Linear PDEs perturbed by Gaussian Noise

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Hannover, January 2015
General Case

Let \((B_t)\) be a continuous centered Gaussian process on a complete probability space with sigma-algebra generated by the process, suppose that its covariance \(R(t, s), s, t \in [0, T]\) may be expressed as

\[
R(t, s) = \int_0^{\min(s, t)} K(t, r)K(s, r)dr,
\]

where \(K\) is square integrable and

\[
\sup_{t \in [0, T]} \int_0^t K(t, s)^2 ds < \infty.
\]

Furthermore, assume that there exists a Wiener process \((W_t)\) such that

\[
B_t = \int_0^t K(t, s)dW_s, \quad t \in [0, T].
\]
General Case

Assume further

(K1) For all $s \in (0, T]$, $K(\cdot, s)$ has a bounded variation on $(s, T]$ and

$$
\int_0^T |K|((s, T], s)^2 ds < \infty.
$$

Set

$$(K^* \varphi)(s) = \varphi(s)K(T, s) + \int_s^T (\varphi(t) - \varphi(s)) K(dt, s). \quad (1)$$

for $\varphi \in \mathcal{E}$, the space of $V$-valued deterministic step functions.
For $x, y \in V$ define

$$\langle x^{1[0, t]}, y^{1[0, s]} \rangle_H := \langle x, y \rangle_V R(t, s), \quad (t, s) \in [0, T]^2.$$  \hspace{1cm} (2)

The inner product $\langle \cdot, \cdot \rangle_H$ can be extended (by linearity) to $E$ and $(E, \langle \cdot, \cdot \rangle_H)$ forms a pre-Hilbert space. The completion of $E$ with respect to the above scalar product is denoted by $H$. Stochastic integral w.r.t. $\beta$ is defined in the standard way on $E$ and extended to $H$. We have

$$\|\varphi\|_H^2 = \|K^*\varphi\|_{L^2([0, T], V)}^2.$$  \hspace{1cm} (3)

and

$$\beta(\varphi) = \int_0^T (K^*\varphi)(t) dW_t, \quad P - a.s..$$
(K2) For some $\alpha \in (0, \frac{1}{2})$, the kernel $K$ satisfies the following:

- For all $s \in (0, T)$ the function $K(\cdot, s) : (s, T] \to \mathbb{R}$ is differentiable in the interval $(s, T)$ and both $K(t, s)$ and the derivatives $\frac{\partial K}{\partial t}(t, s)$ are continuous at every $t \in (s, T)$.

- There exist a constant $c > 0$ such that

$$\left| \frac{\partial K}{\partial t}(t, s) \right| \leq c(t - s)^{\alpha - 1} \left( \frac{s}{t} \right)^{-\alpha},$$

(4)

$$\int_s^t K(t, r)^2 \, dr \leq c(t - s)^{2\alpha + 1}$$

for $0 \leq s < t \leq T$. 

\[ \int_s^t K(t, r)^2 \, dr \leq c(t - s)^{2\alpha + 1} \]
Let $(K1)$ be satisfied. Consider the seminorm
\[
\|\varphi\|_{\mathcal{H}_R}^2 := \int_0^T |\varphi(s)|_V^2 K(s^+, s)^2 ds + \int_0^T \left( \int_s^T |\varphi(t)|_V |\mathcal{K}|(dt, s) \right)^2 ds,
\]
defined on $\mathcal{E}$. Denote by $\mathcal{H}_R$ the completion of $\mathcal{E}$ with respect to $\| \cdot \|_{\mathcal{H}_R}$. Then $\mathcal{H}_R$ is continuously embedded in $\mathcal{H}$.

There exists a finite constant $c_1 > 0$ such that
\[
\|\varphi\|_{\mathcal{H}} \leq c_1 \|\varphi\|_{bB([0, T]; V)}
\]
for all $\varphi \in bB([0, T]; V)$. 

Theorem

Suppose further (for simplicity) that $K(s^+, s) = 0$ for all $0 < s < T$. If (K2) is satisfied then there exists a finite constant $c_3(\alpha) > 0$ such that

$$
\|\varphi\|_\mathcal{H} \leq c_3(\alpha)\|\varphi\|_{L^{\frac{2}{1+2\alpha}}([0, T]; V)}
$$

for each $\varphi \in L^{\frac{2}{1+2\alpha}}([0, T]; V)$. 
Let (K1) be satisfied. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \(U\) be a real separable Hilbert space equipped with an inner product \(\langle \cdot, \cdot \rangle_U\) and \(T > 0\). Given an ONB basis \((e_n)\) of the space \(U\) we define the cylindrical Gaussian Volterra process (with covariance kernel \(K\)) as a formal sum

\[ B_t = \sum_{n=1}^{\infty} \beta_n(t)e_n, \]

where \((\beta_n(t))\) is a sequence of pairwise independent one-dimensional Gaussian Volterra processes with the same covariance kernel. The series does not converge in the space \(U\) but may be understood as usual as a family of random linear functionals (or may be shown to be convergent in any Hilbert space \(U_1\) such that the embedding \(U \hookrightarrow U_1\) is Hilbert-Schmidt).
Let $G : [0, T] \to \mathcal{L}(U, V)$ be an operator-valued function such that $G(\cdot) e_n \in \mathcal{H}$ for $n \in \mathbb{N}$, and $B$ be a standard cylindrical Gaussian Volterra process in $U$.

Define

$$\int_0^T G \, dB^H := \sum_{n=1}^{\infty} \int_0^T Ge_n \, d\beta_n$$

provided the infinite series converges in $L^2(\Omega, V)$. 

Stochastic integral
Linear equations

\begin{align}
\left\{ \begin{array}{l}
\mathrm{d}X_t &= AX_t \, dt + \Phi \, dB_t, \quad t \geq 0 \\
X_0 &= x,
\end{array} \right. \tag{6}
\end{align}

where $A : \text{Dom}(A) \to V$, $\text{Dom}(A) \subset V$, an infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$ on $V$, $\Phi \in \mathcal{L}(U, V)$ and $x \in V$.

$$X_t = S(t)x + \int_0^t S(t-s)\Phi dB_s =: S(t)x + Z(t), \quad \mathbb{P} \text{ - a.s.} \tag{7}$$

for $t \geq 0$. 
For all $T > 0$, $K$ satisfies $(K1)$ on $[0, T]$ and induces a non-atomic measure $\mathcal{K}$. Moreover, $\Phi \in L_2(U, V)$.

For all $T > 0$, $K$ satisfies $(K2)$ on $[0, T]$ and for all $s \in (0, T]$, $S(s)\Phi$ is a Hilbert-Schmidt operator such that

$$\|S(\cdot)\Phi\|_{L_2(U,V)} \in L^{\frac{2}{1+2\alpha}}(0, T).$$ (8)
Linear equations

Proposition

If at least one of the conditions (A1) and (A2) holds, then the process \( Z = (Z_t, t \geq 0) \), is well defined \( V \)-valued Gaussian process and its sample paths are \( \mathbb{P} \)-almost surely in \( L^2([0, T]; V) \) for all \( T > 0 \).
Proposition

Assume that for all $T > 0$, $K$ satisfies (K2) on $[0, T]$ and for all $s \in [0, T]$, $S(s)\Phi$ is a Hilbert-Schmidt operator such that

$$t \rightarrow t^{-\beta} \|S(t)\Phi\|_{L^2(U, V)} \in L^{\frac{2}{1+2\alpha}}(0, T). \quad (9)$$

for some $\beta > 0$. Then the process $Z$ has a Hölder continuous version in $V$. 

Linear equations
Corollary (Sufficient condition for (A2))

If for all $T > 0$ there exist finite constants $c > 0$ and $0 \leq \gamma < \frac{1}{2} + \alpha$ such that

$$\|S(t)\Phi\|_{L^2(U,V)} \leq ct^{-\gamma}, \quad t \in (0, T]$$

then there exists a Hölder continuous version of the process $Z$ in $V$. 
Definition

Let $H$ be an element of $(0, 1)$ (the Hurst parameter). A continuous centered Gaussian process $\beta^H(t)$, $t \in \mathbb{R}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called fractional Brownian motion if

$$\mathbb{E}\beta^H(t)\beta^H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \quad t, s \in \mathbb{R}. \quad (10)$$
Let $K_H(t, s)$ for $0 \leq s \leq t \leq T$ be the kernel function

$$K_H(t, s) = c_H(t - s)^{H - \frac{1}{2}} + c_H \left(\frac{1}{2} - H\right) \int_s^t (u - s)^{H - \frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2} - H}\right) \, du$$

where $c_H = \left[\frac{2H\Gamma(H + \frac{1}{2})\Gamma(\frac{3}{2} - H)}{\Gamma(2 - 2H)}\right]^{\frac{1}{2}}$ and $\Gamma(\cdot)$ is the gamma function.

The operator $K^*_H$ is given by

$$K^*_H \varphi(t) := \varphi(t)K_H(T, t) + \int_t^T (\varphi(s) - \varphi(t)) \frac{\partial K_H}{\partial s}(s, t) \, ds$$

for $\varphi \in \mathcal{E}$. 
Example (Parabolic)

Consider the initial boundary value problem for stochastic parabolic equation

\[
\frac{\partial u}{\partial t}(t, x) = Lu(t, x) + \xi(t, x), \quad (t, x) \in \mathbb{R}_+ \times D,
\]

\[
u(0, x) = u_0(x), \quad x \in D,
\]

\[
u(t, x) = 0, \quad t \in \mathbb{R}_+, x \in \partial D,
\]

where \( D \subset \mathbb{R}^d \) is a bounded domain with a smooth boundary, \( L \) is a second order uniformly elliptic operator on \( D \) and \( \eta \) is a noise process that is the formal time derivative of a space dependent fractional Brownian motion.

- rewrite the parabolic system as an infinite dimensional stochastic differential equation:

\[
U = L^2(D), \ V = L^2(D), \ \Phi = Id; \ we \ get \ (A1) \ with \ \rho = d/4, \ so \\
Z \in \mathcal{C}^\beta([0, T], D^\delta_A) \ for \ \delta + \beta + \frac{d}{4} < H.
\]
Boundary and Pointwise Noise

\[ \frac{\partial u}{\partial t}(t, \xi) = \Delta u(t, \xi), \quad (t, \xi) \in D \subset \mathbb{R}^n, \]

\[ u(0, \xi) = x(\xi), \]

\[ \frac{\partial u}{\partial \nu}(t, \xi) = \eta^H(t, \xi), \quad (t, \xi) \in \partial D \]

(Neumann type boundary noise), or

\[ u(t, \xi) = \eta^H(t, \xi), \quad (t, \xi) \in \partial D \]

(Dirichlet type boundary noise).
Boundary and Pointwise Noise

Modelled as

\[ Z^x(t) = S(t)x + \int_0^t S(t - r)\Phi dB^H(r), \quad t \geq 0, \]

where \( \Phi = (A - \hat{\beta}I)N \), \( N \) is the Neumann (or Dirichlet) map, the state space is \( V = L^2(D) \), and \( B^H \) is a cylindrical fBm on a separable Hilbert space \( U \subset L^2(\partial D) \).

Conditions for existence and time Hölder continuity of the solution:

- \( d = 1 \): \( \frac{1}{4} < H \) (Neumann) and \( \frac{3}{4} < H \) (Dirichlet).
- \( d \geq 2 \): \( \frac{1}{2} + \frac{1}{4}(d - 1) < H \) (Neumann).
Boundary and Pointwise Noise

\[ \frac{\partial u}{\partial t}(t, \xi) = \Delta u(t, \xi) + \delta_z \eta_t^H, \quad (t, \xi) \in D \]

\[ u(0, \xi) = x(\xi), \]

\[ \frac{\partial u}{\partial \nu}(t, \xi) = 0, \quad (t, \xi) \in \partial D \]

(pointwise noise, \( \delta_z \) - Dirac distribution at \( z \in D \)).

Modelled as

\[ Z^x(t) = S(t)x + \int_0^t S(t - r)\Phi d\beta^H(r), \quad t \geq 0, \]

in \( V = L^2(D) \), where \( \Phi \) is a distribution, i.e. \( \Phi \in (D_A^\delta)^* \) for \( \delta > \frac{d}{4} \). We have a (Hölder) continuous solution for \( \delta < H \), i.e. for \( \frac{d}{4} < H \).
Consider the equation with finite-dimensional (fBm)

\[ dX(t) = A(t)X(t)dt + \sum_{k=1}^{m} B_k X(t)d\beta^H_k(t) \]  \hspace{1cm} (12)

\[ X(0) = x_0 \]

where \((A(t))\) generates a strongly continuous family of operators \((U_0(t, s))\), \(t \geq s\),

\[ \frac{\partial}{\partial s} U_0(t, s) = -U_0(t, s)A(s) \] \hspace{1cm} (13)

\[ \frac{\partial}{\partial t} U_0(t, s) = A(t)U_0(t, s) \] \hspace{1cm} (14)
(H1) The family of closed operators \((A(t), t \in [0, T])\) defined on a common domain \(D := \text{Dom}(A(t))\) for \(t \in [0, T]\) generates a strongly continuous evolution operator \((U_0(t, s), 0 \leq s \leq t \leq T)\) on \(V\).

(H2) The collection of linear operators \((B_1, \ldots, B_m)\) generate mutually commuting strongly continuous groups \((S_1(s), \ldots, S_m(s), s \in \mathbb{R})\) which commute with \(A(t)\) on \(D\) for each \(t \in [0, T]\). For \(i, j \in \{1, \ldots, m\}\), \(\text{Dom}(B_i B_j) \supset D\), \(\text{Dom}(A^*(t)) = D^*\) is independent of \(t\) and \(D^* \subset \bigcap_{i,j=1}^m \text{Dom}(B_i^* B_j^*)\) where \(*\) denotes the topological adjoint.

(H3) The family of linear operators \((\tilde{A}(t), t \in [0, T])\) where \(\tilde{A}(t) = A(t) - H t^{2H-1} \sum_{j=1}^m B_j^2\), \(\text{Dom}(\tilde{A}(t)) = D\) for each \(t \in [0, T]\), generates a strongly continuous evolution operator on \(V\), \((U(t, s), 0 \leq s \leq t \leq T)\).
A $\mathcal{B}([0, T]) \otimes \mathcal{F}$ measurable stochastic process $(X(t), t \in [0, T])$ is said to be

(i) a strong solution of (12) if $X(t) \in D$ a.s. $\mathbb{P}$ and

$$X(t) = x_0 + \int_0^t A(s)X(s)ds + \sum_{j=1}^m \int_0^t B_jX(s)d\beta_j^H(s) \quad \text{a.s. (15)}$$

for $t \in [0, T]$.

(ii) a weak solution of (12) if for each $z \in D^*$

$$< X(t), z > = < x_0, z > + \int_0^t < X(s), A^*(s)z > ds$$

$$+ \sum_{j=1}^m \int_0^t < X(s), B_j^*z > d\beta_j^H(s) \quad \text{a.s. (17)}$$

for $t \in [0, T]$ and
(iii) a *mild solution* of (12) if

\[
X(t) = U_0(t, 0)x_0 + \sum_{j=1}^{m} \int_0^t U_0(t, s)B_jX(s)d\beta_j^H(s) \quad \text{a.s.} \quad (18)
\]

for \( t \in [0, T] \),

where the stochastic integrals in (15)–(18) are defined in the Skorokhod sense.
Theorem

Assume that $H > \frac{1}{2}$ and $(H1)-(H3)$ are satisfied. There is a weak solution of (12). If $x_0 \in D$, then there is a strong solution of (12). If $B_j \in \mathcal{L}(\mathcal{V})$ for $j \in \{1, \ldots, m\}$, then there is a mild solution of (12) which is unique in the space $\text{Dom}\delta_H \cap L^2(\Omega; \tilde{H})$, where $\delta_H$ denotes the divergence operator based on $\beta^H$. In each case the solution $(X(t), t \in [0, T])$ is given as follows

$$X(t) = \prod_{j=1}^{m} S_j(\beta_j^H(t))U(t, 0)x_0$$

(19)

for $t \in [0, T]$.

For $H < \frac{1}{2}$ there exists a weak solution given by formula (19) in the ”‘parabolic’” case (by approximations, using Cheredito-Nualart result on closedness of the extension of Skorokhod integral operator).
Proof: Existence in the ”‘strong’” case: By fractional Ito formula, the other cases by approximations of the initial value (Malliavin derivatives in the Ito formula may be easily calculated).

Uniqueness (for simplicity, from now on $m = 1, \beta_H^1 =: \beta^H, B_j =: B, S_1 =: S$).

\[
X_t = U_0(t, 0)x + \int_0^t U_0(t, r)BX_r d\beta^H_r,
\]

\[
Y_t = U_0(t, 0)x + \int_0^t U_0(t, r)BY_r d\beta^H_r,
\]

Define the process $Z = \{Z_t, t \in [0, T]\}$ as

\[
Z_t = X_t - Y_t, \; t \in [0, T].
\]
Equations with Multiplicative Noise - Existence and Uniqueness

Let
\[ X_t = \sum_{n=0}^{+\infty} X_n(t), \quad Y_t = \sum_{n=0}^{+\infty} Y_n(t), \quad t \in [0, T], \]
be the respective Wiener chaos decompositions. Show (by induction) \( Z_n = X_n - Y_n = 0 \). We have \( Z_0 = 0 \) hence
\[ \sum_{n=1}^{+\infty} Z_n(t) = \sum_{n=0}^{+\infty} \int_0^t U_0(t, s) BZ_n(s) d\beta_s^H. \]
Since \( Z_0 \in H_0 \) then
\[ H_1 \ni \int_0^t U_0(t, s) BZ_0(s) d\beta_s^H = 0, \quad t \in [0, T], \]
and consequently
\[ Z_1(t) = \int_0^t U_0(t, s) BZ_0(s) d\beta_s^H = 0 \]
for any \( t \in [0, T] \) because \( Z_1 \in H_1 \).
Suppose $Z_n = 0$ for some fixed $n \in \mathbb{N}$. By commutativity

$$\int_0^t U_0(t, s) B Z_n(s) d\beta^H_s = \int_0^t \int_0^{t_{n-1}} \ldots \int_0^{t_1} U_0(t, s) B^n Z_0(s) d\beta^H_s d\beta^H_{t_1} \ldots d\beta^H_{t_{n-1}}$$

is zero for any $t \in [0, T]$ and the expression belongs to $\mathcal{H}_{n+1}$. Moreover, $Z_{n+1} \in \mathcal{H}_{n+1}$ thus

$$Z_{n+1}(t) = \int_0^t \int_0^{t_{n-1}} \ldots \int_0^{t_1} U_0(t, s) B^n Z_0(s) d\beta^H_s d\beta^H_{t_1} \ldots d\beta^H_{t_{n-1}} = 0$$

for $t \in [0, T]$. 
Examples

Let
\[ dX_t = AX_t \, dt + bX_t \, d\beta_t^H, \quad t > 0, \]
\[ X_0 = x, \]

where \( A : \text{Dom}(A) \subset V \rightarrow V \) is the generator of a strongly continuous semigroup \( \{S_A(t), t \geq 0\} \) and \( b \in \mathbb{R} \setminus \{0\} \). Then

\[ X_t = \exp \left\{ b\beta_t^H - \frac{1}{2} b^2 t^{2H} \right\} S_A(t)x, \quad 0 \leq s \leq t < +\infty, \]

and since there exist some constants \( M > 0, \omega \in \mathbb{R} \) such that

\[ \|S_A(t)\|_{\mathcal{L}(V)} \leq M e^{\omega t}, \quad t \geq 0, \]

we have that

\[ |X_t|_V \leq M \exp \left\{ b\beta_t^H - \frac{1}{2} b^2 t^{2H} + \omega t \right\} |x|_V \rightarrow 0 \]

(21)
a.s. as \( t \rightarrow \infty \) (the solution is pathwise stabilized by noise)
However, for any $p > 0$, taking for simplicity $V = \mathbb{R}$, $A = \omega$, $x \neq 0$

$$\mathbb{E}|X_t|^p = |x|^p \exp \left\{ p\omega t - \frac{1}{2} b^2 pt^{2H} + pbB_t^H \right\}, \quad t \geq 0, \quad p > 1,$$

hence for each $\epsilon > 0$ there exists $\tilde{C}_\epsilon > 0$ such that

$$\mathbb{E}[|X_t|^p] = |x|^p \exp \left\{ \hat{c} t^{2H} + p\omega t \right\} \geq \tilde{C}_\epsilon \exp\{ (\hat{c} - \epsilon) t^{2H} \}, \quad t \geq 0,$$

where $\hat{c} = \frac{1}{2} b^2 (p^2 - p)$, so for $p > 1$ the $p$-th moment of the solution is destabilized by noise.
Examples

\[ \frac{\partial u}{\partial t}(t, \xi) = L(t, \xi)u(t, \xi) + b \frac{d\beta^H}{dt} u \] (22)

\[ u(0, \xi) = x_0(\xi) \]

for \( (t, \xi) \in [0, T] \times \mathcal{O} \)

\[ \left( \frac{\partial u}{\partial \xi} \right)^\alpha(t, \xi) = 0, \quad (t, \xi) \in [0, T] \times \partial \mathcal{O}, \ \alpha \in \{1, \ldots, k - 1\} \]

where \( k \in \mathbb{N}, \ \mathcal{O} \subset \mathbb{R}^d \) is a bounded domain of class \( C^k \), \( b \in \mathbb{R} \setminus \{0\} \) and

\[ L(t, \xi) := \sum_{|\alpha| \leq 2k} a_\alpha(t, \xi) D^\alpha \] (23)

is a strongly elliptic operator on \( \mathcal{O} \), uniformly in \( (t, \xi) \in [0, T] \times \overline{\mathcal{O}} \) and \( a_\alpha(t, \cdot) \in C^{2k}(\overline{\mathcal{O}}) \) for each \( t \in [0, T] \).
The equation (22) is rewritten in the form

\[ dX(t) = A(t)X(t)dt + BX(t)d\beta^H(t) \]  
\[ X(0) = x_0 \in V \]

for \( t \in [0, T] \), where \( V = L^2(\mathcal{O}) \), \( (A(t)u)(\xi) = L(t, \xi)u(t, \xi) \), \( \text{Dom}(A(t)) = D = H^{2k}(\mathcal{O}) \cap H_0^k(\mathcal{O}) \) and \( B = bI \in \mathcal{L}(V) \). It is assumed that

\[ \sup_{\xi \in \mathcal{O}} |a_\alpha(t, \xi) - a_\alpha(s, \xi)| \leq M|t - s|^\gamma \]  
(25)
The ellipticity condition (H3) is satisfied if $a > Ht^{2H-1}b^2$. The solution may be expressed

$$X(t) = S(\beta^H(t))S_{\Delta}\left(at - \frac{1}{2}b^2 t^{2H}\right)x_0.$$  

So the problem is "well posed" for $0 \leq t \leq T$, where $T = \left(\frac{2a}{b^2}\right)^{1/(2H-1)}$. 

**Examples**

\[
\frac{\partial u}{\partial t}(t, \xi) = a \frac{\partial^2 u}{\partial \xi^2}(t, \xi) + b \frac{\partial u}{\partial \xi}(t, \xi) \frac{d\beta^H}{dt}(t) \tag{26}
\]

\[
[S(t)x_0](\xi) = x_0(\xi + bt) \tag{27}
\]
\[
\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial \xi^4} - \alpha u + \frac{\partial u}{\partial \xi} \frac{d\beta^H(t)}{dt} \tag{30}
\]

\[u(0, \xi) = x_0(\xi) = \sin \xi\]

\[
\tilde{A}(t) = L - tH^{2H-1}B^2 = -\frac{\partial^4}{\partial \xi^4} - \alpha I - tH^{2H-1} \frac{\partial^2}{\partial \xi^2} \tag{31}
\]

The solution has the form

\[
X(t) = S(\beta^H(t))U(t, 0)x_0. \tag{32}
\]
Setting \([U(t, 0)x_0](\xi) = \varphi(t)\sin \xi\) we obtain

\[
\dot{\varphi}(t) \sin \xi = -\varphi(t) \sin \xi - \alpha \varphi(t) \sin \xi + Ht^{2H-1} \varphi(t) \sin \xi
\]

\[
\varphi(0) = 1.
\]

and hence

\[
X(t) = \sin (\xi + \beta^H(t)) \exp \left[ - (1 + \alpha)t + \frac{1}{2}t^{2H} \right]. \tag{33}
\]

It follows that

\[
\lim_{t \to \infty} |X(t)| = \infty, \quad \text{a.s.}
\]

so the noise destabilizes the equation.
Assume (K1) and let $F \in C^{1,2}([0, T] \times R)$ has at most exponential growth in the second variable, uniform in $t$. Then $F(t, B_t)$ belongs to $\mathbb{D}^{1,2}$ and we have

$$F(t, B_t) = F(0, 0) + \int_0^t D_t F(s, B_s)ds + \int_0^t D_x F(s, B_s)dB_s$$

$$+ \frac{1}{2} \int_0^t D_x^2 F(s, B_s)dR(s),$$

where $R(s) := R(s, s)$ (under (K1) $R$ has bounded variation).
The natural candidate for the evolution system $U(t, s)$ would be the one corresponding to the equation

$$y(t) = y_0 + \int_0^t A(s)y(s)ds - \int_0^t B^2y(s)dR(s), \quad t \in [0, T].$$

If we additionally assume that $R \in C^1([0, T])$ all results stated above (in the regular case) remain true with $t^{2H}$ replaced by $R(t)$ and $Ht^{2H-1}$ by $R'(t)$. 
Consider
\[ dY_t = AY_t dt + BY_t d\beta_t^H, \quad t > s, \]
\[ Y_s = x, \]  \hspace{1cm} (34)

assume that \((\tilde{A}(t))\) generates the "'parabolic’’ strongly evolution system \(\{U(t, s), 0 \leq s \leq t \leq T\}\) on \(V\).

\[ (U(t, s)(V) \subset D, \]
\[ \|U(t, s)\|_{\mathcal{L}(V)} \leq C_U, \]
\[ \left\| \frac{\partial}{\partial t} U(t, s) \right\|_{\mathcal{L}(V)} = \|\tilde{A}(t)U(t, s)\|_{\mathcal{L}(V)} \leq \frac{C_U}{t - s}, \]
\[ \|\tilde{A}(t)U(t, s)(\tilde{A}(s) - \bar{\omega}I)^{-1}\|_{\mathcal{L}(V)} \leq C_U \]

for some constant \(C_U > 0\) and any \(0 \leq s < t \leq T\).
Random Evolution System

What is the random evolution system defined by the equation (34)? It may be verified that the equation has a weak solution \( \{ U_Y(t, s)x, s \leq t \leq T \} \) given by a formula

\[
U_Y(t, s)x = S(B_t^H - B_s^H)U(t - s, 0)x, \quad s \leq t \leq T,
\]

for any initial value \( x \in V \). Note that \( U_Y(t, s) \) is not the same as

\[
\tilde{U}_Y(t, s) = S(B_t^H - B_s^H)U(t, s).
\]
In one-dimensional case, \( A = a, \ B = b \) we have

\[
\bar{U}_Y(t, s) = S(B_t^H - B_s^H)U(t, s) = \exp \left\{ b(B_t^H - B_s^H) - \frac{1}{2} b^2(t^{2H} - s^{2H}) \right\},
\]

while

\[
U_Y(t, s) = \exp \left\{ b(B_t^H - B_s^H) - \frac{1}{2} b^2(t - s)^{2H} \right\}, \quad 0 \leq s \leq t \leq T.
\]

\( U_Y(t, s) \) does not posses the composition (cocycle) property (the equation does not define RDS) while \( \bar{U}_Y(t, s) \) does.
Theorem

Let \( F : [0, T] \times V \rightarrow V \) be a measurable function satisfying

(i) \( F \) there exists a function \( \bar{L} \in L^1([0, T]) \) such that

\[
\| F(t, x) - F(t, y) \|_V \leq \bar{L}(t) \| x - y \|_V, \ x, y \in V, \ t \in [0, T].
\]

(ii) \( F \) for some function \( \bar{K} \in L^1([0, T]) \)

\[
\| F(t, 0) \|_V \leq \bar{K}(t), \ t \in [0, T].
\]

Then the equation

\[
y(t) = U_Y(t, 0)x + \int_0^t U_Y(t, r)F(r, y(r)) \, dr
\]

(39)

has a unique solution in the space \( C([0, T]; V) \) for a.e. \( \omega \in \Omega \) and any initial value \( x \in V \).
In the Wiener case $H = 1/2$ the solution to the equation (39) is the so-called mild solution to the equation

\[
\begin{align*}
    dX_t &= AX_t dt + F(t, X_t) dt + BX_t dW_t, \\
    X_0 &= x \in V.
\end{align*}
\]

and is known to coincide with the weak solution. What can we say in the general case?
Theorem

Let the assumptions of Theorem 4 hold and \( \{ X_t, t \in [0, T] \} \) be the solution to the equation mild.rce such that there exists a constant \( C_X < +\infty \)

\[
\max \left\{ \sup_{t \in [0, T]} \mathbb{E} \| X_t \|_V^4, \sup_{t \in [0, T]} \sup_{v \in [0, T]} \mathbb{E} \| D_v^H X_t \|_V^4 \right\} \leq C_X. \tag{40}
\]

In addition, let \( F \) be Fréchet differentiable with respect to the space variable for any time \( t \in [0, T] \). Suppose that there exists a function \( C \in L^4([0, T]) \) such that

\[
\max \{ \| F(t, x) \|_V, \| F'_x(t, x) \| \} \leq C(t), \quad t \in [0, T], \tag{41}
\]

holds. Then \( \{ X_t, t \in [0, T] \} \) is a solution to the integral equation
Affine equation

Theorem

\[ X_t = x + \int_0^t A X_r \, dr + \int_0^t F(r, X_r) \, dr + \int_0^t B X_r \, d \beta_r^H \]
\[ + \int_0^t \alpha_H \int_0^T \int_r^t |v - w|^{2H-2} BU_Y(v, r) F'_x(r, X_r) D_w^H X_r \, dv \, dw \, dr \]

in a weak sense, i.e. for any \( y \in D^*, t \in [0, T] \),

\[ \langle X_t, y \rangle_V = \langle x, y \rangle_V + \int_0^t \langle X_r, A^* y \rangle_V \, dr \]
\[ + \int_0^t \langle F(r, X_r), y \rangle_V \, dr + \int_0^t \langle X_r, B^* y \rangle_V \, d \beta_r^H \]
\[ + \int_0^t \alpha_H \int_0^T \int_r^t |v - w|^{2H-2} \langle U_Y(v, r) F'_x(r, X_r) D_w^H X_r, B^* y \rangle_V \, dv \, dw \, dr \]
Consider a one-dimensional equation

\[ dX_t = aX_t dt + bX_t d\beta_t^H, \quad X_0 = 1, \]  

(42)

where \( a, b \in \mathbb{R} \) are nonzero constants. In the previous notation,

\[ dX_t = F(t, X_t)dt + BX_t d\beta_t^H, \quad X_0 = 1, \]

where \( F(t, x) = ax, \ A = 0 \) and \( B = bl \). Recall that

\[ \bar{Y}(t, s) = S(\beta_t^H - \beta_s^H)U(t, s) = \exp \left\{ b(\beta_t^H - \beta_s^H) - \frac{1}{2} b^2(t^{2H} - s^{2H}) \right\}. \]

Then

\[ X_t = \bar{Y}(t, 0) + \int_0^t \bar{Y}(t, r)F(r, X_r)dr \]  

(43)
Let the assumptions of Theorem 4 be satisfied and $F : [0, T] \rightarrow V$ be a measurable function independent of a space variable such that $\|F\|_V \in L^2([0, T])$. Then the solution $\{X^M_t, t \in [0, T]\}$ to the affine equation (39) obtained in Theorem 4 having the form

$$X^M_t = U_Y(t, 0)x + \int_0^t U_Y(t, r)F(r)dr$$

is a weak solution to the equation

$$dX_t = (AX_t + F(t))dt + BX_t d\beta^H_t,$$

$X_0 = x \in V.$
Corollary

For each \( p \geq 1 \) there exists a constant \( c_p > 0 \) depending only on \( p \) such that

\[
\mathbb{E}\left[ \| X_t \|^p \right] \leq c_p M \exp \left\{ \frac{(p^2 - p)b^2}{2} t^{2H} + p\omega t \right\} \| x \|^p \\
+ M t^{p-1} \int_0^t \exp \left\{ \frac{(p^2 - p)b^2}{2} (t - s)^{2H} \right\} \| F(s) \|^p \, ds, \quad t \geq 0.
\] (46)

\[
+ p\omega(t - s) \right\} \| F(s) \|^p \, ds, \quad t \geq 0.
\] (47)

In particular, if \( F(t) \equiv F \) does not depend on \( t \geq 0 \), for each \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) such that

\[
\mathbb{E}\left[ \| X_t \|^p \right] \leq C_\epsilon \exp\{ (\hat{c} + \epsilon) t^{2H} \}, \quad t \geq 0,
\] (48)

holds with \( \hat{c} = 1/2b^2(p^2 - p) \).
Some References

- B. Maslowski and J. Šnupárková, Stochastic equations with multiplicative fractional Gaussian noise in Hilbert space, submitted
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, 
\[U = (U, \langle \cdot, \cdot \rangle_U, |\cdot|_U)\] be a separable Hilbert space.

A cylindrical process \(\langle B^H, \cdot \rangle: \Omega \times \mathbb{R} \times U \to \mathbb{R}\) on \((\Omega, \mathcal{F}, \mathbb{P})\) is called a 
standard cylindrical fractional Brownian motion with Hurst parameter \(H \in (0, 1)\) if

1. For each \(x \in U \setminus \{0\}, \frac{1}{|x|_U} \langle B^H(\cdot), x \rangle\) is a standard scalar fractional
   Brownian motion with Hurst parameter \(H\).
2. For \(\alpha, \beta \in \mathbb{R}\) and \(x, y \in U\),

\[
\langle B^H(t), \alpha x + \beta y \rangle = \alpha \langle B^H(t), x \rangle + \beta \langle B^H(t), y \rangle \quad \text{a.s. } \mathbb{P}.
\]

- \(\langle B^H(t), x \rangle\) has the interpretation of the evaluation of the functional \(B^H(t)\) at \(x\),
- For \(H = \frac{1}{2}\) it is standard cylindrical Wiener process in \(U\).
Cylindrical FBM

We can associate \((B^H(t), t \in \mathbb{R})\) with a standard cylindrical Wiener process \((W(t), t \in \mathbb{R})\) in \(U\) formally by \(B^H(t) = \mathbb{K}_H \left( \dot{W}(t) \right)\). For \(x \in U \setminus \{0\}\), let \(\beta^H_x(t) = \langle B^H(t), x \rangle\). It is elementary to verify from (??) that there is a scalar Wiener process \((w_x(t), t \in \mathbb{R})\) such that

\[
\beta^H_x(t) = \int_0^t K_H(t, s) \, dw_x(s) \tag{49}
\]

for \(t \in \mathbb{R}\).

Furthermore, if \(V = \mathbb{R}\), then \(w_x(t) = \beta^H_x \left( (\mathcal{K}_H^*)^{-1} 1_{[0,t]} \right)\) where \(\mathcal{K}_H^*\) is given by (16). Thus we have a formal series

\[
W(t) = \sum_{n=1}^{\infty} w_n(t) e_n. \tag{50}
\]
Stochastic integral

Let \((e_n, n \in \mathbb{N})\) be a complete orthonormal basis in \(U\).
Let \(G : [0, T] \rightarrow \mathcal{L}(U, V)\) be an operator-valued function such that \(G(\cdot)e_n \in \mathcal{H}\) for \(n \in \mathbb{N}\), and \(B^H\) be a standard cylindrical fractional Brownian motion in \(U\).
Define
\[
\int_0^T G \, dB^H := \sum_{n=1}^{\infty} \int_0^T Ge_n \, d\beta^H_n
\]
provided the infinite series converges in \(L^2(\Omega, V)\).
Note that by condition 2 in the definition above the scalar processes \(\beta^H_n(t) := \langle B^H(t), e_n \rangle\), \(t \in \mathbb{R}, n \in \mathbb{N}\) are independent.
Consider the linear equation

\[ dZ^x(t) = AZ^x(t) \, dt + \Phi \, dB^H(t), \]

\[ Z(0) = x, \]

where \((B^H(t), t \geq 0)\) is a standard cylindrical fractional Brownian motion with Hurst parameter \(H \in (0, 1)\) in \(U\) and \(U\) is a separable Hilbert space, \(A : \text{Dom}(A) \rightarrow V, \text{Dom}(A) \subset V\), \(A\) is the infinitesimal generator of a strongly continuous semigroup \((S(t), t \geq 0)\) on \(V\), \(\Phi \in \mathcal{L}(U, V)\) and \(x \in V\) is generally random. Let \(Q = \Phi \Phi^* \in \mathcal{L}(V)\).
Linear equations

A solution \((Z^x(t), t \geq 0)\) to (51) is considered in the mild form

\[ Z^x(t) = S(t)x + Z(t), \quad t \geq 0, \tag{52} \]

where \((Z(t), t \geq 0)\) is the convolution integral

\[ Z(t) = \int_0^t S(t - u)\Phi dB^H(u). \tag{53} \]

If \((S(t), t \geq 0)\) is analytic, then there is a \(\hat{\beta} \in \mathbb{R}\) such that the operator \(\hat{\beta}I - A\) is uniformly positive on \(V\).

For each \(\delta \geq 0\), let us define \((V_\delta, |\cdot|_\delta)\) a Banach space, where

\[ V_\delta = \text{Dom} \left( (\hat{\beta}I - A)^\delta \right) \]

with the graph norm topology such that

\[ |x|_\delta = \left| (\hat{\beta}I - A)^\delta x \right|_V. \]

The space \(V_\delta\) does not depend on \(\hat{\beta}\) because the norms are equivalent for different values of \(\hat{\beta}\) satisfying the above condition.
Assumptions

Let \((S(t), t \geq 0)\) be an analytic semigroup such that

\[ |S(t)\Phi|_\gamma \leq ct^{-\rho} \quad \text{(A1)} \]

for \(t \in [0, T], \, c \geq 0\) and \(\rho \in [0, H)\).
Regularity

**Theorem**

If (A1) is satisfied, then \((Z(t), t \in [0, T])\) is a well-defined \(V_\delta\)-valued process in \(C^\beta([0, T], V_\delta)\), a.s.-\(P\) for \(\beta + \delta + \gamma < H, \beta \geq 0, \delta \geq 0\).

- Analyticity not necessary for \(H > 1/2\).

Conjecture: Consider the general case \(B_t = \sum \beta_n(t)\) where \(\beta_n\) are continuous centered Gaussian processes defined by (the same) kernel \(K\) satisfying \((K1)\). Then the stochastic convolution integral exists and as a process has a version with sample paths in \(L^2(0, T; V)\) a.s. provided (A1) is satisfied with \(\rho = 0\). If moreover we have for some \(H > 1/2\)

\[
\frac{\partial K}{\partial t}(t, s) \leq (s/t)^{1/2-H}(t - s)^{H-3/2}
\]

the same holds true under weaker condition \(\rho < H\).