Bowley vs. Pareto Optima in Reinsurance Contracting

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- Typically, every equilibrium allocation is efficient in the Pareto sense.

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- However, not every efficient allocation is an equilibrium allocation. But, under some standard conditions on preferences, every efficient allocation can be obtained as an equilibrium allocation if appropriate lump-sum transfers of initial endowments are arranged.

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- We show that:

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- We interpret the latter result as indicative of the limitations of the Bowley equilibrium concept in this literature.

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$$X := \max\left(0, \sum_{i=1}^{n} \left(\phi_i(Y_i) - \pi_i\right)\right).$$

- The insurer, in turn, seeks to cede a part $I(X)$ of the exposure X to a reinsurer, in exchange for a premium payment π.

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- Here, we assume that the first stage of the market has already been optimally determined, and we focus on optimal reinsurance arrangements arising in the second stage.

Setting

- An insurer faces the portfolio loss $X \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$, with $M := \|X\|_{\infty} < +\infty$.
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- Let $\mathcal{I}$ be a collection of *ex ante* admissible indemnity functions.
    - We assume that:

$$\mathcal{I} \subset \mathcal{I}_0 := \{I : \mathbb{R} \to \mathbb{R} \mid I \text{ is Borel-measurable, } I(X) \in L^1(\Omega, \mathcal{F}, \mathbb{P}), \text{ and } 0 \leqslant I(X) \leqslant X\}.$$

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  - For instance, $\mathcal{I}$ could be the customary collection $\mathcal{I}_L$ of indemnities that satisfy the so-called *no-sabotage* condition:

    $$\mathcal{I}_L := \left\{ I \in \mathcal{I}_0 \mid 0 \leqslant I(x_1) - I(x_2) \leqslant x_1 - x_2, \forall\, x_2 \leqslant x_1 \in [0, M] \right\}.$$

$\Longrightarrow$ $\mathcal{I}_L$ is convex and $\|\cdot\|_{sup}$-compact.

Preferences and Pricing Kernels

- The reinsurer prices indemnity functions $I \in \mathcal{I}$ using a premium principle Π, defined as the functional $\Pi : L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}) \times \mathcal{I} \to \mathbb{R}$ given by

$$\Pi(\xi, I) := \int I(X)\, \xi \, d\mathbb{P}, \ \forall (\xi, I) \in L^{\infty}(\Omega, \mathcal{F}, P) \times \mathcal{I},$$

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- For a given $I \in \mathcal{I}$ and $\xi \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$, the risk exposure of the insurer is given by

$$X - I(X) + \Pi(\xi, I),$$

  and the risk exposure of the reinsurer is given by

$$I(X) - \Pi(\xi, I).$$

Preferences and Pricing Kernels

- Assume that the preferences of the insurer and the reinsurer are respectively represented by risk measures

$$\rho^{In} : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^+ \quad \text{and} \quad \rho^{Re} : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^+,$$

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- Define the auxiliary functionals

$$\rho_1^{In}, \rho_1^{Re} : \mathbb{R} \times \mathcal{I} \to \mathbb{R} \quad \text{and} \quad \rho_2^{In}, \rho_2^{Re} : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \times \mathcal{I} \to \mathbb{R}$$

  by:

$$\rho_1^{In}(\pi, I) := \rho^{In}(X - I(X) + \pi) \quad \text{and} \quad \rho_2^{In}(\xi, I) := \rho^{In}(X - I(X) + \Pi(\xi, I)).$$

$$\rho_1^{Re}(\pi, I) := \rho^{Re}(I(X) - \pi) \quad \text{and} \quad \rho_2^{Re}(\xi, I) := \rho^{Re}(I(X) - \Pi(\xi, I)).$$

Preferences and Pricing Kernels

Definition

A risk measure $\rho : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is said to be:

- **Translation-invariant** if $\rho(X + c) = \rho(X) + c$, for all $(X, c) \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \times \mathbb{R}$.
- **Convex** if $\rho(\alpha X + (1-\alpha) Y) \leqslant \alpha \rho(X) + (1-\alpha)\rho(Y)$, for all $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\alpha \in [0, 1]$
- **Comonotonic-additive** if $\rho(X + Y) = \rho(X) + \rho(Y)$, for all $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ that are comonotonic, that is, such that

$$[X(\omega_1) - X(\omega_2)]\,[Y(\omega_1) - Y(\omega_2)] \geqslant 0, \;\; \forall \omega_1, \omega_2 \in \Omega.$$

- **Continuous** if it is L^1-continuous.

Pricing Kernels as Subgradients

- The norm dual of $L^1(\Omega, \mathcal{F}, \mathbb{P})$ is (isometrically isomorphic to) $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.
- Using this standard duality, one can define subgradients of risk measures.

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Definition

A **subgradient** of a risk measure ρ at some $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ is some $\xi \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\rho(Z) \geqslant \rho(Y) + E[\xi(Z - Y)], \ \forall Z \in L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

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The **subdifferential** of ρ at some $Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, denoted by $\partial\rho(Y)$, is the collection of all subgradients of ρ at Y:

$$\begin{aligned} \partial\rho(Y) &:= \left\{\xi \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \,\middle|\, \rho(Z) \geqslant \rho(Y) + E[\xi(Z - Y)], \forall Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})\right\} \\ &= \left\{\xi \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \,\middle|\, \rho(Z) - \Pi(\xi, Z) \geqslant \rho(Y) - \Pi(\xi, Y), \forall Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})\right\}. \end{aligned}$$

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If ρ is convex and continuous, then $\partial\rho(Y) \neq \varnothing$ for all $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Optima – Definitions

Definition (Individual Rationality)

A pair $(\pi, I) \in \mathbb{R} \times \mathcal{I}$ is said to satisfy the individual rationality constraints if

$$\rho_1^{In}(\pi, I) \leqslant \rho_1^{In}(0,0) = \rho^{In}(X) \quad \text{and} \quad \rho_1^{Re}(\pi, I) \leqslant \rho_1^{Re}(0,0) = \rho^{Re}(0) = 0.$$

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- Let $\mathcal{IR} \subset \mathbb{R} \times \mathcal{I}$ denote the collection of all contracts that satisfy the individual rationality constraints.
- $(0,0) \in \mathcal{IR}$ is the status quo.
- If ρ^{In} and ρ^{Re} are translation-invariant, then $\pi \geqslant 0$ for any $(\pi, I) \in \mathcal{IR}$.

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Definition (**Optimality)**

- A pair $(\pi^*, I^*) \in \mathcal{IR}$ is said to be Pareto-Optimal (PO) if there is no other pair $(\tilde{\pi}, \tilde{I}) \in \mathcal{IR}$ such that

$$\rho_1^{In}\left(\tilde{\pi}, \tilde{I}\right) \leqslant \rho_1^{In}\left(\pi^*, I^*\right) \quad \text{and} \quad \rho_1^{Re}\left(\tilde{\pi}, \tilde{I}\right) \leqslant \rho_1^{Re}\left(\pi^*, I^*\right),$$

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  with at least one strict inequality.

- A pair $(\xi^*, I^*) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \times \mathcal{I}$ is said to be Bowley-Optimal (BO) if

  1. $I^* \in \underset{I \in \mathcal{I}}{\arg\min}\, \rho_2^{In}(\xi^*, I)$.
  2. $\rho_2^{Re}(\xi^*, I^*) \leqslant \rho_2^{Re}(\tilde{\xi}, \tilde{I})$ for all $(\tilde{\xi}, \tilde{I}) \in L^\infty(\Omega, \mathcal{F}, P) \times \underset{I \in \mathcal{I}}{\arg\min}\, \rho_2^{In}(\tilde{\xi}, I)$.

Pareto-Optimal Contracts

Lemma (**Pareto Optimality**)

Suppose that ρ^{In} *and* ρ^{Re} *are translation-invariant. A pair* $(\pi^*, I^*) \in \mathcal{IR}$ *is PO if and only if it is optimal for the problem*

$$(\mathcal{P}_1) \qquad \inf_{(\pi, I) \in \mathcal{IR}} \left\{ \rho_1^{In}(\pi, I) + \rho_1^{Re}(\pi, I) \right\}.$$

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$$(\mathcal{P}_2) \qquad \inf_{I \in \mathcal{I}} \left\{ \rho_1^{In}(0, I) + \rho_1^{Re}(0, I) : (\pi, I) \in \mathcal{IR}, \text{ for some } \pi \in \mathbb{R} \right\}$$

if and only if (π^*, I^*) *is optimal for Problem* $(\mathcal{P}_1)$*, for some* $\pi^* \in \mathbb{R}$.

Pareto Vs. Bowley Optima

Lemma

- *If ρ^{Re} is translation-invariant, convex, and continuous, then for every $I \in \mathcal{I}$, there exist $\tilde{\xi}^{Re} \in L^{\infty}(\Omega, \mathcal{F}, P)$ such that*

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- *If ρ^{In} is comonotonic-additive, convex, and continuous, then for each $I \in \mathcal{I}$,*
  - $\Pi(\xi, X - I(X)) = \rho^{In}(X - I(X))$, *for all* $\xi \in \partial\rho^{In}(X - I(X))$.
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Then the following hold:

- *If* (ξ^*, I^*) *is BO, then* $(\Pi(\xi^*, I^*), I^*)$ *is PO.*
- *If, in addition,* ρ^{Re} *is convex and continuous, then for any* (ξ^*, I^*) *that is BO, we have* $\rho_2^{In}(\xi^*, I^*) = \rho_2^{In}(\xi^*, 0)$ $\left(= \rho^{In}(X)\right)$.

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If (π^*, I^*) *is PO and such that* $\rho_1^{In}(\pi^*, I^*) = \rho_1^{In}(0, 0)$, *then there exists some* $\xi^* \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ *such that* (ξ^*, I^*) *is BO and* $\pi^* = \Pi(\xi^*, I^*)$.

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If (π^*, I^*) *is PO and such that* $\rho_1^{In}(\pi^*, I^*) = \rho_1^{In}(0, 0)$, *then there exists some* $\xi^* \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ *such that* (ξ^*, I^*) *is BO and* $\pi^* = \Pi(\xi^*, I^*)$.

Moreover, ξ^* *can be chosen randomly in* $\partial\rho^{In}(I^*(X)) \cap \partial\rho^{In}(X - I^*(X)) \neq \emptyset$.

Pareto Vs. Bowley Optima

Theorem ("Second Welfare Theorem")

Suppose that:

- $\mathcal{I} = \mathcal{I}_L$, *the set of all indemnities in* $\mathcal{I}_0$ *that satisfy the no-sabotage condition.*
- ρ^{In} *is comonotonic-additive, convex, and continuous.*
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Any Pareto efficient contract for which the insurer is indifferent is a Bowley optimum for some pricing kernel.

PO, BO, and Competitive Equilibria: The Case of Convex Distortion Risk Measures

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- These are risk measures of the form

$$\rho_g(Y) = \int_{-\infty}^{0} [g(S_Y(z)) - 1]dz + \int_{0}^{\infty} g(S_Y(z))dz, \ \forall\, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P}),$$

where:

  - $g : [0,1] \to [0,1]$ is non-decreasing and concave, with $g(0) = 0$ and $g(1) = 1$.
  - S_Y denotes the survival function of Y.

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- A convex DRM is monotone, comonotonic-additive, translation-invariant, and convex. If, in addition it is finite, then it is also continuous.

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- A convex DRM is monotone, comonotonic-additive, translation-invariant, and convex. If, in addition it is finite, then it is also continuous.
- Hereafter, let $\rho^{In} = \rho_{g_1}$ and $\rho^{Re} = \rho_{g_2}$, for given concave distortion functions g_1, g_2.

PO, BO, and Competitive Equilibria: The Case of Convex Distortion Risk Measures

We consider competitive equilibria in two reinsurance market settings:

PO, BO, and Competitive Equilibria: The Case of Convex Distortion Risk Measures

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1. In a *complete reinsurance market*, the set of admissible allocations is given by

$$\mathbb{A}(X) := \Big\{(X_1, X_2) \in \left(L^1\left(\Omega, \mathcal{F}, \mathbb{P}\right)\right)^2 : X_1 + X_2 = X\Big\}.$$

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2. In a *comonotone reinsurance market* (a special type of an incomplete market), allocations are confined to the set $C(X)$ of comonotonic allocations, namely,

$$C(X) := \left\{Y \in L^1\left(\Omega, \mathcal{F}, \mathbb{P}\right) : (Y, X - Y) \text{ is comonotonic}\right\},$$

and the resulting set of admissible allocations is then given by

$$\mathbb{A}^c(X) := \left\{(X_1, X_2) \in \left(C(X)\right)^2 : X_1 + X_2 = X\right\}.$$

PO, BO, and Competitive Equilibria: The Case of Convex Distortion Risk Measures

Definition (**Unconstrained Competitive Equilibrium)**

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Definition (**Unconstrained Competitive Equilibrium)**

In a complete reinsurance market, a competitive equilibrium is a pair $((X_1, X_2), \xi) \in \mathbb{A}(X) \times L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies:

1. $\Pi(\xi, X_1) \leqslant \Pi(\xi, X)$.
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3. $\rho^{In}(X_1) = \min\left\{\rho^{In}(Y_1) : \Pi(\xi, Y_1) \leqslant \Pi(\xi, X)\right\}$.
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PO, BO, and Competitive Equilibria: The Case of Convex Distortion Risk Measures

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$\Longrightarrow$ Such a competitive equilibrium is called an **Unconstrained Competitive Equilibrium (UCE)**.

PO, BO, and Competitive Equilibria: The Case of Convex Distortion Risk Measures

Definition (**Constrained Competitive Equilibrium)**

PO, BO, and Competitive Equilibria: The Case of Convex Distortion Risk Measures

Definition (Constrained Competitive Equilibrium)

In a comonotone reinsurance market, a competitive equilibrium is a pair $((X_1, X_2), \xi) \in \mathbb{A}^c(X) \times L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies:

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Definition (**Constrained Competitive Equilibrium)**

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PO, BO, and Competitive Equilibria: The Case of Convex Distortion Risk Measures

Proposition (Competitive Equilibria and Pareto Efficiency)

(i) *The equilibrium price in UCE exists and is unique, and it is given by* $\xi := \frac{d\mathbb{Q}}{d\mathbb{P}}$, *where* $\mathbb{Q}$ *is defined by* $\mathbb{Q}(X > z) := \max\{g_1(S_X(z)), g_2(S_X(z))\}, \ \forall z \in \mathbb{R}$.

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(ii) *Any UCE* $\left(\left(X_1^*, X_2^*\right), \xi^*\right)$ *yields a PO risk transfer, and we have* $\Pi\left(\xi^*, X_2^*\right) = \Pi\left(\xi^*, 0\right) = 0$. *Hence* $\rho^{Re}\left(X_2^*\right) = \rho^{Re}\left(0\right) = 0$.

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(iv) *If* (π^*, I^*) *is PO, then there exists some* ξ^* *such that* $((X_1^*, X_2^*), \xi^*)$ *is a CCE, where* $X_1^* := X - I^*(X) + \Pi(\xi^*, I^*)$ *and* $X_2^* := I^*(X) - \Pi(\xi^*, I^*)$.

PO, BO, and Competitive Equilibria: The Case of Convex Distortion Risk Measures

To sum up, for convex distortion risk measures, the following holds:

1. In any UCE, the risk transfer is PO and the **reinsurer** will be indifferent between selling reinsurance and not selling reinsurance.

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We also examine the relationship with Nash bargaining solutions for convex DRM...

An Example: PO and BO for TVaR

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- Consider a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a concave distortion function g.

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- By the Fenchel-Moreau theorem, the convex DRM ρ_g admits the dual representation

$$\rho_g(X) = \sup \left\{ E(XZ) : Z = g'(U), \; U \text{ has a uniform distribution on } [0,1] \right\}.$$

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$$\rho_g(X) = \sup \Big\{ E(XZ) : Z = g'(U),\ U \text{ has a uniform distribution on } [0,1] \Big\}.$$

- Moreover, by the concavity of g, it follows from Carlier and Dana (2003) that the subdifferential of ρ_g at X is given by

$$\partial \rho_g(X) = \overline{co} \Big\{ g'(1-U) : U \sim Unif(0,1),\ (U, X) \text{ is comonotonic} \Big\} \qquad (\bigstar),$$

where $\overline{co}$ denotes the L^1-closed convex hull.

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- Here we provide an illustrative example for the special case in which the convex DRMs are given by the Tail Value-at-Risk (TVaR) risk measure.

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- The TVaR at level $\alpha \in (0,1)$ is a continuous DRM for which the (concave) distortion function is given by

$$g_\alpha(t) := \min\left\{\frac{t}{1-\alpha}, 1\right\}, \ \forall t \in [0,1].$$

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- The dual representation of TVaR is given by

$$TVaR_{\alpha}\left(X\right) = \sup\left\{E\left(X\,Z\right) : E\left(Z\right) = 1,\ 0 \leqslant Z \leqslant \frac{1}{1-\alpha}\right\}.$$

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- The dual representation of TVaR is given by

$$TVaR_\alpha(X) = \sup\left\{E(XZ) : E(Z) = 1,\ 0 \leqslant Z \leqslant \frac{1}{1-\alpha}\right\}.$$

- Additionally, by $(\bigstar)$,

$$\partial TVaR_\alpha(X) = \overline{co}\left\{\left(\frac{1}{1-\alpha}\right) 1_{[U<1-\alpha]} : U \sim Unif(0,1),\ (U,X) \text{ is comonotonic}\right\}.$$

An Example: PO and BO for TVaR

- Therefore, if X is a continuous random variable, then $F_X(X) \sim Unif(0,1)$ and

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- Therefore, if X is a continuous random variable, then $F_X(X) \sim Unif(0,1)$ and

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- More generally, $\partial TVaR_\alpha(X) \neq \varnothing$ for $\alpha \in (0,1)$, since $\xi^* \in \partial TVaR_\alpha(X)$, where

$$\xi^* := \left(\frac{1}{1-\alpha}\right) 1_{[X > VaR_\alpha(X)]} + \left(\frac{1-\alpha-P(X > VaR_\alpha(X))}{P(X \geqslant VaR_\alpha(X)) - P(X > VaR_\alpha(X))}\right) 1_{[X = VaR_\alpha(X)]}.$$

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Proposition

Suppose that ρ^{In} *and* ρ^{Re} *are TVaR risk measures at respective levels* $\alpha, \beta \in (0,1)$*:*

$$\rho^{In} = TVaR_{\alpha} \;\; and \;\; \rho^{Re} = TVaR_{\beta}.$$

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Then the indemnity function I^* defined below is optimal for Problem $(\mathcal{P}_2)$:

$$I^* = \begin{cases} 0 & \text{if } \alpha < \beta, \\ \in \mathcal{I} & \text{if } \alpha = \beta, \\ Id & \text{if } \alpha > \beta, \end{cases}$$

where Id denotes the identity function.

An Example: PO and BO for TVaR

Hence, we obtain the following result.

Proposition

Suppose that ρ^{In} *and* ρ^{Re} *are TVaR risk measures at respective levels* $\alpha, \beta \in (0,1)$, *and that there exists* $\xi_0 \in L^{\infty}(\Omega, \mathcal{F}, P)$ *such that for each* $I \in \mathcal{I}$,

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Suppose that ρ^{In} and ρ^{Re} are TVaR risk measures at respective levels $\alpha, \beta \in (0,1)$, and that there exists $\xi_0 \in L^{\infty}(\Omega, \mathcal{F}, P)$ such that for each $I \in \mathcal{I}$,

$$\rho_2^{In}(\xi_0, I) \geqslant \rho^{In}(0).$$

Then, the following holds:

- *If $\alpha < \beta$, then $(0,0)$ is PO and $(\xi_0, 0)$ is BO.*
- *If $\alpha = \beta$, then for any $I \in \mathcal{I}$, $(TVaR_{\alpha}(I(X)), I)$ is PO and (ξ, I) is BO, where $\xi \in \partial TVaR_{\alpha}(X)$.*
- *If $\alpha > \beta$, then $(TVaR_{\alpha}(X), X)$ is PO and (ξ, Id) is BO, where $\xi \in \partial TVaR_{\alpha}(X)$.*

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In Conclusion...

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2. Any Pareto efficient contract for which the insurer is indifferent is a Bowley optimum for some pricing kernel.
3. For convex distortion risk measures, there is a tight relationship between competitive equilibria and Pareto Efficiency.
4. For the special case of TVaR, we provided a closed-form characterization of optima.