An Academic Response to Basel 3.5

References:


talk/papers available on my web: https://sites.google.com/site/giovannipuccetti/
8. What are the likely operational constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?
An academic response to Basel 3.5

1. Measuring dependence uncertainty: the DNB case
2. Asymptotic equivalence of VaR/ES worst case estimates
3. Adding extra dependence assumptions
1. A real example: the DNB case. See [3].

DNB risk portfolio used for ICAAP

\[ L^+_d = L_1 + \cdots + L_d \] total loss exposure (for DNB: \( d=6 \))
1. A real example: the DNB case. See [3].

**DNB risk portfolio used for ICAAP**

<table>
<thead>
<tr>
<th>$L_1$</th>
<th>$L_2$</th>
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<th>$L_4$</th>
<th>$L_5$</th>
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<tr>
<td><strong>Credit Risk</strong></td>
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$$L_d^+ = L_1 + \cdots + L_d$$

**total loss exposure (for DNB: d=6)**

Basel II(I) requirement: compute and reserve based on

$$\text{VaR}_\alpha(L_d^+) \quad \text{or} \quad \text{ES}_\alpha(L_d^+)$$
Value-at-Risk (VaR)

\[ \text{VaR}_\alpha(L_d^+) = \inf\{x \in \mathbb{R} : F_{L_d^+}(x) > \alpha\}, \quad \alpha \in (0, 1). \]

\[ P(L_d^+ > \text{VaR}_\alpha(L_d^+)) \leq 1 - \alpha. \]

Expected Shortfall (ES)

\[ \text{ES}_\alpha(L_d^+) = \frac{1}{1 - \alpha} \int_\alpha^1 \text{VaR}_q(L_d^+) \, dq, \quad \alpha \in (0, 1). \]

\[ \text{ES}_\alpha(L_d^+) = E[L_d^+|L_d^+ > \text{VaR}_\alpha(L^+)], \quad \text{if } L_d^+ \text{ is continuous.} \]
ES is a coherent risk measure

\[ ES_\alpha(L_d^+) \leq ES^+_\alpha(L_d^+) := \sum_{i=1}^{d} ES_\alpha(L_i) \]

VaR fails to be subadditive

\[ \text{VaR}_\alpha(L_d^+) > \text{VaR}^+_\alpha(L_d^+) := \sum_{i=1}^{d} \text{VaR}_\alpha(L_i). \]

comonotonic dependence (maximal correlation)
General problem

<table>
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<tr>
<th>$L_1$</th>
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one period risks with statistically estimated marginals

DU-spread for VaR

$$\text{VaR}_\alpha(L^+_d)$$

$$\sum_{i=1}^{d} \text{VaR}_\alpha(L_i)$$

$$\text{VaR}_\alpha(L^+_d)$$

DU-spread for ES

$$\text{ES}_\alpha(L^+_d)$$

$$\text{ES}_\alpha(L^+_d)$$
General problem

DU-spread for VaR

\[ \text{Var}_\alpha (L^+_d) \]

\[ \text{Var}_\alpha (L^+_d) := \sup \{ \text{Var}_\alpha (L_1 + \cdots + L_d) : L_i \sim F_i, 1 \leq i \leq d \} , \]

\[ \text{Var}_\alpha (L^+_d) := \inf \{ \text{Var}_\alpha (L_1 + \cdots + L_d) : L_i \sim F_i, 1 \leq i \leq d \} . \]

superadditive models

\[ \sum_{i=1}^{d} \text{Var}_\alpha (L_i) \]

\[ \text{Var}_\alpha (L^+_d) \]

DU-spread for ES

\[ \text{ES}_\alpha (L^+_d) \]

\[ \text{ES}_\alpha (L^+_d) \]

and unknown dependence structure
General problem

one period risks with statistically estimated marginals and unknown dependence structure

\[ \text{DU-spread for VaR} \]

\[ \text{DU-spread for ES} \]

\[ \text{superadditive models} \]

\[ \text{DU-spread for VaR} \]

\[ \text{DU-spread for ES} \]

\[ \text{superadditive models} \]
How can we compute the bounds?

\[
\text{VaR}_\alpha(L^+_d) \quad \text{and} \quad \overline{\text{VaR}}_\alpha(L^+_d)
\]

\[
\text{ES}_\alpha(L^+_d) \quad \text{and} \quad \sum_{i=1}^{d} \text{ES}_\alpha(L_i) = \overline{\text{ES}}_\alpha(L^+_d)
\]

For general inhomogenous marginals, there does not exist an analytical tool to compute.\[x\]
How can we compute the bounds?

\[ \text{VaR}_\alpha(L^+_d) \]

\[ \text{ES}_\alpha(L^+_d) \]

For general inhomogenous marginals, there does not exist an analytical tool to compute.\[ \text{ES}_\alpha(L^+_d) \]

Then use the Rearrangement Algorithm; see [3] for a step-by-step implementation.
With $N=10^5$, we obtain the first three decimals of $\overline{\text{VaR}}_\alpha(L_3^+) = 45.9898$ in 0.2 sec.
Model uncertainty: the DNB example

DNB risk portfolio (figures in million NOK)

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Quantile level used: $\alpha = 99.97\%$

$\text{VaR}_{\alpha}(L_d^+)$

$\sum_{i=1}^{d} \text{VaR}_{\alpha}(L_i)$

$\overline{\text{VaR}}_{\alpha}(L_d^+)$

$\frac{\overline{\text{VaR}}_{\alpha}(L_d^+)}{\sum_{i=1}^{d} \text{VaR}_{\alpha}(L_i)} = 1.136$

62,156.4

93,152.7

105,878.2
The worst diversification ratio: definition; see [3].

The worst superadditivity (or diversification) ratio for $L_d^+$ is defined as

$$
\Delta_\alpha(L_d^+) := \frac{\text{VaR}_\alpha(L_d^+)}{\text{VaR}_\alpha^+(L_d^+)} = \frac{\text{VaR}_\alpha(L_d^+)}{\sum_{i=1}^d \text{VaR}_\alpha(L_i)}
$$

**worst-possible dependence**

**comotonic dependence**
The worst diversification ratio: definition; see[3].

The worst superadditivity (or diversification) ratio for $L_d^+$ is defined as

$$
\overline{\Delta}_\alpha(L_d^+) := \frac{\text{VaR}_\alpha(L_d^+)}{\text{VaR}_\alpha^+(L_d^+)} = \frac{\text{VaR}_\alpha(L_d^+)}{\sum_{i=1}^{d} \text{VaR}_\alpha(L_i)} \quad \text{worst-possible dependence}
$$

$$
\overline{\Delta}_\alpha(L_d^+) \quad \text{comotonic dependence}
$$

Examples:

- $\overline{\Delta}_\alpha(L_d^+) = 1$: the aggregate position is always less risky than the sum of the marginal exposures. Examples: $(L_1, \ldots, L_d)$ has a multivariate Gaussian or multivariate Student’s t (in general elliptical) distribution.

- $\overline{\Delta}_\alpha(L_d^+) > 1$: superadditivity of VaR. It typically occurs with heavy-tailed and/or skew marginals and/or non-elliptical portfolios.
Explicit upper bound in the **homogeneous** case (all risks have df $F$):

$$
\Delta_{\alpha}(L_d^+):= \frac{\bar{\text{VaR}}_{\alpha}(L_d^+)}{\text{VaR}_{\alpha}^+(L_d^+)} \leq \frac{\bar{\text{ES}}_{\alpha}(L_d^+)}{\text{VaR}_{\alpha}^+(L_d^+)} = \frac{d\text{ES}_{\alpha}(L_1)}{d\text{VaR}_{\alpha}(L_1)} = \frac{\text{ES}_{\alpha}(L_1)}{\text{VaR}_{\alpha}(L_1)}.
$$
Explicit upper bound in the **homogeneous** case (all risks have df F):

\[
\Delta_{\alpha}(L_d^+) := \frac{\text{VaR}_{\alpha}(L_d^+)}{-\text{VaR}_{\alpha}(L_d^+)} \leq \frac{\text{ES}_{\alpha}(L_d^+)}{-\text{VaR}_{\alpha}(L_d^+)} = \frac{d\text{ES}_{\alpha}(L_1)}{d\text{VaR}_{\alpha}(L_1)} = \frac{\text{ES}_{\alpha}(L_1)}{-\text{VaR}_{\alpha}(L_1)}.
\]

**Theorem:** Under some general marginal conditions (including all the continuous distributional models used in QRM) + LOSSES WITH FINITE MEAN, we have

\[
\lim_{d \to \infty} \Delta_{\alpha}(L_d^+) = \frac{\text{ES}_{\alpha}(L_1)}{-\text{VaR}_{\alpha}(L_1)} \quad \text{(homogeneous case)}
\]
### Values for the limit for Pareto(θ) distributions

<table>
<thead>
<tr>
<th>α</th>
<th>θ = 1.1</th>
<th>θ = 1.5</th>
<th>θ = 2</th>
<th>θ = 3</th>
<th>θ = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
<td>11.154337</td>
<td>3.097350</td>
<td>2.111111</td>
<td>1.637303</td>
<td>1.487492</td>
</tr>
<tr>
<td>0.995</td>
<td>11.081599</td>
<td>3.060242</td>
<td>2.076091</td>
<td>1.603135</td>
<td>1.454080</td>
</tr>
<tr>
<td>0.999</td>
<td>11.018773</td>
<td>3.020202</td>
<td>2.032655</td>
<td>1.555556</td>
<td>1.405266</td>
</tr>
</tbody>
</table>

### Values for the limit for LogNormal(0,θ) distributions

<table>
<thead>
<tr>
<th>α</th>
<th>θ = 0.5</th>
<th>θ = 1</th>
<th>θ = 1.5</th>
<th>θ = 2</th>
<th>θ = 2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
<td>1.200364</td>
<td>1.487037</td>
<td>1.920334</td>
<td>2.621718</td>
<td>3.858599</td>
</tr>
<tr>
<td>0.995</td>
<td>1.184949</td>
<td>1.443519</td>
<td>1.823195</td>
<td>2.415980</td>
<td>3.415242</td>
</tr>
<tr>
<td>0.999</td>
<td>1.158988</td>
<td>1.372433</td>
<td>1.670393</td>
<td>2.107238</td>
<td>2.787941</td>
</tr>
</tbody>
</table>

### Values for the limit for Exponential(θ) distributions

<table>
<thead>
<tr>
<th>α</th>
<th>θ = 0.5</th>
<th>θ = 1</th>
<th>θ = 1.5</th>
<th>θ = 2</th>
<th>θ = 2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
<td>1.217147</td>
<td>1.217147</td>
<td>1.217147</td>
<td>1.217147</td>
<td>1.217147</td>
</tr>
<tr>
<td>0.995</td>
<td>1.188739</td>
<td>1.188739</td>
<td>1.188739</td>
<td>1.188739</td>
<td>1.188739</td>
</tr>
<tr>
<td>0.999</td>
<td>1.144765</td>
<td>1.144765</td>
<td>1.144765</td>
<td>1.144765</td>
<td>1.144765</td>
</tr>
</tbody>
</table>
What if the losses have infinite mean?

Under some general marginal conditions, we have that

$$\lim_{d \to \infty} \bar{\Delta}_\alpha(L_d^+) = \infty.$$  

This means that the VaR for a sum can be arbitrarily large with respect to the corresponding VaR estimate for comonotonic risks.
An academic response to Basel 3.5

2. Asymptotic equivalence of VaR/ES worst case estimates
Model uncertainty: the DNB example

DNB risk portfolio

$L_1$  
Credit Risk  
2.5e06 simulations

$L_2$  
Market Risk  
2.5e06 simulations

$L_3$  
Ownership Risk  
2.5e06 simulations

$L_4$  
Operational Risk  
LogNormal distribution

$L_5$  
Business Risk  
LogNormal distribution

$L_6$  
Insurance Risk  
LogNormal distribution

quantile level used: $\alpha = 99.97\%$

$\text{VaR}_\alpha(L_d^+)$  
62,156.4

$\sum_{i=1}^{d} \text{VaR}_\alpha(L_i)$  
93,152.7

$\overline{\text{VaR}}_\alpha(L_d^+)$  
105,878.2

$\text{ES}_\alpha(L_d^+)$  
74,354.7

$\sum_{i=1}^{d} \text{VaR}_\alpha(L_i)$  
110,588.8

$\overline{\text{ES}}_\alpha(L_d^+)$  
110,588.8
In general, we have

\[
\frac{\text{VaR}_\alpha(L_d^+)}{\text{ES}_\alpha(L_d^+)} \leq 1.
\]
Equivalence of worst VaR and ES estimates

In general, we have

\[
\frac{\text{VaR}_\alpha(L^+_d)}{\text{ES}_\alpha(L^+_d)} \leq 1.
\]

**Theorem:** Under some general marginal conditions (including all the continuous inhomogeneous models used in QRM) + LOSSES WITH FINITE MEAN we have

\[
\lim_{d \to \infty} \frac{\text{VaR}_\alpha(L^+_d)}{\text{ES}_\alpha(L^+_d)} = 1.
\]
Equivalence of worst VaR and ES estimates

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\text{VaR}<em>{0.999} \left( \sum</em>{i=1}^{d} L_i \right)$</th>
<th>$\text{ES}<em>{0.999} \left( \sum</em>{i=1}^{d} L_i \right)$</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>640.0679</td>
<td>668.7629</td>
<td>1.0448</td>
</tr>
<tr>
<td>10</td>
<td>2225.8490</td>
<td>2229.2100</td>
<td>1.0015</td>
</tr>
<tr>
<td>50</td>
<td>11146.0300</td>
<td>11146.0500</td>
<td>1.0000</td>
</tr>
<tr>
<td>100</td>
<td>22292.1000</td>
<td>22292.1000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Sum of $d$ LogNormal(2,1) marginals.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\text{VaR}<em>{0.999} \left( \sum</em>{i=1}^{d} L_i \right)$</th>
<th>$\text{ES}<em>{0.999} \left( \sum</em>{i=1}^{d} L_i \right)$</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>186.49</td>
<td>237.33</td>
<td>1.2726</td>
</tr>
<tr>
<td>9</td>
<td>687.09</td>
<td>711.98</td>
<td>1.0362</td>
</tr>
<tr>
<td>30</td>
<td>2370.39</td>
<td>2373.26</td>
<td>1.0012</td>
</tr>
<tr>
<td>99</td>
<td>7831.72</td>
<td>7831.75</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Sum of $d$ different Pareto, LogNormal, Exponential marginals.

- The limit is evident also for relatively small dimensions;

- Important consequences wrt the forthcoming Basel 3+ accords.
An academic response to Basel 3.5

3. Adding extra dependence assumptions
WORST VAR SCENARIO

2d projections to $[0.999,1]^2$ of the support of the 3d-copula merging the upper 99.9%-tails of the three Pareto(2) distributed random variables maximising the 99.9%-VaR of their sum.

The black area represents a completely mixable part, see [1].
In quantitative risk management, the components of a risk portfolio often have some positive dependence structure.

For $X$ and $Y$ in $\mathbb{R}^d$, we define the *concordance order* $Y \leq_{co} X$, if both

$$
\overline{F}_Y(x) \leq \overline{F}_X(x) \quad \text{and} \quad F_Y(x) \leq F_X(x)
$$

hold for all $x \in \mathbb{R}^d$.

The concordance order $Y \leq_{co} X$ implies

$$
\text{Cov}(Y_i, Y_j) \leq \text{Cov}(X_i, X_j); \quad \rho_S(Y_i, Y_j) \leq \rho_S(X_i, X_j); \quad \tau(Y_i, Y_j) \leq \tau(X_i, X_j);
$$

where $\rho_S$ is Spearman’s and $\tau$ is Kendall’s rank correlation coefficient.

**Typical assumption: POD risks, i.e.** $L \perp \leq_{co} L$
Adding positive dependence; see [2]

Our assumptions: \( \alpha = 99.9\% \)
Adding positive dependence; see [2]

Our assumptions: \( \alpha = 99.9\% \)

\[
\begin{aligned}
L_1, L_2, L_3, L_4 & \quad \text{Pareto(2) marginals} \\
\downarrow \text{independence between} \\
L_5, L_6, L_7, L_8 & \quad \text{Exp(1) marginals} \\
\downarrow \text{comonotonicity within} \\
\leq_{\text{co}} (L_1, \ldots, L_8)
\end{aligned}
\]
Adding positive dependence I; see [2]

Our assumptions: $\alpha = 99.9\%$

![Diagram showing the relationships between different variables under different assumptions.]

- $L_1, L_2, L_3, L_4$ with Pareto(2) marginals and comonotonicity within.
- $L_5, L_6, L_7, L_8$ with Exp(1) marginals and comonotonicity within.
- Independence between $L_1, \ldots, L_8$.

**DU-S with marginals info only**

- $\text{VaR}_\alpha(L^+_d)$
- Value: 30.62

**DU-S with marginals info AND $\leq_{co}$ assumption**

- $\sum_{i=1}^{d} \text{VaR}_\alpha(L_i)$
- Value: 122.49

- $\sum_{i=1}^{d} \text{VaR'}_\alpha(L_i)$
- Value: 150.12

- $\text{VaR'}_\alpha(L^+_d)$
- Value: 205.27

- $\text{VaR}_\alpha(L^+_d)$
- Value: 150.12

- $\text{VaR'}_\alpha(L^+_d)$
- Value: 248.24
Why positive dependence does not help

This copula is POD!
**Positive dependence assumption:**

If \((L_1^\parallel, L_2^\parallel) \leq_{co} (L_1, L_2)\) then \(\text{ES}_\alpha(L_1^\parallel + L_2^\parallel) \leq \text{ES}_\alpha(L_1 + L_2)\)
Why positive dependence does not help

Positive dependence assumption:

If \((L_1^\perp, L_2^\perp) \preceq_{\text{co}} (L_1, L_2)\) then \(\text{ES}_\alpha(L_1^\perp + L_2^\perp) \leq \text{ES}_\alpha(L_1 + L_2)\)

Negative dependence assumption:

If \((L_1^\perp, L_2^\perp) \preceq_{\text{co}} (L_1^\perp, L_2^\perp)\) then \(\text{ES}_\alpha(L_1 + L_2) \leq \text{ES}_\alpha(L_1^\perp + L_2^\perp)\)

These ordering results can be generalized to arbitrary dimensions and law invariant, convex risk measure using the weakly conditional increasing in sequence order or the supermodular order between vectors; see a variety of examples in [2].
Our assumptions: $\alpha = 99.9\%$

(Diagram showing different types of assets and their dependencies)

- Market
- Credit
- Insur.
- Busin.
- Asset
- Non life
- Reput.
- Life

Gaussian marginals

No dependence assumption within

- Reinsurance
- Operational
- Catastrophic

- LogN
- LogN
- Pareto

(Diagram showing dependencies between different types of assets)

- Independence between
- Independence between
- Independence between

121.5

Du-S with marginals info only

$\text{Var}_\alpha (L_d^+)$

304.63

$\sum_{i=1}^{d} \text{Var}_\alpha (L_i)$

367.70

$\overline{\text{Var}}_\alpha (L_d^+)$

VaR upper bound with marginal info AND independent subgroups

$\overline{\text{VaR}}_\alpha = 256.04$
- **Adding positive dependence info is not useful to reduce worst bounds:** One should instead assume some independence/negative dependence structure in order to reduce the upper bound on a risk.

- **VaR vs ES:** If you take a worst-case perspective, they are asymptotically equivalent.

- **Superadditivity of VaR:** We have analytical and numerical techniques available for the computation of VaR/ES uncertainty range.

- **There’s more under the top of the iceberg:** The risk assessment of a multivariate bank portfolio cannot be reduced to a single VaR number. The superadditivity ratio and the VaR/ES uncertainty range might help to assess the implied model risk.
Forthcoming event and some initiatives on dependence

http://www.degruyter.com/view/j/demo

https://sites.google.com/site/refereeplus/

https://sites.google.com/site/deppapersevents/