Assessing financial model risk
and an application to electricity prices

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Model risk in risk management and the Basel multiplier

”Absolute” measure of model risk

”Relative” measure of model risk

”Local” measure of model risk

Application to electricity prices
Outline

- Model risk in risk management and the Basel multiplier
  - "Absolute" measure of model risk
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Model risk

- Model risk is a recurrent theme in Economics and Finance.
- It broadly refers to the (bad) impact the choice of a wrong model can have.
- It is difficult to define or measure. Also, it is questionable what the correct model is (and whether it exists)
- It certainly affects areas such as portfolio choice, pricing, hedging and measurement of risk
- A distinction should be made between model (or misspecification) risk and estimation risk.
- In general, two broad approaches have been pursued: model averaging (Bayesian or not) and worst-case approach.
Model risk

- Model risk has a strong impact when assessing the risk of a portfolio.

- Several **model assumptions** affect the final VaR (or ES) figure:
  - Volatilities
  - Other marginal distributions
  - Correlations
  - Other joint distributions (copulae)
  - Pricing models for derivatives and choice of the relevant factors

- **Worst-case risk measures** under different sets of models (incomplete information) have been intensively studied, also in connection with robust portfolio optimization. See Ghaoui, Oks and Oustry (2003) among many others.

- Kerkhof, Melenberg and Schuhmacher (2010) introduce a measure of model risk which is based on the worst-case risk figure. It is different from what we propose here.
A motivation: the Basel multiplier

- Within the Basel framework, financial institutions are allowed to use internal models to assess the capital requirement due to market risk.

- The term that measures risk in *usual* conditions is given by:

\[
CC = \max \left\{ VaR^{(0)}, \frac{\lambda}{60} \sum_{i=1}^{60} VaR^{(-i)} \right\},
\]

where

- \( VaR^{(0)} \) is the portfolio’s **Value-at-Risk** (order 1% and 10-day horizon) computed/estimated today
- \( VaR^{(-i)} \) is the VaR we obtained \( i \) days ago
- \( \lambda \) is the *multiplier*

- Remind:

\[
VaR_\alpha(X) = -F_X^{-1}(\alpha) \quad \text{if } F_X \text{ is invertible} \\
= -\inf\{x : F_X(x) \geq \alpha\} \quad \text{more in general}
\]
A motivation: the Basel multiplier

- Remind

\[ CC = \max\{ VaR_0, \lambda \overline{VaR} \} \]

(\( \overline{VaR} \) average of the last 60 VaR’s)

- The **multiplier** (\( \lambda \)) is assigned to each institution by the regulator

- It depends on back-testing performances of the system (poor performance yields higher \( \lambda \)) and it is revised on a periodical basis

- It is in the interval [3, 4]

- As \( \lambda \geq 3 \), it is apparent that in normal conditions the second term is the leading one in the maximum giving the capital charge \( CC \)
A motivation: the Basel multiplier

- Stahl (1997) offered a simple theoretical justification for $\lambda$ to be in the range $[3, 4]$.

- Let $X$ be the portfolio Profits-and-Losses (r.v.) due to market risk.

- As the time-horizon is short, we can assume $\mathbb{E}[X] = 0$. From Chebyshev inequality:

\[
P(X \leq -q) \leq P(|X| \geq q) \leq \frac{\sigma^2}{q^2}, \quad q > 0.
\]

- It immediately follows

\[
\text{VaR}_\alpha(X) \leq \frac{\sigma}{\sqrt{\alpha}}
\]

- The r.h.s. provides an upper bound for the VaR of a r.v. having mean 0 and variance $\sigma^2$. 
A motivation: the Basel multiplier

- This bound can be compared with the VaR we obtain under the normal hypothesis ($\alpha < 0.5$)

\[
VaR_\alpha(X) = \sigma |z_\alpha| \quad (z_\alpha = \Phi^{-1}(\alpha))
\]

- Here are the two VaRs (normal: black, upper bound: red, $\sigma = 1$)
A motivation: the Basel multiplier

- Here is the ratio (upper bound/normal)
The Chebishev inequality can be used to obtain an upper bound for the Expected Shortfall

$$ES_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_u(X) \, du$$

under $\mathbb{E}[X] = 0$ and $\sigma(X) = \sigma$.

Integrating we have

$$ES_\alpha(X) \leq \frac{1}{\alpha} \int_0^\alpha \frac{\sigma}{\sqrt{u}} \, du = \frac{2\sigma}{\sqrt{\alpha}}$$
A Chebyshev bound for the Expected Shortfall

- Under the **normal** hypothesis for $X$ we have

$$ES_\alpha(X) = \frac{\sigma \varphi(z_\alpha)}{\alpha}$$

where $\varphi$ is the density of a standard normal.

- Here are the two ESs (normal: black, upper bound: red, $\sigma = 1$)
A Chebyshev bound for the Expected Shortfall

- Here is the ratio (upper bound/normal)
Cantelli upper bounds

- Even though Chebyshev inequalities are sharp, the upper bounds on VaR and ES are not.
- A single-tail **sharp** inequality is the **Cantelli**’s one:

\[
P(X \leq -q) \leq \frac{\sigma^2}{\sigma^2 + q^2} \quad (q < 0)
\]

- It follows a **sharp bound** for VaR

\[
VaR_\alpha(X) \leq \sigma \sqrt{\frac{1 - \alpha}{\alpha}}
\]

and a slightly improved (but still not sharp) bound for ES

\[
ES_\alpha(X) \leq \sigma \left( \sqrt{\frac{1 - \alpha}{\alpha}} + \frac{1}{\alpha} \arctan \sqrt{\frac{1 - \alpha}{\alpha}} \right)
\]
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- "Relative" measure of model risk
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An absolute measure of model risk: definition

- We want to generalize the notion of multiplier as the ratio between the worst-case risk and the risk computed under a reference model.

- So, we need to consider:
  - a risk measure
  - a reference model
  - a set of alternative models
A risk measure $\rho$ is given, defined on some set of random variables.

We assume $\rho$ is
- **law-invariant**, i.e. $\rho(X) = \rho(Y)$ whenever $X \sim Y$
- **positive homogeneous**: $\rho(aX) = a\rho(X)$ for any $a \geq 0$
- **translation invariant**: $\rho(X + b) = \rho(X) - b$ for any $b \in \mathbb{R}$

Both VaR and ES satisfy these properties (but also spectral and other r.m.)
An absolute measure of model risk: definition

- Let $X_0$ be a reference r.v. Assume $\rho(X_0) > 0$.

- By law-invariance, what really matters is the distribution of $X_0$.

- Let $\mathcal{L}$ be a set of alternative r.v.’s, with $X_0 \in \mathcal{L}$.

- Define

$$
\underline{\rho}(\mathcal{L}) = \inf_{X \in \mathcal{L}} \rho(X), \quad \overline{\rho}(\mathcal{L}) = \sup_{X \in \mathcal{L}} \rho(X)
$$

and assume they are finite.
The absolute measure of model risk is defined as

\[ AM = AM(\rho, X_0, \mathcal{L}) = \frac{\bar{\rho}(\mathcal{L})}{\rho(X_0)} - 1. \]

- \( AM \geq 0 \) with equality if and only if \( X_0 \) has already a worst-case distribution, i.e. \( \rho(X_0) = \bar{\rho}(\mathcal{L}) \)

- We see that \( AM + 1 \) can be interpreted as a generalized multiplier

- The larger is \( \mathcal{L} \), the higher is \( AM \) (hence absolute measure)
An absolute measure of model risk: properties

▸ **(scale invariance)** For any $a > 0$ it holds

$$AM(aX_0, a\mathcal{L}) = AM(X_0, \mathcal{L})$$

where $a\mathcal{L} = \{aX : X \in \mathcal{L}\}$.

▸ **(translation)** For $b \in \mathbb{R}$ it holds

$$AM(X_0 + b, \mathcal{L} + b) \begin{cases} > AM(X_0, \mathcal{L}), & \text{for } b > 0 \\ < AM(X_0, \mathcal{L}), & \text{for } b < 0 \end{cases}$$

where $\mathcal{L} + b = \{X + b : X \in \mathcal{L}\}$. 
An absolute measure of model risk: an example

- For $X$ having mean $\mu$ and variance $\sigma^2$, consider the set of alternative models

$$\mathcal{L}_{\mu,\sigma} = \{X : \mathbb{E}[X] = \mu, \sigma(X) = \sigma\}$$

- Set, as before, $\mu = 0$.

- By scale invariance, w.l.o.g. we concentrate on the case $\sigma = 1$.

- If $X_0 \in \mathcal{L}_{0,1}$, we have

$$AM = \frac{\bar{\rho}(\mathcal{L}_{0,1})}{\rho(X_0)} - 1$$
We already know (sharp Cantelli ineq.) that
\[ \sup_{X \in \mathcal{L}_{0,1}} \text{VaR}_\alpha(X) = \sqrt{\frac{1 - \alpha}{\alpha}} \]

Bertsimas et al (2004), using convex programming techniques, proved that
\[ \sup_{X \in \mathcal{L}_{0,1}} \text{ES}_\alpha(X) = \sqrt{\frac{1 - \alpha}{\alpha}} \]

(a much lower bound than that derived using Cantelli)

Therefore
\[ \text{AM} = \frac{1}{\rho(X_0)} \sqrt{\frac{1 - \alpha}{\alpha}} - 1 \]

for \( \rho = \text{VaR}_\alpha \) or \( \text{ES}_\alpha \)
An absolute measure of model risk: an example

- $X_0$ standard normal. Black: VaR, red: ES.
An absolute measure of model risk: an example

- $X_0 \text{ student-t}$ with $\nu = 3$ degrees of freedom. Black: VaR, red: ES.
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A relative measure of model risk: definition

- The **relative measure of model risk** is defined as
  \[
  RM = RM(\rho, X_0, L) = \frac{\bar{\rho}(L) - \rho(X_0)}{\bar{\rho}(L) - \underline{\rho}(L)}.
  \]

- For instance
  \[\begin{array}{c}
  \underline{\rho} \\
  \rho(X_0) \\
  \bar{\rho}
  \end{array}\]
  
  \[RM = 0.75\]

- 0 ≤ RM ≤ 1 and RM = 0 or 1 precisely when \(\rho(X_0) = \bar{\rho}(L)\) (**no model risk**) or \(\rho(X_0) = \underline{\rho}(L)\) (**full model risk**)

- RM need not be increasing in \(L\), thus providing a *relative* assessment of model risk
A relative measure of model risk: properties

- **(scale and translation invariance)** For any \( a > 0 \) and \( b \in \mathbb{R} \) it holds

\[
RM(aX_0 + b, a\mathcal{L} + b) = RM(X_0, \mathcal{L}).
\]

- As \( \mathcal{L}_{\mu,\sigma} = \sigma \mathcal{L}_{0,1} + \mu \) it follows

\[
RM(X_0, \mathcal{L}_{\mu,\sigma}) = RM(\tilde{X}_0, \mathcal{L}_{0,1}),
\]

where

\[
\tilde{X}_0 = \frac{X - \mu}{\sigma}
\]

- A more general result holds for \( \mathcal{L} \subset \mathcal{L}_{\mu,\sigma} \), with \( \tilde{\mathcal{L}} = \{\tilde{X} : X \in \mathcal{L}\} \) replacing \( \mathcal{L}_{0,1} \)

- Therefore, w.l.o.g. we can concentrate on **standard** r.v. (i.e. in \( \mathcal{L}_{0,1} \))
A relative measure of model risk: an example

- We already know that

\[
\sup_{X \in L_{0,1}} \text{VaR}_\alpha(X) = \sup_{X \in L_{0,1}} \text{ES}_\alpha(X) = \sqrt{\frac{1 - \alpha}{\alpha}}
\]

- Using bi-atomic distributions it is quite easy to see that

\[
\inf_{X \in L_{0,1}} \text{VaR}_\alpha(X) = -\sqrt{\frac{\alpha}{1 - \alpha}} \quad \text{(negative!)}
\]

- Using tri-atomic distributions we can also prove that

\[
\inf_{X \in L_{0,1}} \text{ES}_\alpha(X) = 0
\]

(see also Bertsimas et al, 2004)
Let $X_0$ standard.

For VaR we immediately find

$$RM = 1 - \alpha - \sqrt{\alpha(1 - \alpha)} \text{VaR}_\alpha(X_0)$$

In a similar way, for ES

$$RM = 1 - \sqrt{\frac{\alpha}{1 - \alpha}} \text{ES}_\alpha(X_0)$$
A relative measure of model risk: an example

- $X_0$ standard normal. Black: VaR, red: ES.
A relative measure of model risk: an example

- $X_0$ student-$t$ with $\nu = 3$ degrees of freedom. Black: VaR, red: ES.
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A local measure of model risk: definition

- Finally, we want to assess model risk **locally** around $X_0$
- Let $(\mathcal{L}_\varepsilon)_{\varepsilon > 0}$ a decreasing (as $\varepsilon$ decreases) family of alternative distributions sets, meaning that
  - $\mathcal{L}_\varepsilon$ is a set of r.v. for any $\varepsilon > 0$
  - if $\varepsilon < \varepsilon'$, then $\mathcal{L}_\varepsilon \subset \mathcal{L}_{\varepsilon'}$
  - $\bigcap_\varepsilon \mathcal{L}_\varepsilon = \{X_0\}$
- Examples (assume $X_0 \in \mathcal{L}_{0,1}$)
  - If $d$ is some **distance** between distributions (Levy, Kolmogorov, Kullback-Leibler divergence, etc) consider
    \[
    \mathcal{L}_\varepsilon = \{X : d(X, X_0) \leq \varepsilon\}
    \]
  - In particular, Kullback-Leibler (or relative entropy) is considered by Alexander and Sarabia (2012) and by Glasserman and Xu (2013)
  - As before, with all $X$ in $\mathcal{L}_{0,1}$
  - if $F_0$ is the distribution of $X_0$,
    \[
    \mathcal{L}_\varepsilon = \{X : F_X = (1 - \theta)F_0 + \theta G, \ G \in \mathcal{L}_{0,1}, \ \theta \in (0, \varepsilon)\}
    \]
The local measure of model risk is

\[ LM \ = \ \lim_{\varepsilon \to 0} \ RM(L_\varepsilon) = \lim_{\varepsilon \to 0} \ \frac{\rho(L_\varepsilon) - \rho(X_0)}{\rho(L_\varepsilon) - \rho(L_0)} \]

The limit is in the form 0/0.

If it exists, then it is in [0, 1].

It describes the relative position of \( \rho(X_0) \) w.r.t. the worst and best case for infinitesimal perturbations.
A local measure of model risk: an example

- Consider again the set of \( \varepsilon \)-mixtures

\[
\mathcal{L}_\varepsilon = \{ X : F_X = (1 - \theta)F_0 + \theta G, \ G \in \mathcal{L}_{0,1}, \ \theta \in (0, \varepsilon) \}
\]

- It is immediate to see that \( \mathcal{L}_\varepsilon \subset \mathcal{L}_{0,1} \)

- Using results from the theory of Markov-Chebishev extremal distributions we can prove that for \( \rho = \text{VaR}_\alpha \) we have

\[
\rho(\mathcal{L}_\varepsilon) = \inf_{X \in \mathcal{L}_\varepsilon} \text{VaR}_\alpha(X) = \text{VaR}_{\frac{\alpha}{1-\varepsilon}}(X_0)
\]

provided \( \alpha \) is small enough \( (\alpha < (1 - \varepsilon)F_0(0)) \)

- Also, \( r = \bar{\rho}(\mathcal{L}_\varepsilon) \) is solution of the following equation

\[
(1 - \varepsilon)F_0(-r) + \frac{\varepsilon}{1 + r^2} = \alpha
\]

which can easily be treated numerically.
A local measure of model risk: an example

- $X_0$ standard normal, $\rho = \text{VaR}_\alpha$. Black: $\varepsilon = 0.2$, red: $\varepsilon = 0.05$. $\alpha$ on the x-axis
A local measure of model risk: an example

- $X_0$ standard normal, $\rho = \text{VaR}_{\alpha}$. Black: $\alpha = 1\%$, red: $\alpha = 3\%$. $\varepsilon$ on the x-axis
Using de l’Hôpital and the particular form of the extremal distribution we can also explicitly compute (remind: \( \rho = \text{VaR}_{\alpha} \))

\[
LM = \lim_{\varepsilon \to 0} RM(L_{\varepsilon}) = 1 - \alpha(1 + q_{\alpha}(X_0)^2)
\]

provided \( X_0 \) is a.c.

If \( X_0 \) is standard normal

\[
LM = 1 - \alpha(1 + z_{\alpha}^2)
\]
A local measure of model risk: an example

- $X_0$ standard normal, $\rho = VaR_\alpha$. LM as a function of $\alpha$ on the x-axis
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The portfolio

- As a numerical example, we apply the relative measure of model risk to daily Value-at-Risk (1% and 5%) estimation for a portfolio investing in the (German) electricity market.

- In particular, let $P_t$ be the price at day $t$ for 1 MWh in the day-ahead market (this is considered as a spot market). Notice that $P_t$ may become negative in Germany!

- At every day, the portfolio invests in 1 unit (i.e. 1 MWh), so that its daily Profit and Loss is

\[ PL_{t+1} = \Delta_{t+1} P = P_{t+1} - P_t. \]

- If $\mathcal{F}_t$ is the information up to time $t$, we want to estimate

\[ \text{VaR}_{\alpha, t+1} = \text{VaR}_\alpha (PL_{t+1} | \mathcal{F}_t), \]

where $\alpha = 1\%$ or 5\%.
Electricity prices

- Day-ahead price for 1 MWh (in Euro) in the German market
GARCH modeling

- Price differences $X_t = \Delta_t P$ are usually assumed to be nearly stationary. However, contrarily to equity prices, the mean component is not negligible.

- We estimate (daily) an AR(5)-GARCH(1,1) model for $X_t$, meaning that

$$X_t = \mu_t + \varepsilon_t = \mu_t + \sigma_t Z_t,$$

where

- $\mu_t = \mathbb{E} [X_t | \mathcal{F}_{t-1}]$ is defined according to the AR(5)

$$\mu_t = c + \phi_1 X_{t-1} + \ldots + \phi_5 X_{t-5}$$

- $\sigma_t = \sigma(X_t | \mathcal{F}_{t-1})$ is defined according to the GARCH(1,1) model

$$\sigma^2_t = \omega + \alpha \varepsilon^2_{t-1} + \beta \sigma^2_{t-1}$$

- the innovations $(Z_t)$ are IID $\sim D(0, 1)$ (i.e. standard)

- According to this specification

$$\text{VaR}_\alpha(PL_{t+1} | \mathcal{F}_t) = -\mu_{t+1} + \sigma_{t+1} \text{VaR}_\alpha(Z)$$
In the classical "normal GARCH", we assume $Z \sim N(0, 1)$. This is our reference distribution.

Then we consider as alternative distributions:

- Skew Normal
- t-Student and Skew t-Student
- Generalized Error Distributions (GED) and Skew GED
- Johnson’s SU;
- Normal Inverse Gaussian (NIG);
- Generalized Hyperbolic (a superclass including some of the previous classes).

These are distributions of common use in risk management. Note that some allow for asymmetry, some for heavy tails and for both.

The parameters of alternative distributions are fitted with ML using observed innovations (i.e. the series $\hat{Z}_t = (X_t - \mu_t)/\sigma_t$)
Relative measure of model risk

- Day by day we compute

\[
\overline{\text{VaR}}_{\alpha, t} = \sup_{Z \in \text{Models}} \text{VaR}_\alpha(PL_t | F_{t-1}) = -\mu_t + \sigma_t \sup_{Z \in \text{Models}} \text{VaR}_\alpha(Z),
\]

where ”Models” is a suitable class of models for \( Z \).

- We define \( \underline{\text{VaR}}_{\alpha, t} \) similarly.

- Therefore, the empirical measure of relative model risk at date \( t \) is

\[
RM_t = \frac{\overline{\text{VaR}}_{\alpha, t} - \text{VaR}^{\text{Normal}}_{\alpha, t}}{\overline{\text{VaR}}_{\alpha, t} - \underline{\text{VaR}}_{\alpha, t}} = \frac{\sup_Z \text{VaR}_\alpha(Z) - \text{VaR}_\alpha(N(0, 1))}{\sup_Z \text{VaR}_\alpha(Z) - \inf_Z \text{VaR}_\alpha(Z)}
\]

- Notice that the rhs depends on day \( t \) as the parameters for the distributions of innovations are fitted to past observations.
Results

- Relative measure of model risk for $\text{VaR}_{1\%}$
Results

- Rel. measure of model risk for $\text{VaR}_{5\%}$ (red) and $\text{VaR}_{1\%}$ (black)
References

- Glasserman P., Xu X. (2013) ”Robust risk measurement and model risk” Quantitative Finance
- Stahl G. (1997) ”Three cheers” Risk Magazine 10 (5)