M}_c}\int_0^1 \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\,\mu(d\lambda) &= \int_0^1 \Phi^{-1}(\alpha)\,\nu^*(d\alpha)\\ &=\Big(\int_0^1 e^{\beta\Phi^{-1}(\alpha)}\,d\alpha\Big)^{-1}\int_0^1 \Phi^{-1}(\alpha)e^{\beta\Phi^{-1}(\alpha)}\,d\alpha\\ &=\Big(\int_{-\infty}^\infty e^{\beta x}\varphi(x)\,dx\Big)^{-1}\int_{-\infty}^\infty xe^{\beta x}\varphi(x)\,dx\\ &= \int_{-\infty}^\infty xe^{\beta x-\frac{1}{2}\beta^2}\varphi(x)\,dx\\ &=\beta,\end{aligned} \tag{64}$$

where we have used the substitution $x = \Phi^{-1}(\alpha)$ in the third line. On the other hand, $H(\nu^*|\lambda_{[0,1]}) = \frac{1}{2}\beta^2$, and so the condition $H(\nu^*|\lambda_{[0,1]}) = c$ implies $\beta = \sqrt{2c}$. In view of (64) we have thus shown (63). $\square$

As our last example, we consider a truncated version of the convex entropic risk measure e_γ defined in (3).

Definition 4.1. *For parameters $\gamma > 0$ and $c > 0$, we define the* truncated entropic risk measure $e_{\gamma,c}$ *by*

$$e_{\gamma,c}(X) := \sup_{Q:H(Q|P)\leq c} \{E_Q[-X] - \tfrac{1}{\gamma} H(Q|P)\}. \tag{65}$$

Clearly, $e_{\gamma,c}$ is a convex risk measure such that

$$e_{\gamma,c}(X) \leq \min\{\rho_c(X), e_\gamma(X)\}.$$

Consider now a position X with finite exponential moments and variance $\sigma_P^2(X) > 0$. Let $\{Q_{X,\beta} | \beta \in \mathbb{R}\}$ be the exponential family defined by $-X$ and P, i. e., $Q_{X,\beta}$ is given by the density

$$e^{-\beta X - Z(\beta)}$$

with $Z(\beta) := \log E_P[e^{-\beta X}]$. If $p(X) := P[X = \operatorname{ess\,inf} X] > 0$, then we include as limiting case the measure $Q_{X,\infty} := \lim_{\beta\uparrow\infty} Q_{X,\beta} = P[\cdot | X = \operatorname{ess\,inf} X]$. Note that the expectations $E_{Q_{X,\beta}}[-X]$ only depend on the distribution of X under P. For $a \in (0, -\log p(X))$, let $\beta(a)$ denote the unique parameter $\beta > 0$ such that $H(Q_{X,\beta}|P) = a$, and define $\beta(a) := \infty$ for $a \geq -\log p(X)$.

Lemma 4.2. *For X as above, the supremum in (65) is attained by the measure $Q_{X,\gamma\wedge\beta(c)}$. In particular, the convex risk measure $e_{\gamma,c}$ is law-invariant.*

Proof. 1. Let us first consider the case $c < c^* := -\log p(X)$. For a given value $H(Q|P) = a \leq c$ we then have

$$a = H(Q|P) = H(Q|Q_{X,\beta(a)}) + E_Q[-\beta(a)X] - Z(\beta(a)),$$

hence

$$E_Q[-X] \leq \tfrac{1}{\beta(a)}(a + Z(\beta(a))) = E_{Q_{X,\beta(a)}}[-X].$$

The computation of $e_{\gamma,c}(X)$ is thus reduced to maximizing the function

$$f(a) := m(\beta(a)) - \tfrac{a}{\gamma}$$

on the interval $[0, c]$. Here we denote by $m(\beta)$ the expectation and by $\sigma^2(\beta)$ the variance of $-X$ under $Q_{X,\beta}$. Since $m(\beta) = Z'(\beta)$, $m'(\beta) = \sigma^2(\beta)$, and $H(Q_{X,\beta}|P) = \beta m(\beta) - Z(\beta)$, we get

$$\tfrac{\partial}{\partial\beta} H(Q_{X,\beta}|P) = \beta\sigma^2(\beta),$$

hence

$$1 = \tfrac{\partial}{\partial a} H(Q_{X,\beta(a)}|P) = \beta(a)\sigma^2(\beta(a))\beta'(a).$$

This implies

$$f'(a) = \tfrac{1}{\beta(a)} - \tfrac{1}{\gamma}.$$

The maximum of f over $[0, c]$ is therefore attained in $a(\gamma) := H(Q_{X,\gamma}|P)$ if $\gamma < \beta(c)$, and in c if $\gamma \geq \beta(c)$. The corresponding maximizing measure Q in (65) is thus given by $Q_{X,\gamma\wedge\beta(c)}$.

2. If $p(X) > 0$, then the limiting measure $Q_{X,\infty}$ satisfies $H(Q_{X,\infty}|P) = c^*$ and $E_{Q_{X,\infty}}[-X] = \operatorname{ess\,sup}(-X)$. Note that $p(X) > 0$ implies $\operatorname{ess\,sup}(-X) < \infty$, since X is integrable. For $c \geq c^*$, we thus obtain

$$\sup_{Q:H(Q|P)\in[c^*,c)} \{E_Q[-X] - \tfrac{1}{\gamma} H(Q|P)\} = \operatorname{ess\,sup}(-X) - \tfrac{c^*}{\gamma},$$

where the maximum is attained for $Q_{X,\infty}$. In view of step 1 this implies

$$e_{\gamma,c}(X) = \max\{\sup_{a\in[0,c^*)} f(a), \operatorname{ess\,sup}(-X) - \tfrac{c^*}{\gamma}\}.$$

Since $\lim_{a \uparrow c^*} \beta(a) = \infty$, we have

$$\lim_{a \uparrow c^*} f(a) = \lim_{a \uparrow c^*} \left(m(\beta(a)) - \tfrac{a}{\gamma} \right) = \operatorname{ess\,sup}(-X) - \tfrac{c^*}{\gamma},$$

and step 1 shows that the maximum of f over $[0, c^*)$ is attained in $a(\gamma) := H(Q_{X,\gamma}|P)$. This completes the proof. $\square$

The asymptotic behavior of the premia

$$\pi_{\gamma,c,n} := \tfrac{1}{n} e_{\gamma,c}(S_n)$$

now follows immediately from Theorem 3.3 and Proposition 4.3:

Corollary 4.1. *The premia $\pi_{\gamma,c,n}$ induced by the truncated entropic risk measure $e_{\gamma,c}$ converge to the fair premium $E_P[-X_1]$, and*

$$\lim_{n \uparrow \infty} \sqrt{n}(\pi_{\gamma,c,n} - E_P[-X_1]) = \sigma_P \sqrt{2c}. \tag{66}$$

Proof. The assumptions of Theorem 3.3 are clearly satisfied, and the coherent risk measure associated to $e_{\gamma,c}$ via (52) is the coherent entropic risk measure ρ_c in (1). Thus (66) follows from (51) and Proposition 4.3. $\square$

5 Beyond law-invariance

Let us now consider a situation of model ambiguity where P is replaced by a whole class $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$. As in [8] we will assume that all measures $P \in \mathcal{P}$ are equivalent to some reference measure R on $(\Omega, \mathcal{F})$, and that the family of densities

$$\Phi_{\mathcal{P}} := \{ \tfrac{dP}{dR} | P \in \mathcal{P} \}$$

is convex and weakly compact in $L^1(R)$. Throughout this section, the subscript P indicates the dependence on a specific measure $P \in \mathcal{P}$. In particular, we use the notation

$$\rho_{P,\mathcal{M}}(X) := \sup_{\mu \in \mathcal{M}} \int_{(0,1]} \mathrm{AVaR}_{P,\lambda}(X)\, \mu(d\lambda)$$

for a law-invariant coherent risk measure with respect to $P \in \mathcal{P}$ specified the subset $\mathcal{M} \subseteq \mathcal{M}_1((0,1])$. In the face of model ambiguity, we consider the robust version of $\rho_{P,\mathcal{M}}$ defined by

$$\rho_{\mathcal{P},\mathcal{M}}(X) := \sup_{P \in \mathcal{P}} \rho_{P,\mathcal{M}}(X) = \sup_{P \in \mathcal{P}} \sup_{\mu \in \mathcal{M}} \int_{(0,1]} \mathrm{AVaR}_{P,\lambda}(X)\, \mu(d\lambda).$$

Clearly, $\rho_{\mathcal{P},\mathcal{M}}$ is again a coherent risk measure. Specific examples are the robust Average Value at Risk defined by

$$\mathrm{AVaR}_{\mathcal{P},\lambda}(X) := \sup_{P \in \mathcal{P}} \mathrm{AVaR}_{P,\lambda}(X)$$

for $\lambda \in (0,1]$, and the robust extension $\rho_{\mathcal{P},c}$ of the coherent entropic risk measure (57) given by

$$\rho_{\mathcal{P},c}(X) := \sup_{P \in \mathcal{P}} \rho_{P,c}(X) = \sup_{Q \in \mathcal{M}_1 : \inf_{P \in \mathcal{P}} H(Q|P) \le c} E_Q[-X]$$

for parameters $c > 0$; cf. [8], Section 5.

In this section we look at the behavior of the robust premia

$$\pi_{\mathcal{P},\mathcal{M},n} := \tfrac{1}{n} \rho_{\mathcal{P},\mathcal{M}}(S_n).$$

For this purpose, we introduce the following assumption:

Assumption 5.1. *We assume that the random variables $X_1, X_2, \dots$ are i. i. d. under any $P \in \mathcal{P}$, that the exponential moments of X_1 are bounded uniformly in $P \in \mathcal{P}$, i. e.,*

$$\sup_{P\in\mathcal{P}} E_P[e^{\alpha|X_1|}] < \infty \quad \textit{for any } \alpha > 0, \tag{67}$$

and that the variances σ_P^2 of X_1 under P satisfy

$$\inf_{P\in\mathcal{P}} \sigma_P > 0. \tag{68}$$

For the class of mixing measures $\mathcal{M}$ we impose the integrability condition

$$\sup_{\mu\in\mathcal{M}} \int_{(0,1]} \log\left(\tfrac{1}{\lambda}\right) \mu(d\lambda) < \infty. \tag{69}$$

Remark 5.1. *Only the distribution of X_1 is subject to model ambiguity, since we retain the structural i. i. d. assumption for any $P \in \mathcal{P}$. Here again, the i. i. d. assumption could be replaced by weaker conditions of homogeneity and weak dependence, as pointed out in Remark 3.1.*

The following theorem yields an upper bound for the asymptotics of the robustified risk premia $\pi_{\mathcal{P},\mathcal{M},n} - \sup_{P\in\mathcal{P}} E_P[-X_1]$ as the portfolio becomes large. As to a lower bound, see Remark 5.2 below.

Theorem 5.1. *We have*

$$\lim_{n\uparrow\infty} \pi_{\mathcal{P},\mathcal{M},n} = \sup_{P\in\mathcal{P}} E_P[-X_1],$$

and the decay of the risk premia satisfies

$$\overline{\lim_{n\uparrow\infty}} \sqrt{n}(\pi_{\mathcal{P},\mathcal{M},n} - \sup_{P\in\mathcal{P}} E_P[-X_1]) \le \sup_{P\in\mathcal{P}} \sigma_P \sup_{\mu\in\mathcal{M}} \int_{(0,1]} \tfrac{1}{\lambda}\varphi(\Phi^{-1}(\lambda))\, \mu(d\lambda). \tag{70}$$

Proof. 1. In order to verify (70), we use the estimate

$$\begin{aligned}
\sqrt{n}(\pi_{\mathcal{P},\mathcal{M},n} - \sup_{P\in\mathcal{P}} E_P[-X_1]) &\le \sup_{P\in\mathcal{P}} \sqrt{n}(\tfrac{1}{n}\rho_{P,\mathcal{M}}(S_n) - E_P[-X_1]) \\
&= \sup_{P\in\mathcal{P}} \left(\sigma_P \sup_{\mu\in\mathcal{M}} \int_{(0,1]} \mathrm{AVaR}_{P,\lambda}(S^*_{P,n})\, \mu(d\lambda) \right) \\
&\le \sup_{P\in\mathcal{P}} \sigma_P \cdot \sup_{P\in\mathcal{P}} \sup_{\mu\in\mathcal{M}} \int_{(0,1]} \mathrm{AVaR}_{P,\lambda}(S^*_{P,n})\, \mu(d\lambda)
\end{aligned}$$

in terms of the P-standardized random variables

$$S^*_{P,n} = \tfrac{S_n - nE_P[X_1]}{\sqrt{n}\sigma_P}.$$

Thus (70) follows if we can prove the uniform convergence

$$\lim_{n\uparrow\infty} \sup_{P\in\mathcal{P}} \sup_{\mu\in\mathcal{M}} \left| \int_{(0,1]} \mathrm{AVaR}_{P,\lambda}(S^*_{P,n})\, \mu(d\lambda) - \int_{(0,1]} \mathrm{AVaR}_{P,\lambda}(Z_P)\, \mu(d\lambda) \right| = 0, \tag{71}$$

where Z_P is standard normally distributed under $P \in \mathcal{P}$. Indeed, denoting by $q_{P,n}$ any quantile function of $S^*_{P,n}$ with respect to P, the inequality (38) yields

$$\begin{aligned}
&\sup_{P\in\mathcal{P}} \sup_{\mu\in\mathcal{M}} \left| \int_{(0,1]} \mathrm{AVaR}_{P,\lambda}(S^*_{P,n})\, \mu(d\lambda) - \int_{(0,1]} \mathrm{AVaR}_{P,\lambda}(Z_P)\, \mu(d\lambda) \right| \\
\le& 2 \sup_{P\in\mathcal{P}} \|q_{P,n} - \Phi^{-1}\|_h \sup_{\mu\in\mathcal{M}} \int_{(0,1]} \frac{1}{\lambda (h^*)^{-1}\left(\frac{1}{\lambda}\right)}\, \mu(d\lambda).
\end{aligned}$$

By condition (69) and Lemma 3.1 the last term at the right-hand side is finite. In the second step of this proof we are going to show that

$$\lim_{n\uparrow\infty}\sup_{P\in\mathcal{P}}\|q_{P,n}-\Phi^{-1}\|_h=0. \tag{72}$$

This implies (71), and we have thus verified the upper bound (70) for the rate of decay.

2. In order to check (72), it suffices to show that

$$\lim_{n\uparrow\infty}\sup_{P\in\mathcal{P}}\int_0^1 e^{\beta|q_{P,n}(\alpha)-\Phi^{-1}(\alpha)|}\,d\alpha\le 1 \tag{73}$$

for all $\beta>0$, in analogy to the proof of Theorem 3.1. In part 3 we are going to show that for all $\epsilon,\delta>0$ there exists $n_0(\epsilon,\delta)$ such that

$$\sup_{P\in\mathcal{P}}\sup_{\delta\le\alpha\le 1-\delta}|q_{P,n}(\alpha)-\Phi^{-1}(\alpha)|\le\epsilon \tag{74}$$

for all $n\ge n_0(\epsilon,\delta)$. Using (74) for given $\epsilon,\delta>0$ and applying Cauchy-Schwarz, we obtain the estimate

$$\begin{aligned}\sup_{P\in\mathcal{P}}\int_0^1 e^{\beta|q_{P,n}(\alpha)-\Phi^{-1}(\alpha)|}\,d\alpha \le& e^{\beta\epsilon}+\sup_{P\in\mathcal{P}}\int_0^1 e^{\beta|q_{P,n}(\alpha)|}e^{\beta|\Phi^{-1}(\alpha)|}1_{[0,\delta]\cup[1-\delta,1]}(\alpha)\,d\alpha\\ \le& e^{\beta\epsilon}+\sup_{P\in\mathcal{P}}\Big(\int_0^1 e^{2\beta|q_{P,n}(\alpha)|}\,d\alpha\Big)^{\frac12}\Big(2\int_0^\delta e^{2\beta|\Phi^{-1}(\alpha)|}\,d\alpha\Big)^{\frac12}\\ \le& e^{\beta\epsilon}+\Big(\sup_{n\in\mathbb{N}}\sup_{P\in\mathcal{P}}E_P[e^{2\beta|S^*_{P,n}|}]\Big)^{\frac12}\Big(2\int_0^\delta e^{2\beta|\Phi^{-1}(\alpha)|}\,d\alpha\Big)^{\frac12}\end{aligned}$$

for all $n\ge n_0(\epsilon,\delta)$. As shown in part 4, the first factor is finite. For any $a>1$, we can therefore choose constants $\epsilon,\delta>0$ such that the right-hand side is less than a for all $n\ge n_0(\epsilon,\delta)$. But this translates into (73).

3. Under our assumptions (67) and (68) the Berry-Esseen theorem applies and yields uniform convergence of the distribution functions $F_{P,n}$ of $S^*_{P,n}$ to the standard normal distribution function Φ. More precisely,

$$\sup_{x\in\mathbb{R}}|F_{P,n}(x)-\Phi(x)|\le\tfrac{C}{\sqrt{n}}\tfrac{E_P[|X_1|^3]}{\sigma_P^3}$$

with some constant C. In view of (67) and (68), the bound

$$\tfrac{C}{\sqrt{n}}\sup_{P\in\mathcal{P}}E_P[|X_1|^3]\big(\inf_{P\in\mathcal{P}}\sigma_P^3\big)^{-1}<\infty.$$

is valid uniformly for all $P\in\mathcal{P}$. It is now easy to check that the corresponding quantile functions $q_{p,n}$ converge to Φ^{-1} uniformly on each interval $[\delta,1-\delta]$ and uniformly in $P\in\mathcal{P}$, i. e., we obtain (74).

4. It remains to show that

$$\sup_{n\in\mathbb{N}}\sup_{P\in\mathcal{P}}E_P[e^{\alpha|S^*_{P,n}|}]<\infty\quad\text{for any }\alpha>0.$$

Indeed, condition (67) ensures that both $\sup_{P\in\mathcal{P}}E_P[e^{\alpha X_1}]$ and $\sup_{P\in\mathcal{P}}E_P[e^{-\alpha X_1}]$ are finite for any $\alpha>0$, and we may assume without loss of generality that $E_P[X_1]=0$ for any $P\in\mathcal{P}$. Then the functions $Z_P(\lambda):=\log E_P[e^{\lambda X_1}]$, $\lambda\ge 0$, are smooth and satisfy

$$Z_P(\lambda)=Z_P(0)+Z_P'(0)\lambda+\tfrac12 Z_P''(\tilde\lambda)\lambda^2=\tfrac12 Z_P''(\tilde\lambda)\lambda^2$$

for some $\tilde{\lambda} \in [0, \lambda]$. Since $E_P[e^{\tilde{\lambda} X_1}] \geq e^{\tilde{\lambda} E_P[-X_1]} = 1$, we see that

$$\begin{aligned} Z_P''(\tilde{\lambda}) \leq & E_P[e^{\tilde{\lambda} X_1} X_1^2](E_P[e^{\tilde{\lambda} X_1}])^{-1} \\ \leq & (E_P[e^{2\tilde{\lambda} X_1}])^{\frac{1}{2}} (E_P[X_1^4])^{\frac{1}{2}} \\ \leq & \sup_{P \in \mathcal{P}} (E_P[e^{2\lambda |X_1|}] E_P[X_1^4])^{\frac{1}{2}} =: c(\lambda), \end{aligned}$$

and this implies $\sup_{P \in \mathcal{P}} Z_P(\lambda) \leq \frac{1}{2} \lambda^2 c(\lambda)$ for the increasing function c. Thus we have

$$\sup_{P \in \mathcal{P}} \log E_P[e^{\alpha \frac{S_n}{\sqrt{n}}}] = n \sup_{P \in \mathcal{P}} Z_P(\tfrac{\alpha}{\sqrt{n}}) \leq \tfrac{1}{2} \alpha^2 c(\tfrac{\alpha}{\sqrt{n}}).$$

For fixed $\alpha > 0$, the right-hand side decreases to a finite limit as n tends to ∞, and this yields a bound which is uniform in n. Applying the same argument also to $-X_1$, we finally obtain that

$$\sup_{P \in \mathcal{P}} E_P[e^{\alpha |S_n^*|}] \leq \sup_{P \in \mathcal{P}} E_P[e^{\alpha (\inf_{P \in \mathcal{P}} \sigma_P)^{-1} \frac{S_n}{\sqrt{n}}}] + \sup_{P \in \mathcal{P}} E_P[e^{-\alpha (\inf_{P \in \mathcal{P}} \sigma_P)^{-1} \frac{S_n}{\sqrt{n}}}]$$

remains bounded uniformly in n, and this completes the proof of Theorem 5.1. □

Remark 5.2. *Suppose that*

$$\sup_{P \in \mathcal{P}} E_P[-X_1] = E_{P^*}[-X_1] \tag{75}$$

for some $P^* \in \mathcal{P}$. Then we have

$$\begin{aligned} \varliminf_{n \uparrow \infty} \sqrt{n}(\pi_{\mathcal{P},\mathcal{M},n} - \sup_{P \in \mathcal{P}} E_P[-X_1]) \geq & \sigma_{P^*} \sup_{\mu \in \mathcal{M}} \int_{(0,1]} \tfrac{1}{\lambda} \varphi(\Phi^{-1}(\lambda))\, \mu(d\lambda) \\ \geq & \inf_{P \in \mathcal{P}} \sigma_P \sup_{\mu \in \mathcal{M}} \int_{(0,1]} \tfrac{1}{\lambda} \varphi(\Phi^{-1}(\lambda))\, \mu(d\lambda). \end{aligned}$$

Indeed, since $\pi_{\mathcal{P},\mathcal{M},n} \geq \frac{1}{n} \rho_{P^*,\mathcal{M}}(S_n)$, we obtain

$$\sqrt{n}(\pi_{\mathcal{P},\mathcal{M},n} - \sup_{P \in \mathcal{P}} E_P[-X_1]) \geq \sqrt{n}(\tfrac{1}{n} \rho_{P^*,\mathcal{M}}(S_n) - E_{P^*}[-X_1]),$$

and so the lower bound for the rate of decay follows immediately from Theorem 3.2. Note that, due to our compactness assumption on $\mathcal{P}$, condition (75) does hold if X_1 is bounded.

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