On capped product designs within variable annuities

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Variable annuities: unit-linked life insurance contracts

- **Basic ingredient**: buyer participates in the evolution of an investment in mutual funds
- In addition, the buyer can *choose à la carte*
  - Which protection feature he wants: the guarantee
  - Which protection style he wants: the guarantee types
  - Whether he wants to buy additional options (riders): fund switching rights, surrender rights, locally lock-in rights

▶ Recently, contracts with an additional cap on the investments at maturity of the contract became popular
Payoff variable annuity with guarantee and cap
Motivation

Complex interplay between options

The perspective of the insurance company

- The buyer’s choices influence the implicit charges
  - Thus, the pricing and risk management of the insurance company is affected

The perspective of the insured

- Additional riders/options influence the utility gained from the contract
  - But, the insured is also affected by the insurance companies pricing of the options
Research Question:

Why do we observe capped product designs?
- facilitates the risk management?
- brings benefits to the insured?

This talk: Mahayni and Schneider (2012b)

- Comparison of capped and uncapped product designs taking account of the rider to switch investment decisions (fund switching right)
- Equivalently: comparison of different manners of charging for the investment guarantee
- Implications of the contract design on the pricing of the insurance company and the utility of the insured
Payoffs and Pricing

**Payoff Profiles Simple Guarantee Contract**

\[ A_T^1 = \max\{P e^{gT}, \alpha P R_T\} \]

- \( R_T \): return on the investment at maturity
- \( P \): premium paid by the insured (normalized to one)
- \( G_T = P e^{gT} \): exogenous guarantee
- \( \alpha \): investment fraction
Payoffs and Pricing

Payoff Profiles Capped Contract

\[ A^2_T = \min \left\{ \max \left\{ P e^{gT}, PR_T \right\}, P e^{cT} \right\} \]

- \( R_T \): return on the investment at maturity
- \( P \): premium paid by the insured (normalized to one)
- \( G_T = P e^{gT} \): exogenous guarantee
- \( \alpha \): investment fraction
- \( C_T = P e^{cT} \): cap
Payoffs and Pricing

Payoff Profiles  General Contract

\[ A_{Gen}^T = \min \left\{ \max \left\{ P e^{gT}, \alpha P R_T \right\}, P e^{cT} \right\} \]

- \( R_T \): return on the investment at maturity
- \( P \): premium paid by the insured (normalized to one)
- \( G_T = P e^{gT} \): exogenous guarantee
- \( \alpha \): investment fraction
- \( C_T = P e^{cT} \): cap

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Without rider: Symmetric Information

- Commitment: Buyer of VA commits himself a priori to an investment strategy.
- Commitment strategy is a constant mix in two risky and one risk-free asset (borrowing and short-selling constrained)

\[
\frac{dV}{V_t} = \mu dt + \sigma_{\text{Inv}} dW_t
\]

- Insurance company prices contracts fairly, i.e., present value of benefits must coincide with present value of contributions such that guarantee costs are calculated fairly.
Without rider: Symmetric Information

A fair contract satisfies

\[ E_{P^*} \left[ e^{-rT} A^\text{Gen}_T(\sigma_{\text{Inv}}, g, \alpha, c) \right] = P = 1 \]

- \( \sigma_{\text{Inv}} \): the portfolio volatility of the investment the insured commits himself to at contract initiation
- \( g \): exogenously given guarantee rate
- \( \alpha \): investment fraction
- \( c \): cap rate
Investment premium

Investment fraction without rider

Fair investment fraction

Parameter values: \( r = 0.039; \sigma_1 = 0.30; \sigma_2 = 0.15; g = 0.00; T = 10; \rho = \pm 0.26; \)

\[
\sigma_{\text{Inv}} = \sqrt{\pi_1^2 \sigma_1^2 + \pi_2^2 \sigma_2^2 + 2\pi_1 \pi_2 \sigma_1 \sigma_2 \rho}
\]
With rider: Asymmetric Information

- Insurer faces additional risk through uncertainty about insured’s risk tolerance and future decisions.
- Decisions typically depend on non-contractible information.
- Insurance company prices contracts to be on the safe side.
- We interpret the information asymmetry along the lines of an uncertain volatility model.
- The provider knows that $\sigma_{t,\text{Inv}} \in [\sigma_{\text{min}}, \sigma_{\text{max}}]$ for all $t \in [0, T]$.

$$dV_t^{\text{UVM}} = V_t^{\text{UVM}} \left( \mu_t^{\text{UVM}} dt + \sigma_t^{\text{UVM}} dW_t \right) \text{ where } V_0^{\text{UVM}} = \alpha^{\text{Switch}}.$$
With rider: Asymmetric information

- The arbitrage-free price of $A_T$ is not defined uniquely.
- We consider the superhedging strategy which allows the provider to be on the safe side.

\[ A_{t}^{\text{Gen, Switch}} = \nu(t, V_{t}; A^{\text{Gen}}). \]

- $\nu$ denotes the lowest upper price bound at time $t$.
- $\nu$ is the solution of a Black-Scholes-Barenblatt equation.
- Generally, $\nu$ cannot be obtained in closed-form.
  - For the simple guarantee contract the above reduces to a Black-Scholes put price on $\sigma_{\text{max}}$.
  - For the cap we rely on the results of Vanden (2006).
Investment fraction with rider

Parameter values: \( r = 0.039; \sigma_1 = 0.30; \sigma_2 = 0.15; g = 0.00; T = 10; \rho = \pm 0.26; \)

\[
\sigma_{\text{Inv}} = \sqrt{\pi_1^2 \sigma_1^2 + \pi_2^2 \sigma_2^2 + 2\pi_1 \pi_2 \sigma_1 \sigma_2 \rho}
\]
Sunk cost ratio

\[
\frac{\alpha^* - \alpha_{\text{Switch}}}{\alpha_{\text{Switch}}}
\]

Parameter values: \( r = 0.039; \sigma_1 = 0.30; \sigma_2 = 0.15; g = 0.00; T = 10; \rho = \pm 0.26; \)

\[
\sigma_{\text{Inv}} = \sqrt{\pi_1^2 \sigma_1^2 + \pi_2^2 \sigma_2^2 + 2\pi_1 \pi_2 \sigma_1 \sigma_2 \rho}
\]
Expected utility

Perspective of the insured

Review: Mahayni and Schneider (2012a)

Findings: Simple Guarantee Contract

- The rider to switch gives an incentive to invest more aggressively (higher volatility than without)
- True benefits of the rider to switch are revealed in the presence of background risk
- Benefits of flexibility offset losses in investment premium
Optimization problems for a given cap rate

Optimization problem of the CRRA-insured: $c$ fixed

**Symmetric information**

$$\max_{\pi \in \Pi} \mathbb{E} \left[ u(A^\text{Gen}_T(\alpha, c)) \right] \text{ s.t. } \alpha = \alpha^{*, \text{Gen}}(c, \sigma_{\text{Inv}})$$

$$\sigma_{\text{Inv}} = \sqrt{\pi_1^2 \sigma_1^2 + \pi_2^2 \sigma_2^2 + 2 \rho \pi_1 \pi_2 \sigma_1 \sigma_2}.$$

**Asymmetric information**

$$\max_{\pi \in \Pi} \mathbb{E} \left[ u(A^\text{Gen}_T(\alpha, c)) \right] \text{ s.t. } \alpha = \alpha^{*, \text{Gen,Switch}}(c).$$

Expected utility can be represented in closed-form but optimization needs numerical method
Distortion effects of stylized contracts

Distortions on the volatility

\[ v - \text{ratio} = \frac{\sigma_{*, \text{Switch}}(c)}{\sigma^*(c)} \]

Parameter values: \( r = 0.039; \sigma_1 = 0.29; \mu_1 = 0.08; \sigma_2 = 0.15; \mu_2 = 0.10; g = 0.00; T = 10; \rho = -0.26; \)

\[ \sigma_{\text{Inv}} = \sqrt{\pi_1^2 \sigma_1^2 + \pi_2^2 \sigma_2^2 + 2 \rho \pi_1 \pi_2 \sigma_1 \sigma_2}, \gamma = 2 \]

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Distortions on the volatility

\[ v - ratio = \frac{\sigma^{*, \text{Switch}} (c)}{\sigma^* (c)} \]

Certainty Equivalent:

\[ CE_T = \left( (1 - \gamma)E_P \left[ u(A^{\text{Gen}}_T) \right] \right)^{\frac{1}{1 - \gamma}} \]

<table>
<thead>
<tr>
<th>( A^{*, \text{Gen}} (c) )</th>
<th>( A^{*, \text{Gen}, \text{Switch}} (c) )</th>
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<tbody>
<tr>
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<td>g=0.00</td>
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<td>c=10%</td>
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<td>( \mu_{\text{Inv}} ) p.a.</td>
<td>8.66%</td>
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<td>( \sigma_{\text{Inv}} ) p.a.</td>
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<td>( \mu_{\text{Inv}} ) p.a.</td>
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<td>( \sigma_{\text{Inv}} ) p.a.</td>
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<tr>
<td>CE</td>
<td>2.07</td>
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Optimization problem of the CRRA-insured: c endogenous

Symmetric information

- Optimizing over $c$ yields that the optimal contract is a simple guarantee contract ▶ El Karoui et al. (2005)
- The optimal commitment strategy is the Merton strategy, i.e., a constant mix strategy
- Additional cap leads to loss in utility

Asymmetric information

- No closed-form solution possible
- Numerically tractable relying on closed-form representation of expected utility
Expected utility
Endogenous cap rate

Predictions on distortion effects

Theoretical predictions

- Symmetric information

\[
\max_c \max_{\pi \in \Pi} E \left[ u(A_{T}^{\text{Gen}}(\alpha, c)) \right] = \max_{\pi \in \Pi} E \left[ u(A_{T}^{\text{Gen}}(\alpha, \infty)) \right] = \max_{\pi \in \Pi} E \left[ u(A_{T}^{(1)}(\alpha)) \right].
\]

- Asymmetric information

\[
\max_c \max_{\pi \in \Pi} E \left[ u(A_{T}^{\text{Gen,Switch}}(\alpha, c)) \right] > \max_{\pi \in \Pi} E \left[ u(A_{T}^{\text{Gen,Switch}}(\alpha, \infty)) \right] = \max_{\pi \in \Pi} E_P \left[ u(A_{T}^{(1),\text{Switch}}(\alpha)) \right].
\]
Loss Rate

$L_T^{\text{Switch}} = \frac{1}{T} \ln \left( \frac{CE^{*, \text{Gen, Switch}}}{CE^{*, \text{Gen}}} \right)$

$\Rightarrow L_T^{\text{Switch}} = L_T^{\text{Sunk}} + L_T^{\text{Dis}}$
### Utility loss of sunk costs and payoff distortions

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$C^*$</th>
<th>$C^{*,\text{Switch}}$</th>
<th>$\sigma_{\text{inv}}^*$</th>
<th>$\sigma_{\text{inv}}^{*,\text{Switch}}$</th>
<th>$\nu$</th>
<th>$L_{T\text{Switch}}$</th>
<th>$\frac{L_T^{\text{Sunk}}}{L_{T\text{Switch}}}$</th>
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</tbody>
</table>

Panel $g = 0.00$
Discussion and Conclusion

Capped contract has merits under asymmetric information due to the rider to switch
- Offsets unwanted distortions of the investment decision
- Without cap the investor conducts a riskier strategy due to the worst case pricing
- Cap gives an opportunity to mitigate the sunk costs

Implication beyond the rider to switch
- Seemingly suboptimal change in contract design can be beneficial to offset utility losses due to other sources

What about the risk management?