Set-Valued Risk Measures and Systemic Risk

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Outline

- Idea: Capital requirements for financial firms to control the risk to the outside economy

1. Overview

- Idea: Capital requirements for financial firms to control the risk to the outside economy
- Model the financial system via a network of obligations
- Introduce (random) stresses into the system and find payment structure
- System is "acceptable" as measure of net payments to the outside economy (1 dimensional), but capital requirements separated by institution

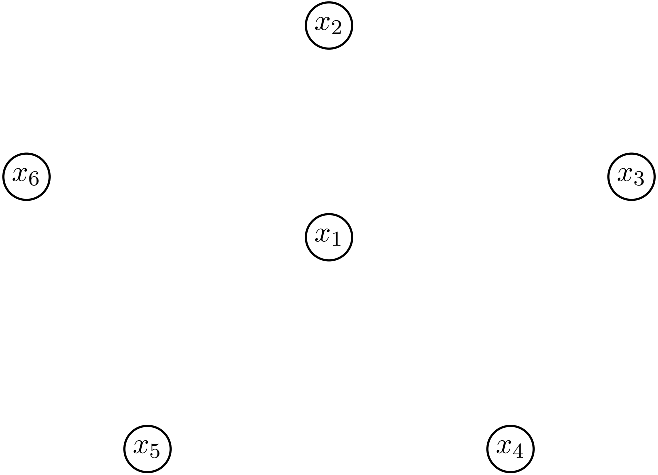

Figure: Independent financial firms

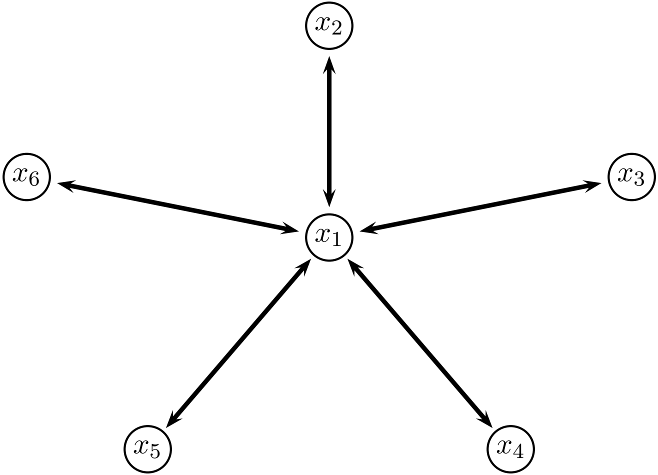

Figure: Network with single systemically important firm

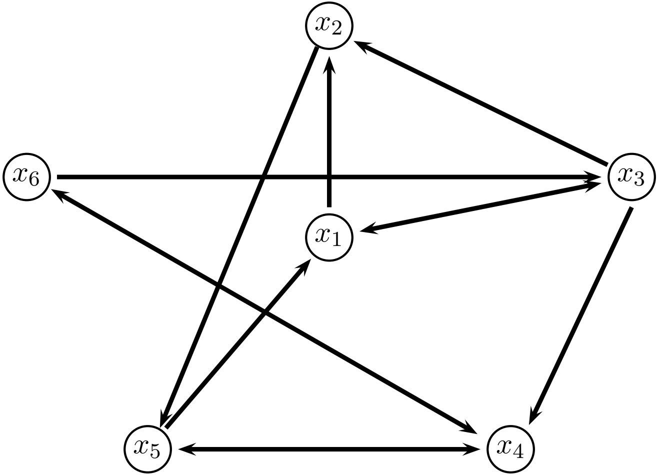

Figure: Network with no clear systemically important firm

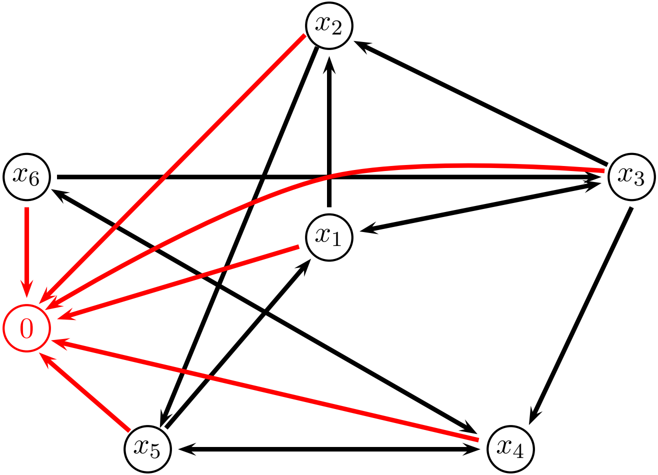

Figure: Network with node for "real" economy

- n financial firms

3. Systemic risk measures

- n financial firms
- Equity and loss (E&L) function: $e : \mathbb{R}^n_+ \to \mathbb{R}^{n+1} \cup \{-\infty\}$
- Pre-image: Vector of bank endowments before network effects
- Image: Vector of bank equity after network effects

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- Pre-image: Vector of bank endowments before network effects
- Image: Vector of bank equity after network effects
- e_0 is the equity value of the outside economy from the financial system
- Assume:
  - e is nondecreasing
  - $e(y) = -\infty$ for all $y \notin \mathbb{R}^n_+$
  - e_0 is bounded from above
  - $e_0(y) \geq 0$ for all $y \in \mathbb{R}^n_+$

Systemic Risk Measures

$R_{\mathcal{A}}^{sys}: L^0(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n; \mathbb{R}^n_+) = \{D \subseteq \mathbb{R}^n \mid D = D + \mathbb{R}^n_+\}$ is a ***systemic risk measure*** if for some acceptance set $\mathcal{A} \subseteq L^\infty(\mathbb{R})$ of a scalar risk measure:

$$R_{\mathcal{A}}^{sys}(X) = \{k \in \mathbb{R}^n \mid e_0(k + X) \in \mathcal{A}\}.$$

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Equivalently:

$$R_{\mathcal{A}}^{sys}(X) = \{k \in \mathbb{R}^n \mid k + X \in \mathcal{A}^e\}$$

$$\mathcal{A}^e = e_0^{-1}[\mathcal{A}] := \{Y \in L^0(\mathbb{R}^n_+) \mid e_0(Y) \in \mathcal{A}\}$$

Assume: e_0 is concave and continuous
$\mathcal{A}$ is convex, closed, and law-invariant

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Properties: Let $X, Y \in L^0(\mathbb{R}^n)$, $k \in \mathbb{R}^n$, and $\alpha \in [0, 1]$

- *Translative*: $R^{sys}_{\mathcal{A}}(X + k) = R^{sys}_{\mathcal{A}}(X) - k$
- *Monotone*: $R^{sys}_{\mathcal{A}}(X) \supseteq R^{sys}_{\mathcal{A}}(Y)$ if $X \geq Y$ a.s.

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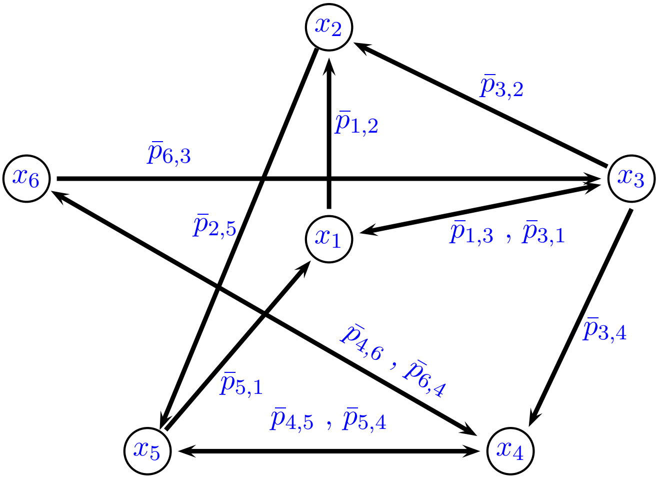
- *Law-invariant:* $R_{\mathcal{A}}^{sys}(X) = R_{\mathcal{A}}^{sys}(Y)$ if $X \stackrel{d}{=} Y$

3.1 Systemic risk measures: Eisenberg-Noe network model



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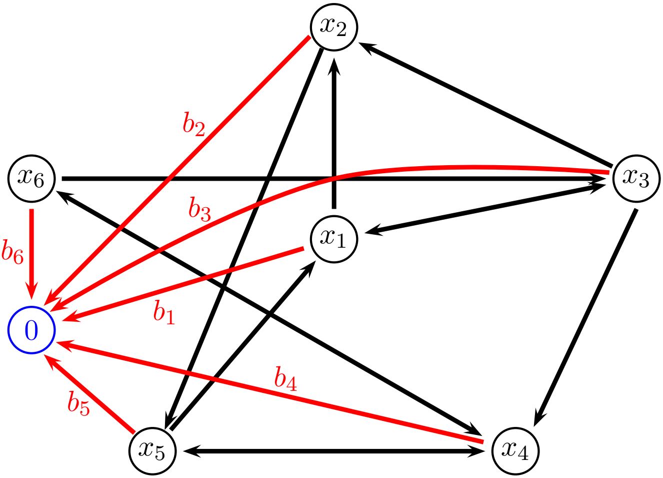

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- n financial firms + outside economy:
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- Liability of firm i to j is given by $\bar{p}_{ij} \geq 0$
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- Liability of firm i to outside economy is given by $b_i \geq 0$
- Total liabilities for firm i given by $\bar{p}_i = b_i + \sum_{j \neq i} \bar{p}_{ij}$
- Relative liabilities for firm i to j is given by $a_{ij} = \frac{\bar{p}_{ij}}{\bar{p}_i}$

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- Realized sum of debt and equity minus promised payments

$$e_i(x) := x_i + \sum_{j=1}^{n} a_{ji} \cdot p_j(x) - \bar{p}_i$$

is the value of firm i (if positive) or losses from default (if negative)

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- e_0 is concave, nondecreasing, and Lipschitz continuous

- To compute: approximate expectations by Monte Carlo simulation
- Approximate via smart grid search for boundary of set
- Idea: draw a grid over area of interest (e.g. box around $\mathcal{C}(X)$), and find the grid points in the set
- Possible improvement with parallel computing

Sample Acceptance Sets:

- *Average value at risk*: for $\lambda \in (0,1)$

$$\mathcal{A}^\lambda = \{Z \in L^\infty(\mathbb{R}) \mid \inf_{r \in \mathbb{R}} (\mathbb{E}\left[(r - Z)^+\right] - r\lambda) \leq 0\}$$

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- *Optimized certainty equivalents*: for concave utility $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$

$$\mathcal{A}^{u} = \{Z \in L^{\infty}(\mathbb{R}) \mid \sup_{\eta \in \mathbb{R}} (\eta + \mathbb{E}\left[u(Z - \eta)\right]) \geq 0\}$$

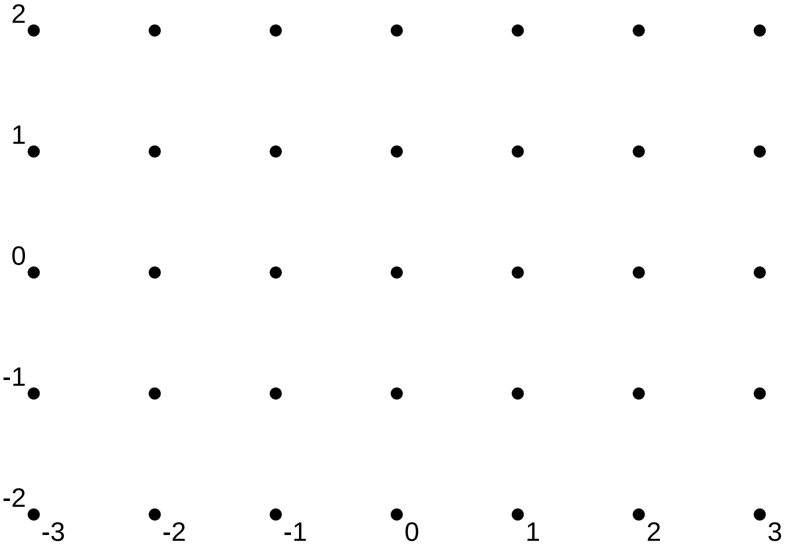
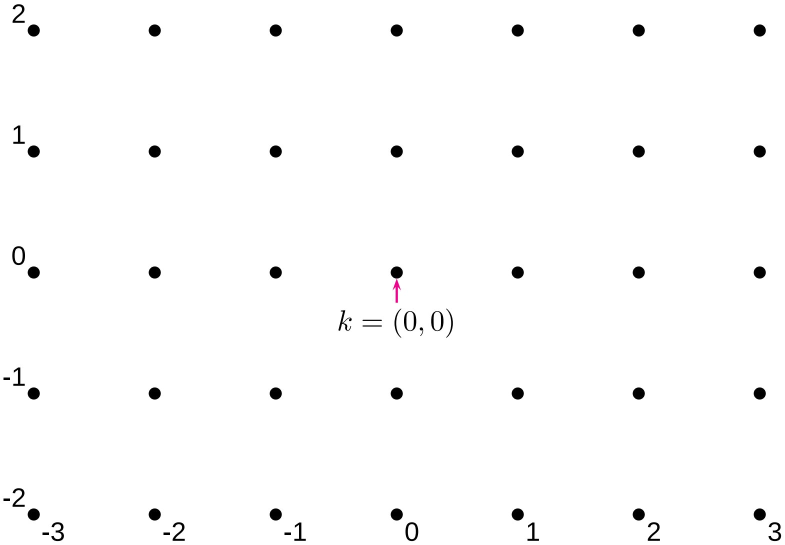


Figure: Grid search



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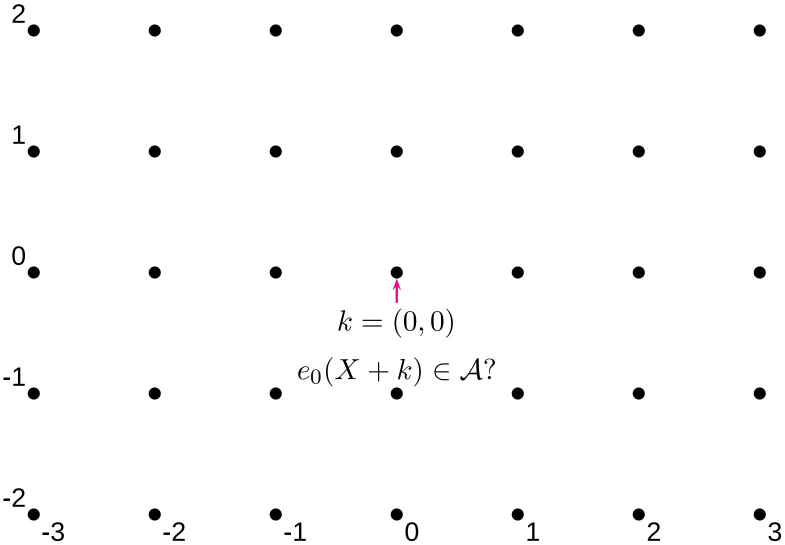

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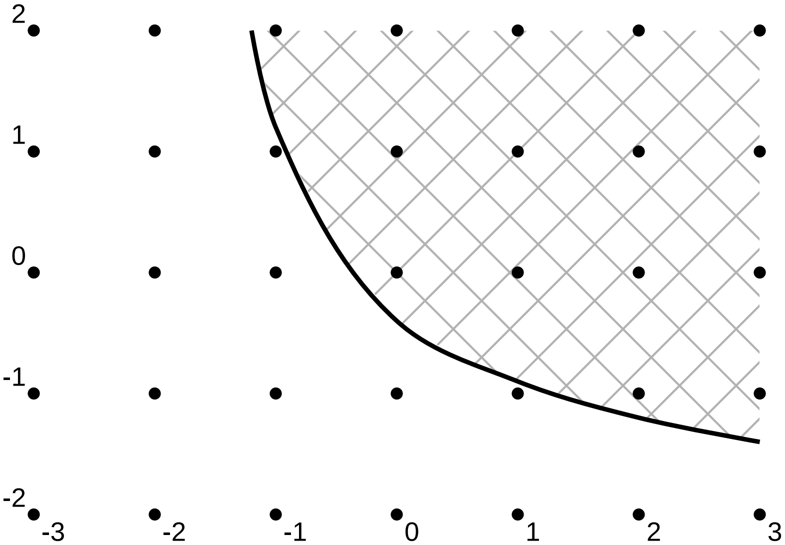

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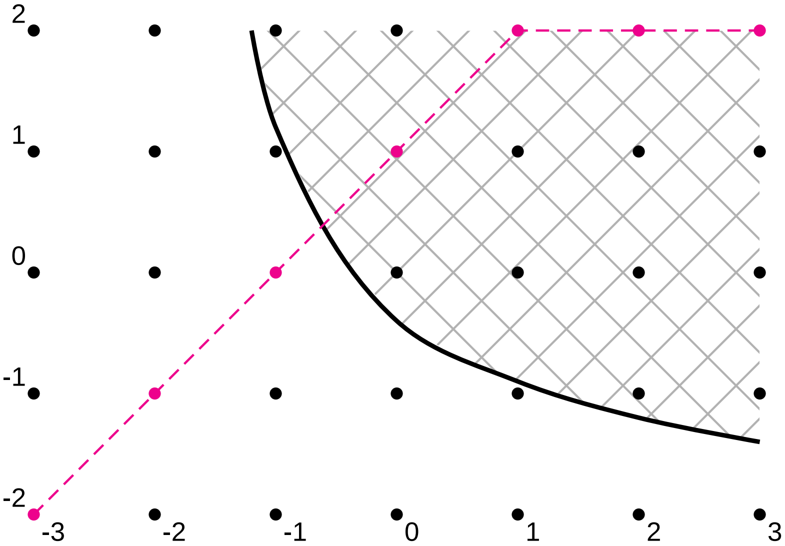

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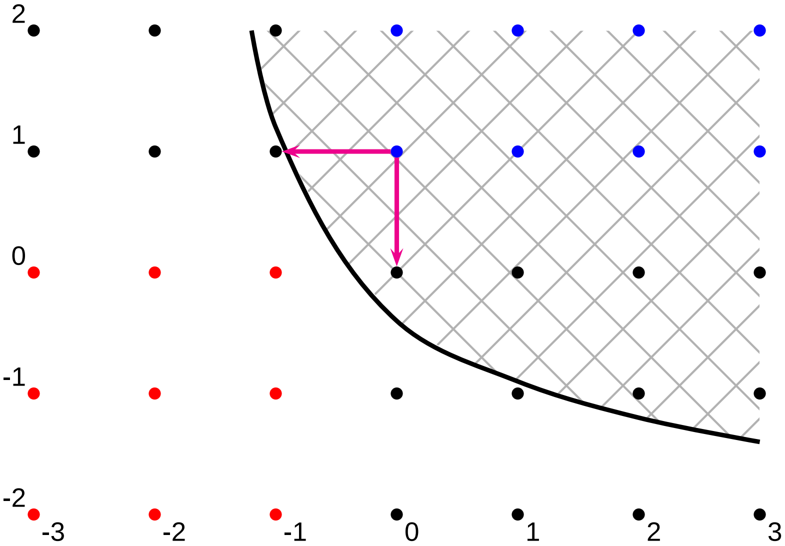

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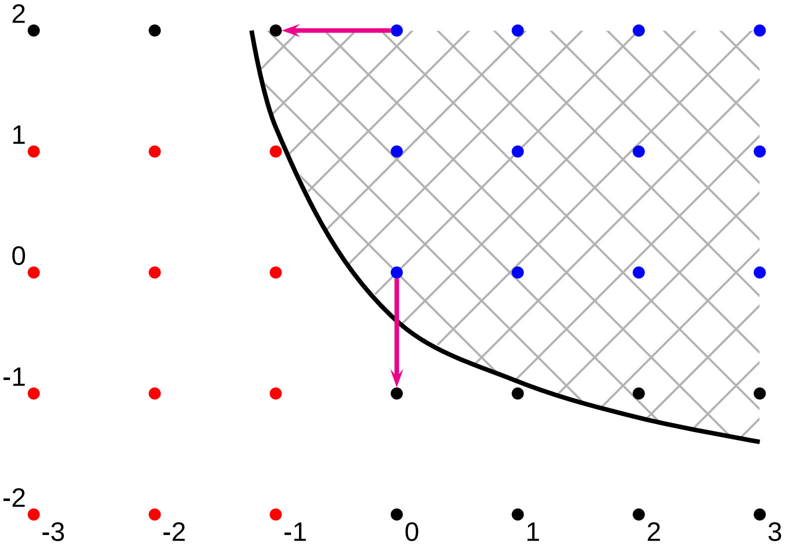
4. Computation



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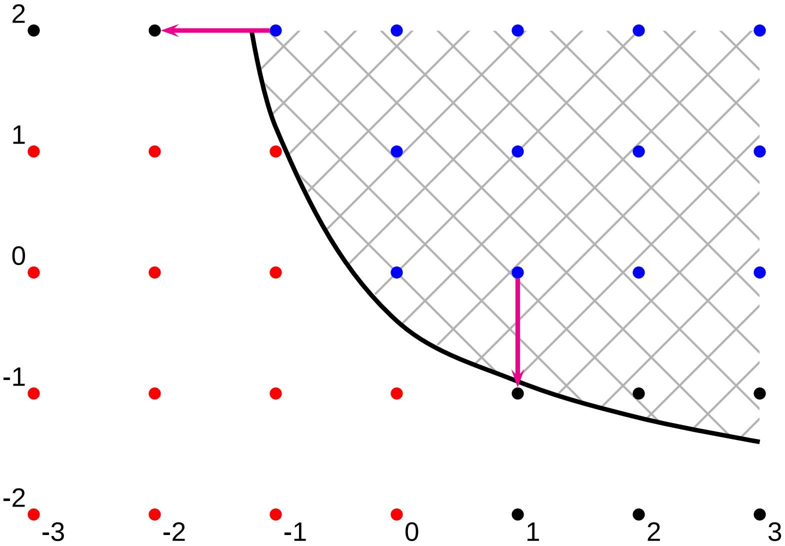

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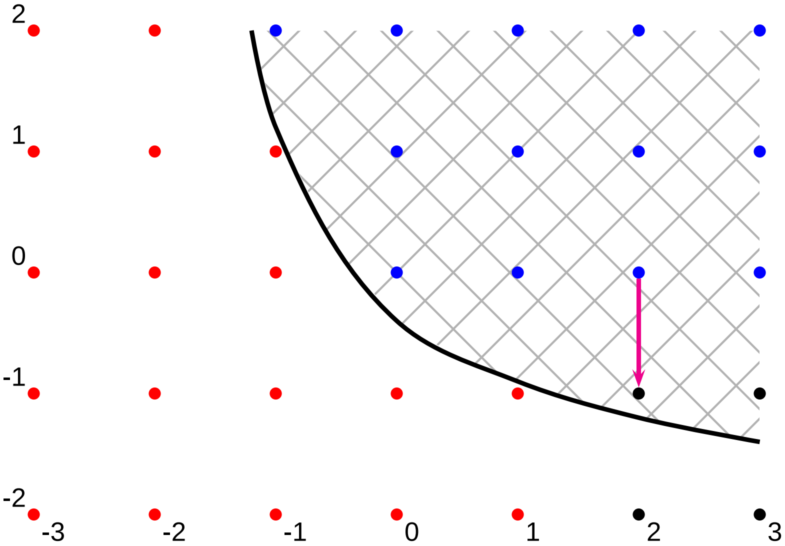

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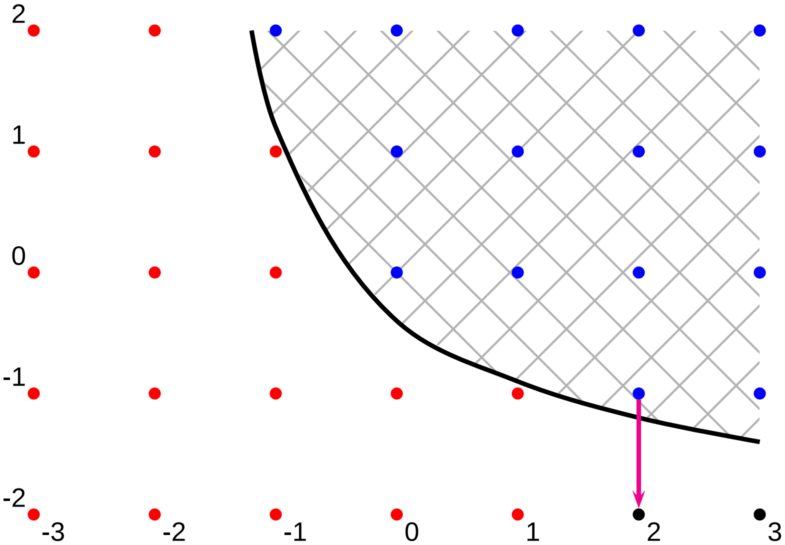

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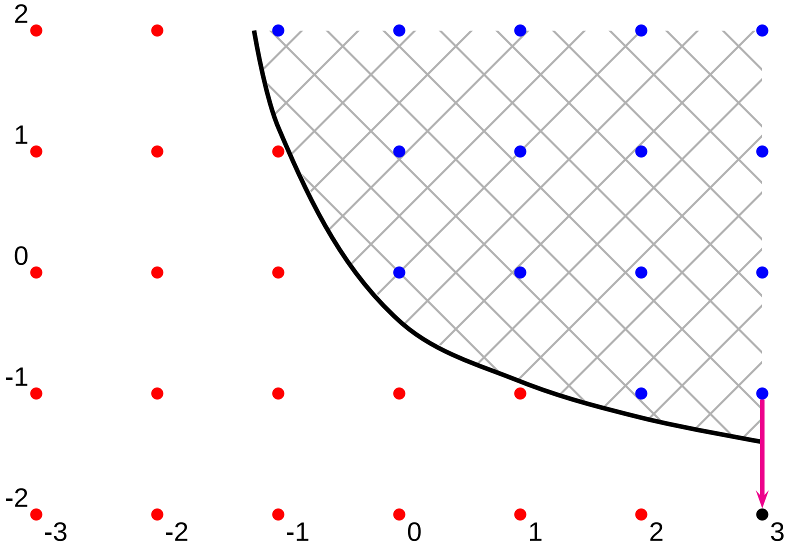

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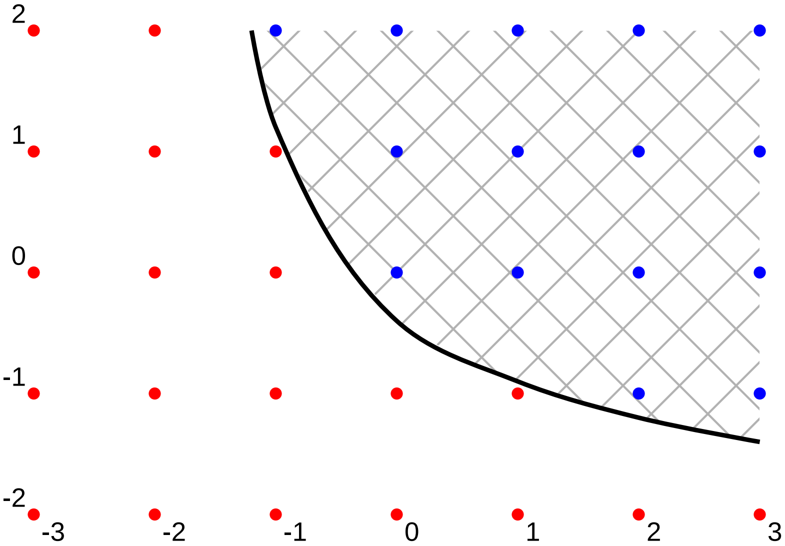

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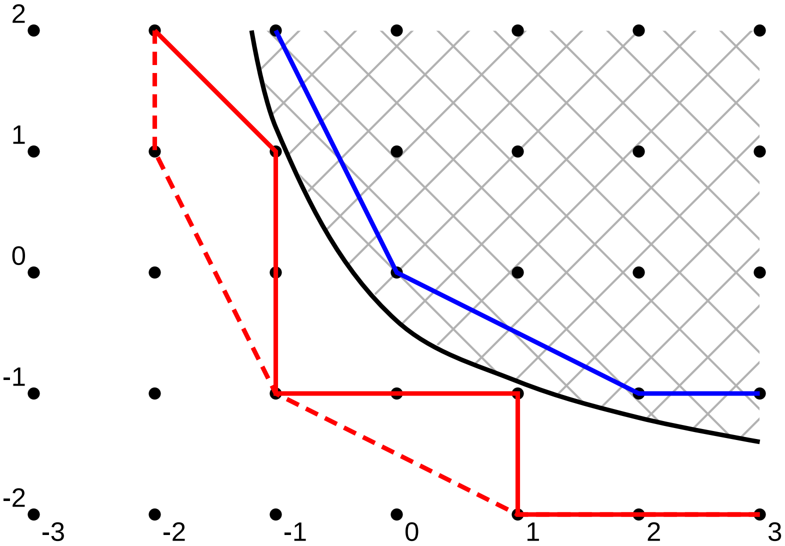

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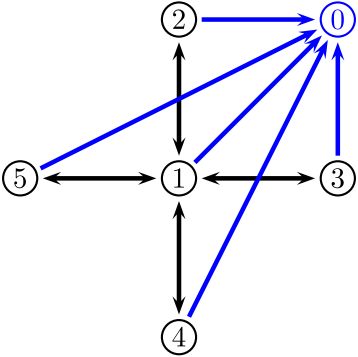


Figure: Centrally connected network

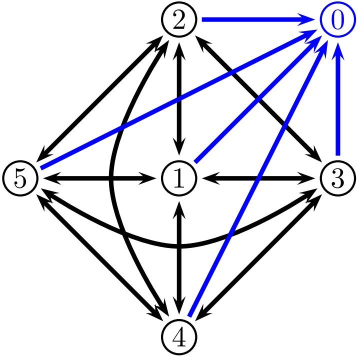


Figure: Completely connected network

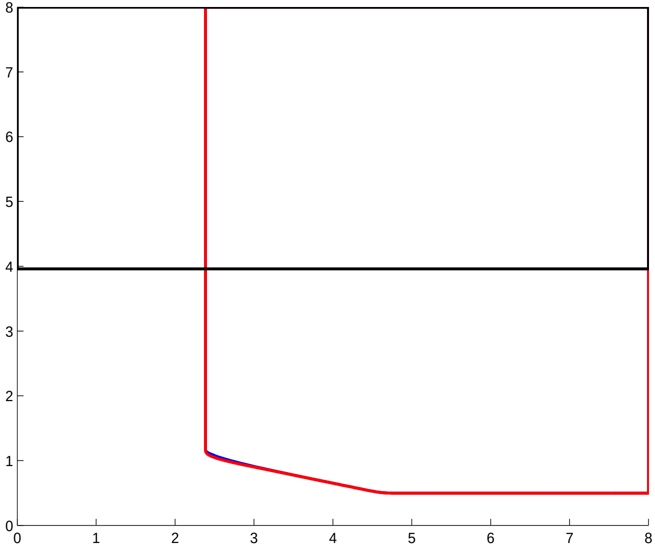


Figure: Centrally connected (×2) vs. completely connected

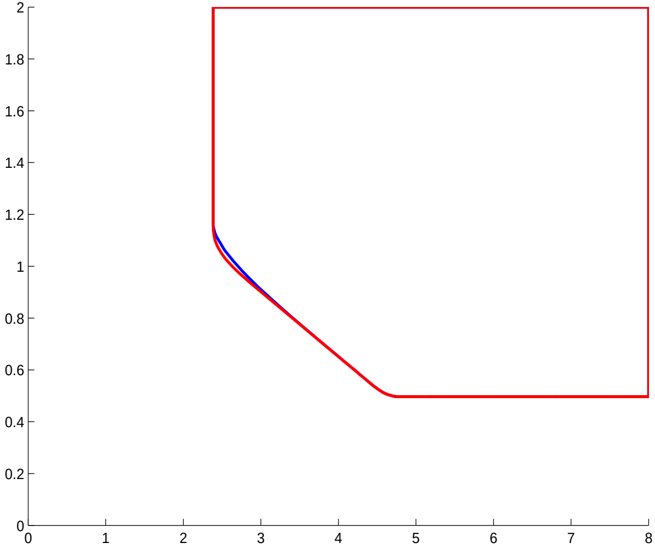
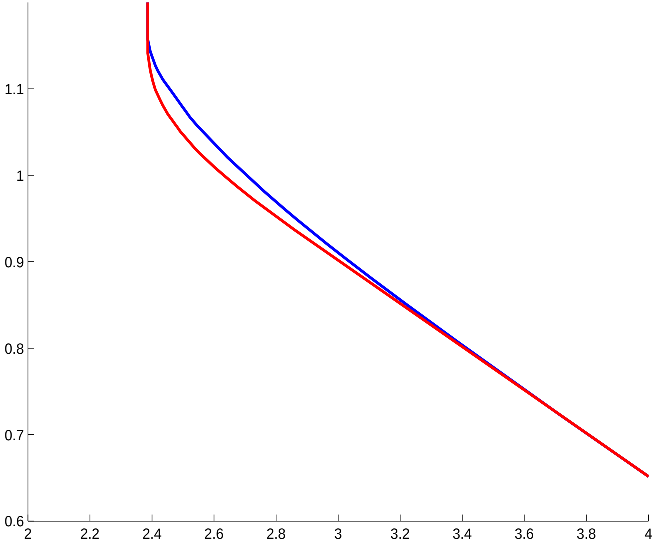


Figure: Centrally connected vs. completely connected



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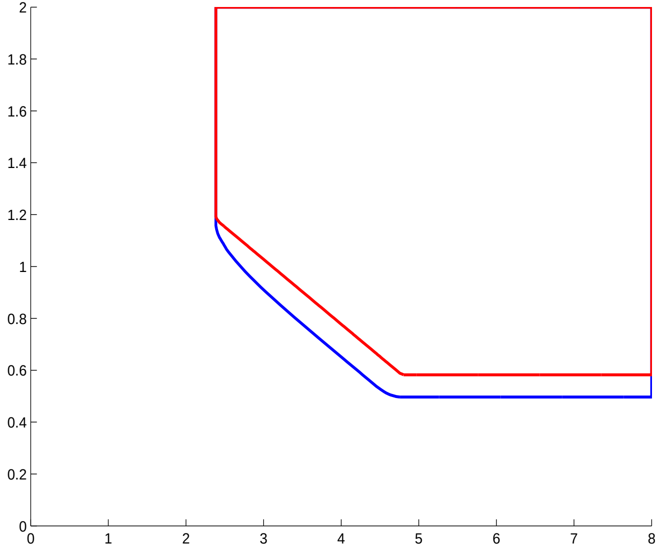

Figure: Centrally connected: independent vs. comonotonic

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- Capital requirements for each firm $k_i \geq k_i^*$, i.e. capital requirements are $k^* + \mathbb{R}^n_+$

Orthant risk measure

$\underline{\mathrm{k}}^{sys}_{\mathcal{A}} : L^0(\mathbb{R}^n) \to \mathbb{R}^n$ is an ***orthant risk measure*** associated with the systemic risk measure $R^{sys}_{\mathcal{A}}$ if for all $X, Y \in L^0(\mathbb{R}^n)$, $k \in \mathbb{R}^n$, and $\alpha \in [0,1]$

- *Minimal valued*: $\underline{\mathrm{k}}^{sys}_{\mathcal{A}}(X) \in \operatorname{Min} R^{sys}_{\mathcal{A}}(X)$
- *Translative*: $\underline{\mathrm{k}}^{sys}_{\mathcal{A}}(X + k) = \underline{\mathrm{k}}^{sys}_{\mathcal{A}}(X) - k$
- *Monotone*: $\underline{\mathrm{k}}^{sys}_{\mathcal{A}}(X) + \mathbb{R}^n_+ \ni \underline{\mathrm{k}}^{sys}_{\mathcal{A}}(Y)$ if $X \geq Y$ a.s.
- *Quasi-convex*: $\underline{\mathrm{k}}^{sys}_{\mathcal{A}}(\alpha X + (1-\alpha)Y) + \mathbb{R}^n_+ \ni \underline{\mathrm{k}}^{sys}_{\mathcal{A}}(X) \vee \underline{\mathrm{k}}^{sys}_{\mathcal{A}}(Y)$
- *Law-invariant*: $\underline{\mathrm{k}}^{sys}_{\mathcal{A}}(X) = \underline{\mathrm{k}}^{sys}_{\mathcal{A}}(Y)$ if $X \overset{d}{=} Y$

- Fix $w \in \operatorname{int}(\bigcap_Z \operatorname{recc}\left(R_{\mathcal{A}}^{sys}(Z)\right)^+)$ (typically $w \in \mathbb{R}^n_{++}$)

$$\underline{\mathrm{k}}_{\mathcal{A}}^{sys}(X) = \arg\min\left\{\sum_{i=1}^n w_i \cdot k_i \mid k \in R_{\mathcal{A}}^{sys}(X)\right\}$$

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5. Orthant risk measures

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  - $w_i = 1/\bar{p}_i$: minimize total capital weighted by obligation
  - $w_i = \max(1/AV@R_{0.5\%}(X_i + \sum_{j \neq i} \bar{p}_j \cdot a_{ji} - \bar{p}_i), \epsilon)$: minimize total capital weighted by individual risk (neglecting counterparty risk)

Thank you

Zachary Feinstein, Birgit Rudloff, and Stefan Weber.
Measures of systemic risk.
Preprint, 2014.